

## MATH 327: NINTH PROBLEM SET

Due Friday, June 5

1. Fix a prime  $p$  and a torsion free chain complex  $C$  of Abelian groups. Assume that each  $H_q(C)$  is a finitely generated Abelian group. We can construct an exact couple of graded (not bigraded!) Abelian groups from the short exact sequence

$$0 \longrightarrow C \xrightarrow{p} C \longrightarrow C/pC \longrightarrow 0.$$

(Details in the notes, or supply them yourself.) The resulting spectral sequence,  $\{E^r C\}$ , is called the “mod  $p$  Bockstein spectral sequence of  $C$ ”. Its differentials are denoted  $\beta^r$ , with  $\beta^1 = \beta$ . It is functorial in  $C$ .

- (i) Give an elementwise chain level description of the  $\beta^r$ .
- (ii) Construct a new chain complex  $C'$  that is the direct sum of chain complexes each of which have either only one or two non-zero terms (in adjacent degrees) and a quasi-isomorphism  $C' \rightarrow C$ .
- (iii) Show that  $E_\infty C$  is isomorphic to  $H_*(C)/\text{Tor}H_*(C) \otimes \mathbb{F}_p$ .
- (iv) Suppose that each  $E^r C$  is known. Explain how to read off the structure of the Abelian groups  $H_q(C)$  from this knowledge.

This is the standard tool that algebraic topologists use to compute integral homology  $H_*(X; \mathbb{Z})$  from the mod  $p$  homologies  $H_*(X; \mathbb{F}_p)$  for all primes  $p$ . The universal coefficient theorem tells how to compute the  $H_*(X; \mathbb{F}_p)$  from  $H_*(X; \mathbb{Z})$ , but that in practice is a harder way to go.

2. Compute the mod  $p$  homology and the mod  $p$  Bockstein spectral sequence of  $K(C_{p^q}, 1)$  for all  $q$ . Equivalently, this can be viewed as the Bockstein spectral sequence of a chain complex for the computation of  $H_*(C_{p^q}; \mathbb{Z})$ .

3. Show that  $H_*(G; \mathbb{Z})$  and  $H^*(G; \mathbb{Z})$  are functorial in  $G$ , and describe how to compute the maps  $f_*$  and  $f^*$  induced by a group homomorphism  $f$ .

4. For  $H \subset G$  and  $g \in G$ , conjugation by  $g$ ,  $h \mapsto ghg^{-1}$ , is a group homomorphism  $c_g: H \rightarrow gHg^{-1}$ .

- (i) When  $H = G$ , show that  $(c_g)_*$  and  $c_g^*$  are the identity map.
- (ii) Let  $H$  be a normal subgroup of  $G$ . Deduce that conjugation by elements  $g \in G$  induces an action of  $G/H$  on  $H_*(H; \mathbb{Z})$  and  $H^*(H; \mathbb{Z})$ .
- (iii) Let  $D_n = C_n \rtimes C_2$  as in Problem 6 of Homework 8. Compute the action of  $C_2$  on  $H_*(C_n; \mathbb{Z})$  and  $H^*(C_n; \mathbb{Z})$ . Hint: consider  $g = (0, \sigma_2) \in D_n$  and construct a map from the canonical  $\mathbb{Z}[C_n]$ -free resolution  $W$  to  $W$  regarded as a  $\mathbb{Z}[C_n]$ -module by pullback along  $c_g: C_n \rightarrow C_n$ . Then compute.

In the next problem take it for granted that for an extension

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

there are spectral sequences

$$E_{p,q}^2 = H_p(G/H; H_q(H; \mathbb{Z})) \implies H_{p+q}(G; \mathbb{Z})$$

and

$$E_2^{p,q} = H^p(G/H; H^q(H; \mathbb{Z})) \implies H^{p+q}(G; \mathbb{Z}).$$

They can be constructed either algebraically or as Serre spectral sequences of a bundle  $K(G, 1) \longrightarrow K(G/H, 1)$  with fiber  $K(H, 1)$ .

5. Compute the graded Abelian groups  $H_*(D_n; \mathbb{Z})$  and  $H^*(D_n; \mathbb{Z})$  when  $n = 2m$  and when  $n = m$ , where  $m$  is odd.

6. Let  $A$  be a DGA over a field  $K$  and let  $M$  and  $N$  be right and left DG  $A$ -modules, all concentrated in non-negative degrees (to avoid convergence problems). Ignoring the given differentials on  $M$ ,  $A$ , and  $N$ , we have a bigraded chain complex

$$B_{*,*} \equiv B(M, A, N) = B(M, A, A) \otimes_A N$$

with “external” differential  $d' : B_{p,q} \longrightarrow B_{p-1,q}$  as if we were computing the usual Tor. Remembering the given differentials but ignoring  $d'$ , we get an “internal” differential  $d'' : B_{p,q} \longrightarrow B_{p,q-1}$  induced by the differentials on the  $M \otimes A^p \otimes N$ . Notice that  $d'd'' = d''d' : B_{p,q} \longrightarrow B_{p-1,q-1}$ . Regrade by total degree

$$B_n(M, A, N) = \sum_{p+q=n} B_{p,q}$$

with total differential  $d = d' + (-1)^p d''$  on the  $B_{p,q}$ . Then define

$$\text{Tor}^A(M, N) = H_*(B(M, A, N)).$$

Construct a spectral sequence of differential coalgebras such that

$$E_{p,q}^2 = \text{Tor}_{p,q}^{H_*(A)}(H_*(M), H_*(N)) \implies \text{Tor}_{p+q}^A(M, N).$$

On the left,  $p$  gives the homological degree ( $p$ th Tor term) and  $q$  gives the internal degree coming from the grading of  $H_*(A)$ ,  $H_*(M)$ , and  $H_*(N)$ .

For a topological group  $G$ , one can show that the singular chains  $C_*(X; K)$  form a DGA and prove that  $H_*(BG; K) \cong \text{Tor}_*^{C_*(G)}(K, K)$  as coalgebras. If  $G$  is compact and connected and  $K$  has characteristic zero, then  $H_*(G)$  is an exterior algebra on odd degree generators (structure theorem for Hopf algebras) and the  $E^2$ -term is a divided polynomial coalgebra on corresponding even degree generators (direct calculation), so that its dual is a polynomial algebra on even degree generators. The spectral sequence must collapse at  $E^2$ , and therefore  $H^*(BG; K)$  must be a polynomial algebra on even degree generators.