

MATH 327: THIRD PROBLEM SET

Due Monday, April 20

A ring is *left hereditary* if every left ideal is a projective module. PID's give the obvious examples.

1. Prove the following theorem of Kaplansky. If R is left hereditary, then every submodule of a free left R -module F is isomorphic to a direct sum of ideals. (Hint: give F a totally ordered basis S and consider the submodules F_β and \overline{F}_β generated by all basis elements $\alpha < \beta$ and all basis elements $\alpha \leq \beta$). Deduce that R is hereditary if and only if $l.gl.dim(R) \leq 1$, that is, every submodule of a projective module is projective or, equivalently by the two ways of calculating Ext, every quotient module of an injective module is injective.

2. Prove that a fractional ideal I of an integral domain R is invertible iff it is a projective R -module. (Hint: use the dual basis theorem). Then deduce that an integral domain is a Dedekind ring iff it is hereditary.

3. A commutative Noetherian ring R is said to be quasi-Frobenius if R is injective as an R -module. Prove that (a) and (b) below are equivalent and that they imply (c). Then try to prove that (c) implies (b). (This is true, but hard).

(a) R is quasi-Frobenius.

(b) Every projective R -module is injective.

(c) Every injective R -module is projective.

I think there should be an elementary proof that (c) \Rightarrow (a), but I don't know one. You are invited to try to find one. (Notes on this problem are on the web page).

4. If R is quasi-Frobenius, then either R is a product of fields or the global dimension of R is ∞ .

(It is actually unnatural to restrict to commutative rings in defining quasi-Frobenius. The dichotomy should be semi-simple versus infinite global dimension.)

5. If R is a Dedekind ring and I is a non-zero proper ideal, show that $S = R/I$ is quasi-Frobenius. (Hint: by the Chinese remainder theorem, reduce to the case when R is a DVR and $I = (t^s)$ for a UP t . The only ideals in S are of the form (t^r) , and it suffices to show that any S -map $f: (t^r) \rightarrow S$ extends over S .)

The following problems are worked in some standard texts in basic algebra. I've given them in the spring quarter of our honors level undergraduate algebraic course. If this is old hat to you, give it short shrift and help others. If it is new to you, then do it carefully. In any case, this is illustrative of basic algebraic number theory that every educated mathematician should see.

In (6) and (7), let R be the integral closure of \mathbb{Z} in the quadratic field $\mathbb{Q}[\sqrt{D}]$, where D is a square free integer.

6. Find a \mathbb{Z} -basis for R . [The answer depends on $D \pmod{4}$].

7. Let $p \in \mathbb{Z}$ be a prime and consider $(p) \subset R$. Show that $R/(p)$ has p^2 elements and that one of the following three possibilities describes the factorization of (p) as a product of prime ideals in R .

- (i) (p) is a prime ideal [p is “inert”].
- (ii) $(p) = P_1 P_2$ for distinct prime ideals P_1 and P_2 [(p) is “split”]
- (iii) $(p) = P^2$ for a prime ideal P [(p) is “ramified”].

In fact, p ramifies iff either p divides D or $p = 2$ and $D \equiv 3 \pmod{4}$.

In (8), (9), and (10), specialize to the case $D = -5$, so that $R = \mathbb{Z}[\sqrt{-5}]$.

8. Let $I = (2, 1 + \sqrt{-5})$, $J = (3, 2 + \sqrt{-5})$ and $J' = (3, 2 - \sqrt{-5})$. Show that none of these ideals is principal, but that

$$I^2 = (2), \quad IJ = (1 + \sqrt{-5}), \quad IJ' = (1 + \sqrt{-5}).$$

Conclude that $I^2 J J' = (6)$.

9. Let $P \subset R = \mathbb{Z}[\sqrt{-5}]$ be a prime ideal. Show that either $2 \notin P$ and $I_P = R_P$ or $P = I$ and $I_P = (1 + \sqrt{-5})R_P$. Conclude that I_P is isomorphic to R_P for all primes P , but that I is not isomorphic to R .

The point is that a map $M \rightarrow N$ of R -modules is an isomorphism iff it localizes to an isomorphism at all prime ideals P , but that a map must be given to obtain that conclusion.

10. Determine the prime ideal factorizations of $(p) \subset R$ when $p = 2, 3$, and 5 .