

## MATH 327: SEVENTH PROBLEM SET

Due Monday, May 16

Recommended reading: Weibel, pages 216–227, my notes;  
see also the standard books by Jacobson and Humphreys

Let  $k$  be a field in this problem set.

1. Show that, up to isomorphism, there is only one non-Abelian 2-dimensional Lie algebra over  $k$ .

In the next three problems, let  $L$  be a non-Abelian Lie algebra of dimension 3 and let  $L' = [L, L]$ . Prove the following statements.

2. If  $\dim(L') = 1$ , there are two non-isomorphic choices for  $L$ , one with  $L'$  contained in the center of  $L$  and the other not.

3. If  $\dim(L') = 2$ , then  $L'$  is Abelian and, taking  $k$  to be algebraically closed for simplicity, there are infinitely many non-isomorphic  $L$ .

4. If  $k$  is algebraically closed with  $\text{char}(k) \neq 2$ , then, up to isomorphism, there is a unique  $L$  such that  $L = L'$ . It can be chosen to have basis  $\{e, f, h\}$  such that

$$[e, h] = 2e, \quad [f, h] = -2f, \quad [e, f] = h.$$

5. Let  $k$  be a field of positive characteristic  $p$ . Consider the non-commutative twisted polynomial ring  $k[t; \phi]$  in an indeterminate  $t$  with twisted scalar multiplication  $tk = \phi(k)t$ , where  $\phi$  is the Frobenius,  $\phi(k) = k^p$ , which is an automorphism of  $k$  when  $k$  is perfect. Prove that the category of Abelian restricted Lie algebras over  $k$  is equivalent to the category of (left)  $k[t; \phi]$ -modules.

(The structure theory for modules over PID's extends to rings such as  $k[t; \phi]$ ).

6. Let  $L$  be a restricted Lie algebra such that  $\xi(x) = \lambda^p x$  for all  $x \in L$  and some fixed non-zero scalar  $\lambda$ . Prove that  $L$  is Abelian.

7. Let  $T$  denote the free (or tensor) algebra functor on (ungraded) vector spaces and assume that  $\text{char}(k) = 0$ . Let  $L(V) = PT(V)$  be the Lie algebra of primitive elements of  $T(V)$ ; elements of  $L(V)$  are called Lie elements of  $T(V)$ .

(i) Show that  $T(V)$  is a primitively generated Hopf algebra.

(ii) Show that  $L(V)$  is the free Lie algebra generated by  $V$ : a map of  $k$ -spaces  $V \rightarrow M$ , where  $M$  is a Lie algebra, extends uniquely to a map of Lie algebras  $L(V) \rightarrow M$ .

8. Give  $T(V)$  the grading by monomial degree. Show that  $L(V)$  is spanned by its homogeneous elements

$$\sigma(x_{i_1} \cdots x_{i_m}) = [\cdots [[x_{i_1}, x_{i_2}]x_{i_3}] \cdots x_{i_m}],$$

where the  $x_i$  run through a basis for  $V$ . Define a map of  $k$ -spaces

$$\sigma: IT(V) \longrightarrow L(V)$$

via the function  $\sigma$  on monomials, starting with  $\sigma(x_i) = x_i$ . Prove the *Specht-Wever Theorem*: A homogeneous element of degree  $m$  in  $T(V)$  is a Lie element if and only if  $\sigma(x) = mx$ .