A graded vector space $V = \{ V_n \}$ is finite dimensional if $V_n = 0$ for all but finitely many $n$ and all $V_n$ are finite dimensional. Define the Euler characteristic $\chi(V)$ of such a $V$ to be $\sum (-1)^n \dim V_n$.

1. (a) Let $\cdots \longrightarrow V'_n \longrightarrow V_n \longrightarrow V''_n \longrightarrow \cdots$ be a long exact sequence of finite dimensional graded vector spaces. Prove that $\chi(V) = \chi(V') + \chi(V'')$.

(b) If $\{ V_n, d_n \}$ is a finite dimensional chain complex, show that $\chi(V) = \chi(H_*(V))$.

2. Let $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ be an exact sequence of Abelian groups and let $C$ be a chain complex of flat (= torsion free) Abelian groups. Write $H_*(C; A) = H_*(C \otimes A)$. Construct a natural long exact sequence

$$\cdots \longrightarrow H_q(C; A') \xrightarrow{f_*} H_q(C; A) \xrightarrow{g_*} H_q(C; A'') \xrightarrow{\beta} H_{q-1}(C; A') \longrightarrow \cdots.$$ 

The connecting homomorphism $\beta$ is called a Bockstein operation.

3. Prove the following:

$$\text{Tor}_1^R(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}.$$ 

$$\text{Ext}_1^R(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}.$$ 

4. Let $M$ be an abelian group. Prove that the torsion subgroup of $M$ can be identified with $\text{Tor}_1^\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$.

In the following, $R$ is a ring and modules are left $R$-modules.

5. Sketch a proof that $\text{Ext}_1^R(M, N)$ is isomorphic to the Abelian group of equivalence classes of extensions ($N$ and $M$ fixed) $0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$.

6. Show that an $R$-module $Q$ is injective if and only if for each ideal $I$ and $R$-map $f : I \longrightarrow Q$, there is an $R$-map $\tilde{f} : R \longrightarrow Q$ that extends $f$. Note that $\tilde{f}$ must be given by $\tilde{f}(1) = q$ where $f(r) = rq$ for $r \in I$. (Hint: set up an application of Zorn’s lemma.)

7. A module $M$ over an integral domain $R$ is divisible if for each $m$ in $M$ and $r \neq 0$ in $R$, there exists $n$ in $M$ such that $m = rn$. Show that an Abelian group $M$ is divisible iff $M$ is an injective $\mathbb{Z}$-module. (Hint: use the previous problem.)

8. Prove that the following conditions on $R$ are equivalent.

(i) $R$ is semi-simple as an $R$-module.
(ii) Every ideal is a direct summand of $R$.
(iii) Every ideal is an injective module.
(iv) All short exact sequences of $R$-modules split.
(v) All $R$-modules are projective.
(vi) All $R$-modules are injective.
(vii) All $R$-modules are semi-simple.
Hint: Presumably (i) $\iff$ (ii) was done in the first quarter. The key point is (ii) implies (vii), for which use the previous problem. Note: we are using left $R$-modules. It is equivalent to use right $R$-modules for this result. Why? Observe that this problem gives equivalent characterizations of rings $R$ such that $gl.dim R = 0$.

8. Let $n = qr$ where $q$ and $r$ are greater than 1. Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (which is convenient poor notation). Show the following.

(i) There is a short exact sequence

$$0 \rightarrow q\mathbb{Z}_n \rightarrow \mathbb{Z}_n \rightarrow r\mathbb{Z}_n \rightarrow 0.$$ 

(ii) The sequence splits iff $(q, r) = 1$.

(iii) $r\mathbb{Z}_n$ is $\mathbb{Z}_n$-projective iff the sequence splits.

Observe that this gives examples of projective modules which are not free.

(iv) There is a long exact sequence of $\mathbb{Z}_n$-modules

$$\cdots \rightarrow \mathbb{Z}_n \rightarrow \mathbb{Z}_n \rightarrow \mathbb{Z}_n \rightarrow r\mathbb{Z}_n \rightarrow 0$$

(v) The projective dimension of $r\mathbb{Z}_n$ is 0 if $(q, r) = 1$ and $\infty$ if not.

(vi) $\mathbb{Z}_n$ is semi-simple iff $n$ is a product of distinct primes.

(vii) Either $\mathbb{Z}_n$ is semi-simple or $gl.dim \mathbb{Z}_n$ is $\infty$. 