

## MATH 327: FOURTH PROBLEM SET

Due Monday, April 29

1. Let  $R = k[x, y, s, t]/(xs - yt)$  and  $S = R/(x, y) \cong k[s, t]$ . Let  $P = (s, t) \subset R$  and let  $Q$  be its image in  $S$ . Show that  $ht(P) = 1$  but  $ht(Q) = 2$ .

2. Let  $R = k[x, y, z]/(xz - z)$  and consider the set  $\{x, xy - y\}$ . Notice that the ideal  $(x, xy - y)$  in  $R$  is  $(x, y)$ . Show that  $\{x, xy - y\}$  is a regular sequence but  $\{xy - y, x\}$  is not a regular sequence: permutations of regular sequences need not be regular.

3. Let  $\{a, b\}$  be non-zero elements in an integral domain  $R$ .

- (a) Show that the ideal  $(ax - b)$  of  $R[x]$  is prime if and only if  $H_1(K(a, b)) = 0$  (which holds if  $\{a, b\}$  is a regular sequence).
- (b) When  $(ax - b)$  is prime, show that  $R[x]/(ax - b)$  is isomorphic to the subring  $R(a/b)$  of the quotient field of  $R$ .

Recall that an associated prime of an  $R$  module  $M$  is a prime ideal that is the annihilator of a nonzero element of  $M$ . An associated prime  $P$  of an ideal  $I$  is defined (confusingly!) to be an associated prime of the  $R$ -module  $R/I$ , so that there is an  $x \in R - I$  such that  $P = \{r | rx \in I\}$ .

All given rings  $R$  are local (and Noetherian) in the rest of the problems.

4. Let  $\{a_1, \dots, a_n\}$  be elements of the maximal ideal of  $R$ .

- (a) Show that  $\{a_1, \dots, a_n\}$  is a regular sequence if and only if  $a_i$  is not in any associated prime ideal of  $(a_1, \dots, a_{i-1})$  for  $1 \leq i \leq n$ .
- (b) Show that  $\{a_1, \dots, a_n\}$  is part of a system of parameters of  $R$  such that each  $(a_1, \dots, a_i)$  is of height  $i$  if and only if  $a_i$  is not in any minimal prime ideal of  $R/(a_1, \dots, a_{i-1})$  for  $1 \leq i \leq n$ .

5. Let  $M$  be a finitely generated  $R$ -module,  $P$  a prime ideal. Show that  $\text{depth}_P(M) \leq \text{depth}_{PR_P}(M_P)$  and construct an example to show that the inequality can be strict.

6. Let  $\dim(R) = 0$ . Show the following.

- (a)  $R$  is a Cohen Macaulay (CM) ring.
- (b)  $R$  is regular if and only if  $R$  is a field.

7 – 8. Let  $\dim(R) = 1$ .

- (a) If  $R$  has no nilpotent elements, show that  $R$  is CM.
- (b) Construct an  $R$  such that  $\dim(R) = 1$  but  $R$  is not CM.
- (c) If  $R$  is regular, then  $R$  is a DVR.
- (d) For a field  $k$ , show that the subring of the DVR  $k[[x]]$  generated by  $x^2$  and  $x^3$  is CM of dimension 1 but is not regular.

9. Let  $R = S/I$ , where  $S$  is a regular local ring and  $I$  is generated by a regular sequence in  $S$ . Show that any localization of  $R$  is also a quotient of a regular local ring by an ideal generated by a regular sequence.

A local ring  $R$  is called a “complete intersection” if its completion at its maximal ideal is the quotient of a regular local ring by an ideal generated by a regular sequence. The rings of problem 9 are examples.

10. Let  $\mathbf{a} = (a_1, \dots, a_n)$  be any sequence of elements in  $R$ , let  $S = R[x_1, \dots, x_n]$  and let  $f: S \rightarrow R$  be the ring homomorphism that sends  $x_i$  to  $a_i$ . Let  $M$  be an  $R$ -module and regard  $M$  as an  $S$ -module by pullback along  $f$ ,  $sm = f(s)m$ . Regard  $R$  as the quotient  $S$ -module  $S/(x_1, \dots, x_n)$ . Prove that

$$H_*(K(\mathbf{a}) \otimes_R M) \cong \mathrm{Tor}_*^S(R, M).$$