

MATH 327: SIXTH PROBLEM SET

Due Monday, May 9

Recommended reading: Weibel, pages 160–193 (or even 160–206)

BUT: please do not just copy from Weibel and write it down. Think things through.

In the following problems, G is a group, M is a G -module (= $\mathbb{Z}[G]$ -module), and I is the augmentation ideal $\text{Ker}(\varepsilon: \mathbb{Z}[G] \rightarrow \mathbb{Z})$.

1. Show that $H_0(G; M) \cong M_G \equiv M/IM$ and $H^0(G; M) \cong M^G \equiv \{m \mid Im = 0\}$.
2. Show that $H_1(G; \mathbb{Z}) \cong G/[G, G]$ and relate that to the Hurewicz homomorphism for the space $K(G, 1)$.
3. Define a crossed homomorphism $C: G \rightarrow M$ to be a function such that $C(gh) = gC(h) + C(g)$. For $m \in M$, define a crossed homomorphism C_m by $C_m(g) = gm - m$; C_m is said to be principal. Let $Z(G; M)$ be the Abelian group of crossed homomorphisms under pointwise addition and let $B(G; M)$ be the subgroup of principal crossed homomorphisms. Show that $H^1(G; M) \cong Z(G; M)/B(G; M)$. Show that if G acts trivially on M , then this is the Abelian group of homomorphisms of groups $G \rightarrow M$.

Remark: Let K be a Galois extension of a field F with Galois group G , and let K^\times be the group of units (non-zero elements) of K ; note that K^\times is a G -module. A result known as Hilbert's Theorem 90 says that $H^1(G; K^\times) = 0$. see Weibel, 6.4.7.

4. Let $E = M \rtimes G$, so that we have an extension of G -modules

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1.$$

Let $\text{Aut}(G; M)$ be the group of automorphisms α of E that stabilize M and G , in the sense that that $\alpha(m) = m$ for $m \in M$ and the induced automorphism of G is the identity. Say that α is inner if $\alpha(n, g) = m(n, g)m^{-1} = (m + n - hm, g)$ for some $m \in M$ and let $\text{Inn}(G; M)$ be the subgroup of inner automorphisms. For a crossed homomorphism $C: G \rightarrow M$, define $\alpha_C(m, g) = (m + C(g), g)$. Show that $\alpha_C \in \text{Aut}(G; M)$ and that $C \mapsto \alpha_C$ gives an isomorphism $Z(G; M) \rightarrow \text{Aut}(G; M)$ that restricts to an isomorphism $B(G; M) \rightarrow \text{Inn}(G; M)$. Thus $H^1(G; M)$ is the group $\text{Out}(G; M) \equiv \text{Aut}(G; M)/\text{Inn}(G; M)$ of outer automorphisms of E .

5. Define an equivalence relation on the set of those extensions of groups

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1$$

(where 0 and 1 serve to remind that M but not G is required to be Abelian) such that the given action of G on M coincides with the action induced by the short exact sequence: gm is specified by $\iota(gm) = e\iota(m)e^{-1}$ for any e such that $\pi(e) = g$.

Sketch a proof that the set of equivalence classes is an Abelian group and can be identified with $H^2(G; M)$.

6. Let C_n be the cyclic group of order n with generator σ_n and let C_2 act on the Abelian group C_n by $\sigma_2\sigma_n = -\sigma_n$. The semi-direct product $D_n = C_n \rtimes C_2$ is the dihedral group of order $2n$. Compute $H_*(D_2; \mathbb{Z})$ and $H^*(D_2; \mathbb{Z})$. Then compute $H_i(D_3; \mathbb{Z})$ and $H^i(D_3; \mathbb{Z})$ for $i = 1$ and $i = 2$.

7. Show that $H_*(G; \mathbb{Z})$ and $H^*(G; \mathbb{Z})$ are functorial in G , and describe how to compute the maps f_* and f^* induced by a group homomorphism f .

8. For $H \subset G$ and $g \in G$, conjugation by g , $h \mapsto ghg^{-1}$, is a group homomorphism $c_g: H \rightarrow gHg^{-1}$.

- (i) When $H = G$, show that $(c_g)_*$ and c_g^* are identity maps.
- (ii) Let H be a normal subgroup of G . Deduce that conjugation by elements $g \in G$ induces an action of G/H on $H_*(H; \mathbb{Z})$ and $H^*(H; \mathbb{Z})$.
- (iii) Let $D_n = C_n \rtimes C_2$ as in Problem 7. Compute the action of C_2 on $H_*(C_n; \mathbb{Z})$ and $H^*(C_n; \mathbb{Z})$. Hint: consider $g = (0, \sigma_2) \in D_n$ and construct a map from the canonical $\mathbb{Z}[C_n]$ -free resolution W to W regarded as a $\mathbb{Z}[C_n]$ -module by pull-back along $c_g: C_n \rightarrow C_n$. Then compute. If you get stuck, see Weibel 6.7.10.

Remark: The computation does not depend on knowing $H_*(D_n; \mathbb{Z})$ and $H^*(D_n; \mathbb{Z})$. Rather, it is a tool towards that calculation, making it easy by use of the “Lyndon–Hochschild–Serre spectral sequence”, which can be viewed as the specialization of the Serre spectral sequence to a fibration

$$K(N, 1) \rightarrow K(G, 1) \rightarrow K(G/N, 1)$$

for a normal subgroup N of a group G . See Weibel, 6.8.5.

9. Compute $\text{Ext}_k^*[G](k, k) \cong H^*(G; k)$ as an algebra, where G is the cyclic group of prime order p and k is the field with p elements. (Do the cases $p = 2$ and $p > 2$ separately. You can use either algebra or topology.)