

**MATH 327: EIGHTH PROBLEM SET**

Due Monday, May 25

In the following problems,  $G$  is a group,  $M$  is a  $G$ -module (=  $\mathbb{Z}[G]$ -module), and  $I$  is the augmentation ideal  $\text{Ker}(\varepsilon: \mathbb{Z}[G] \rightarrow \mathbb{Z})$ .

1. Show that  $H_0(G; M) \cong M/IM$  and  $H^0(G; M) \cong M^G \equiv \{m \mid Im = 0\}$ .
2. Show that  $H_1(G; \mathbb{Z}) \cong G/[G, G]$  and relate that to the Hurewicz homomorphism for the space  $K(G, 1)$ .
3. Define a crossed homomorphism  $C: G \rightarrow M$  to be a function such that  $C(gh) = gC(h) + C(g)$ . For  $m \in M$ , define a crossed homomorphism  $C_m$  by  $C_m(g) = gm - m$ ;  $C_m$  is said to be principal. Let  $Z(G; M)$  be the Abelian group of crossed homomorphisms under pointwise addition and let  $B(G; M)$  be the subgroup of principal crossed homomorphisms. Show that  $H^1(G; M) \cong Z(G; M)/B(G; M)$ . Show that if  $G$  acts trivially on  $M$ , then this is the Abelian group of homomorphisms of groups  $G \rightarrow M$ .

Remark: Let  $K$  be a Galois extension of a field  $F$  with Galois group  $G$ , and let  $K^\times$  be the group of units (non-zero elements) of  $K$ ; note that  $K^\times$  is a  $G$ -module. A result known as Hilbert's Theorem 90 says precisely that  $H^1(G; K^\times) = 0$ .

5. Let  $E = M \rtimes G$ , so that we have an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1.$$

(See # 7.) Let  $\text{Aut}(G; M)$  be the group of automorphisms  $\alpha$  of  $E$  that stabilize  $M$  and  $G$ , in the sense that that  $\alpha(m) = m$  for  $m \in M$  and the induced automorphism of  $G$  is the identity. Say that  $\alpha$  is inner if  $\alpha(n, g) = m(n, g)m^{-1} = (m+n+hm^{-1}, g)$  for some  $m \in M$  and let  $\text{Inn}(G; M)$  be the subgroup of inner automorphisms. For a crossed homomorphism  $C: G \rightarrow M$ , define  $\alpha_C(m, g) = (m + C(g), g)$ . Show that  $\alpha_C \in \text{Aut}(G; M)$  and that  $C \mapsto \alpha_C$  gives an isomorphism  $Z(G; M) \rightarrow \text{Aut}(G; M)$  that restricts to an isomorphism  $B(G; M) \rightarrow \text{Inn}(G; M)$ . Thus  $H^1(G; M)$  is the group  $\text{Out}(G; M) \equiv \text{Aut}(G; M)/\text{Inn}(G; M)$  of outer automorphisms of  $E$ .

6. Let  $C_n$  be the cyclic group of order  $n$  with generator  $\sigma_n$  and let  $C_2$  act on the Abelian group  $C_n$  by  $\sigma_2\sigma_n = -\sigma_n$ . The semi-direct product  $D_n = C_n \rtimes C_2$  is the dihedral group of order  $2n$ . Compute  $H^1(C_2; C_n)$ .

7. Define an equivalence relation on the set of those extensions of groups

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1$$

(where 0 and 1 serve to remind that  $M$  but not  $G$  is required to be Abelian) such that the given action of  $G$  on  $M$  coincides with the action induced by the short exact sequence:  $gm$  is specified by  $\iota(gm) = e\iota(m)e^{-1}$  for any  $e$  such that  $\pi(e) = g$ .

Prove that the set of equivalence classes is an Abelian group and can be identified with  $H^2(G; M)$ .

In the following problems,  $L$  is an (ungraded) Lie algebra over a field  $K$  and  $M$  is an  $L$ -module (=  $U(L)$ -module).

8. Show that  $H_0(L; M) \cong M/LM$  and  $H^0(L; M) \cong M^L \equiv \{m \mid Lm = 0\}$ .

9. Show that  $H_1(L; K) \cong L/[L, L]$ . Define a derivation  $D: L \rightarrow M$  to be a map of  $K$ -spaces such that  $D([x, y]) = xD(y) - yD(x)$  for  $x, y \in L$ . For  $m \in M$ , define a derivation  $D_m: L \rightarrow M$  by  $D_m(x) = xm$ ;  $D_m$  is said to be inner. Let  $Z(L; M)$  be the  $K$ -space of derivations and  $B(L; M)$  be the subspace of inner derivations. Show that  $H^1(L; M) \cong Z(L; M)/B(L; M)$ .

10. View  $M$  as an Abelian Lie algebra. Define an equivalence relation on the set of those extensions of Lie algebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 1$$

such that the given action of  $L$  on  $M$  coincides with the action induced by the short exact sequence. Prove that the set of equivalence classes is a  $K$ -space and can be identified with  $H^2(L; M)$ .