

MATH 327: SECOND PROBLEM SET

Due Monday, April 13

A graded vector space $V = \{V_n\}$ is *finite dimensional* if $V_n = 0$ for all but finitely many n and all V_n are finite dimensional. Define the *Euler characteristic* $\chi(V)$ of such a V to be $\sum (-1)^n \dim V_n$.

1 (a) Let $\cdots \rightarrow V'_n \rightarrow V_n \rightarrow V''_n \rightarrow V'_{n-1} \rightarrow \cdots$ be a long exact sequence of finite dimensional graded vector spaces. Prove that $\chi(V) = \chi(V') + \chi(V'')$.

(b) If $\{V_n, d_n\}$ is a finite dimensional chain complex, show that $\chi(V) = \chi(H_*(V))$.

2. Let $0 \rightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \rightarrow 0$ be an exact sequence of Abelian groups and let C be a chain complex of flat (= torsion free) Abelian groups. Write $H_*(C; A) = H_*(C \otimes A)$. Construct a natural long exact sequence

$$\cdots \rightarrow H_q(C; A') \xrightarrow{f_*} H_q(C; A) \xrightarrow{g_*} H_q(C; A'') \xrightarrow{\beta} H_{q-1}(C; A') \rightarrow \cdots$$

The connecting homomorphism β is called a *Bockstein operation*.

3 Prove the following:

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\mathrm{gcd}(m, n)\mathbb{Z}.$$

$$\mathrm{Ext}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\mathrm{gcd}(m, n)\mathbb{Z}.$$

4. Let M be an abelian group. Prove that the torsion subgroup of M can be identified with $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

In the following, R is a ring and modules are left R -modules.

5. Sketch a proof that $\mathrm{Ext}_R^1(M, N)$ is isomorphic to the Abelian group of equivalence classes of extensions (N and M fixed) $0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$.

6. Show that an R -module Q is injective if and only if for each ideal I and R -map $f: I \rightarrow Q$, there is an R -map $\tilde{f}: R \rightarrow Q$ that extends f . Note that \tilde{f} must be given by $\tilde{f}(1) = q$ where $f(r) = rq$ for $r \in I$. (Hint: set up an application of Zorn's lemma).

7. A module M over an integral domain R is *divisible* if for each m in M and $r \neq 0$ in R , there exists n in M such that $m = rn$. Show that an Abelian group M is divisible iff M is an injective \mathbb{Z} -module. (Hint: use the previous problem.)

8. Prove that the following conditions on R are equivalent.

- (i) R is semi-simple as an R -module.
- (ii) Every ideal is a direct summand of R .
- (iii) Every ideal is an injective module.
- (iv) All short exact sequences of R -modules split.
- (v) All R -modules are projective.
- (vi) All R -modules are injective.
- (vii) All R -modules are semi-simple.

Hint: Presumably (i) \Leftrightarrow (ii) was done in the first quarter. The key point is (ii) implies (vii), for which use the previous problem. Note: we are using left R -modules. It is equivalent to use right R -modules for this result. Why? (Observe that this result characterizes rings R such that $gl.dim R = 0$).

8. Let $n = qr$ where q and r are greater than 1. Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Show the following.

(i) There is a short exact sequence

$$0 \longrightarrow q\mathbb{Z}_n \longrightarrow \mathbb{Z}_n \longrightarrow r\mathbb{Z}_n \longrightarrow 0.$$

(ii) The sequence splits iff $(q, r) = 1$.

(iii) $r\mathbb{Z}_n$ is \mathbb{Z}_n -projective iff the sequence splits.

Observe that this gives examples of projective modules which are not free.

(iv) There is a long exact sequence of \mathbb{Z}_n -modules

$$\cdots \longrightarrow \mathbb{Z}_n \longrightarrow \mathbb{Z}_n \longrightarrow \mathbb{Z}_n \longrightarrow r\mathbb{Z}_n \longrightarrow 0$$

(v) The projective dimension of $r\mathbb{Z}_n$ is 0 if $(q, r) = 1$ and ∞ if not.

(vi) \mathbb{Z}_n is semi-simple iff n is a product of distinct primes.

(vii) Either \mathbb{Z}_n is semi-simple or $gl.dim \mathbb{Z}_n$ is ∞ .