

## MATH 327: NINTH PROBLEM SET

Due Friday, June 6

1. Prove that the opposite category of a triangulated category is triangulated.

2. Prove that in any triangulated category  $\mathcal{C}$ , the functor  $\mathcal{C}(X, -)$  takes distinguished triangles to long exact sequences. Then deduce formally that the functor  $\mathcal{C}(-, Y)$  also takes distinguished triangles to long exact sequences.

We have derived tensor products  $M \otimes^{\mathbb{L}} N$  and derived Hom objects  $\mathbb{R}Hom(M, N)$  in  $\mathcal{D}_R$  for chain complexes  $M$  and  $N$ . Take  $M$  and  $N$  to be  $k$ -modules, regarded as chain complexes concentrated in degree 0.

2. With  $A = R$ , prove that  $H_*(M \otimes^{\mathbb{L}} N)$  and  $H_*(\mathbb{R}Hom(M, N))$ , the latter regraded cohomologically, satisfy the defining axioms to be  $Tor_*^R(M, N)$  and  $Ext_R^*(M, N)$ .

For a DGA  $A$ , this dictates the definition of Tor and Ext for general chain complexes  $M$  and  $N$ .

3. Show that the "negative triangles"  $(-f, -g, -h)$  associated to a triangulation of an additive category  $\mathcal{T}$  with suspension  $\Sigma$  give a new triangulation, in general different from the original one.

Here " $\_$ " is an additive automorphism of the category  $\mathcal{T}$  (it is the identity on objects) that commutes with  $\Sigma$ . Any such automorphism gives a new triangulation, which we say is equivalent to the original one. Research problem: can a given additive category  $\mathcal{T}$  with suspension  $\Sigma$  have inequivalent triangulations? [As far as I know, the question is open.]

4. Fix a prime  $p$  and a torsion free chain complex  $C$  of Abelian groups. Assume that each  $H_q(C)$  is a finitely generated Abelian group. Construct an exact couple of graded (not bigraded!) Abelian groups from the short exact sequence

$$0 \longrightarrow C \xrightarrow{p} C \longrightarrow C/pC \longrightarrow 0.$$

The resulting spectral sequence,  $\{E^r C\}$ , is called the "mod  $p$  Bockstein spectral sequence of  $C$ ". Its differentials are denoted  $\beta^r$ , with  $\beta^1 = \beta$ . It is functorial in  $C$ .

- (i) Give an elementwise chain level description of the  $\beta^r$ .
- (ii) Construct a new chain complex  $C'$  that is the direct sum of chain complexes each of which have either only one or two non-zero terms (in adjacent degrees) and a quasi-isomorphism  $C' \rightarrow C$ .
- (iii) Show that  $E_{\infty} C$  is isomorphic to  $H_*(C) \otimes \mathbb{Q}$ .

- (iv) Suppose that each  $E^r C$  is known. Explain how to read off the structure of the Abelian groups  $H_q(C)$  from this knowledge.

This is the standard tool that algebraic topologists use to compute integral homology  $H_*(X; \mathbb{Z})$  from the mod  $p$  homologies  $H_*(X; \mathbb{F}_p)$  for all primes  $p$ . The universal coefficient theorem tells how to compute the  $H_*(X; \mathbb{F}_p)$  from  $H_*(X; \mathbb{Z})$ , but that in practice is a harder way to go.

5. Compute the mod 2 Bockstein spectral sequence of  $\mathbb{R}P^n$  for  $n \geq 2$ .

6. Let  $A$  be a DGA over a field  $K$  and let  $M$  and  $N$  be right and left DG  $A$ -modules, all concentrated in non-negative degrees (to avoid convergence problems). Construct a spectral sequence such that  $E_{p,q}^2 = \text{Tor}_{p,q}^{H_*(A)}(H_*(M), H_*(N))$  which converges in total degrees  $p+q$  to  $\text{Tor}_*^A(M, N)$ . Here  $p$  gives the homological degree ( $p$ th Tor term) and  $q$  gives the internal degree coming from the grading of  $H_*(A)$ ,  $H_*(M)$ , and  $H_*(N)$ .

For a topological group  $G$ , one can show that the singular chains  $C_*(X; K)$  form a DGA and prove that  $H_*(BG; K) \cong \text{Tor}_*^{C_*(G)}(K, K)$  as coalgebras. If  $G$  is compact and connected and  $K$  has characteristic zero,  $H_*(G)$  is an exterior algebra on odd degree generators (structure theorem for Hopf algebras) and the  $E^2$ -term is a divided polynomial coalgebra on corresponding even degree generators (so that its dual is a polynomial algebra). The spectral sequence must collapse at  $E^2$ , and therefore  $H^*(BG; K)$  must be a polynomial algebra on even degree generators.