Math 176
Basic Geometry
Script 3

February 10, 2011

1 Absolute Geometry

Recall the first four postulates of Euclidean geometry from Script 1.

Postulates

Postulate 1. (It is possible) to draw a straight line from any point to any point, and this line is unique.

Postulate 2. (It is possible) to produce a finite straight line continuously in a straight line.

Postulate 3. (It is possible) to describe a circle with any center and distance.

Postulate 4. All right angles are equal.

Additional Axioms

Axiom 1 (SSS)**.** Two triangles are congruent if their corresponding sides all have the same length.

Axiom 2 (SAS)**.** Two triangles are congruent if two corresponding sides and the angle in between are the same.

Axiom 3 (ASA)**.** Two triangles are congruent if one corresponding side and the two angles surrounding it are the same.

Note that we have no parallel postulate. This set of axioms is often called Absolute Geometry. Later we will assume a fifth postulate to get Hyperbolic Geometry.

Definition 1.1. Given a line ℓ, a line *parallel* to ℓ is a line that does not intersect ℓ.

The Script

Please take a moment to remind yourself which of the exercises and theorems from Script 1 hold without the parallel postulate. Here are a few more results of absolute geometry.

Proposition 1.2. Given a triangle $\triangle ABC$ with one side extended along a straight line to D as shown, $\angle BCD$ is greater than $\angle A$ and greater than $\angle B$. (Hint: First bisect the side CB and use congruent triangles to show $\angle BCD$ is greater than $\angle B$.)

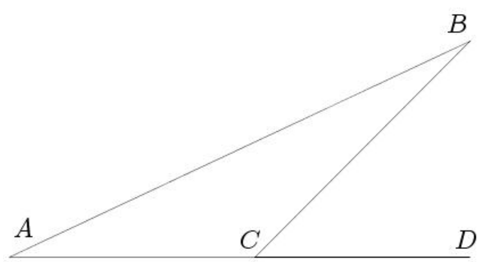

Figure 1: Diagram for Proposition 1.1.

Proposition 1.3. The sum of any two angles of a triangle must be less than π.

Exercise 1.4. Given a triangle $\triangle ABC$ construct a triangle $\triangle A'B'C'$ such that the angle $\angle A'$ is smaller than $\frac{1}{2}\angle A$ and both triangles have the same angle sum.

Proposition 1.5. The sum of the angles of a triangle cannot be greater than π. (Hint: use the construction from exercise 1.4 to contradict proposition 1.3.)

Remark. If the sum of the angles of any triangle is π, then the sum of the angles of all triangles is π.

Exercise 1.6. Assuming that all triangles have angle sum π, prove that it is possible to perform the following construction. Given an isosceles triangle $\triangle ABC$ with sides AB and BC equal, extend BC in the direction of C to a point D to create a new triangle $\triangle ABD$ such that angle $\angle BAD$ is equal to $3/2$ of angle $\angle BAC$.

Theorem 1.7. Assume for this theorem that all triangles have angle sum π. Given three lines ℓ, m and n such that m is perpendicular to ℓ and parallel to n, then n is perpendicular ℓ. (Hint: given any triangle with one side on m and another on ℓ, the third side lies below n.)

Playfair's Postulate says that there is a unique parallel to a given line through each point not on that line.

Corollary 1.8. If the sum of the angles of any triangle is π, then Playfair's postulate holds.

Playfair's postulate is equivalent to the parallel postulate from Script 1. Thus, Euclidean geometry is the only geometry satisfying Euclid's first four postulates where triangles have angle sum π.

Hyperbolic Geometry

> You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave this science of parallels alone. You should detest it just as much as lewd intercourse; it can deprive you of all your leisure, your health, your rest, and the whole happiness of your life. This abysmal darkness might perhaps devour a thousand towering Newtons.
>
> –Wolfgang Bolyai

We now introduce an alternative fifth postulate. For the remainder of this sheet we will be assuming Euclid's first four postulates as well as this new postulate. The geometry that results is called Hyperbolic Geometry.

Postulate 5. The sum of the angles of any triangle is less than π.

Lemma 1.9. Given two lines and a transverse, if the opposite interior angles of the transverse are equal then the lines must be parallel.

Theorem 1.10. Given a line ℓ and a point P not on ℓ, there are an infinite number lines parallel to ℓ through P.

Lemma 1.11. The sum of the angles of any quadrilateral is less than 2π.

Proposition 1.12. Given a line ℓ and a distinct line m that shares a perpendicular with ℓ (i.e. some line perpendicular to ℓ is also perpendicular to m), then no other perpendicular of ℓ is perpendicular to m.

Theorem 1.13. Triangles are congruent if their corresponding angles are the same.

Definition 1.14 (Distance)**.** The distance $d(P, Q)$ between two points P and Q is the length of the unique line segment connecting them. The distance between two lines ℓ and m is

$$d(\ell, m) = \inf\{d(P, Q) \mid P \text{ on } \ell \text{ and } Q \text{ on } m\}$$

Proposition 1.15. Let m and ℓ be a pair of parallel lines, and let n be a transversal intersecting m at a point P and ℓ at a point Q. Prove that if the opposite interior angles of the transversal n are equal, then any line passing through the midpoint of the segment $\overline{PQ}$ satsifies one of the two following conditions:

1. is parallel to both m and ℓ.

2. is a transversal of m and ℓ with equal opposite interior angles.

Corollary 1.16. If ℓ and m are parallel and n is a transversal as in Proposition 1.15, then ℓ and m have a common perpendicular.

Proposition 1.17. Given a quadrilateral whose base angles are right and whose opposite sides extending from those base angles are of equal length, the 'top' angles of the quadrilateral are congruent and their sum is less than π. This is a called a *Saccheri Quadrilateral*.

Proposition 1.18. Given a Saccheri quadrilateral, the line formed by constructing a perpendicular bisector to the base (AB in figure 2) meets the top (CD in figure 2) of the quadrilateral at a right angle.

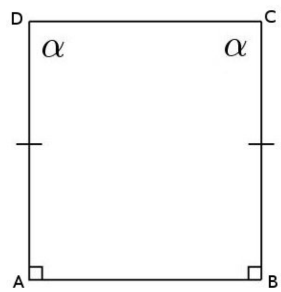

Figure 2: A Saccheri Quadrilateral with 'top' angle α.

Proposition 1.19. Given a quadrilateral whose base angles are right, if one top angle is greater than the other, then the side whose top angle is smaller has greater length.

Lemma 1.20. Let P be a point not on ℓ. Then the distance from P to ℓ is realized by the perpendicular from P to ℓ.

Corollary 1.21. If two parallel lines share a perpendicular, then the length of that perpendicular is the distance between two lines.

Theorem 1.22. Let ℓ be a line, and P a point not on ℓ. Let m be a perpendicular to ℓ through P, and n a perpendicular to m through P. There exists an angle α, as shown, such that if a line q makes an angle of α with n at P then the following are true:

1. q is parallel to ℓ
2. any line through P that makes an angle less than α with n is parallel to ℓ
3. any line through P that makes an angle greater than α with n intersects ℓ

Remark. Call the lines from Theorem 1.22 the left and right *boundary parallels*. Note that these lines is unique in each direction through a given point.

Exercise 1.23. Given a line ℓ and a point P not on ℓ, the perpendicular from P to ℓ bisects the angle between the left boundary parallel and the right boundary parallel.

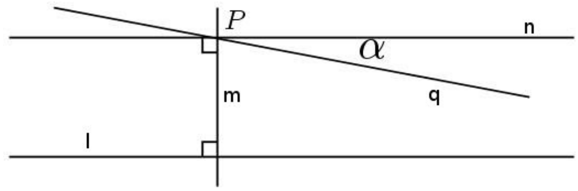

Figure 3: Setup for Theorem 1.22.

Proposition 1.24. Let ℓ be a line and P be a point not on ℓ. Also let m be the right (respectively the left) boundary parallel to ℓ through P and Q be any point on m. Then m is also the right (respectively the left) boundary parallel to ℓ through Q.

Remark. Thus we can say that a line m is a right (resp. left) boundary parallel to another line ℓ without specifying a point on m.

Proposition 1.25. If a line ℓ is a boundary parallel to another line m, then m is a boundary parallel to ℓ.

(Hint: Pick a point Q on m and draw points P and R on ℓ such that PQ is perpendicular to m and RQ is perpendicular to ℓ. Then use similar triangles to show that any line through Q that lies between ℓ and m in the direction where these lines get close intersects ℓ.)
Note: if you get stuck then move on.

Proposition 1.26. If ℓ is a boundary parallel to a line m and m is a boundary parallel to a line n on the same side, then ℓ is a boundary parallel to n also on the same side.

Lemma 1.27. If a line ℓ is the right boundary parallel to m through a point P, then the distance from any point P' on ℓ to the right of P to m is less than the distance from P to m.

Lemma 1.28. Let ℓ, m, and P be as above, and let Q be the intersection of the perpendicular through P with m. Let R be a point on PQ and let n be the left boundary parallel to m through R. Then n intersects ℓ to the right of P.

Theorem 1.29. If a line ℓ is right (resp. left) boundary parallel to a line m, then ℓ gets arbitraily close to m to the right (resp. left) and arbitraily far from m to the left (resp. right).

Corollary 1.30. Given a line ℓ and a point P not on the line, there exists two unique lines m_1 and m_2 parallel to ℓ through P such that $d(m_1, \ell) = d(m_2, \ell) = 0$. Also m_1 gets arbitrarily close to ℓ to the right of P, and m_2 gets arbitrarily close to ℓ to the left of P.