

## MATH 327: FIRST PROBLEM SET

Due Monday, April 4

1. (Monic and epic morphisms)
  - (i) Prove that the inclusion  $\mathbb{Z} \subset \mathbb{Q}$  is epi in the category of rings.
  - (ii) Show that kernels are monic and are unique up to isomorphism.
  - (iii) In Groups, show that monics are injections and kernels are monics with normal image. What are epis and cokernels?
2. (Colimits and van Kampen)
  - (i) What is the coproduct of groups usually called? Describe it informally.
  - (ii) Let  $H$  be a subgroup of groups  $G$  and  $G'$ . What is the pushout  $G \cup_H G'$  usually called? Describe it informally.
  - (iii) When is the coproduct of groups  $G$  and  $H$  finite?
  - (iv) Give a conceptual statement of van Kampen's theorem in terms of colimits.
3. Let  $R$  and  $S$  be commutative rings. Show that the coproduct of  $R$  and  $S$  in the category of commutative rings is  $R \otimes_{\mathbb{Z}} S$ . The same holds more generally for commutative algebras over a commutative ring. Is the same true for non-commutative rings and algebras?
4. "Left adjoints preserve colimits". Let  $L: \mathcal{C} \rightarrow \mathcal{C}'$  be a left adjoint. Let  $F: \mathcal{D} \rightarrow \mathcal{C}$  be a functor such that  $\text{colim } F$  exists in  $\mathcal{C}$ . Prove that  $L(\text{colim } F)$  is a colimit of the functor  $L \circ F: \mathcal{D} \rightarrow \mathcal{C}'$ . Deduce dually that right adjoints preserve limits.
5. Prove that if  $\mathcal{C}$  has coproducts and coequalizers, then  $\mathcal{C}$  is cocomplete. (Hint: any colimit is a coequalizer of a pair of maps between certain coproducts.) Deduce dually that if  $\mathcal{C}$  has products and equalizers, then  $\mathcal{C}$  is complete.
6. Boolean theory: A ring  $R$  is Boolean if  $x^2 = x$  for all  $x \in R$ . A lattice  $L$  is Boolean if the following three conditions hold.
  - (i)  $L$  has a least element 0 and a greatest element 1.
  - (ii)  $\wedge$  and  $\vee$  each distribute over the other.
  - (iii) Each  $y \in L$  has a complement  $z$  such that  $y \vee z = 1$  and  $y \wedge z = 0$ .

Without giving complete boring details, explain why the categories of Boolean rings (and ring maps) and Boolean lattices (and lattice maps) are isomorphic. [See Atiyah-MacDonald, p.11 #11 and p.14 #24.]

Define the orbit category  $\mathcal{O}_G$  of a group  $G$  to be the category whose objects are the left  $G$ -sets  $G/H$  and whose morphisms are the  $G$ -maps. Observe that the morphisms  $\phi: G/H \rightarrow G/K$  are given by  $\phi(gH) = g\gamma K$  for some element  $\gamma \in G$  such that  $\gamma^{-1}H\gamma \subset K$  (subconjugacy relation).

7. The fundamental theorem of Galois theory: Let  $G = \text{Gal}(E/F)$  be the Galois group of a finite Galois extension  $E/F$ . Let  $\mathcal{E}/F$  be the category whose objects are the fields  $F \subset K \subset E$  and whose morphisms are the field maps  $K \rightarrow L$  that fix  $F$  pointwise. Describe an *isomorphism* of categories between  $\mathcal{O}_G^{\text{op}}$  and  $\mathcal{E}/F$ .

8. Covering space theory: Requiring covering spaces of a (well-behaved) connected topological space  $B$  to be connected, let  $\mathcal{C}ov(B)$  be the category of covering spaces of  $B$  and maps over  $B$ . If  $G$  is the fundamental group of  $B$ , then  $\mathcal{O}_G$  is *equivalent*, not *isomorphic*, to  $\mathcal{C}ov(B)$ . Sketch the proof. (Hint: use a universal cover of  $B$  to construct a skeleton of the category  $\mathcal{C}ov(B)$ .)