

**MATH 327: SEVENTH PROBLEM SET**

Due Monday, May 18

1. A two-sided ideal  $I$  in a bialgebra  $A$  is a bi-ideal if  $\varepsilon(I) = 0$  and

$$\psi(I) \subset A \otimes I + I \otimes A.$$

If  $A$  is a Hopf algebra, then  $I$  is a Hopf ideal if, further,  $\chi(I) \subset I$ . Show:

- (i) If  $I$  is a bi-ideal (Hopf ideal), then  $A/I$  is a bialgebra (Hopf algebra) and the quotient map  $A \rightarrow A/I$  is a map of bialgebras.
- (ii) If  $f: A \rightarrow B$  is a map of bialgebras (Hopf algebras), then  $\text{Ker}(f)$  is a bi-ideal (Hopf ideal).

2. Let  $A$  be a Hopf algebra over  $k$  and define  $G(A)$  to be the set of elements  $g$  such that  $\psi(g) = g \otimes g$ . Show that  $G(A)$  is a group under the product in  $A$  and that the subspace it spans is a sub Hopf algebra isomorphic to the group ring  $k[G(A)]$ .

In the following two problems, let  $A$  be a Hopf algebra with antipode  $\chi$  and let  $\gamma$  be the transposition isomorphism (with sign in the graded case).

3. Show that the following statements hold.

- (i)  $\chi \circ \phi = \phi \circ (\chi \otimes \chi) \circ \gamma: A \otimes A \rightarrow A$ .
- (ii)  $\chi \circ \eta = \eta: k \rightarrow A$ .
- (iii)  $\varepsilon \circ \chi = \varepsilon: A \rightarrow k$ .
- (iv)  $\psi \circ \chi = \gamma \circ (\chi \otimes \chi) \circ \psi: A \otimes A \rightarrow A$ .

4. Show that the following statements are equivalent.

- (i)  $\phi(\chi \otimes \text{id})\gamma\psi = \eta\varepsilon$ . (This can also be written  $\phi\gamma(\text{id} \otimes \chi)\gamma\psi = \eta\varepsilon$ ).
- (ii)  $\phi(\text{id} \otimes \chi)\gamma\psi = \eta\varepsilon$ . (This can also be written  $\phi\gamma(\chi \otimes \text{id})\psi = \eta\varepsilon$ ).
- (iii)  $\chi^2 = \text{id}$ .

Conclude that if  $A$  is either commutative or cocommutative, then  $\chi^2 = \text{id}$ .

5. Let  $T$  be the tensor algebra on (degree 0) variables  $X, Y, Z$  over a field  $k$ . Show

- (i)  $T$  is a bialgebra with augmentation and coproduct given by

$$\varepsilon(X) = 1, \quad \varepsilon(Y) = 1, \quad \varepsilon(Z) = 0$$

$$\psi(X) = X \otimes X, \quad \psi(Y) = Y \otimes Y, \quad \psi(Z) = Z \otimes X + 1 \otimes Z.$$

- (ii) The two-sided ideal  $I$  generated by  $XY - 1$  and  $YX - 1$  is a bi-ideal.
- (iii) Let  $B = A/I$  and let  $x, y, z$  denote the images of  $X, Y, Z$ , so that  $y = x^{-1}$ . Let  $C \subset B$  be the subalgebra generated by  $x$  and  $y$ . As algebras,  $C \cong k[F]$ , where  $F$  is the free Abelian group on one generator  $x$ , and  $B$  is the free product of  $C$  and the polynomial subalgebra  $k[z]$ .
- (iv)  $B$  has the antipode  $\chi$  specified by  $\chi(x) = y$ ,  $\chi(y) = x$ , and  $\chi(z) = -zy$ .
- (v)  $\chi^{2n}(z) = x^n z y^n$  and  $\chi^{2n+1}(z) = -x^n z y^{n+1}$ , hence  $\chi$  has infinite order.

- (vi) Let  $I_n$  be the two-sided ideal of  $B$  generated by  $x^nz y^n - z$  and  $J_n$  be the two-sided ideal generated by  $x^n - 1$ . Then  $I_n$  and  $J_n$  are Hopf ideals, and the antipodes of  $B/I_n$  and  $B/J_n$  have order  $2n$ .

(When  $n = 1$ , these are commutative Hopf algebras and are the coordinate rings of affine algebraic groups; they represent the affine transformation group and the underlying additive group of a 1-dimensional  $k$ -space).

6. If  $A$  is a bialgebra over  $k$ , show that  $\mathrm{Tor}_*^A(k, k)$  is connected graded cocommutative coalgebra over  $k$ , not necessarily a bialgebra. Dually, show that  $\mathrm{Ext}_A^*(k, k)$  is a graded commutative algebra.

7. Compute  $\mathrm{Ext}_A^*(k, k)$  as an algebra when  $A = k[G]$ , where  $G$  is the cyclic group of prime order  $p$  and  $k$  is the field with  $p$  elements. (Do the cases  $p = 2$  and  $p > 2$  separately. You can use either algebra or topology.)

8\*. Show that if  $R$  is a commutative local ring with residue field  $k$ , then  $\mathrm{Tor}_*^R(k, k)$  is a connected graded commutative Hopf algebra over  $k$ , not necessarily cocommutative. Describe this Hopf algebra when  $R$  is regular. [Warning: the structure of commutative algebra is not hard, but it is not so clear how to obtain the coproduct, except, for example, by dualizing the Yoneda product.]