

MATH 327: NINTH PROBLEM SET

Due Friday, June 3

The last few problems, 6-8, in this set may be hard, but I hope interesting. Start early and give the whole set a real try. For sure, hand in all that you can do.

1. A two-sided ideal I in a bialgebra A is a bi-ideal if $\varepsilon(I) = 0$ and

$$\psi(I) \subset A \otimes I + I \otimes A.$$

If A is a Hopf algebra, then I is a Hopf ideal if, further, $\chi(I) \subset I$. Show:

- (i) If I is a bi-ideal (Hopf ideal), then A/I is a bialgebra (Hopf algebra) and the quotient map $A \rightarrow A/I$ is a map of bialgebras.
- (ii) If $f: A \rightarrow B$ is a map of bialgebras (Hopf algebras), then $\text{Ker}(f)$ is a bi-ideal (Hopf ideal).

2. Let A be a Hopf algebra over k and define $G(A)$ to be the set of non-zero elements g such that $\psi(g) = g \otimes g$. Show that $G(A)$ is a group under the product in A and that the subspace it spans is a sub Hopf algebra isomorphic to the group ring $k[G(A)]$.

In the following two problems, let A be a Hopf algebra with antipode χ and let γ be the transposition isomorphism.

3. Show that the following statements hold.

- (i) $\chi \circ \phi = \phi \circ (\chi \otimes \chi) \circ \gamma: A \otimes A \rightarrow A$.
- (ii) $\chi \circ \eta = \eta: k \rightarrow A$.
- (iii) $\varepsilon \circ \chi = \varepsilon: A \rightarrow k$.
- (iv) $\psi \circ \chi = \gamma \circ (\chi \otimes \chi) \circ \psi: A \otimes A \rightarrow A$.

4. Show that the following statements are equivalent.

- (i) $\phi(\chi \otimes \text{id})\gamma\psi = \eta\varepsilon$. (This can also be written $\phi\gamma(\text{id} \otimes \chi)\gamma\psi = \eta\varepsilon$).
- (ii) $\phi(\text{id} \otimes \chi)\gamma\psi = \eta\varepsilon$. (This can also be written $\phi\gamma(\chi \otimes \text{id})\psi = \eta\varepsilon$).
- (iii) $\chi^2 = \text{id}$.

Conclude that if A is either commutative or cocommutative, then $\chi^2 = \text{id}$.

5. Let A be the quotient of the free k -algebra on two generators x and y by the two-sided ideal generated by $x^2 - 1$, y^2 , and $xy + yx$. Show the following.

- (i) A is a four dimensional vector space.
- (ii) A is a Hopf algebra with

$$\begin{aligned}\psi(x) &= x \otimes x, & \psi(y) &= y \otimes x + 1 \otimes y \\ \varepsilon(x) &= 1, & \varepsilon(y) &= 0 \\ \chi(x) &= x, & \chi(y) &= xy\end{aligned}$$

- (iii) χ has order 4: $\chi^4 = \text{id}$ but $\chi^2 \neq \text{id}$.

6. Let A be the tensor algebra on variables X, Y, Z over a field k . Show

- (i) A is a bialgebra with augmentation and coproduct given by

$$\varepsilon(X) = 1, \quad \varepsilon(Y) = 1, \quad \varepsilon(Z) = 0$$

$$\psi(X) = X \otimes X, \quad \psi(Y) = Y \otimes Y, \quad \psi(Z) = Z \otimes X + 1 \otimes Z.$$

- (ii) The two-sided ideal I generated by $XY - 1$ and $YX - 1$ is a bi-ideal.
- (iii) Let $B = A/I$ and let x, y, z denote the images of X, Y, Z , so that $y = x^{-1}$. Let $C \subset B$ be the subalgebra generated by x and y . As algebras, $C \cong k[F]$, where F is the free Abelian group on one generator x , and B is the free product of C and the polynomial subalgebra $k[z]$.
- (iv) B has the antipode χ specified by $\chi(x) = y$, $\chi(y) = x$, and $\chi(z) = -zy$.
- (v) $\chi^{2n}(z) = x^n z y^n$ and $\chi^{2n+1}(z) = -x^n z y^{n+1}$, hence χ has infinite order.
- (vi) Let I_n be the two-sided ideal of B generated by $x^n z y^n - z$ and J_n be the two-sided ideal generated by $x^n - 1$. Then I_n and J_n are Hopf ideals, and the antipodes of B/I_n and B/J_n have order $2n$.

Remark: When $n = 1$, these are commutative Hopf algebras and are the coordinate rings of affine algebraic groups; they represent the affine transformation group and the underlying additive group of a 1-dimensional k -space.

7. If A is a bialgebra over k , show that $\text{Tor}_*^A(k, k)$ is connected graded cocommutative coalgebra over k , not necessarily a bialgebra. Dually, show that $\text{Ext}_A^*(k, k)$ is a graded commutative algebra.

8*. Show that if R is a commutative local ring with residue field k , then $\text{Tor}_*^R(k, k)$ is a connected graded commutative Hopf algebra over k , not necessarily cocommutative. Describe this Hopf algebra when R is regular. [Warning: the structure of commutative algebra is not hard, but it is less clear how to obtain the coproduct. One way is to use the bar construction, another is to dualize the Yoneda product. In the regular case, consider the Koszul complex.]