

MATH 327: EIGHTH PROBLEM SET

Due Monday, May 23

Recommended reading: Weibel, pages 228–253;

BUT, again, please do not just copy anything in Weibel.

In the following problems, L is a Lie algebra over a field k and M is an L -module (= $U(L)$ -module). Note the parallel with the homology and cohomology of groups.

1. Show that $H_0(L; M) \cong M/LM$ and $H^0(L; M) \cong M^L \equiv \{m \mid Lm = 0\}$.
2. Show that $H_1(L; k) \cong L/[L, L]$.

3. Define a derivation $D: L \rightarrow M$ to be a map of k -spaces such that $D([x, y]) = xD(y) - yD(x)$ for $x, y \in L$. For $m \in M$, define a derivation $D_m: L \rightarrow M$ by $D_m(x) = xm$; D_m is said to be inner. Let $Z(L; M)$ be the k -space of derivations and $B(L; M)$ be the subspace of inner derivations. Show that $H^1(L; M) \cong Z(L; M)/B(L; M)$.

4. View M as an Abelian Lie algebra. Define an equivalence relation on the set of those extensions of Lie algebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 1$$

such that the given action of L on M coincides with the action induced by the short exact sequence. Sketch a proof that the set of equivalence classes is a k -space and can be identified with $H^2(L; M)$

- a Directly from the definitions.
- b By use of the standard resolution.

5. Compute $H^*(L; k) = \text{Ext}_{U(L)}^*(k, k)$ as an algebra, where $L = \mathfrak{sl}(2, \mathbb{C})$ is the complex Lie algebra with basis e, f, h and bracket operation determined by

$$[e, h] = 2e, \quad [f, h] = -2f, \quad [e, f] = h$$

(the unique simple 3-dimensional Lie algebra).

Remark: If L is any semi-simple Lie algebra and M any finite dimensional L -module, then $H^1(L; M) = 0$ and $H^2(L; M) = 0$ (“Whitehead lemmas 1 and 2”), but you should just have proven that the analogue for H^3 is false.

6. In analogy with groups, prove that

$$H_*(L; M \otimes_k N) \cong \text{Tor}_*^{U(L)}(M, N) \quad \text{and} \quad H^*(L; \text{Hom}_k(M, N)) \cong \text{Ext}_{U(L)}(M, N)$$

(taking M to be a right or a left $U(L)$ -module, N to be a left $U(L)$ -module).

7. In contrast with groups, prove that if L is a Lie algebra of dimension n over a field k , then $\text{gldim } U(L) = n$.

a For \leq , consider the standard resolution of k .

b For \geq , show that $H^n(L; \Lambda^n L) \cong k$, where $\Lambda^n L$ is the n th exterior power of L with L -module structure given by

$$y(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leq i \leq n} x_1 \wedge \cdots \wedge x_{i-1} \wedge [y, x_i] \wedge x_{i+1} \wedge \cdots \wedge x_n.$$