MATH 327: FOURTH PROBLEM SET

Due Monday, April 25

1. Let \( R = k[x,y,s,t]/(xs - yt) \) and \( S = R/(x,y) \cong k[s,t] \). Let \( P = (s,t) \subset R \) and let \( Q \) be its image in \( S \). Show that \( ht(P) = 1 \) but \( ht(Q) = 2 \).

2. Let \( R = k[x,y,z]/(xz - z) \) and consider the set \( \{ x, xy - y \} \). Notice that the ideal \( (x, xy - y) \) in \( R \) is \( (x, y) \). Show that \( \{ x, xy - y \} \) is a regular sequence but \( \{ xy - y, x \} \) is not a regular sequence: permutations of regular sequences need not be regular.

3. Let \( \{ a, b \} \) be non-zero elements in an integral domain \( R \).
   (a) Show that the ideal \( (ax - b) \) of \( R[x] \) is prime if and only if \( H_1(K(a,b)) = 0 \) (which holds if \( \{ a, b \} \) is a regular sequence).
   (b) When \( (ax - b) \) is prime, show that \( R[x]/(ax - b) \) is isomorphic to the subring \( R(b/a) \) of the quotient field of \( R \).

Recall that an associated prime of an \( R \) module \( M \) is a prime ideal that is the annihilator of a nonzero element of \( M \). An associated prime \( P \) of an ideal \( I \) is defined (confusingly!) to be an associated prime of the \( R \)-module \( R/I \), so that there is an \( x \in R - I \) such that \( P = \{ r | rx \in I \} \). Notes on these notions are posted.

All given rings \( R \) are local (and Noetherian) in the rest of the problems.

4. Let \( \{ a_1, \ldots, a_n \} \) be elements of the maximal ideal of \( R \).
   (a) Show that \( \{ a_1, \ldots, a_n \} \) is a regular sequence if and only if \( a_i \) is not in any associated prime ideal of \( (a_1, \ldots, a_{i-1}) \) for \( 1 \leq i \leq n \).
   (b) Show that \( \{ a_1, \ldots, a_n \} \) is part of a system of parameters of \( R \) such that each \( (a_1, \ldots, a_i) \) is of height \( i \) if and only if \( a_i \) is not in any minimal prime ideal of \( R/(a_1, \ldots, a_{i-1}) \) for \( 1 \leq i \leq n \).

5. Let \( M \) be a finitely generated \( R \)-module, \( P \) a prime ideal.
   (a) Show that \( \text{depth}_P(M) \leq \text{depth}_{PR_P}(M_P) \).
   (b) Show that the inequality can be strict. Hint: Consider \( R = k[x,y,z], M = (xy, y^2, yz), \) and \( P = (x,y) \). Note that this is not an esoteric example.

6. Let \( \dim(R) = 0 \). Show the following.
   (a) \( R \) is a Cohen Macaulay (CM) ring.
   (b) \( R \) is regular if and only if \( R \) is a field.
7 and 8. Let $\dim(R) = 1$.

(a) If $R$ has no nilpotent elements, show that $R$ is CM.
(b) Construct an $R$ such that $\dim(R) = 1$ but $R$ is not CM.
(c) If $R$ is regular, then $R$ is a DVR.
(d) For a field $k$, show that the subring of the DVR $k[[x]]$ generated by $x^2$ and $x^3$ is CM of dimension 1 but is not regular.

9. Let $R = S/I$, where $S$ is a regular local ring and $I$ is generated by a regular sequence in $S$. Show that any localization of $R$ is also a quotient of a regular local ring by an ideal generated by a regular sequence.

A local ring $R$ is called a “complete intersection” if its completion at its maximal ideal is the quotient of a regular local ring by an ideal generated by a regular sequence. The rings of problem 9 are examples.

10. Let $a = (a_1, \ldots, a_n)$ be any sequence of elements in $R$, let $S = R[x_1, \ldots, x_n]$ and let $f: S \rightarrow R$ be the ring homomorphism that sends $x_i$ to $a_i$. Let $M$ be an $R$-module and regard $M$ as an $S$-module by pullback along $f$, $sm = f(s)m$. Regard $R$ as the quotient $S/(x_1, \ldots, x_n)$. Prove that

$$H_*(K(a) \otimes_R M) \cong \text{Tor}_*^S(R, M).$$