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On the Theory and Applications of Differential Torsion Products

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Abstract. A new approach to differential homological algebra is developed, one which exploits more general types of resolutions than the bi-complexes used traditionally. An example of such a generalized resolution is exhibited and is used to prove that the differential torsion product reduces to the classical torsion product in favorable cases. This result is used to compute the cohomology of various spaces. For example, if $H$ is a closed subgroup of a compact connected Lie group $G$ and if $H$ and $G$ have no torsion, then the integral cohomology $H^*(G/H)$ is isomorphic as a graded Abelian group to $\text{Tor}^H_1(Z, H^*(BG))$. The paper also includes proofs (within the new framework) of the results of Eilenberg and Moore which relate differential torsion products to the homology and cohomology of spaces, a discussion of the relationship between differential torsion products and matrix Massey products, and a detailed exposition (required for technical reasons) of the Eilenberg-MacLane-Cartan calculation of the cohomology of $K(s,n)$'s.

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In their fundamental papers [33] and [14], Eilenberg and Moore introduced the subject of differential homological algebra and established its usefulness in algebraic topology. They defined functors \( \text{Tor} \) (and \( \text{Ext} \), \( \text{Cotor} \), and \( \text{Coext} \)) for differential algebras and modules by means of appropriate bicomplexes, with "internal" differentials arising from the given differentials and "external" differentials corresponding to the classical definition of \( \text{Tor} \). The filtrations on such bicomplexes given by their external degrees led to spectral sequences, which, under appropriate flatness hypotheses, converged from classical torsion products to the new "differential" torsion products.

For appropriate algebras and modules of chains or cochains, it was shown that the differential torsion products were isomorphic to the homology or cohomology of certain spaces, and thus spectral sequences converging from classical torsion products to these homology or cohomology groups were obtained.

Subsequently, a number of authors exploited these spectral sequences for calculations. With the single exception of the paper of Baum and Smith [14], which concerns cohomology with coefficients in the real numbers, all of these applications were based on dimensionality and naturality arguments. Typically, certain algebraic hypotheses would be placed on the cohomology algebras and maps which determine the classical torsion products and it would be shown that, under these hypotheses, \( E_2 = E_\infty \) for purely dimensional reasons. While a number of important results were obtained in this way, particularly by Baum [3] and Smith [37], the limitations to this approach were obvious.
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In 1966, the second author described the differentials in these spectral sequences in terms of matric Massey products and used this description to prove that $E_2 = E_\infty$ in many cases to which dimensionality arguments did not apply and to show how to compute $E_\infty$ in other cases where $E_2$ was not equal to $E_\infty$. These results were announced in [26]; however, the original proofs were lengthy and quite ugly, and they were therefore never published. In 1970, the first author, partly influenced by his joint work with Milgram on perturbation methods [16], obtained new, and very much simpler, proofs of these results. The new proofs were based on the use of richer structures than bicomplexes, and it soon became apparent that these structures could be used as the basis for a redevelopment of the subject of differential homological algebra; it is this theory and its applications which will be presented here.

Thus, in the first two sections, we give a new definition of the differential torsion product (and Ext functor), develop its main properties, and prove the algebraic theorem which will imply the geometric results of [26]. The advantage of our theory lies in its greater flexibility—much more general types of resolutions are permitted—and in its closer connections with classical homological algebra. It also has the welcome merit of brevity, although we should admit that this is largely due to the fact that we can offer no categorical justification (in terms of projective objects, etc.) for our definitions.

In the third section, we show how to derive the applications of the differential torsion product to geometry in our framework. The results here are somewhat sharper than those of Eilenberg and Moore in that no flatness hypotheses are required for the identification of the $E_2$-terms of the spectral sequences obtained. For this reason, our theorems 3.3 and 3.9 are in fact generalizations of the classical Künneth theorems for cohomology and homology respectively. To see this, specialize $B$ to a point in 3.3, and $G$ to the trivial group in the second part of 3.9. Precise comparisons with the work of Eilenberg and Moore will appear in the appendix.

The results of [26] will be proven in section 4. We state and discuss two important special cases of Theorem 4.3 here. Let $R$ be a commutative Noetherian ring taken as coefficients in cohomology. Let $G$ be a connected topological group (with integral homology of finite type) such that $H^*(BG)$ is a polynomial algebra and let $H$ be a compact connected Lie group such that $H$ has no $p$-torsion for any prime $p$ which divides the characteristic of $R$.

Under these hypotheses, we have the following theorems A and B.

**Theorem A.** If $H$ is embedded as a closed subgroup of $G$ and if $f: BH \to BG$ denotes the induced map of classifying spaces, then

$$H^*(G/H) = \text{Tor}_{H^*(BG)}(R, H^*(BH))$$

as a graded $R$-module, where $H^*(BH)$ is given a structure of left $H^*(BG)$-module by means of $f^*: H^*(BG) \to H^*(BH)$.

If $G$ is constrained to satisfy the same hypotheses as $H$, then $f^*$ can be computed in terms of the Weyl groups of $H$ and $G$ by the methods described by Borel in [5, §28]. Under this constraint, the theorem was conjectured by Baum [3] and was proven by Cartan [8] when $R$ is the real numbers, by Borel [5] when $R$ is a field and $\text{rank } H = \text{rank } G$ (whence, as we
recall in Theorem 4.10, $H^*(BH)$ is free as an $H^*(BG)$-module, and by
Baum [3] when $R$ is a field and $t^*$ satisfies a certain algebraic condition
(which serves to guarantee that $E_2 = E_\infty$ for dimensional reasons). The
geometry of the embedding of $H$ in $G$ plays no role in our proof of the
theorem. Indeed, the same proof will yield the following result.

**Theorem B.** Let $q: E \to BH$ be a principal $G$-bundle, and let $q$ be
classified by $f: BH \to BG$. Then

$$H^*(E) = \text{Tor}_{H^*(BG)}^1(R, H^*(BH))$$

as a graded $R$-module, where $H^*(BH)$ is given a structure of left $H^*(BG)$-
module by means of $t^*: H^*(BG) \to H^*(BH)$.

It should be emphasized that these are not theorems about spectral
sequences; their proofs will primarily deal directly with the structure of
differential torsion products. In particular, we are left with no additive ex-
tension problem. The force of this statement becomes apparent when it is
recalled that there are examples of embeddings $H \subset G$ of compact connected
Lie groups such that $H$ and $G$ have no torsion (so that Theorem A applies
with $R = \mathbb{Z}$) and yet $G/H$ does have torsion. Multiplicatively, however, we
are left with an extension problem; our results will compute the associated
graded algebras of $H^*(G/H)$ and of $H^*(E)$ with respect to suitable filtra-
tions. Refinements of our algebraic theory could conceivably yield precise
procedures for the computation of these cohomology algebras. When $R = \mathbb{Z}$,
there are examples where the extensions are non-trivial. There are no such
examples known when $R$ is a field of characteristic $\neq 2$.

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We shall also obtain an algorithm for the computation of the mod 2 coho-
mology of any simply connected two-stage Postnikov system in section 4.
Here, by an example discovered by Schochet [34], it is not in general the
case that $E_2 = E_\infty$ in the relevant Eilenberg-Moore spectral sequence.

As noted above, the first proof of these results used Massey products.
The present proof does not. However, we shall show in section 5 that all
previously obtained information about the relationship between Massey pro-
ducts and the Eilenberg-Moore spectral sequence easily follows from the
algebraic theory of sections 1 and 2. As a result, we shall be able to de-
scribe the kernels of various induced maps of homology and cohomology
groups and of various suspension maps in terms of matric Massey products.
We shall also show that the cohomology of a connected algebra is generated
under matric Massey products by its elements of homological degree one.

A lengthy, but quite elementary, appendix contains proofs of certain
technical theorems on the relationship between the cochains and cohomology
of $K(z, n)$'s. These results are required in section 4.
§1. Differential homological algebra

Let $R$ be a commutative ring; all $R$-modules are assumed to be $Z$-graded, by subscripts. Let $U$ be an associative and unital differential $R$-algebra, and let $M$ be a right and $N$ a left differential $U$-module (thus $d(mu) = d(m)u + (1)\deg m d(u)$ for elements $m \in M$ and $u \in U$; all differentials will be denoted by $d$ and will lower degrees by one). Our goal is to obtain a pragmatically convenient and conceptually satisfactory description of the functor $\text{Tor}^U(M,N)$.

We take the position that, however it is defined, $\text{Tor}^U(M,N)$ is of interest only when it is sufficiently closely related to the classical torsion product $\text{Tor}^H(U,M,HN)$, where $H$ denotes the homology functor, since only then are computations practicable. We shall thus redefine $\text{Tor}^U(M,N)$ in terms of this relationship, and we shall then see that projective $HU$-resolutions of $HM$, which are in any case the main computational tool, can be directly exploited for the computation of the differential torsion product.

Let $\mathcal{M}_U$ (resp. $\mathcal{F}_U$) denote the category of right (resp. left) differential $U$-modules. Let $\mathcal{F}_U$ denote the subcategory of $\mathcal{M}_U$ consisting of the filtered objects and filtration-preserving maps. Precisely, an object $X \in \mathcal{F}_U$ is to be a differential $U$-module together with an increasing sequence of subdifferential $U$-modules $F^p X$ such that $X$ is the union of the $F^p X$ and $F^p X = 0$ for $p < 0$; we shall sometimes allow $F^{-1} X$ to be also non-zero, and we denote by $\mathcal{F}^\bot U$ the category of such filtered differential $U$-modules.
Recall that a filtered differential $R$-module $X$ gives rise to a spectral sequence $\{E^r_{pq}\}$ starting from $E^0_{pq} = (E_{p} X / F_{p-1} X)_{p+q}$.

Our approach to $\text{Tor}_U^+(M, U)$ is based on objects $X \in \mathcal{M}_U$ augmented over $M$, by which we mean only that a morphism $\alpha : X \to M$ is given in $\mathcal{M}_U$.

**Definitions 1.1.** Let $\alpha : X \to M$ be a morphism in $\mathcal{M}_U$, where $X \in \mathcal{M}_U$. Define an object $X^\alpha \in \mathcal{M}_U$ as follows:

(i) $X^\alpha_n = M_{n+1} \otimes X_n$ specifies $X^\alpha$ as a $\mathbb{Z}$-graded $R$-module.

(ii) $(m, x)u = (mu, xu)$ for $m \in M$, $x \in X$, and $u \in U$.

(iii) $F_\alpha X = M$ and $F_p X^\alpha = M \oplus F_p X$ if $p \geq 0$.

(iv) $d(m, x) = (\alpha(x) - d(m), d(x))$ for $m \in M$ and $x \in X$.

$X^\alpha$ will be called the mapping cylinder of $\alpha$. Observe that $E_{-q}^{-1, q} X^\alpha = H_q(M)$ and $E_{p}^{-1, p} X^\alpha = E_{p}^{-1} X$ for all $p \geq 0$. Since the right $U$-module structure on $F_\alpha X$ induces a right $H(U)$-module structure on $E_{p}^{-1} X$, $(E_{p}^{-1} X^\alpha, d')$ is a complex of right $H(U)$-modules of the form

$$0 \to E_{p-1}^{-1} X \to E_{p}^{-1} X \to \cdots \to E_{0}^{-1} X \to H(U) = 0.$$

We say that $X^\alpha$ is a resolution of $M$ if the sequence (1) is exact. For $N \in \mathcal{M}_U$, give $X \otimes_U N$ the induced filtration $F_p (X \otimes_U N) = (F_p X) \otimes_U N$ and observe that there is a Künneth map (or homology product) of differential $R$-modules:

$$\kappa : E_{-q}^{-1} X \otimes U H(N) = E_{-q}^{-1} (X \otimes_U N), \quad \kappa((x) \otimes (n)) = (x \otimes n).$$

We say that $X$ is a Künneth object if each $E_p^{-1} X$ is a flat $H(U)$-module and if the maps (2) are isomorphisms for all $N$. We shall define

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$$\text{Tor}_U^+(M; N) = H(U) \otimes_U N$$

and

$$E_{p}^{-1} (M, U, N) = E_p^{-1} (X \otimes_U N), \quad p \geq 2,$$

where $X$ is any Künneth resolution of $M$. Visibly,

$$E^2_{(p, q)}(M, U, N) = \text{Tor}_U^+(H(U), H(U)), \quad (p, q) \geq 1.$$

$(E_p^{-1} (M, U, N))$ is the algebraic Eilenberg-Moore spectral sequence.

Note that $(E_p^{-1} (M, U, N))$ converges to $\text{Tor}_U^+(M, N)$ since the filtration of $X \otimes_U N$ is bounded below and convergent above [22, XI 3.2]. Our notion of a Künneth resolution extracts precisely the minimum amount of information necessary to identify $E^2_{(p, q)}(M, U, N)$. The resolutions used in other treatments of differential homological algebra are Künneth objects whenever they yield the classical torsion product at the $E^2$ level.

It is very far from obvious that $H(U) \otimes_U N$ and $E_p^{-1} (X \otimes_U N)$ are independent of the choice of $X$. We shall prove this shortly, but we must first describe certain special objects of $\mathcal{M}_U$. These objects are precisely the most general filtered differential $U$-modules that can be expected to be of computational value.

**Definition 1.2.** Let $X \in \mathcal{M}_U$. $X$ is said to be a split object if there is a bigraded $R$-module $\overline{X}$, with $\overline{X}_{pq} = 0$ for $p < 0$, such that

(i) $X = \overline{X} \otimes U$ as a right $U$-module; the grading of $X$ is specified by $X_n = \sum_{p+q+r=n} \overline{X}_{pq} \otimes U_r$ for $n \in \mathbb{Z}$.

(ii) $F_p X = \sum_{m \leq p} \overline{X}_{m} \otimes U$ for $p \geq 0$.

$X$ may then be bigraded by $X_{pq} = \sum_{i+j=q} \overline{X}_{ij} \otimes U_j$, additively, $E^0_{pq} X = X_{pq}$.
The differential on $X_{p,q} \subseteq X$ necessarily has the form

$$d = \sum_{r \geq 0} d^r, \quad d^r: X_{p,q} \to X_{p-r, q+r-1},$$

where $\sum_{i+j=r} d^i d^j = 0$.

Since $X$ is a differential $U$-module and each $X_{p,q}^\alpha$ is a sub $U$-module of $X$, the $d^r$ satisfy the formula

$$d^r(xu) = d^r(x)u + (-1)^{\deg x} x d^r(u) \quad \text{and} \quad d^r(xu) = d^r(x)u \quad \text{if} \quad r > 0,$$

where $x \in X$ and $u \in U$.

If $\alpha: X \to M$ is a morphism in $\mathcal{M}_U$, then $X^\alpha$ is said to be an augmented split object. $X^\alpha$ is bigraded by $X_{-1,0}^\alpha = M$, and $X_{p,q}^\alpha = X_{p,q}$ for $p \geq 0$. The notation of (iii) is extended to $X^\alpha$ by the notational convention

$$d^{p+1} = \alpha: X_{p,q}^\alpha \to M_{p+1,q} \quad \text{and} \quad d^0 = -d_1: M_0 \to M_0.$$

Then formulas (iii) and (iv) remain valid with $X$ replaced by $X^\alpha$.

The notion of a split object generalizes the notion of a twisted differential structure, which was exploited by the first author for the study of fibrations in [15].

For historical reasons, differential homological algebra has been developed using only those split objects such that $d^r = 0$ for $r > 1$. ($d^0$ and $d^1$ are usually called "internal" and "external" differentials). This restriction is unnecessary and, in our view, undesirable.

By a morphism of split objects, or of augmented split objects, we understand a morphism in $\mathcal{M}_U^n$, or in $\mathcal{M}_U$. In the latter case, some elaboration may be illuminating.

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**Remarks.** A morphism $g: X^\alpha \to Y^\beta$ in $\mathcal{F}^+ \mathcal{M}_U$ between split objects augmented over $M$ and $M'$ has components

$$g^r: X_{p,q}^\alpha \to Y_{p-r, q+r}^\beta, \quad 0 \leq r \leq p+1.$$

Define $k: M \to M'$, $K: X \to Y$, and $t: X \to M'$ by letting

$$k = g^0 \text{ on } M, \quad K = \sum_{r=0}^{p} g^r \text{ on } X_{p,q}^\alpha, \quad \text{and} \quad t = g^{p+1} \text{ on } X_{p,q}^\alpha.$$

Then $dg = gd$ if and only if $k$ and $K$ are maps in $\mathcal{M}_U$ and $\beta K - k\alpha = dt + td$. Thus $g$ determines and is determined by the homotopy commutative diagram in $\mathcal{M}_U$:

$$\begin{array}{ccc}
X & \xrightarrow{K} & Y \\
\alpha \downarrow & & \beta \downarrow \\
M & \xrightarrow{k} & M'
\end{array}$$

and the specific homotopy $t$ between $\beta K$ and $k\alpha$. Even when $X$ or $Y$ is not split, such a triple $(K,k,t)$ determines a morphism $g: X^\alpha \to Y^\beta$ in $\mathcal{F}^+ \mathcal{M}_U$.

Our theory centers about those split objects singled out in the following definition.

**Definition.** A split object $X$ is said to be distinguished if each $X_{p,q}^\alpha$ is a projective $R$-module and if $d^0 = 0$ on $X$, so that $d^1 = 1 \otimes d$ on $X = X^\alpha \otimes U$. A resolution $X^\alpha$ of $M$ is said to be distinguished if $X$ is a distinguished object of $\mathcal{F} \mathcal{M}_U$.

Distinguished objects play a role in our theory analogous to that played by projective complexes in classical homological algebra. K"unneth objects
are analogous to flat complexes, and we have the following observation.

**Lemma 1.5.** If \( X \) is a split object of \( J_{MU} \) such that \( d^0(X) \subseteq X \) and the boundaries and homology of \( X \) with respect to \( d^0 \) are \( R \)-flat, then \( X \) is a Künneth object. In particular, any distinguished object is a Künneth object.

**Proof.** By the Künneth theorem [22, V. 10.2], if \( N \in U' \mathcal{U} \), then
\[
E^1(X \otimes U N) = E^1(X \otimes N) \cong H(X \otimes N; d^0 \otimes 1 + 1 \otimes d) = H(X; d^0) \otimes HN.
\]
Taking \( N = U \), we see that \( E^1X \) is \( HU \)-flat and that
\[
E^1X \otimes HUHN = H(X; d^0) \otimes HU \otimes HUHN = E^1(X \otimes U N).
\]

**Remark 1.6.** By restricting attention to objects suitably graded so that the spectral Künneth theorem [22, XII. 12.1] applies to \( X \otimes N \), the assumption that the boundaries of \( X \) under \( d^0 \) are \( R \)-flat can be replaced by the weaker assumption that \( X \) is \( R \)-flat.

The following simple result provides an analog to the standard comparison of resolutions.

**Theorem 1.7.** Let \( X^\alpha \) be a distinguished object augmented over \( M \) and let \( Y^\beta \) be a resolution of \( M' \). Let \( k: M \to M' \) be a morphism in \( \mathcal{M}_U \).

(i) There is a map \( g = (k, k, t): X^\alpha \to Y^\beta \) in \( J^+ \mathcal{M}_U \).

(ii) If \( X^\alpha \) is a resolution of \( M \) and \( Y \) is a Künneth object, then, for any map \( g \) as in (i) and for any \( N \in U' \mathcal{U} \),
\[
E^2(X \otimes U N; E^2(X \otimes U N) \to E^2(Y \otimes U N)
\]
coincide with
\[
\text{Tor}^H_{U}(Hk, 1); \text{Tor}^H_{U}(HM, HN) \to \text{Tor}^H_{U}(HM', HN);
\]
therefore \( H(X \otimes U N) \to H(Y \otimes U N) \) is an isomorphism if
\( Hk: HM \to HM' \) is an isomorphism.

**Proof.** (i) Let \( G_p \) denote the set of elements \( y \in F_p Y^\beta \) such that \( d(y) \in F_{p-1} Y^\beta \). Observe that since \( Y \) is a resolution of \( M' \),
\[
F_{p-1} Y^\beta \cap \text{Ker} \ d = d(G_p).
\]
Indeed, if \( z \in F_{p-1} Y^\beta \) and \( d(z) = 0 \), then there exists \( y \in F_{p-1} Y^\beta \) such that \( d(y) = z \) and \( y \in F_{p-1} Y^\beta \). Thus \( y \) is a cycle in \( F_{p-1} Y^\beta \) whose class in \( E^1 Y^\beta \) must be a boundary. Define
\[
g = k \circ M \xi X^\alpha \quad \text{and assume inductively that } \quad g: X^\alpha \to F_{p-1} Y^\beta
\]
has been defined for \( m < p \) with \( dg = gd \). Since \( d^0 = 0 \) on \( X^\alpha \) and \( d(\xi \xi_{p1}) \) is contained in \( F_{p-1} X^\alpha \), thus \( gd \) is defined on \( X^\alpha \) and, since \( gd = gd = 0 \),
\[
gd(\xi \xi_{p1}) \subseteq F_{p-1} Y^\beta \cap \text{Ker} \ d = d(G_p).
\]
Since \( X^\alpha \) is \( R \)-projective, there exists \( g: X^\alpha \to G_p \) such that \( dg = gd \), and we extend this map by \( U \)-linearity to obtain \( g: X^\alpha \to F_p Y \) such that \( dg = gd \). If \( g' \) is another such map, define \( h = 0 \) on \( M \) and assume that
\[
h: X^\alpha \to F_{p+1} Y \text{ has been defined for } m < p \text{ with } dh + hd = g - g'.
\]
Then
(g - g' - hd) + F_y \cap \text{Ker} d = d(G_{p+1}).

Again, h[X_p \to F_{p+1} Y] such that dh = g - g' - hd is obtained by the projectivity of X over R and by U-linearity.

(ii) Since both X^p and Y^p are Künneth resolutions,

\[ E^1(K \otimes i) = E^1(K \otimes i; E^1 X \otimes_{HU} HN) \to E^1 Y \otimes_{HU} HN. \]

Now \[ E^1 K : E^1 X \to E^1 Y \] is a map over \[ Hk : HM \to HM' \] from an HU-projective resolution of HM to an HU-flat resolution of HM'. The induced map on homology is therefore \[ \text{Tor}_H^H(Hk, 1), \] as can be verified by an easy exercise in classical homological algebra. The last statement follows by [22, XI.3.4].

We shall prove in the next section that every \( M \in M_U \) admits a distinguished resolution. Therefore, given two Künneth resolutions of M, we can compare them both to a distinguished resolution (although there need not be any direct comparison map between the given resolutions). Thus we have the following corollary which, finally, validates Definition 1.1.

**Corollary 1.8.** \( \text{Tor}_U^U(M, N) \) and the algebraic Eilenberg-Moore spectral sequence are well-defined functors with domain the category \( \mathcal{A} \) of triples \( (M, U, N) \); the morphisms of \( \mathcal{A} \) are triples \( (k, f, t) : (M, U, N) \to (M', U', N') \), where \( f : U \to U' \) is a morphism of differential algebras and \( k : M \to M' \) and \( f : N \to N' \) are \( f \)-equivariant morphisms of differential R-modules.

If \( Hk, Hf, \) and \( Hf \) are isomorphisms, then \( \text{Tor}_U^U(k, t) \) is also an isomorphism.

**Proof.** Let \( X^a \) and \( Y^p \) be distinguished U and U'-resolutions of M and M'. Regard M' and Y as U-modules via \( f \). There exists a morphism \( g = (K, k, t) : X^a \to Y^p \) in \( \mathcal{A} M_U \) and we define

\[ \text{Tor}_U^2(k, t) = H(K \otimes i) \] and \( E^r(k, f, t) = E^r(K \otimes i), \quad r \geq 2 \)

where \( K \otimes i : X \otimes_{U} N \to Y \otimes_{U} N' \).

The theorem clearly implies that these maps are well-defined and that \( E^2(k, f, t) = \text{Tor}_U^H(Hk, Hf) \).

We shall also need the following corollary, which gives the external product for the differential torsion product.

**Corollary 1.9.** Let \( (M, U, N) \) and \( (M', U', N') \) be objects of \( \mathcal{A} \). Then there is a commutative and associative natural transformation

\[ \pi : \text{Tor}_U^U(M, N) \otimes \text{Tor}_U^U(M', N') \to \text{Tor}_U^U(M \otimes M', N \otimes N'); \]

\( \pi \) is an isomorphism if R is a field (or, more generally, under appropriate flatness hypotheses).

**Proof.** If \( X^a \) and \( (X')^a \) are distinguished U and U'-resolutions of M and M', then \( (X \otimes X')^a \otimes a' \) is a distinguished \( U \otimes U' \)-object augmented over \( M \otimes M' \) (and is a resolution under flatness hypotheses). Let \( Y^p \) be a distinguished \( U \otimes U' \)-resolution of \( M \otimes M' \), let \( g = (K, 1, t) : (X \otimes X')^a \otimes a' \to Y^p \) be a morphism in \( \mathcal{A} M_{U \otimes U'} \), and define \( \pi \) by the commutativity of the following diagram (where T is the interchange):
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\[ H(\bigotimes_{U} N) \otimes H(\bigotimes_{U} N') \xrightarrow{=} H(\bigotimes_{U} N \otimes X \bigotimes_{U} N') \xrightarrow{H(1 \otimes T \otimes 1)} H(\bigotimes_{U} N \otimes X' \bigotimes_{U} N') \xrightarrow{= \ H(1 \otimes T \otimes 1)} H(\bigotimes_{U} N \otimes N') \]

clearly \( \tau \) is well-defined, and the rest follows easily.

For applications of \( \text{Tor}_U^x(M, N) \) to homology, it is convenient to restrict attention to non-negatively graded \( R \)-modules. For application to cohomology, it is convenient to restrict attention to non-positively graded \( R \)-modules, regraded by non-negative superscripts; we then write \( \text{Tor}_U^x \) instead of \( \text{Tor}_U^x \). To form resolutions in this context, \( U \) must be connected, \( U^0 = R \). Then

\[ E_{pq}^r(M, U, N) = \text{Ext}_{r}^{P_q - q}(M, U, N) \]

is non-zero only if \( p \leq 0 \) and \( q \geq -p \). If \( U \) is simply connected, \( U^1 = 0 \), then the spectral sequence converges in the naive sense that the filtration is finite in each degree, since we can then employ split objects \( X \) such that \( X^{pq} = 0 \) for \( q < -2p \), where, of course, \( X^{pq} = X_{-p-q} \).

Our applications deal only with \( \text{Tor}_U \) but, for completeness, we give the analogous development of the differential \( \text{Ext}_U \) functor in the following remarks.

\textbf{Remarks 1.10.} Let \( M, M' \in \mathcal{M}_U \) and let \( \alpha : X \to M \) be a morphism in \( \mathcal{M}_U \). Let \( \mathcal{F} \mathcal{M}_U \). Filter \( \text{Hom}_U(X, M') \) by

\[ F_0 \text{Hom}_U(X, M') = \text{Hom}_U(X, M') \]

and observe that there is a natural map of differential \( R \)-modules

\[ \chi' : \text{Ext}_U^x(M, M') \to \text{Hom}_R^x(E^1_U X, HM') \]

(2) \( \chi'(f) = f \).

Call \( X \) a proper Kähnle object if each \( E^1_U X \) is a projective \( H_U \)-module and if the maps (2) and (2)' are both isomorphisms. By the universal coefficient theorem [10, VI, 3.1a], a split object \( X \) is a proper Kähnle object if \( d^0 X = X \) and if the boundaries and homology of \( X \) with respect to \( d^0 \) are projective \( R \)-modules; in particular, any distinguished object is a proper Kähnle object. We define

\[ \text{Ext}_U^x(M, M') = H(\text{Hom}_U(X, M')) \]

and

\[ E_r^x(M, U, M') = E_r(\text{Hom}_U(X, M')) \]

where \( X \) is any proper Kähnle resolution of \( M \). These are well-defined functors of \( M \) and \( M' \) by Theorem 1.6(ii), the evident analog of Theorem 1.6(ii), and the existence of distinguished resolutions. Visibly,

\[ E_2^x(M, U, M') = \text{Ext}_R^x(HM, HM') \]

Via the standard adjunction isomorphism

\[ \text{Hom}_U(X, \text{Hom}_R(N, R)) \cong \text{Hom}_R(X \bigotimes_{U} N, R) \]

(and maps \( \chi' \)), we obtain natural transformations

\[ \text{Ext}_U^x(M, N^*) \to \text{Tor}_U^x(M, N)^* \]

and

\[ E_r^x(M, U, N^*) \to E_r^x(M, U, N)^* \]

By [10, VI, 5.11], these maps are isomorphisms if \( R \) is self-injective (e.g., if \( R \) is a field or if \( R = Z_n \) for any \( n \)).
§2. Realizations of resolutions

We here complete the theory of the previous section by proving the existence of distinguished resolutions. In fact, we shall prove that for $M \in \mathcal{M}_U$, any HU-free resolution of $HM$ can be realized as the $E^1$-term of a distinguished resolution of $M$. We shall then prepare for the geometric applications by giving an explicit example of such a distinguished resolution and using it to prove the algebraic theorem which underlies our results on the cohomology of homogeneous spaces and principal bundles.

**Theorem 2.1.** Let $M \in \mathcal{M}_U$ and suppose given an HU-projective resolution of $HM$ of the form

$$
\cdots \to X_{p,0} \otimes U \xrightarrow{d} X_{p-1,0} \otimes U \to \cdots \to X_{0,0} \otimes U \xrightarrow{d} HM \to 0,
$$

where each $X_{p,0}$ is a projective $R$-module. Then the filtered right $U$-module $X = \bigoplus_{p} X_{p,0}$ admits a differential $d$ and a map $\alpha : X \to M$ such that $X^\alpha$ is a distinguished resolution of $M$ and the complex $E^1X^\alpha$ coincides with the given resolution of $HM$.

**Proof.** The grading and filtration of $X$ are specified by (i) and (ii) of Definition 1.2, and we define $X^\alpha$ as a $Z$-graded filtered right $U$-module by formulas (i), (ii), and (iii) of Definition 1.1. (This makes sense even though $\alpha$ is not yet defined.) We bigrade $X$ and $X^\alpha$ as in the cited definitions and, as in formulas (iii) and (v) of Definition 1.2, we write $d^r$ for the components of the differential to be constructed on $X^\alpha$. The construction of $d$ on $X^\alpha$ will thus include the construction of $\alpha$. We must define $d^0 = -d$ on $M = X_{-1,0}$.

and $d^0 = 1 \otimes d$ on $X_{p,e} \otimes U = X_{p,e}$ for $p \geq 0$. For $r > 0$, we must have $d^R(xu) = d^R(x)u$, and it therefore suffices to define $d^R$ on $X_{p,e}$ for $1 \leq r \leq p+1$. Let $ZU$ and $ZM$ denote the cycles of $U$ and $M$. By the projectivity of $X_{p,e}$, we can choose maps $d^1$, such that the following diagrams commute, where the maps $\pi$ are the evident epimorphisms:

$$
\begin{array}{ccc}
X_{0,0} & \xrightarrow{d^1} & ZM \\
\otimes U & \xrightarrow{d} & HM \\
\otimes U & \xrightarrow{d} & HM
\end{array}
$$

and, if $p > 0$,

$$
\begin{array}{ccc}
X_{p,e} & \xrightarrow{d^1} & X_{p-1,e} \\
\otimes U & \xrightarrow{d} & X_{p-1,0} \otimes ZU \\
\otimes U & \xrightarrow{d} & X_{p-1,0} \otimes ZU
\end{array}
$$

With this choice for $d^1$, $E^1X^\alpha$ will clearly have the desired form. Note that $d^0 d^1 + d^1 d^0 = 0$ since this holds on each $X_{p,e}$. We now define $d^R$ on $X_{p,e}$ for $r \geq 2$ and $p \geq 1$ by induction on $p$ and, for fixed $p$, by induction on $r$.

We let $d^R = 0$ on $X_{p,e}$ for $r > p+1$, and all $d^R$ are thus defined on $X_{0,e}$.

Assume that all $d^R$ are defined on $X_{m,e}$ for $m < p$ and that $d^R$ is defined on $X_{p,e}$ for $n < r$. Define maps $e : X_{p,e} \to X_{p-r,e}$ and $f : X_{p,e} \to X_{p-r-1,e}$ by the formulas

$$
e = \sum_{ij=r, i > 0} d^i d^j$$

and

$$f = -\sum_{ij=r+1, i > 1} d^i d^j.
$$

Since $d^0 = 0$ on $X_{p,e}$, $e$ and $f$ are defined. We have that

$$d^0 d^a = \sum_{a+b=i, a > 0} d^a d^b$$

and

$$d^1 d^i = -\sum_{a+b=i+1, a \neq 1} d^a d^b$$

on $X_{m,e}$ for $m < p$. By substitution and the induction hypothesis,

$$d^0 e = -\sum_{i+j=r, a+b=1, a > 0} d^a b^a d^j = -\sum_{a > 0} b^{a} \sum_{b+j=r-3} d^a b^j = 0,$$
and
\[ d^1 e = \sum_{i+j=r, a+b=i+1, a \neq 1} d^2 d^1 = d^0 f + \sum_{a \neq 1, b+j=r+1-a} d^a d^b d^1 = d^0 f. \]

Since \( d^0 e = 0 \), \( e(\overline{X}_{p^*}) \subseteq \ker d^0 = \overline{X}_{p-r,*} \otimes ZU \). Consider the following diagram, in which the squares are commutative:

\[ \begin{array}{ccc}
\overline{X}_{p-r+1,*} \otimes ZU & \xrightarrow{\pi} & \overline{X}_{p-r+1,*} \otimes HU \\
\downarrow^{d^1} & & \downarrow^{\delta} \\
\overline{X}_{p,*} & \xrightarrow{e} & \overline{X}_{p-r,*} \otimes ZU & \xrightarrow{\pi} & \overline{X}_{p-r,*} \otimes HU \\
\downarrow^{f} & & \downarrow^{d^1} & & \downarrow^{\delta} \\
X_{p-r-1,*} \otimes ZU & \xrightarrow{\pi} & X_{p-r-1,*} \otimes HU \\
\end{array} \]

(The bottom row must be modified when \( r = p \); the middle row must be modified and the bottom row set equal to zero when \( r = p+1 \).) Since the right column is exact and since \( \pi d^0 = 0 \), we can choose a map \( \tilde{e} \) such that

\[ \pi e = \delta \tilde{e} = \pi d^1 \tilde{e}. \]

(We cannot in general achieve, nor do we desire, \( e = d^1 \tilde{e} \).) Obviously \( d^0 \tilde{e} = 0 \). If \( r = 2 \), then \( e = d^1 \tilde{e} \), hence \( \pi e = \delta \tilde{e} = 0 \), and we agree to choose \( \tilde{e} = 0 \). We now agree to replace \( d^{r-1} \) on \( \overline{X}_{p^*} \) by \( d^{r-1} - \tilde{e} \); observe that this neither changes \( d^1 \) nor affects the validity of the induction hypothesis \( \sum_{i+j=r-1} d^i d^j = 0 \) on \( \overline{X}_{p^*} \). Finally, since \( \pi(\tilde{e} - e) = 0 \) and \( \ker \gamma = \text{Im } d^0 \), we can choose \( d^r : \overline{X}_{p^*} \to \overline{X}_{p-r,*} \) such that \( d^0 d^r = d^1 \tilde{e} - e \).

With our modified value for \( d^{r-1} \), this equation is equivalent to

\[ \sum_{i+j=r} d^i d^j = 0 \]

and the proof is complete.

Example 2.2. Let \( U \) be a differential \( R \)-algebra with a \( \cup_i \)-product (of degree 1) \( U \otimes U \to U \) such that if \( a \in U_p, b \in U_q \) and \( c \in U_r \), then

\begin{align*}
(i) & \quad d(a \cup_i b) = ab - (-1)^p ba - da \cup_i b - (-1)^q a \cup_i db \\
(ii) & \quad (ab) \cup_i c = (-1)^p a(b \cup_i c) + (-1)^q (a \cup_i c)b.
\end{align*}

Formula (i) says that \( U \) is homotopy commutative and formula (ii), the Hirsch formula, says essentially that \( \cup_i : U \to U \) is a derivation. Assume further that \( HU \) is the polynomial algebra \( P \{b_i, i \in I \} \), where \( I \) is a totally ordered indexing set; each \( b_i \) has even degree unless the characteristic of \( R \) is two. Finally, assume that \( U \) is augmented over \( R \) and that the induced augmentation of \( HU \) is the standard one, \( \varepsilon(b_i) = 0 \). Consider the Koszul resolution \( K(HU) \) of \( R \). By definition, \( K(HU) \) is the differential \( HU \)-algebra specified by

\[ \K(HU) = \mathcal{E} \{x_i | i \in I \} \otimes HU, \text{ bideg } x_i = (1, \deg b_i), \text{ with } \delta x_i = -b_i. \]

Here \( \mathcal{E} \) denotes an exterior algebra; clearly Theorem 2.1 applies to the long exact sequence

\[ \cdots \to \mathcal{E}(x_i^\ast) \otimes HU \xrightarrow{\delta} \mathcal{E}(x_i^\ast) \otimes HU \to \cdots \to \mathcal{E}(x_i^\ast) \otimes HU \xrightarrow{\delta} R \to 0. \]

Let \( K(U) = E \{x_i^\ast \} \otimes U \), and let \( \varepsilon : K(U) \to R \) be the evident augmentation. We shall define an explicit differential on \( K(U) \) such that \( K(U)^\varepsilon \) is a distinguished resolution of \( R \) and \( E^1 K(U)^\varepsilon \) is the exact sequence above.
Since $K(U)$ is to be a differential $U$-module, it suffices to specify $d$ on $E \{x_i\}$. Let $\alpha = (i_1, \ldots, i_p)$ be a sequence of indices, $i_1 \prec \cdots \prec i_p$. Define $x_\alpha = x_{i_1} \cdots x_{i_p} \in E \{x_i\}$, with $x_{\emptyset} = 1$ in $R = E_{\emptyset} \{x_i\}$ if $\emptyset$ is the empty sequence. Let $a_\beta \in U$ be a representative for $b_\beta \in HU$. Define $a_{\emptyset} = 0$, $a_{\alpha} = a_{i_1}$ if $\alpha = (i_1)$, and

$$a_{\alpha} = (\cdots (((a_{i_1} \cup a_{i_2}) \cup a_{i_3}) \cup \cdots ) \cup a_{i_p})$$

for $p > 1$.

Write $|\alpha| = p$ and observe that $\deg x_\alpha = |\alpha|$ and $\deg a_\alpha = |\alpha| - 1 \mod 2$ if $\text{char } R + 2$. With these notations, define $d$ on the basis elements $x_\alpha$ of $E \{x_i\}$ by the formula

$$(\nu) \quad d(x_\alpha) = - \sum_{\beta \subset \alpha} \tau(\alpha, \beta)x_{\alpha-\beta} \otimes a_\beta,$$

where the sum is taken over all increasing sub-sequences $\beta = (i_{\beta(1)}, \ldots, i_{\beta(|\beta|)})$ of $\alpha$ and $\alpha-\beta$ is the increasing sub-sequence of $\alpha$ obtained by deleting the $i_{\beta(j)}$. $\beta$ thus denotes both a sub-sequence of $\alpha$ and the corresponding increasing function $\{i_1, \ldots, i_r\} \rightarrow \{1, \ldots, p\}$, and

$$(\nu) \quad (\alpha, \beta) = (-1)^{|\alpha-\beta| + \varepsilon(\beta)}$$

where $\varepsilon(\beta) = \sum_{t=1}^{p} (-1)^{|\beta(t)-t|} \cdot (\alpha, \beta).

Since $a_{\emptyset} = 0$, $d^0 = 0$ on $E \{x_i\}$. A check of signs verifies that

$$d^1(x_{i_1} \cdots x_{i_p}) = - \sum_{k=1}^{p} (-1)^{p-k-1} x_{k} \cdots x_{i_p} \otimes a_{i_k}.$$

Hence $E^1 K(U) = K(HU)$ will surely hold once we have verified that $dd = 0$.

By the definition,

$$dd(x_\alpha) = \sum_{\beta \subset \alpha} \tau(\alpha, \beta) \sum_{\gamma \subset \alpha-\beta} \tau(\alpha-\beta, \gamma)x_{(\alpha-\beta)-\gamma} \otimes a_{\gamma} \quad (-1)^{|\alpha-\beta|} x_{\alpha-\beta} \otimes d(a_\beta).$$

For a fixed $\beta \subset \alpha$, the coefficient in $U$ of $(-1)^{|\alpha-\beta|}\tau(\alpha, \beta)x_{(\alpha-\beta)-\gamma}$ is

$$d(a_\beta) + \sum_{\delta \subset \beta} (-1)^{|\alpha-\beta|+\varepsilon(\delta)} \tau(\alpha, \delta) \tau(\alpha-\delta, \beta-\delta) a_{\beta-\delta},$$

A combinatorial argument, by induction on $|\alpha-\beta|$, proves that

$$(-1)^{|\alpha-\beta|} \tau(\alpha, \delta) \tau(\alpha-\delta, \beta-\delta) = \tau(\beta, \delta), \quad \delta \subset \beta \subset \alpha.$$

(The details of this check of signs are left to the reader.) It therefore suffices to prove that, for all $\beta$,

$$(\nu) \quad d(a_\beta) = \sum_{\delta \subset \beta} \tau(\beta, \delta) a_{\beta-\delta}.$$

If $|\beta| = 0$ or $|\beta| = 1$, this is trivial. Assume it for all $\gamma$ such that $|\gamma| < |\beta|$. Let $\beta = (j_1, \ldots, j_r)$ and define $\gamma = (j_1, \ldots, j_{r-1})$. By (iv),

$$(\nu) \quad a_\beta = a_\gamma \cup_{i} a_{i_r};$$

by (i), the induction hypothesis, and (ii)

$$(\nu) \quad d(a_\beta) = a_\gamma \cup_{i} d(a_{i_r}).$$

By use of (iv), it is easy to verify that formula (vii) coincides with the desired formula (vi). Thus $dd = 0$ and $K(U)^e$ is indeed a distinguished resolution of $R^e$.

Together with our general theory, the example implies the following theorem.
Theorem 2.3. Let $U$ and $N$ be differential $R$-algebras which are homotopy commutative via $\cup_1$-products that satisfy the Hirsch formula.

Let $f: U \to N$ be a morphism of differential algebras which commutes with the $\cup_1$-products. Assume that

(a) $U$ is augmented, and $HU$ is a polynomial algebra.

(b) There exists a morphism $g: N \to HN$ of differential algebras such that $Hg: HN \to HN$ is the identity map and $g$ annihilates all $\cup_1$-products in $N$.

Then $\text{Tor}_n^U(R, N) = \text{Tor}_n^{HU}(R, HN)$ as a graded $R$-module; that is, $\text{Tor}_n^U(R, N) = \sum_{p+q=n} \text{Tor}_{pq}^{HU}(R, HN)$ for all integers $n$.

Proof. Note that we are not only asserting that the spectral sequence collapses; we are also asserting that, additionally, there is no extension problem. Regard $N$ and $HN$ as differential left $U$-modules via $f$ and $g$.

Clearly, by Corollary 1.8, the map

$$\text{Tor}_n^U(R, N) \to \text{Tor}_n^{HU}(R, HN)$$

is an isomorphism. We may compute $\text{Tor}_n^U(R, HN)$ by use of the complex

$$K(U) \otimes_U HN = E \langle x_i \rangle \otimes HN$$

with differential specified by the formula

$$d(x \otimes y) = \sum_{\beta < \alpha} \tau(\alpha, \beta)x_{\alpha-\beta} \otimes gf(a_\beta)^* y, \quad y \in HN.$$

(The unexplained notations are those of the previous example.) Since $f$ commutes with $\cup_1$-products and $g$ annihilates $\cup_1$-products, and since $a_\beta = 0$, the only non-zero terms occur for those $\beta$ such that $|\beta| = 1$. In other words, $d = d^1 \otimes 1$ on $KU \otimes U HN$, which means that

$$KU \otimes U HN = K(HU) \otimes HU HN$$

as complexes. Thus $\text{Tor}_n^U(R, HN) = \text{Tor}_n^{HU}(R, HN)$.

Observe that the conclusion remains valid if $g$ annihilates not all $\cup_1$-products but only those of the form $f(a_\alpha)$. Moreover, even if $g$ does not annihilate these elements, the complex $E \langle x_i \rangle \otimes HN$ with the explicit differential (*) is still suitable for the computation of $\text{Tor}_n^U(R, N)$. If we can effectively compute the elements $gf(a_\alpha) \in HN$ in terms of the homology classes $gf(a_\alpha) = (Hf)(b_\alpha)$, then we have an algorithm for the effective computation of $\text{Tor}_n^U(R, N)$. The following lemma shows that, no matter how $g$ is obtained, the elements $gf(a_\alpha)$ in principle depend only on the $gf(a_\alpha)$.

Lemma 2.4. Let $N$ and $g: N \to HN$ be as in Theorem 2.3, but do not assume that $g$ annihilates $\cup_1$-products. Let $z_i$ and $z'_i$, $1 \leq i \leq p$, be homologous cycles of $N$. Then

$$g(z_1 \cup_1 \cdots \cup_1 z_p) = g(s_1' \cup_1 \cdots \cup_1 s_p')$$

and we can thus define "homology operations" $HN \otimes \cdots \otimes HN \to HN$ by

$$\{s_1 \} \cup_1 \cdots \cup_1 \{s_p \} = g(s_1' \cup_1 \cdots \cup_1 s_p').$$

(Here all $\cup_1$-products are to be associated in the order indicated by the parentheses in formula (iv).)
Theorem 2.3. Let \( U \) and \( N \) be differential \( R \)-algebras which are homotopy commutative via \( \cup^1 \)-products that satisfy the Hirsch formula. Let \( f: U \to N \) be a morphism of differential algebras which commutes with the \( \cup^1 \)-products. Assume that

(a) \( U \) is augmented, and \( HU \) is a polynomial algebra.

(b) There exists a morphism \( g: N \to HN \) of differential algebras such that \( Hg: HN \to HN \) is the identity map and \( g \) annihilates all \( \cup^1 \)-products in \( N \).

Then \( \text{Tor}_n^U(R, N) = \text{Tor}_n^{HU}(R, HN) \) as a graded \( R \)-module; that is,

\[
\text{Tor}_n^U(R, N) = \sum_{p+q=n} \text{Tor}^{HU}_{pq}(R, HN)
\]

for all integers \( n \).

Proof. Note that we are not only asserting that the spectral sequence collapses; we are also asserting that, additively, there is no extension problem. Regard \( N \) and \( HN \) as differential left \( U \)-modules via \( f \) and \( gf \).

Clearly, by Corollary 1.8, the map

\[
\text{Tor}_n^U(1, g): \text{Tor}_n^U(R, N) \to \text{Tor}_n^U(R, HN)
\]

is an isomorphism. We may compute \( \text{Tor}_n^U(R, HN) \) by use of the complex

\[
K(U) \otimes_U HN = E \{ x_i \} \otimes HN
\]

with differential specified by the formula

\[
d(x_i \otimes y) = - \sum_{\beta \subset \alpha} \tau(\alpha, \beta)x_{\alpha-\beta} \otimes gf(\beta) \cdot y, \quad y \in HN.
\]

(The unexplained notations are those of the previous example.) Since \( f \) commutes with \( \cup^1 \)-products and \( g \) annihilates all \( \cup^1 \)-products, and since

\[
a_\emptyset = 0, \text{ the only non-zero terms occur for those } \beta \text{ such that } |\beta| = 1. \text{ In other words, } d = d^1 \otimes 1 \text{ on } KU \otimes_U HN, \text{ which means that } KU \otimes_U HN = K(U) \otimes_HU HN
\]

as complexes. Thus \( \text{Tor}_n^U(R, HN) = \text{Tor}_n^{HU}(R, HN) \).

Observe that the conclusion remains valid if \( g \) annihilates not all \( \cup^1 \)-products but only those of the form \( f(a_\alpha) \). Moreover, even if \( g \) does not annihilate these elements, the complex \( E \{ x_i \} \otimes HN \) with the explicit differential (\(*)\) is still suitable for the computation of \( \text{Tor}_n^U(R, N) \). If we can effectively compute the elements \( g(f(a_\alpha)) \in HN \) in terms of the homology classes \( g(f(a_\alpha)) = (HF)(b_i) \) where \( HU = P \{ b_i \} \), then we have an algorithm for the effective computation of \( \text{Tor}_n^U(R, N) \). The following lemma shows that, no matter how \( g \) is obtained, the elements \( g(f(a_\alpha)) \) in principle depend only on the \( g(f(a_\alpha)) \).

Lemma 2.4. Let \( N \) and \( g: N \to HN \) be as in Theorem 2.3, but do not assume that \( g \) annihilates all \( \cup^1 \)-products. Let \( x_i \) and \( x_i^1 \), \( 1 \leq i \leq p \), be homologous cycles of \( N \). Then

\[
g(x_1 \cup_1 \ldots \cup_1 x_p^1) = g(x_1^1 \cup_1 \ldots \cup_1 x_p^1)
\]

and we can thus define "homology operations" \( \text{HN} \otimes \ldots \otimes HN \to HN \) by

\[
(\{ x_i \} \cup_1 \ldots \cup_1 \{ x_p \}) = g(\{ x_i^1 \} \cup_1 \ldots \cup_1 \{ x_p^1 \}).
\]

(Here all \( \cup^1 \)-products are to be associated in the order indicated by the parentheses in formula (\(*)\).)
Proof. Note that $g$ annihilates both commutators and boundaries and that the Hirsch formula implies the formula
\[
[a, b] \cup_1 c = (-1)^{\deg a \deg b} [a, b \cup_1 c] + (-1)^{\deg b} \deg c [a \cup_1 c, b],
\]
$a, b, c \in N$, where the brackets indicate commutators.

We first show that $g(z_1 \cup_1 \cdots \cup_1 z_p) = 0$ if $z_1 = d(y_1)$ for any $i$. To see this, let $x = z_1 \cup_1 \cdots \cup_1 z_{i-1} \cup_1 y_1 \cup_1 z_{i+1} \cup_1 \cdots \cup_1 z_p$. Then, calculating modulo the commutator ideal of $N$ and neglecting precise control of signs, we find that
\[
d(x) = -d(z_1 \cup_1 \cdots \cup_1 z_{i-1} \cup_1 y_1 \cup_1 z_{i+1} \cup_1 \cdots \cup_1 z_{p-1}) \cup_1 z_p
\]
\[= \cdots = \pm d(z_1 \cup_1 \cdots \cup_1 z_{i-1} \cup_1 y_1 \cup_1 z_{i+1} \cup_1 \cdots \cup_1 z_p)
\]
\[= \pm z_1 \cup_1 \cdots \cup_1 z_p \pm d(z_1 \cup_1 \cdots \cup_1 z_{i-1}) \cup_1 y_1 \cup_1 z_{i+1} \cup_1 \cdots \cup_1 z_p
\]
\[= \cdots = \pm z_1 \cup_1 \cdots \cup_1 z_p.
\]

Thus $g(z_1 \cup_1 \cdots \cup_1 z_p) = \pm g(d(x)) = 0$. Reverting to the hypotheses of the lemma, we conclude that
\[
0 = \sum_{i=1}^p g(z_1 \cup_1 \cdots \cup_1 z_{i-1} \cup_1 z_i \cup_1 z_{i+1} \cup_1 \cdots \cup_1 z_p)
\]
\[= g(z_1 \cup_1 \cdots \cup_1 z_p) - g(z_1 \cup_1 \cdots \cup_1 z_p).
\]

3. The differential torsion product and geometry

We here show how to identify the cohomology and homology of certain topological spaces with differential torsion products. The arguments we shall give are essentially those of Eilenberg and Moore [14 (dualized) and 33], but in our framework these arguments lead to somewhat sharper results.

For the application of Tor to cohomology, we shall need two easy technical lemmas. Recall that a commutative ring $R$ is Noetherian if every ideal of $R$ is finitely generated. The following result is a direct consequence of [10, VI exer. 3, 4].

Lemma 3.1. Let $R$ be a commutative Noetherian ring. Then any finitely generated flat $R$-module is projective and the dual of any projective $R$-module is flat. In particular, the singular cochains $C^*(X; R)$ are flat (but not in general projective) for any space $X$.

Lemma 3.2. Let $C$ be a non-negatively graded $\mathbb{Z}$-free complex such that $HC$ is of finite type, let $R$ be a commutative Noetherian ring, and let $G$ be an $R$-module. Then the morphism of differential $R$-modules
\[
\mu : \text{Hom}_\mathbb{Z}(C, R) \otimes_R G \to \text{Hom}_\mathbb{Z}(C, G)
\]
defined by $\mu(f \otimes g)(c) = f(c) \cdot g$ induces an isomorphism on homology.

Proof. Since $HC$ is of finite type, there exists a $\mathbb{Z}$-free complex $C'$ of finite type and a morphism $\psi : C' \to C$ of complexes such that $Hy$ is an isomorphism. Since $\mu$ is natural, we therefore have a commutative diagram
\[
\text{Hom}_Z(C, R) \otimes_R G \xrightarrow{\mu} \text{Hom}_Z(C, G) \\
\text{Hom}(\nu, 1) \otimes_1 \text{Hom}(\nu, 1)
\]

The bottom arrow is easily seen to be an isomorphism. The vertical arrows induce isomorphisms on homology by the classical universal coefficient theorem and the spectral Künneth Theorem; the latter result applies to the vertical arrow since
\[
\text{Hom}_Z(C, R) \cong \text{Hom}_Z(C \otimes_1 R, R)
\]

flat over \(R\) by the previous lemma.

**Theorem 3.3.** Let \(p: Y \to B\) be a Serre fibration with fibre \(F\) and let \(A \to B\) be a map. Consider the fibre product diagram

\[
\begin{array}{ccc}
F & \to & F \\
\downarrow & & \downarrow \\
E & \to & Y \\
\downarrow q & & \downarrow p \\
A & \to & B
\end{array}
\]

such that \(A\) and \(B\) are connected and have integral homology of finite type. Let \(R\) be a commutative Noetherian ring taken as coefficients in homology and assume that \(\pi_1 B\), and therefore also \(\pi_1 A\), acts trivially on \(H^* F\). Let \(C^*\) denote the normalised singular cochains with coefficients in \(R\). Then there exists an isomorphism of \(R\)-modules
\[
\delta: \text{Tor}^C_{*B}(C^*_A, C^*_Y) \to H^* E
\]

which is natural on commutative diagrams of the form

\[
\begin{array}{ccc}
E & \to & Y \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}
\]

where both the front and back faces satisfy the stated hypotheses.

**Proof.** Choose base-points \(a \in A\) and \(b \in B\) such that \(f(a) = b\) and take \(p^{-1}(b) = Y = q^{-1}(a)\). Replace the total singular complexes of \(A\) and \(B\) by the subcomplexes consisting of those simplices with all vertices at the base-points and replace the total singular complexes of \(E\) and \(Y\) by the inverse images of these subcomplexes. Chains and cochains are to be formed from these altered simplicial sets; by Corollary 1.8, this does not change the differential torsion product. We require some recollections about the Serre spectral sequence. Filter \(C^*_Y\) by letting \(F^p C^*_Y\) be the submodule generated by those simplices \(y\) such that \(p(y)\) is an iteration of degeneracy operators applied to a simplex of \(C^*_Y\). Give \(C^*_Y\) the dual filtration \(F^p C^*_Y = (C^*_Y/F^p C^*_Y)^*\). Filter \(C^*_B\) by degrees,
\[
F^p C^*_B = \sum_{i \geq p} C^i B,
\]

and give \(C^*_B \otimes C^*_Y\) the tensor product filtration. A simple calculation demonstrates that the module product
\[
C^*_B \otimes C^*_Y \xrightarrow{F^p \otimes 1} C^*_Y \otimes C^*_Y \xrightarrow{\gamma} C^*_Y
\]
is filtration-preserving, where \(\gamma\) is the standard (Alexander-Whitney) cup product. There is thus an induced map
\[
\gamma^*_0: C^*_B \otimes E^0 C^*_Y \to E^p C^*_Y.
\]
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early $E^0_\eta Y = C^*B$ and $d_0 \circ \tau_0 = \tau_0 \circ (1 \otimes \delta)$, hence $\tau_0$ induces

$$\tau_1 : C^{P_q} \otimes H^q F \rightarrow E^{P_q}_{1} C^*Y.$$

A dualization argument, $\tau_1$, factors as a composite

$$C^{P_q} \otimes H^q F \rightarrow C^{P_q} \otimes (H^q F) \rightarrow E^{P_q}_{1} C^*Y.$$

Serre's calculation (carried out semi-simplicially) [35, 17] and by the viality of the local coefficients, $\tau_1$ is an isomorphism of complexes (we all only need that $H_1$ is an isomorphism). By the previous lemma, $H_1$ also is an isomorphism. Of course, this discussion applies equally well to $E \rightarrow A$. Now let $X^*$ be a distinguished $C^*$-resolution of $C^*A$. Define

$$X \otimes_{C^*B} C^*Y \rightarrow C^*E$$

to be the composite

$$X \otimes_{C^*B} C^*Y \rightarrow X \otimes_{C^*B} C^*E \rightarrow C^*A \otimes_{C^*B} C^*E \rightarrow C^*E \otimes_{C^*B} C^*E \rightarrow C^*E.$$

It is trivial to verify that $\theta_\tau$ is a well-defined morphism of filtered differential $R$-modules, where $C^*Y$ and $C^*E$ are given the Serre filtration, $X$ filtered by degree, and $X \otimes_{C^*B} C^*Y$ is given the tensor product filtration.

te that, additively,

$$X \otimes_{C^*B} C^*Y = \{X \otimes C^*B, Y, \text{ where } X = X \otimes C^*B.$$

early $E_0^{P_q}(X \otimes_{C^*B} C^*Y) = \sum_{\Delta \in \Delta^Q} X \otimes E_0^{P_q} C^*Y$ and $d_0 = 1 \otimes d_0 \Delta$ (since the algebra on $X$ raises degree by one). Thus

$$E_1(X \otimes_{C^*B} C^*Y) = \overline{X} \otimes E_1 C^*Y = X \otimes_{C^*B} E_1 C^*Y$$

the K"unneth theorem, since $\overline{X}$ is $R$-flat. Now consider the following gram:

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$$\begin{array}{c}
X \otimes_{C^*B} (C^*A \otimes H^*Y) = X \otimes H^*F \xrightarrow{\alpha \otimes 1} C^*A \otimes H^*Y \\
\end{array}$$

Since $H_1$ is an isomorphism, $H(1 \otimes \tau_1)$ is an isomorphism by the algebraic Eilenberg-Moore spectral sequence. Since $H_2$ is an isomorphism (because $X^*$ is a resolution), $H(\alpha \otimes 1)$ is an isomorphism by the spectral K"unneth theorem. Thus $E_2$ is an isomorphism and $\theta = H_0$ gives the desired natural isomorphism of $R$-modules.

In the following three corollaries, we give the main properties of the resulting "geometric Eilenberg-Moore spectral sequence" defined by

$$E_\tau = E_2(C^*A, C^*B, C^*Y);$$

corporate the hypotheses of the theorem. We shall need the following important technical lemma, which is proven by Eilenberg and Moore in [14, § 17].

Lemma 3.4. For spaces $X$ and $Y$, the shuffle product

$$\eta : C^*Y \otimes C^*Y \rightarrow C^*(X \times Y)$$

is a morphism of differential coalgebras.

Corollary 3.5. $(E_\tau)$ is a spectral sequence of differential algebras such that $E_2 = \text{Tor}_{H^*B}(H^*A, H^*Y)$ as an algebra and $(E_\tau)$ converges to the algebra $H^*_X$. The algebra $H^*_X$.

Proof. Only the statements about products require proof. We give $\text{Tor}_{C^*B}(C^*A, C^*Y)$ the following composite product, where $\tau$ is as defined in Corollary 1.9, $\Delta$ is the diagonal map, and $\gamma$ is the natural map.
from the tensor product of duals to the dual of a tensor product:

\[ \text{Tor}_{\mathbb{C}^*} \left( \mathbb{C}^* A, \mathbb{C}^* Y \right) \otimes \text{Tor}_{\mathbb{C}^*} \left( \mathbb{C}^* A, \mathbb{C}^* Y \right) \]

\[ \xrightarrow{\pi} \mathbb{C}^* (\mathbb{C}^* A \otimes \mathbb{C}^* A, \mathbb{C}^* Y \otimes \mathbb{C}^* Y) \]

\[ \text{Tor}_{\mathbb{C}^*} (\mathbb{C}^* A \otimes \mathbb{C}^* A, \mathbb{C}^* Y \otimes \mathbb{C}^* Y) \]

\[ \xrightarrow{\text{Tor}_{\mathbb{C}^*} \left( \mathbb{C}^* (A \times A), \mathbb{C}^* (Y \times Y) \right)} \]

\[ \text{Tor}_{\mathbb{C}^*} (\mathbb{C}^* A, \mathbb{C}^* Y) \]

Since the \( \gamma, \eta, \) and \( \Delta \) are all morphisms of differential algebras, the stated maps are well-defined by Corollary 1.8. \( \text{Tor}_{\mathbb{C}^*} (\mathbb{C}^* A, \mathbb{C}^* Y) \) is an isomorphism by the algebraic Eilenberg-Moore spectral sequence. It is trivial to verify that all maps are induced by filtration-preserving maps on the chain level and, visibly, the induced product on \( E_2 \) is that determined by the external product (with \( \mathbb{C}^* \) replaced by \( H^* \)) and by the (commutative) cup products. A laborious diagram chase, in which the key fact is the homotopy commutativity of \( \mathbb{C}^* E \), demonstrates that \( \theta \) is an isomorphism of algebras.

The result above is stronger than that of Eilenberg and Moore [14, dualized] in that we require no flatness hypotheses on cohomology for the identification of the \( E_2 \)-term. In particular, no hypotheses as to torsion are required in the case \( R = \mathbb{Z} \). The following corollary is an easy consequence of the definition of \( \theta^* \).

**Corollary 3.6.** The following diagram is commutative, where the maps \( e^* \) are the cohomology algebra units and the unlabelled arrows are the evident epimorphisms:

\[ \begin{array}{ccc}
R \otimes H^* Y & \to & H^* Y \\
\varepsilon^* \otimes 1 & \downarrow & \downarrow e^* \\
H^* A \otimes H^* Y & \to & H^* A \otimes H^* B^* \otimes H^* Y = E_\infty^0 \to E_\infty^0 \to \text{Tor}_{\mathbb{C}^*} (\mathbb{C}^* A, \mathbb{C}^* Y) \to H^* E \\
1 \otimes e^* & \downarrow & \downarrow q^* \\
H^* A \otimes R & \to & H^* A
\end{array} \]

The following addendum to our general theory will be used to obtain the analog for suspension of the preceding corollary.

**Definition 3.7.** Let \( (M, U, N) \in \mathcal{A} \) and let \( m \in M_0 \) and \( n \in N_0 \) be fixed cycles. Let \( \mu : U \to M \) and \( \nu : U \to N \) be the morphisms of differential \( U \)-modules given by \( \mu(u) = mu \) and \( \nu(u) = un \). Define the suspension homomorphism

\[ \sigma : \text{Ker} H\mu \cap \text{Ker} H\nu \to E_1 \otimes (M, U, N) \cong \text{Tor}^U (M, N) / H^0 \text{Tor}^U (M, N) \]

as follows. Let \( u \) be a cycle of \( U \) such that the homology class \( \{u\} \) is in \( \text{Ker} H\mu \cap \text{Ker} H\nu \); choose \( m' \in M \) and \( n' \in N \) such that \( d(m') = mu \) and \( d(n') = un \). Choose an \( HU \)-resolution \( X \otimes HU \to HM \) of \( HM \), where \( X \) is \( R \)-projective, and realize it by a distinguished resolution \( X^0 \) of \( M \), \( X = \overline{X} \otimes U \). By the opening remark in the proof of Theorem 1.7, there is an
element \( x \in F_0 X^2 \) such that \( d(x) = m \) (that is, by 1.2, \( \sigma(x) = m \)) and, since \( d(xu) = d(m') \), there is then an element \( y \in F_1 X^2 \) such that \( d(y) = xu - m' \).

Identify \( X \) with \( X^2/M \) as a filtered differential \( U \)-module and let \( \overline{x} \) and \( \overline{y} \) denote the images of \( x \) and \( y \) in \( X \). Then

\[
d(\overline{x} \otimes n) = \overline{xu} \otimes n = \overline{xu} \otimes \overline{u} = \overline{x} \otimes d(n') = d(\overline{x} \otimes n')
\]

in \( X \otimes U \mathbb{N}_1 \), and \( \tau \{ u \} \) is defined to be the image in \( E_{1,1}^0(M, U, N) \) of \( \{ \overline{x} \otimes n - \overline{x} \otimes n' \} \) in \( F_1 \text{Tor}^U(M, N) \). It is easily verified that \( \tau \) is well-defined (although the class obtained in \( F_1 \text{Tor}^U(M, N) \) is not) and depends only on \( m \) and \( n \). If \( d = 0 \) on \( M, U, \) and \( N \), then \( \tau \) specializes to the classical suspension \( \sigma; \ker m \cap \ker n \to \text{Tor}^U_{1,1}(M, N) \) and, in the general case, the following diagram is easily verified to be commutative:

\[
\begin{array}{ccc}
\text{Ker } H_\mu \cap \text{Ker } H_\nu & \xrightarrow{\sigma} & E_{1,1}^0(M, U, N) \\
\downarrow & & \downarrow \\
\text{Tor}^U_{1,1}(M, N) & \text{cohomological suspension} & E_{1,1}^0(M, U, N)
\end{array}
\]

By the diagram, the general definition of \( \tau \) is in fact not logically required; it should be thought of as an explicit description of the composite specified in the diagram. With this description, the following corollary is again an easy consequence of the definition of \( \theta_\sigma \).

**Corollary 3.5.** Let \( \phi; \ker p^* \cap \ker f^* \to \text{H}_d^* \mathbb{E} \) be the additive relation determined by \( \phi^*(b) = \{ q \cdot a - g \cdot y \} \), where \( b \in C^\cdot B \) is a cocycle such that \( p^*(b) = \delta(y) \) and \( f^*(b) = \delta(a) \). Then \( \phi^* \) coincides with the composite additive relation

\[
\begin{align*}
\text{Ker } p^* \cap \text{Ker } f^* & \xrightarrow{\sigma} E_{1,1}^0 \to E_{1,1}^0 \to \text{F}_1 \text{Tor}^U_{1,1}(C^\cdot A, C^\cdot Y) \\
& \xrightarrow{\theta} \text{H}_d^* \mathbb{E}
\end{align*}
\]

(where \( \sigma \) is defined with respect to the identity elements of \( \text{H}_d^* \mathbb{E} \) and \( \text{H}_d^* \mathbb{E} \)).

We turn next to the application of \( \text{Tor} \) to homology. The following theorem is conceptually dual to Theorem 3.3.

**Theorem 3.9.** Let \( G \) be a topological monoid. Let \( G \) act from the right on a space \( A \) and let \( p; A \to B \) be a Serre fibration such that \( B \) is connected, \( p(a) = p(a') \) for \( a \in A \) and \( a' \in G \), and the map \( \mu; G \to p^{-1}(b) \) defined by \( \mu(g) = ag \) is a weak homotopy equivalence for all \( b \in B \) and \( a \in p^{-1}(b) \). Then, for any commutative ring \( R \) of coefficients, there is an isomorphism of \( R \)-modules

\[
\phi; \text{Tor}^U_{1,1}(C_\phi A, R) \to H_d^* B
\]

which is natural on pairs of commutative diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow & & \downarrow \\
P & \xrightarrow{p^*} & B'
\end{array}
\quad \text{and} \quad \begin{array}{ccc}
A \times G & \xrightarrow{f \times h} & A' \times G' \\
\downarrow & & \downarrow \\
P & \xrightarrow{p^*} & B'
\end{array}
\]

where \( p \) and \( p' \) satisfy the stated hypotheses with respect to monoids \( G \) and \( G' \) and \( h; G \to G' \) is a morphism of monoids. Suppose further that \( G \) acts from the left on a space \( Y \) and that the map \( q; A \times G Y \to B \) defined by \( q(a) = p(a) \) for \( a \in A \) and \( y \in Y \) is a Serre fibration such that \( \nu; Y \to q^{-1}(b) \), \( \nu(y) = (a, y) \), is a weak homotopy equivalence for all \( b \in B \).
and a \( p^{-1}(b) \). Then there is a natural isomorphism of \( R \)-modules
\[
\psi : \text{Tor}^{C_{s}G}_{n} (C_{s}A, C_{s}Y) \to H_{n}(A \times G).
\]

**Proof.** Recall that \( A \times G = (A \times Y)/(\sim) \), where \( \sim \) is the equivalence relation generated by \((ag, y) \sim (a, gy)\). Choose a base-point \( b \in B \), replace the total singular complex of \( B \) by the subcomplex consisting of those simplices with all vertices at \( b \), and replace the total singular complexes of \( A \) and \( A \times G \) by the inverse images under \( p \) and \( q \) of this subcomplex.

Chains are to be formed from these altered simplicial sets; this does not change the differential torsion products. Of course, the \( C_{s}G \)-module product on \( C_{s}A \) is the composite
\[
C_{s}A \otimes C_{s}G \xrightarrow{\gamma} C_{s}(A \times G) \xrightarrow{\cong} C_{s}A.
\]

Give \( C_{s}A \) the Serre filtration; a trivial calculation demonstrates that each \( F_{p}C_{s}A \) is a sub \( C_{s}G \)-module of \( C_{s}A \). Observe that
\[
E_{pq}^{1} C_{s}A = C_{s}B \otimes H_{G}^{Q} = (C_{s}B \otimes H_{0}G) \otimes_{H_{0}G} H_{G}^{Q} = E_{pq}^{0} C_{s}A \otimes H_{0}G^{Q}.
\]

Let \( X^{\sigma} \) be a distinguished \( C_{s}G \)-resolution of \( C_{s}A \). Filter \( X \) by
\[
F_{p}X = \sum_{1 \leq p} X^{p} \otimes C_{s}G,
\]
where \( X = \sum X^{p} \otimes C_{s}G \);

thus \( F_{p}X \) is the sub \( C_{s}G \)-module of \( X \) generated by \( \sum_{1 \leq p} X^{p} \). Clearly, \( \sigma : X \to C_{s}A \) is a filtration-preserving homotopy isomorphism.

\[
E_{pq}^{0} X = \sum_{1 \leq p} X^{p} \otimes C_{s} G^{Q} \text{ and } d^{0} = 1 \otimes d;
\]

therefore
\[
E_{pq}^{1} X = \sum_{1 \leq p} X^{p} \otimes C_{s} G^{Q} = (\sum_{1 \leq p} X^{p} \otimes H_{0}G^{Q}) \otimes_{H_{0}G} H_{G}^{Q} = E_{pq}^{0} C_{s}A \otimes H_{0}G^{Q}.
\]

Now \( E_{s0}^{1} = E_{s0}^{1} \otimes H_{0}G^{Q} : E_{s0}^{1} C_{s}B \otimes H_{0}G^{Q} \to E_{s0}^{1} C_{s}A \otimes H_{0}G^{Q} \). By the comparison theorem [22, XII. 11.1] applied (after a verification that the possible non-commutativity of \( H_{0}G \) presents no difficulties) to the ground ring \( H_{0}G \), \( E^{1}_{s0} \psi \) induces an isomorphism on homology. By the spectral Künneth theorem [22, XII. 12.1], it follows that:
\[
E_{s0}^{1} \otimes H_{0}G^{Q} C = (X \otimes H_{0}G)^{Q} \otimes_{H_{0}G} H_{G}^{Q} = (C_{s}B \otimes H_{0}G)^{Q} \otimes_{H_{0}G} H_{G}^{Q} = C_{s}B
\]
induces an isomorphism \( \overline{\phi} \) on homology. Since the complex \( \overline{X} \) may here be identified with \( X \otimes C_{s}G^{Q} \), this proves the first part. For the second part filter \( X \) as above, give \( X \otimes C_{s}G \) the induced filtration
\[
F_{p}(X \otimes C_{s}G) = F_{p}X \otimes C_{s}G,
\]
and give \( C_{s}(A \times G) \) the Serre filtration.

Define \( \psi_{a} \) to be the composite
\[
X \otimes C_{s}G \otimes C_{s}Y \xrightarrow{\cong} C_{s}A \otimes C_{s}G \otimes C_{s}Y \xrightarrow{\gamma} C_{s}(A \times G).
\]

where \( \gamma \) is induced from the shuffle product. Then \( \psi_{a} \) is filtration-preserving and an easy calculation demonstrates that \( E_{s1}^{1} \psi_{a} \) coincides with the map
\[
E_{s1}^{1} \otimes H_{0}G^{Q} \bar{C} = E_{s1}^{1} X \otimes H_{0}G^{Q} H_{G}^{Q} = E_{s0}^{1} C_{s}A \otimes H_{0}G^{Q} H_{G}^{Q}.
\]

By the first part, \( E_{s0}^{1} \psi \) induces an isomorphism on homology. By the spectral Künneth theorem, so does \( E_{s1}^{1} \psi_{a} \); \( \psi = H \psi_{a} \) is the desired isomorphism.

The following three corollaries give the main properties of the "geometric Eilenberg-Moore spectral sequence" defined by
\[
E^{F} = E^{F}(C_{s}A, C_{s}G, C_{s}Y) \text{ under the hypotheses of the theorem. Additively}
\]
Corollary 3.10. If $H_\bullet(A \times_G Y)$ and each $E^r$, $r \geq 2$, is $R$-flat, then $(E^r)$ is a spectral sequence of differential coalgebras which converges as a $H_\bullet^G$ coalgebra from $\operatorname{Tor}(H_\bullet A, H_\bullet Y)$ to $H_\bullet(A \times_G Y)$.

Proof. This is easily verified by use of the following composite coproduct on $\operatorname{Tor}^G(C_\bullet A, C_\bullet Y)$:

$$
\begin{align*}
\operatorname{Tor}^G(C_\bullet A, C_\bullet Y) & \xrightarrow{\pi^{-1}} \operatorname{Tor}^G(C_\bullet A \otimes C_\bullet A, C_\bullet Y) \\
& \xrightarrow{\hat{\mu} \otimes 1} \operatorname{Tor}^G(C_\bullet A, C_\bullet Y) \otimes \operatorname{Tor}^G(C_\bullet A, C_\bullet Y)
\end{align*}
$$

Here $\hat{\mu} = \xi A_\bullet$ is the standard chain level coproduct; Lemma 3.4, together with simple diagram chasing, implies that $(\hat{\mu}, \hat{\nu}, \hat{\pi})$ is a morphism in the category $\mathcal{A}$ of Corollary 1.8. The map $\pi$ is an isomorphism by Corollary 1.9.

In the remaining corollaries, we consider the commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\nu} & Y \\
\mu \downarrow & & \downarrow 1 \\
A & \xrightarrow{j} & A \times_G Y
\end{array}
$$

where $\mu(g) = a_0 g$ and $(g) = s \gamma_0^{i+1}$ for fixed base-points $a_0 \in A$ and $\gamma_0 \in Y$.

and where $i(y) = (a_0, y)$ and $j(a) = (a, \gamma_0)$.

Corollary 3.11. The following diagram is commutative:

$$
\begin{array}{ccc}
R \otimes H_\bullet Y & \xrightarrow{(a_0) \otimes 1} & H_\bullet A \\
\downarrow & & \downarrow i_a \\
H_\bullet A \otimes H_\bullet Y & \xrightarrow{H_\bullet A \otimes H_\bullet Y} & H_\bullet A \otimes H_\bullet Y
\end{array}
$$

(Here $a_0$ and $\gamma_0$ are regarded as the evident inclusion maps).

Corollary 3.12. Let $\sigma_\bullet: \ker \mu_\bullet \cap \ker \nu_\bullet \rightarrow H_\bullet(A \times_G Y)$ be the additive relation determined by $\sigma_\bullet(g) = (j_\bullet(a) - i_\bullet(y))$, where $g: C_\bullet G$ is a cycle such that $\mu_\bullet(g) = d(a)$ and $\nu_\bullet(g) = dy$. Then $\sigma_\bullet$ coincides with the composite additive relation

$$
\ker \mu_\bullet \cap \ker \nu_\bullet \xrightarrow{\sigma_\bullet} E^2_{1,1} \xrightarrow{F_1} E^\infty \leftarrow F_1 \operatorname{Tor}^G(C_\bullet A, C_\bullet Y) \xrightarrow{\pi} H_\bullet(A \times_G Y).
$$

Of course, analogous results apply to the spectral sequence $(E^r(C_\bullet A, C_\bullet G, R))$ which converges from $\operatorname{Tor}^G(H_\bullet A, R)$ to $H_\bullet B$ under the hypotheses of the theorem; here the diagram above should be replaced by the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\nu} & Y \\
\mu \downarrow & & \downarrow 1 \\
A & \xrightarrow{p} & B
\end{array}
$$
§4. Principal bundles, homogeneous spaces, and Postnikov systems

In this section, we combine the algebraic results of section 2 with the geometric results of section 3 to study the cohomology of various spaces of interest. Throughout, we shall assume given one of the following diagrams

\[
\begin{array}{c}
\text{Figure 1} \\
E \quad R \quad Y \\
q \\
A \quad f \quad B
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\text{Figure 2} \\
E \\
A \quad f \quad B
\end{array}
\]

and we shall tacitly assume the following data relative to a fixed commutative Noetherian ring \( R \) of coefficients.

\text{Figure 1: } p \text{ is a fibration with fibre } F, \ q \text{ is the fibration induced by } f, \ A \text{ and } B \text{ are connected and have integral homology of finite type, } \pi_1 B \text{ and } \pi_1 A \text{ act trivially on } H^* F, \text{ and } Y \text{ is } R\text{-acyclic, } H^* Y = R.

\text{Figure 2: } f \text{ is a fibration with fibre } E, \ B \text{ is connected and has integral homology of finite type, and } \pi_1 B \text{ acts trivially on } H^* E.

In both cases, Theorem 3.3 clearly implies that \( H^* E \) is isomorphic to \( \text{Tor}_{C*A}^* (R, C*A) \). We wish to use the results of section 2 to study \( H^* E \). In the appendix, we shall rederive the well-known facts that \( C*B \) and \( C*A \) are differential \( R \)-algebras which are homotopy commutative via \( \cup_1 \)-products that satisfy the Hirsch formula and that \( i^*: C*B \rightarrow C*A \) is a morphism of differential algebras which commutes with the \( \cup_1 \)-products. In order to apply Theorem 2.3 (or Example 2.2 and Lemma 2.4) to the computation of \( H^* E \), we shall of course have to assume that \( H^* B \) is a polynomial algebra.
and we shall have to make use of spaces $A$ for which there exists a homology isomorphism $g: C^*_A \to H^*_A$ of differential algebras. Fortunately, such spaces $A$ do exist; in fact, we shall prove the following two theorems in the Appendix.

**Theorem 4.1.** Let $BT^n$ be the classifying space of the $n$-torus $T^n$. Then, for any commutative ring $R$, there is a morphism

$$g: C^*(BT^n; R) \to H^*(BT^n; R)$$

of differential algebras such that $g$ induces the identity on homology.

Moreover, $g$ annihilates all $\cup_i$-products.

**Theorem 4.2.** Let $\pi_i$ be a finitely generated Abelian group and let $n_i \geq 1$, $1 \leq i \leq q$. Assume that $\pi_i$ has no 4-torsion (elements of order 4) if $n_i = 1$. Then there is a morphism

$$g: C^*(\prod_{i=1}^q K(\pi_i, n_i); Z_2) \to H^*(\prod_{i=1}^q K(\pi_i, n_i); Z_2)$$

of differential Hopf algebras such that $g$ induces the identity on homology.

Moreover, there is an explicit algorithm for the computation of $g(x_1 \cup \cdots \cup x_p)$ in terms of $g(x_1), \ldots, g(x_p)$ for cocycles $x_j \in C^*(\prod_{i=1}^q K(\pi_i, n_i); Z_2)$.

With these facts at our disposal, we can proceed to the applications. We begin with the following theorem.

**Theorem 4.3.** Assume that $H^*_B$ is a polynomial algebra and that, for some $n$, there is a map $e: BT^n \to A$ such that $H^*_B T^n$ is a free $H^*_A$-module via $e^*$. Then $H^*_B$ is isomorphic to $\text{Tor}_{H^*_B}(R, H^*_A)$ as a graded $R$-module via $e^*$. Then $H^*_B$ is isomorphic to $\text{Tor}_{H^*_B}(R, H^*_A)$ as a graded $R$-module.

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via $e^*$. Then $H^*_B$ is isomorphic to $\text{Tor}_{H^*_B}(R, H^*_A)$ as a graded $R$-module

(where the bigraded torsion product is regraded by total degree), and $H^*_B$ admits a filtration such that its associated graded algebra is isomorphic to $\text{Tor}_{H^*_B}(R, H^*_A)$ as a bigraded $R$-algebra.

**Proof.** The second statement will follow from the first in view of Corollary 3.5. We must prove that

$$\text{Tor}_{C^*_B}(R, C^*_A) = \text{Tor}_{H^*_B}(R, H^*_A)$$

as a graded $R$-module. If $A = BT^n$, the result follows immediately from Theorems 2.3 and 4.1. The reduction of the general case to this special case is due to Baum [3] and goes as follows. $e^*: C^*_A \to C^*_B$ induces a morphism of graded $R$-modules

$$\text{Tor}(1, e^*); \text{Tor}_{C^*_B}(R, C^*_A) \to \text{Tor}_{C^*_B}(R, C^*_B T^n)$$

and a morphism of spectral sequences

$$E_1^{r, 1, e^*}; E_1^{r}(R, C^*_B, C^*_A) \to E_1^{r}(R, C^*_B, C^*_B T^n),$$

where $C^*_B T^n$ is a $C^*_B$-module via the composite $e^* e^*$. Since $H^*_B T^n$ is a free $H^*_A$-module via $e^*: H^*_A \to H^*_B T^n$, there exists a free graded $R$-module $S$ with unit $\eta: R \to S$ and an isomorphism of $H^*_A$-modules $j: H^*_B T^n \to H^*_A \otimes S$ such that the following diagram is commutative:

$$\begin{array}{ccc}
H^*_A & \xrightarrow{e^*} & H^*_B T^n \\
\text{Tor}_{H^*_B}(R, H^*_A) & \xrightarrow{j} & H^*_A \otimes S
\end{array}$$
Clearly \( \text{Tor}_H^*(R, H^*A \otimes S) = \text{Tor}_H^*(R, H^*A) \otimes S \), and we thus have the further commutative diagram

\[
\begin{array}{ccc}
\text{Tor}_H^*(R, H^*A) & \xrightarrow{\text{Tor}(1, e^\eta)} & \text{Tor}_H^*(R, H^*B) \\
\downarrow & & \downarrow \\
\text{Tor}_H^*(R, H^*A) \otimes R & \xrightarrow{1 \otimes \eta} & \text{Tor}_H^*(R, H^*A) \otimes S
\end{array}
\]

Therefore \( E_2(1, 1, e^\eta) \) is a monomorphism. It follows both that

\[E_2(R, C^*B, C^*A) = E_\infty(R, C^*B, C^*A)\]

and that the additive extensions from the \( E_\infty \)-term \( \text{Tor}_H^*(R, H^*A) \) to \( \text{Tor}_C^*(R, C^*A) \) are trivial, since any non-trivial differential or extension would quite obviously force the existence of a non-trivial differential or extension in the spectral sequence \( E_0(R, C^*B, C^*B) \).

The hypothesis on \( A \) is very often satisfied by the classifying space \( BG \) of a compact connected Lie group \( G \). To show this in full generality, we shall need the following elementary lemma.

**Lemma 4.1.** Let \( H^*E \) be a flat \( R \)-module. Then \( E_2 = H^*B \otimes H^*E \) in the Serre spectral sequence of Figure 2; moreover, \( E_2 = E_\infty \) if and only if \( q^*: H^*A \to H^*E \) is an epimorphism, in which case \( H^*A = H^*B \otimes H^*E \) as a left \( H^*B \)-module via \( f^* \).

**Proof.** The first statement follows from Lemma 3.2 and the Kunneth theorem (see the discussion of the Serre spectral sequence in the proof of Theorem 3.3 for more details). The second statement holds since \( q^* \) is the edge homomorphism \( H^*A \to E_\infty \to E_2^0 \to H^*E \) and since there clearly can be no \( H^*B \)-module extension problem when \( E_2 = E_\infty \).

The following key theorem was proven by Bott and Samelson [7] when \( R = Z \). The result for arbitrary \( R \) is an obvious consequence.

**Theorem 4.5.** Let \( T \) be a maximal torus in a compact connected Lie group \( G \). Then \( G/T \) has no torsion, hence \( H^*(G/T) \) is a free \( R \)-module, and \( H^*(G/T) \) is generated as an \( R \)-algebra by its elements of degree two.

**Corollary 4.6.** Let \( e: BT \to BG \) be the map of classifying spaces induced by the inclusion of a maximal torus \( T \) in a compact connected Lie group \( G \). Assume that \( G \) has no \( p \)-torsion for any prime \( p \) which divides the characteristic of \( R \). Then \( H^*BT = H^*BG \otimes H^*(G/T) \) as a left \( H^*BG \)-module via \( e^* \).

**Proof.** The hypothesis on \( G \) is equivalent to the statement that \( H^*BG \) is a polynomial algebra on even degree generators. By the lemma and theorem, \( E_2 = H^*BG \otimes H^*(G/T) \) in the Serre spectral sequence of \( e \), and \( E_2 = E_\infty \) since all elements of \( E_2 \) have even degree. The results follows.

In view of the corollary, Theorems A and B of the introduction are indeed special cases of Theorem 4.3.

In [26], a discussion of Borel's results on the cohomology of homogeneous spaces was promised. Baum's paper [3] has since appeared and covers much of the same ground. The relevant statements (Baum's 5.3, 3.10, and 6.2) deserve to be better known, and will therefore also be given here. The
The following obvious result applies to either Figure 1 or Figure 2.

**Lemma 4.7.** Let $H^*A$ be a flat $H^*B$-module. Then $H^*E = R \otimes_{H^*B} H^*A$ as an algebra, and $q_2^*: H^*A \to H^*E$ is the evident quotient map.

**Proof.** $E_2^{**} = E_2^0 = R \otimes_{H^*B} H^*A$ is the Eilenberg-Moore spectral sequence, and the result follows from Corollaries 3.5 and 3.6.

The lemma above should be compared with Lemma 4.4.

Our proof of the following algebraic lemma is rather different from that of Baum, which uses power series rings and Poincaré series; in both proofs, the basic ideas go back to Macaulay [20, published in 1916].

**Lemma 4.8.** Let $\Lambda = P(x_1, \ldots, x_n)$ and $\Gamma = P(y_1, \ldots, y_n)$ be polynomial algebras on $n$ positive degree generators over a field $R$. Let $f: \Lambda \to \Gamma$ be a morphism of $R$-algebras and assume that $\Gamma$ is finitely generated as a $\Lambda$-module via $f$. Then $\Gamma$ is a free $\Lambda$-module.

**Proof.** Let $x_i = f(x_i)$ and let $\{w_1, \ldots, w_q\}$ be a set of generators for $\Gamma$ over $\Lambda$. Let $s = \min \deg(y_j)$ and $t = \max \deg(w_j)$. If $rs + t$, then any product of $r$ elements of the maximal ideal $M = (y_1, \ldots, y_n)$ of $\Gamma$ is a linear combination $\sum a_j x_j w_j$, where $a_j \in \Lambda$ and $\deg a_j > 0$. Thus $a_j \in (x_1, \ldots, x_n)$ and $M^r$ is contained in the ideal $I = (x_1, \ldots, x_n)$. Therefore $M$ is the only prime ideal of $\Gamma$ which contains $I$. The rest follows easily from the results of Auslander and Buchsbaum [2]. $\Lambda$ has global dimension $n$ by the Hilbert syzygy theorem, hence $\Lambda$ is regular by [2, Proposition 4.5]. We have just shown that $I$ has rank $n$ [see 2, p.390], and it follows by [2, Proposition 4.11] that $\mathcal{I}_1, \ldots, \mathcal{I}_n$ is a $\Gamma$-sequence. This means that $x_i$ is not a zero divisor in $\Gamma/(x_1, \ldots, x_n-1)$ and it follows that $x_1, \ldots, x_n$ is a $\Gamma$-sequence [2, p.391] in the sense that $x_i$ is not a zero divisor for the $\Lambda$-module $\Gamma/(x_1, \ldots, x_{i-1})\Gamma$. By [2, Proposition 1.4] and the definition [2, p.390] of homological dimension,

$$n = \text{gl. dim } \Lambda \geq \text{hd } \Lambda \left(\Gamma/(x_1, \ldots, x_n)\right) = n + \text{hd } \Gamma.$$

Therefore $\text{hd } \Lambda = 0$ and $\Gamma$ is $\Lambda$-projective. By a standard argument [10, Theorem VIII.6.1 or 22, Lemma VII.6.2], it follows that $\Gamma$ is $\Lambda$-free. The reader worried about our use of [2] for the study of graded rings is invited either to verify for himself that the results of [2] go through in the graded case or to take direct sums over degrees and to then observe that $\text{hd } \Lambda = 0$ in the resulting ungraded situation implies $\text{hd } \Lambda = 0$ in the original graded situation.

**Lemma 4.9.** In Figure 2, if $H^*E$ is a finitely generated $R$-module and $H^*B$ is a Noetherian ring, then $H^*A$ is a finitely generated $H^*B$-module.

**Proof.** It follows from Lemma 3.2 and the spectral Kunneth theorem that the $E_2$-term $H^*(B; H^*E)$ of the Serre spectral sequence is finitely generated as an $H^*B$-module. Its differentials are morphisms of $H^*B$-modules, and $E_{\infty}$ is the associated graded object of $H^*A$ with respect to a filtration by sub $H^*B$-modules, hence the conclusion.
Of course, by the Hilbert basis theorem, a polynomial algebra on finitely many generators over a commutative Noetherian ring is itself a Noetherian ring. The following theorem is now an immediate consequence of Lemma 4.4 and the preceding three lemmas. It contains Borel's results [5, §29 and §30] on subgroups of maximal rank and also his results in [6] on subgroups of maximal 2-rank. Thus this theorem explains algebraically the analogy described by Borel [6] between maximal torii in real cohomology and maximal elementary 2-groups in mod 2 cohomology.

**Theorem 4.10.** Let \( H \) be a closed subgroup of a compact Lie group \( G \) and let \( R \) be a field such that \( H^*B^G \) and \( H^*B^H \) are polynomial algebras on the same number of generators and such that \( \tau_1^G \) acts trivially on \( H^*(G/H) \). Then \( H^*(G/H) = R \otimes_{H^*B^G} H^*B^H \) as an algebra and \( H^*B^H = H^*B^G \otimes H^*(G/H) \) as a left \( H^*B^G \)-module.

Returning to the discussion of applications of section 2, we note that the essential content of Theorem 4.3 is that \( H^*E \) is determined as an \( R \)-module by \( f^*: H^*B \to H^*A \) and that there is an explicitly given complex, namely \( K(H^*B) \otimes_{H^*B} H^*A \), whose differential is determined by \( f^* \) and whose homology is \( H^*E \). Seen in this light, the following theorem is just as satisfactory as Theorem 4.3. For the remainder of this section, we tacitly assume that \( R = Z_2 \). We retain the assumptions about Figures 1 and 2.

**Theorem 4.11.** Let \( A = \bigoplus_{i=1}^{q} K(\tau_1, n_i) \), where \( n_i \geq 1 \), \( \tau_1 \) is a finitely generated Abelian group, and \( \tau_1 \) has no 4-torsion if \( n_i = 1 \). Assume that \( H^*B \) is a polynomial algebra. Then \( H^*E \) is determined as a \( Z_2 \)-vector space by \( f^*: H^*B \to H^*A \), and \( H^*E \) admits a filtration such that its associated graded algebra is also determined by \( f^* \).

**Proof.** As in the proof of Theorem 2.3, \( Tor^R_{C*B}(Z_2, C^*A) \cong Tor^R_{C*B}(Z_2, H^*A) \) may be computed by means of the complex \( K(C^*B) \otimes C^*B H^*A \) with differential specified by formula (\( s \)); here \( f \) is to be replaced by \( f^*: C^*B \to C^*A \) and \( g: C^*A \to H^*A \) is given by Theorem 4.2. If \( H^*B = P(h_i) \) and if \( a_i \in C^*B \) represents \( h_i \), then the elements \( g^p(a_i, \cup a_i \ldots \cup a_i) \) in (\( s \)) are explicitly computable in terms of the elements \( g^p(a_i) = f^p(b_i) \) by the methods explained at the end of the appendix. Thus we have an explicitly given complex whose differential is determined by \( f^* \) and whose homology is \( H^*E \). The last statement holds since we can keep track of filtration and so compute \( E_\infty \) as an algebra.

If \( B \) in Theorem 4.11 satisfies the same hypotheses as does \( A \), then \( E \) (in Figure 1) is called a 2-stage space, or a generalized two-stage Postnikov system. We have thus determined how to compute the mod 2 cohomology of such spaces. If \( E \) is stable, in the sense that \( f^*: H^*B \to H^*A \) is a morphism of Hopf algebras, then, as shown by Smith [37, Corollary 4.12], \( Tor^R_{H^*B}(Z_2, H^*A) \) is generated as an algebra by its elements of homological degrees zero and minus one and therefore \( E_2 = E_\infty \) in the Eilenberg-Moore spectral sequence. In the unstable case, the following example of Schochet [34] shows that \( E_2 \) need not equal \( E_\infty \) in general.
Example 4.12. In Figure 1, let \( B = K(Z_2, 4) \) with fundamental class \( i \), let \( A = K(Z_2 \oplus Z_2, 2) \) with fundamental classes \( j \) and \( k \), and let \( f^*(i) = jk \).

Then \( E_2 \neq E_3 \). In fact, Schochet defines an element \( \overline{c} \in K(C^* B) \otimes_{C^* B} H^* A \) such that, in the notation of our formula 1.2 (iii), \( d^1(\overline{c}) = 0 \) and \( d^2(\overline{c}) \) is not in the image of \( d^1 \) (of course, \( d^0 \) is identically zero). It follows that \( \overline{c} \) represents an element \( c \in E_2 \) such that \( d_2(c) \neq 0 \) in the spectral sequence.

Actually, of course, Schochet's example precedes our theory and he works in \( B(C^* B) \otimes_{C^* B} H^* A \); if the elements \( [a] \otimes [a'] \in B(C^* B) \) in his notation are replaced by the exterior products \( xx' \in K(C^* B) \), where \( x \) and \( x' \) are the exterior algebra generators of \( K(C^* B) \) corresponding to the polynomial generators \( \{a\} \) and \( \{a'\} \) of \( H^* B \), then the above description of his argument is correct. His calculation of \( d^2(\overline{c}) \) provides a good illustration of the use of the formulas developed in the appendix for the computation of \( \cup \)-products.

Remarks 4.13. Assume that \( H^* B \) is a polynomial algebra. Let \( A' = \bigotimes_{i} K(\tau_i, n_i) \), with \( \tau_i \) and \( n_i \) as in Theorem 4.11, and assume that there is a map \( e : A' \to A \) such that \( H^* A' \) is a free \( H^* A \)-module via \( e^* \). Then by the proof of Theorem 4.3,

\[ \text{Tor}(1, e^*) : \text{Tor}_{H^* B}(Z_2, H^* A) \to \text{Tor}_{H^* B}(Z_2, H^* A') \]

is a monomorphism, and Theorem 4.14 provides an algorithm for the computation of \( \text{Tor}_{C^* B}(Z_2, C^* A') \). Unfortunately, this information does not appear to be sufficient to determine \( H^* E \). For this reason, the claims made by the second author in part (b) of the theorem and corollaries of [26] must be
§ 5. Matric Massey products

We here use the theory of sections 1 and 2 to study the relationship between differential torsion products and matric Massey products, as defined by the second author in [27]. Our results will demonstrate that matric Massey products provide the conceptually appropriate notion of decomposability in a variety of geometric and algebraic situations.

It will be expedient first to record analogs for the differential torsion products of some of the standard properties of the classical torsion product.

Lemma 5.1. Let \((M, U, N) \in \mathcal{O}\) and let \(X^\alpha\) and \(Y^\beta\) be distinguished resolutions of \(M\) and \(N\). Then

\[
\text{Tor}_n^U(M, N) = H(X \otimes_U N) = H(X \otimes_U Y) = H(M \otimes U Y). \]

Proof. Given \(X \otimes_U Y\) the tensor product filtration and observe that \(E^2(X \otimes_U Y) = E^4(X \otimes_U HU Y)\) as a differential R-module. The maps

\[
\alpha \otimes Y: X \otimes_U Y \to X \otimes_U N \quad \text{and} \quad \alpha \otimes Y: X \otimes_U Y \to M \otimes Y, \]

are filtration-preserving and induce isomorphisms on the \(E^2\)-level by the classical case of the present result.

Lemma 5.2. If \(0 \to N' \to N \xrightarrow{f} N'' \to 0\) is a short exact sequence in \(\text{Tor}_n^U(M, N)\), then there is a long exact sequence

\[
\cdots \to \text{Tor}_n^U(M, N') \to \text{Tor}_n^U(M, N) \to \text{Tor}_n^U(M, N'') \to \cdots.
\]

The connecting homomorphism \(\partial\) is natural and filtration-preserving; it is filtration-decreasing if \(g\) restricts to an epimorphism on cycles.

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Proof. Let \(X^\alpha\) be a distinguished resolution of \(M\). Since

\[
X \otimes_U N = \bar{X} \otimes N \quad \text{as an R-module and} \quad X \quad \text{is R-projective}, \quad \text{we have a short exact sequence}
\]

\[
0 \to X \otimes_U N' \to X \otimes_U N \to X \otimes U N'' \to 0.
\]

The desired natural long exact sequence follows. A cycle \(y^n \in \text{F}_p(X \otimes_U N^n)\) can certainly be lifted to an element \(y \in \text{F}_p(X \otimes_U N); \) obviously \(d^0 y^n = 0\) and, if \(g\) is an epimorphism on cycles, then \(y\) can be so chosen that \(d^0 y = 0\) and therefore \(dy \in \text{F}_{p-1}(X \otimes_U N)\).

Definition 5.3. Define the augmentation \(\lambda: \text{Tor}_n^U(M, N) \to \text{H}(M \otimes U N)\) by \(\lambda = \text{H}(a \otimes 1)\), where \(X^\alpha\) is any distinguished resolution of \(M\); \(\lambda\) is independent of the choice of \(X^\alpha\) and is natural by Theorem 1.7. When \(d = 0\) on \(M, U, N\), \(\lambda\) specializes to the classical isomorphism between

\[
\text{Tor}_n^U(M, N) \quad \text{and} \quad M \otimes U N.
\]

We shall generalize the evaluation of \(\lambda\) in the classical case by proving (Corollary 5.9) that the image of \(\lambda\) is the set of all elements of all matric Massey products in \(\text{H}(M \otimes U N)\). Thus, although we do not have an axiomatization of \(\text{Tor}_n^U(M, N)\), we do have generalizations to the differential torsion product of all of the axioms for the classical torsion product.

The map \(\lambda\) is of geometric interest since the maps \(\theta\) of Theorem 3.3 and \(\psi\) of Theorem 3.9 all factor through it. In fact, \(\theta\) is the composite

\[
\text{Tor}_{C^B}(C^A, C^* Y) \xrightarrow{\lambda} \text{H}(C^A \otimes C^* Y) \xrightarrow{H^* E} H^* E.
\]


where $\xi^*: C_E^* \otimes E \to C_E^*$ is given by $\xi^*(a \otimes y) = q^*(a)g^*(y)$; in
more suggestive notation, write the fibre product $E$ as $A \times B_Y$ and think
of $\xi^*$ as the map induced by the dual of the Alexander-Whitney map
\[\Psi: C_S(A \times Y) \to C_S A \otimes C_S Y.\]
Similarly, $\psi$ is the composite
\[\text{Tor}^C C_S G \xrightarrow{\lambda} H(C_S A \otimes C_S Y) \xrightarrow{H \pi} H_S(A \times Y),\]
where $\pi$ is induced from the shuffle map $\eta: C_S A \otimes C_S Y \to C_S(A \times Y)$. Our
description of $\lambda$ will thus imply that every element of $H^k(A \times Y)$ or of
$H_S(A \times Y)$ is the image under $H^k_S$ or of $H \pi$ of an element of a matrix
Massey product.

Remarks 5.4. By use of twisted Cartesian products and twisted tensor pro-
ducts, the first author has proven in [15, §6 (dualized)] that, under the
hypotheses of Theorem 3.3 (and suitable flatness hypotheses), $C^*_E Y$ can be
replaced by a $C^*_E B$-module for which $\lambda$ is actually an isomorphism; in that
context, Theorem 3.3 can easily be proven without use of the Serre spectral
sequence. Note, however, that it does not follow that $\lambda$ is an isomorphism
for $C^*_E Y$. In contrast to the behavior of $\text{Tor}^U(M,N)$, it is not true that a
morphism in $\mathcal{Q}$ which induces isomorphisms on $HM$, $HU$, and $HN$
necessarily induces an isomorphism on $H(M \otimes U N)$. On the other hand, our
evaluation of $\lambda$, coupled with [27, Theorem 1.5], will imply that the image
of $\lambda$ does have this invariance property.

In our discussion of matric Massey products, we shall assume
familiarity with the first few pages of [27]; we shall thus recall only that
minimum of notation which is required for intelligibility. By a matrix $A$
with entries in a differential $R$-module $E$, we understand an infinite array
$(a_{ij})$, $i \geq 1$ and $j \geq 1$, of elements of $E$ such that all but finitely many
entries in each column are zero. We write

\[\overline{A} = (-1)^{i+j} a_{ij}, \quad \text{where } \epsilon(i,j) = 1 + \deg a_{ij}, \quad \text{and } \quad dA = (da_{ij});\]

if $dA = 0$, we write $\{A\}$ for the matrix $\{a_{ij}\}$ of homology classes.

Given a pairing $E \otimes E' \to F$ of differential $R$-modules, the product $AA'$ of
matrices $A$ and $A'$ with entries in $E$ and $E'$ is defined as usual provided
that $\deg a_{ik} + \deg a_{kj}$ is independent of $k$ for each $i$ and $j$. Clearly

\[dA = \overline{dA}, \quad AA' = \overline{A} \overline{A'}, \quad \text{and } \quad d(\overline{A}) = d(A)A' - \overline{A} d(A').\]

Let $(M, U, N) \in \mathcal{Q}$. We assume given a distinguished resolution $X^\varphi$
of $M$ such that each $X_p$ is a countably generated free $R$-module. When $R$
is Noetherian and $HU$ and $HM$ are of finite type, Theorem 2.1 implies that $M$
admits such a resolution. Choose an ordered basis $A_p$ for $X_p$ and regard
$A_p$ as a row matrix with entries in $X$. The matrix equations

\[(1) \quad \alpha A_j = A_{-1,j} \quad \text{if } j > 0 \quad \text{and} \quad d^{j-1} A_j = \overline{A}_{-1,j} \quad \text{if } j > i \geq 0\]

define row matrices $A_{-1,j}$ with entries in $M$ and matrices $A_{ij}$ for $i \geq 0$
with entries in $U$. The equation $dA = \overline{dA}$, formulas 1.2 (iii) and (iv), and
the fact that $d^0 = 0$ on $X$ imply

\[d^j A_j = dA_j = dA_{j-1} \quad \text{and} \quad \sum_{k=0}^{j-1} \overline{A}_{-1,k} A_{kj} = \alpha dA_j = dA_{j-1,j},\]

and
\[ 0 = \sum_{k=1}^{i-1} d^{k-1}d^{-k}A_j = \sum_{k=1}^{i-1} d^{k-i}A_{k}A_{j} = \sum_{k=1}^{i-1} A_{k}A_{k}A_{j} - A_{1}A_{1}A_{j} \]

Since the $A_p$ are basis matrices, we conclude that, for all $i \geq -1$,

\[ (2) \quad dA_{i+1,i+1} = 0 \quad \text{and} \quad dA_{i,j} = \overline{A}_{i,j} \quad \text{if} \quad j > i, \quad \text{where} \quad \overline{A}_{i,j} = \sum_{k=1}^{i-1} A_{k}A_{k}A_{j}. \]

Define $W_i = \{ A_{i-1,i} \}$ for $i \geq 0$; $W_0$ is a row matrix with entries in HM and $W_i$, $i > 0$, is a matrix with entries in HU. If we compare (2) to [27, Definition 1.2], one connection between differential torsion products and matrix Massey products becomes obvious: for $n > m \geq 0$,

\[ \{ A_{i,j} \mid m \leq i \leq j \leq n, j-i \leq n-m \} \]

is a defining system for the matrix Massey product $<W_m \ldots, W_n>$ such that the associated element $\{ \overline{A}_{m-1,n} \}$ of $<W_m \ldots, W_n>$ is zero.

Remarks 5.5. As observed in [27, p. 538], the defining system $\{ A_{i,j} \mid i \geq 1 \geq 1 \}$ can itself be regarded as an upper triangular matrix $A$ (the Massey matrix), and then formula (2) can be abbreviated to the matrix equation $dA = \overline{A}A$.

This equation exhibits the close connections between defining systems and "twisting cochains" (see, for example, [15] or [25]) and between distinguished objects and twisted tensor products.

Now consider the spectral sequence $\{ E^r \} = \{ E^r(\mathcal{M}_U N) \}$. For $r \geq 2$, this is of course the algebraic Eilenberg-Moore spectral sequence of $(\mathcal{M}, U, \mathcal{N})$. We recall the standard definition of the spectral sequence of a filtered object, namely

\[ E^r_{p*, s} = Z^r_{p*}((Z^{r+s}_{p*}, B^r_{p*}) \quad \text{where} \quad Z^r_{p*} = F_p \cap d^{-1}F_{p-r} \quad \text{and} \quad B^r_{p*} = dZ^r_{p*}. \]

Write the evident pairings $\mathcal{M} \otimes N \rightarrow \mathcal{M} \otimes U N$ and $X \otimes N \rightarrow X \otimes U N$ by juxtaposition. Then any element of $F_p$ may be written in the form

\[ f = \sum_{k=0}^{p} \overline{A}_{k}C_k, \]

where $C_k$ is a column matrix with entries in $N$, and

\[ df = \sum_{i=0}^{p} \overline{A}_i \left( \sum_{k=1}^{p} \overline{A}_{ik}C_k \cdot dC_k \right). \]

Therefore $f \in E^r_{p*}$ for $r \geq 1$ if and only if

\[ (3) \quad dC_0 = 0 \quad \text{and} \quad dC_i = \sum_{k=1}^{p} \overline{A}_{ik}C_k \quad \text{if} \quad p+1-r \leq i \leq p-1. \]

We then write $f' = \sum_{i \leq p-r} \overline{A}_iC_i$; $df$ and $df' \cdot d'f$ represent the same element of $E^r_{p-r,*}$, since $df' \cdot e_{p-r}F^r_{p-r,*}$, and

\[ d(f' - f) = \sum_{1 \leq p-r} A_i \left( \sum_{k=1}^{p-r+1} \overline{A}_{ik}C_k \right). \]

Looking at the images of $f$ in $E^r_{p*}$ and of $d(f' - f)$ in $E^r_{p*,f}$ and observing that a typical element of $E^r_{p*} = \overline{A}_p \otimes \mathcal{N}$ may be written in the form

\[ \overline{A} \otimes V_{p+1}, \]

where $V_{p+1}$ is a column matrix with entries in $HN$, we see that the arguments above prove the following theorem.

Theorem 5.6. A typical element $\overline{A}_p \otimes V_{p+1}$ of $E^r_{p*}$ survives to an element $\mathcal{S}$ of $E^r_{p*}$, $s \leq p+1$, if and only if there exist column matrices $C_i$...
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with entries in $\mathbb{N}$ for $p+1-r \leq i \leq p$ such that (3) holds and $\{C_i\} = V_{p+1}$.

If $r \leq p$, the column matrix

$$\sum_{k=0}^{p} \overline{A}_{p-r, k} C_k$$

of cycles in $\mathbb{N}$ then represent a column matrix

$$V_{p-r, p+1}\epsilon <W_{p+1-r}, \ldots, W_p, V_{p+1}>,$$

and the element $A_{p-r, p+1}$ of $E_{p-r, 0}^1$ is a permanent cycle which survives to $d^r(\delta)$ in $E_{p-r, 0}^r$. If $r = p+1$, so that $\overline{A}_{p+1}$ is a permanent cycle, the cycle

$$\sum_{k=0}^{p} \overline{A}_{-1, k} C_k$$

in $M \otimes U_N$ then represents an element

$$V_{-1, p+1}\epsilon <W_{0}, \ldots, W_p, V_{p+1}>,$$

and $\lambda : \text{Tor}(M, N) \to \text{H}(M \otimes U, N)$ is specified by the formula

$$\lambda \left( \sum \overline{A}_{k} C_k \right) = V_{-1, p+1}.$$

The theorem focuses attention on "canonically defined" matrix Massey products, by which we understand those elements of

$$<W_{0}, \ldots, W_p, V_{p+1}> \subset \text{H}(M \otimes U, N)$$

which are obtained from defining systems which extend the fixed set $\{A_{ij} \mid -1 \leq i < j \leq p, j-i \leq p\}$ of matrices by column matrices $C_i$, $0 \leq i \leq p$, with entries in $\mathbb{N}$ which satisfy (3). By abuse, we agree to call $\{C_0, \ldots, C_p\}$ a canonical defining system and we agree that the symbol $<W_{0}, \ldots, W_p, V_{p+1}>$ shall henceforward denote only

the set of canonically defined elements $V_{-1, p+1} = \{ \sum_{k=0}^{p} \overline{A}_{-1, k} C_k \}$. The following lemma shows that these canonical matrix Massey products are quite well-behaved. Recall the definition of indeterminancy,

$$\text{In}<W_0, \ldots, W_p, V_{p+1}> = \{ x \cdot x' \mid x, x' \in <W_0, \ldots, W_p, V_{p+1}> \}.$$

Lemma 5.7. $\text{In}<W_0, \ldots, W_p, V_{p+1}> = \bigcup <W_0, \ldots, W_{p-1}, V_p>$, where the union is taken over all $V_p$ such that $<W_0, \ldots, W_{p-1}, V_p>$ is defined and the degree of the $i$th entry of $V_p$ is one greater than the degree of the $i$th entry of the product $W_p V_{p+1}$. Moreover, $\text{In}<W_0, \ldots, W_p, V_{p+1}>$ is an $R$-module and $<W_0, \ldots, W_p, V_{p+1}>$ is a coset in the sense that for each $x \in <W_0, \ldots, W_p, V_{p+1}>

$$x + \text{In}<W_0, \ldots, W_p, V_{p+1}> = \{ x + y \mid y \in \text{In}<W_0, \ldots, W_p, V_{p+1}> \} = <W_0, \ldots, W_p, V_{p+1}>

$$

Proof. Let $x$ and $x'$ in $<W_0, \ldots, W_p, V_{p+1}>$ be determined by canonical defining systems $\{C_0, \ldots, C_p\}$ and $\{C_0', \ldots, C_p'\}$. We may assume that $C_p = C_p'$ since otherwise we could replace $C_p$ by $C_p'$ and $C_i$ by $C_i' - A_i D$ for $i < p$, where $A D = C_p - C_p'$, without changing the class $x'$. But then

$$x - x' = \{ \sum_{k=0}^{p-1} \overline{A}_{-1, k} (C_{-1, k} - C_{-1, k}') \} \epsilon <W_0, \ldots, W_{p-1}, \{C_{-1, k}' - C_{-1, k} \}>

The rest of the proof consists of similar, but even simpler, manipulations of canonical defining systems.

We consider next a general matrix Massey product

$$<V_0, \ldots, V_{p+1}> \subset \text{H}(M \otimes U, N),$$

where $V_0$ is a row matrix with entries in $\text{H}_p,$
\[ V_1 \text{ is a matrix with entries in } HU \text{ for } 1 \leq i \leq p \text{ and } V_{p+1} \text{ is a column matrix with entries in } HN. \] Let \( \{ B_{ij} | -1 \leq i < j \leq p+1, j-i \leq p+1 \} \) be a defining system for \( \langle V_0', \ldots, V_{p+1} \rangle \) and let \( B_{-1,p+1} \) be the associated cycle of \( M_{X \otimes U} N \). Thus

\[ dB_{-1,i} = 0, \{ B_{-1,i} \} = V_1, \quad \text{and} \quad dB_{ij} = \sum_{k=1+i}^{i-1} \overline{B}_{ik} \overline{B}_{kj} \quad \text{for} \quad -1 \leq i < j \leq p. \]

Clearly we may choose countably generated free \( R \)-modules \( \overline{Y}_j \) for \( 0 \leq j \leq p \) with basis row matrices \( B_j \) such that if \( \overline{Y}_j = 0 \) for \( j > p \) and if \( Y = Y \otimes U \), then \( Y^\beta \) becomes a well-defined distinguished object over \( M \) with \( \beta \) and \( d^r \), \( r > 0 \), defined on \( Y_j^{\beta} \) by the matrix equations

\[ \beta B_j = B_{-1,j} \quad \text{if} \quad j \geq 0 \quad \text{and} \quad d^{j-i} B_j = \overline{B}_{i,j} B_{-1} \quad \text{if} \quad j > i \geq 0. \]

By Theorem 1.7 and Remarks 1.3, there is a map \( K r Y \to X \) in \( F_{M_{X \otimes U} N} \) such that \( \alpha K \) is homotopic to \( \beta \). Now

\[ g = \sum_{k=0}^{p} \overline{B}_{k} \overline{B}_{k+1} \]

is a cycle in \( Y \otimes U N \). Its image \( f \) under \( K \times 1 \) may be written

\[ f = \sum_{k=0}^{p} K(\overline{B}_k) \overline{B}_{k+1} = \sum_{k=0}^{p} \overline{A}_k \overline{C}_k \]

since \( K \) is filtration-preserving. Since \( f \) is a cycle in \( X \otimes U N \), \( \{ C_0, \ldots, C_p \} \) is a canonical defining system. The cycle \( \tilde{Z}_{-1,p+1} = (\beta \otimes 1)(g) \) is homologous to \( (\alpha \otimes 1)(0) \), since \( \alpha K \) is homotopic to \( \beta \), and therefore

\[ \{ \tilde{Z}_{-1,p+1} \} \in \langle W_0', \ldots, W_p, (C_p) \rangle. \]

We have proven the following curious result.

**Proposition 5.6.** Every element of every \((p+2)\)-fold matric Massey product \( \langle V_0', \ldots, V_{p+1} \rangle \subset H(M \otimes U N) \) is an element of some canonically defined \((p+2)\)-fold matric Massey product.

**Corollary 5.3.** The image of \( \lambda : \text{Tor}^U(M, N) \to H(M \otimes U N) \) is the set \( D(M, U, N) \) of all elements of all matric Massey products \( \langle V_0', \ldots, V_{p+1} \rangle \) for all \( p \geq 0 \).

**Proof.** By the proof of Theorem 5.6, all elements of all canonically defined matric Massey products are in the image of \( \lambda \).

It should be recalled here that a two-fold matric Massey product \( \langle V_0' V_4 \rangle = \overline{V}_0 \overline{V}_4 \) is just a typical element of the image of \( H(M \otimes U N) \to H(M \otimes U N) \).

One can easily derive further relationships between matric Massey products and differential torsion products. For example, \( \text{Tor}^U(M, N) \) can be described in terms of equivalence classes of defining systems, and each \( E^r(M, U, N) \) can be described in terms of "partial" defining systems. Such results are now seen to be merely notationally awkward first approximations to the theory presented here; they add no useful information.

We now assume that \( U \) is an augmented differential \( R \)-algebra with augmentation \( \varepsilon : U \to R \). We write \( IU = \text{Ker} \varepsilon \) and we let \( i \) denote the inclusion of \( IU \) in \( U \). Classically, if \( N \in M_{X \otimes U} N \) and \( d = 0 \) on \( U \) and \( N \), then the set of decomposable elements of \( N \) is defined to be \( IU \cdot N \). In view
of the previous corollary, the following definition gives the appropriate notion of decomposability in the presence of differentials.

Definition 5.10. Define $D(HN; HU)$ to be the image of $D(IU, U, N)$ under

$$H(1 \otimes 1): H(IU \otimes_U N) \rightarrow H(U \otimes_U N) = HN;$$

that is, $D(HN; HU)$ is the set of all elements of the left $HU$-module $HN$ which are decomposable in terms of matric Massey products. Define $D(HM; HU)$ by symmetry. Define $DHU$ to be the image of $D(IU, U, IU)$ under

$$H \cdot H(1 \otimes 1): H(IU \otimes_U IU) \rightarrow HU,$$

where $H: IHU = HU \rightarrow HU$ is the evident inclusion. Equivalently,

$$DHU = D(HI_U; HU),$$

where $IU$ is regarded either as a left or as a right $HU$-module.

We need the following analog to Definition 3.7 in order to relate $D(HN; HU)$ to $\text{Tor}^U(R, N)$.

Definition 5.11. Define $\pi: HN \rightarrow \text{Tor}^U(R, N)$ by $\pi(n) = (x \otimes n)$, where $x$ denotes an element of $X_{0,0}$ such that $\alpha(x) = 1 \in R$ in some distinguished resolution $X^\alpha$ of $R$. Equivalently, $\pi$ is the composite $HN = R \otimes HN \rightarrow R \otimes_{HU} HN = E_0^\alpha(R, U, N) \rightarrow E_0^\alpha(R, U, N) = F_0 \text{Tor}^U(R, N)$. By Lemma 5.1, we may define $\phi: HN \rightarrow \text{Tor}^U(M, R)$ by symmetry.

Corollary 5.12. $\text{Ker} \pi = D(HN; HU)$ and $\text{Ker} \phi = D(HM; HU)$.

Proof. We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Tor}^U(IU, N) & \xrightarrow{\text{Tor}(1, \xi)} & \text{Tor}^U(U, N) & \xrightarrow{\text{Tor}(\xi, 1)} & \text{Tor}^U(R, N) \\
\downarrow{\lambda} & & \downarrow{\lambda} & & \downarrow{\lambda} \\
H(IU \otimes_U N) & \xrightarrow{H(1 \otimes 1)} & H(U \otimes_U N) & \xrightarrow{\phi} & HN \\
\end{array}
\]

The middle map $\lambda$ is an isomorphism since $U$ is itself a distinguished resolution of $U$. The upper row is exact by Lemma 5.2, hence $\text{Ker} \pi = \text{im} H(1 \otimes 1) \cdot \lambda$. The first part follows by Corollary 5.9 and the symmetric conclusion holds in view of Lemma 5.1.

Since $R = F_0 \text{Tor}^U(R, R)$ is a direct summand of $\text{Tor}^U(R, R)$, we may regard $E_1^\alpha(R, U, R)$ as contained in $\text{Tor}^U(R, R)$. Thus the suspension $\sigma: IHU = \text{Ker} H \phi \rightarrow \text{Tor}^U(R, R)$ is defined by Definition 3.7.

Corollary 5.13. $\text{Ker} \sigma = DHU$; that is, $\text{Ker} \sigma$ is the set of all elements of $HU$ which are decomposable as matric Massey products.

Proof. Consider the following diagram:

\[
\begin{array}{c}
0 \rightarrow R = \text{Tor}^U(R, U) \xrightarrow{\text{Tor}(1, \xi)} \text{Tor}^U(R, R) \xrightarrow{\phi} \text{Tor}^U(R, IU) \rightarrow 0 \\
\downarrow{\sigma} & \downarrow{\pi} & \downarrow{\pi} \\
\text{IHU} & & \\
\end{array}
\]

The row is exact by Lemma 5.2 since $\text{Tor}(1, \xi)$ is clearly a monomorphism onto the direct summand $F_0 \text{Tor}^U(R, R)$. The triangle is commutative by a trivial comparison of Definitions 3.7 and 5.7. Thus $\text{Ker} \sigma = \text{Ker} \pi$, and the result follows from the previous corollary in view of Definition 5.10.
Application of the previous two corollaries to the geometric situations of Theorems 3.3 and 3.9 yields the following results (in view of Corollaries 3.6, 3.8, 3.11 and 3.12).

Corollary 5.14. Assume the hypotheses of Theorem 3.3. Then

(i) If $A$ is a point, so that $g$ is the inclusion of $F$ in $Y$, then
\[
\ker g^\# = D(H^\# Y; H^\# B).
\]

(ii) If $Y$ is acyclic, so that $f$ is a classifying map, then
\[
\ker q^\# = D(H^\# A; H^\# B).
\]

(iii) If $A$ is a point and $Y$ is acyclic, so that $F$ has the weak homotopy type of the loop space $\Omega B$, then $\ker g^\# = DH^\# B$.

Corollary 5.15. Assume the hypotheses of the first part of Theorem 3.9 (under which \( \phi \) is defined) in (i) and (iii) and of the second part in (ii). Then

(i) $\ker p^\# = D(H_\# A; H_\# G)$.

(ii) If $A$ is acyclic, then $\ker i^\# = D(H_\# Y; H_\# G)$, i.e., $Y \to A \times G$.

(iii) If $A$ is acyclic, so that $B$ has the weak homotopy type of the classifying space $BG$, then $\ker q^\# = DH_\# G$.

Finally, we discuss an application of the results above to the cohomology of algebras. For simplicity, let $R$ be a field and assume that all $R$-modules in sight are non-negatively graded and of finite type. Let $A$ be a connected $R$-algebra and let $V$ be a right $A$-module. Let $\text{CA}$ denote the dual of $\overline{BA}$ and let $C(A; V)$ denote the dual of $B(V, A, R)$; equivalently
\[
C(A; V) = \text{Hom}_A(B(V, A, A), R).
\]

The definition of the two-sided bar construction used here is recalled in the appendix. Write
\[
H^\# A = HCA = \text{Ext}_A(R, R) \quad \text{and} \quad H^\#(A; V) = HC(A; V) = \text{Ext}_A(Y, R).
\]

$B(V, A, R)$ is a right differential $\overline{BA}$-comodule with structure map $D$ defined, in analogy with formula 3.8, by
\[
D(v[a_1|\ldots|a_p]) = \sum_{i=0}^{p} (-1)^{(p-i)} v[a_1|\ldots|a_i] \otimes [a_{i+1}|\ldots|a_p],
\]
\[
q_i = \text{deg } v + \sum_{j=1}^{i} \text{deg } a_j.
\]

Dually, $C(A; V)$ is a right differential $\text{CA}$-module. Now $\text{Tor}_{\text{CA}}^r(C(A; V), R)$ is defined, and $E_\infty V = \text{Tor}_{H^\# A}^r(C^\#(A; V), R)$ in the algebraic Eilenberg-Moore spectral sequence $(E_\infty V)$ of $(C(A; V), \text{CA}, R)$. Observe that the dual $\eta^\#: C(A; A) \to R$ of the unit $\eta: R \to B(A, A, R)$ is a morphism of differential $\text{CA}$-modules which induces an isomorphism on homology. Thus $(E_\infty A)$ is isomorphic to the algebraic Eilenberg-Moore spectral sequence of $(R, \text{CA}, R)$. Of course, we are using cohomological notations, with gradings by superscripts. Note that $\text{CA}$ and $C(A; V)$ are bigraded. The differentials only change the homological degree, and the internal degree arising from the grading of $A$ and $V$ may safely be ignored; it gives rise to an undenoted extra grading on all objects in sight. We evaluate $\text{Tor}_{\text{CA}}^r(C(A; V), R)$ and $E_\infty V$ in the following proposition; the case $V = A$ determines $\text{Tor}_{\text{CA}}^r(R, R)$. 


Proposition 5.16. Filter $V$ by $F_p V = V$ for $p \geq 0$ and $F_{-p} V = (IA)^p$ for $p > 0$, and give $V^*$ the dual filtration $F^p V^* = (V/F_{-p} V)^*$. Then $\text{Tor}_{CA}^0(C(A, V), R)$ is isomorphic to $V^*$ as a filtered $R$-module and $\text{Tor}_{CA}^n(C(A; V), R) = 0$ for $n \neq 0$. Therefore $E_{-p, p}^\infty = F^{-p} V^*/F^{-1, 0} V^*$ for $p \geq 0$, and $E_{-p, q}^\infty = 0$ for $q < -p$.

Proof. Filter $B(C(A, V), CA, R)$ by the opposite filtration to that leading to the algebraic Eilenberg-Moore spectral sequence, namely

$$E^q B^n(C(A, V), CA, R) = \bigoplus_{j \geq 0} B^{n-j, j}(C(A, V), CA, R).$$

In the resulting spectral sequence $(E_r)$, $d^1$ is induced by the external (or simplicial) differential and therefore $E_1^1$ is the classical torsion product $\text{Tor}_{CA}^1(C(A, V), R)$, with internal differentials ignored. By (4), $C(A; V) = V^* \otimes CA$ is free as a right $CA$-module. Therefore $E_1^\infty = E_1^0 = V^*$ (as a graded $R$-module), and $E_1 = E_1^\infty$. The verification that the filtration on $\text{Tor}_{CA}^0(C(A, V), R)$ agrees with that on $V^*$ is similar to the argument given in [24, p. 336]; we omit the details since there is certainly some filtration on $V^*$ for which the statement is true and its precise form is not important for our present purposes.

As explained in [24, p. 335], the spectral sequence $(E_r)$ is itself an algorithm for the computation of $H^*(A; V)$. Now consider the following commutative diagrams:

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\text{IH}^\# A \xrightarrow{\psi} \text{Tor}_{CA}^*(C(A; V)), R) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H^*(A; V) \quad \xrightarrow{\phi} \quad \text{Tor}_{CA}^*(C(A; V), R) \\
\end{array}
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
QH^\# A = E_2^{-1, 0} A \quad \quad \xrightarrow{e} \quad E_2^{-1, 0} A \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
QH^*(A; V) = E_2^{0, 0} \quad \xrightarrow{\beta} \quad E_2^{0, 0} \\
\end{array}
\end{array}
\end{align*}
\]

Here $QH^\# A = IH^\# A/(IH^\# A)^2$ and $QH^*(A; V) = H^*(A, V)/H^*(A; V)$. $IH^\# A$ are the respective sets of indecomposable elements. Recall that

$$\text{Tor}_{CA}^1(R, R) = QA = IA/(IA)^2$$

and $\text{Tor}_{CA}^0(V, R) = QV = V/V \cdot IA$.

The edge homomorphisms $e$ are isomorphisms on $E_2^{-1, 0} A$ and on $E_2^{0, 0} V$; indeed, no element of any $E_2^{p, q}$ can bound since all $E_2^{-1, q} V = 0$ for $q < p$.

By the proposition, we therefore have identifications

$$H^1 A = (QA)^\# = IH^\# A/Ker \sigma$$

and $H^0(A; V) = (QV)^\# = H^*(A; V)/Ker \beta$. By Corollaries 5.12 and 5.13, the right sides are the sets of elements which are indecomposable in terms of matric Massey products. We have proven the following result.

Corollary 5.17. $H^1 A$ is generated under matric Massey products by $H^1 A$ and $H^*(A; V)$ is generated under matric Massey products by $H^0(A; V)$.

That is, every element of higher (homological) degree is decomposable as an element of a matric Massey product.

The phenomenon recorded in the corollary was first observed empirically in the second author's computations of the cohomology of the Steenrod algebra [23]. The analogous results for stable homotopy groups, in terms of matric Toda brackets, have been proven by Cohen [44], and the
relationship between matric Massey products and matric Toda brackets in the Adams spectral sequence has been studied by Lawrence [19].

Appendix

We must prove the theorems stated in § 4 which relate the cochains to the cohomology of $K(\pi, n)$'s. To do this, we shall have to redevelop the machinery of Eilenberg and Mac Lane [12, 13], Moore [32], and Cartan [9], keeping close track of the products and $\omega_1$-products on the cochain level. We shall work in rather greater generality than is needed for the immediate applications in order to delineate precisely what is and is not true about the cochains of $K(\pi, n)$'s. Some of our results about products (but none of their proofs) and most of our results about $\omega_1$-products (and all of their proofs) are new; otherwise, the present material is contained, at least implicitly, in the cited sources.

We shall study simplicial differential algebra in statements A.1 through A.7. By explicit use of this theory (which is implicit in the work of Eilenberg and Mac Lane [12, 13]), we shall be able to obtain a uniform treatment of internal structures, such as $\omega_1$-products, in the singular cochains of spaces and in the duals of the bar and W constructions. The bar construction (which is due to Eilenberg and Mac Lane [12]) and its internal structures are discussed in A.8 through A.13. In particular, A.9 and A.10 contain a precise comparison between the bar construction approach to differential homological algebra and our approach. The W construction (as defined by Mac Lane [21] and Moore [32], generalizing the original definition of Eilenberg and Mac Lane [12]) and its internal structures are discussed in A.14 through A.19. The bar and W constructions are compared in A.20, and
this comparison is used to study $K(n,n)'s$ in A.21 and A.22. Cartan's "little constructions" [9], which are minimal resolutions of the ground ring of commutative Hopf algebras, are studied and compared to the bar construction after A.23 and A.24. Finally, in A.25, A.26, and the subsequent discussion of $\cup_1$-products mod 2, this material is combined to yield the results required in section 4.

Our work will be simplified by use of the simplicial formulation of the bar construction given by the second author in [29, §9 and §10]. For any category $\mathcal{U}$, let $\textbf{dU}$ denote the category of simplicial objects in $\mathcal{U}$. Let $\mathcal{M} = \mathcal{M}_R$ be the category of $Z$-graded differential $R$-modules and let $\mathcal{A}$ be the category defined in Corollary 1.8.

**Definition A.1.** Define a functor $B_p: \mathcal{A} \to \mathcal{M}$ as follows. On objects $(M, U, N) \in \mathcal{A}$, define the differential $R$-module $B_p(M, U, N)$ of $p$-simplices by

$$B_p(M, U, N) = M \otimes U \otimes \cdots \otimes U \otimes N, \quad p \text{ factors } U.$$

Define the face and degeneracy operators on $B_p(M, U, N)$ by

$$\delta_0 = \lambda \otimes 1^P, \quad \lambda: M \otimes U \to M; \quad \delta_i = 1^i \otimes \mu \otimes 1^{P-1} \quad \text{if } 0 < i < p, \quad \mu: U \otimes U \to U;$$

$$\delta_p = 1^P \otimes \xi, \quad \xi: U \otimes N \to N; \quad \varepsilon_i = 1^{i+1} \otimes \eta \otimes 1^{P+1-i} \quad \text{if } 0 \leq i \leq p, \quad \eta: R \otimes U.$$

Here $\lambda, \mu, \xi,$ and $\eta$ are the structure maps entailed by the definition of $\mathcal{A}$. Define $B_p(k, i, f) = k \otimes f \otimes f$ on morphisms $(k, i, f) \in \mathcal{A}$.

We shall define the bar construction $B: \mathcal{A} \to \mathcal{M}$ to be the composite $C \circ B_p$, where $C: \mathcal{M} \to \mathcal{M}$ is the following condensation functor (which combines the usual processes of totalization and normalization).

**Definition A.2.** Let $X \in \mathcal{M}$ and write $X_{pq}$ for the $R$-module of $p$-simplices of degree $q$. Define $CX \in \mathcal{M}$ by letting $C X$ be the quotient of

$$\sum_{p+q = n} X_{pq}$$

by the sub $R$-module generated by the degenerate simplices (the elements of $\cup_i X_{i-1,q}, 0 \leq i \leq p$) and giving $CX$ the differential induced by passage to quotients from

$$d = (-1)^p \partial + \sum_{i=0}^p (-1)^i \delta_i$$

on $X_{pq}$,

where $\delta_i: X_{pq} \to X_{q-1}$ is the given differential on $X_{pq}$. Clearly $d^2 = 0$ since $\delta_0^2 = 0$, $\delta_0 \delta_1 = \delta_1 \delta_0$, and $\delta_i \delta_j = \delta_j \delta_i$ if $i < j$. With the evident definition of $C$ on morphisms, $C$ becomes a functor $\mathcal{M} \to \mathcal{M}$. If $h: f \to g$ is a homotopy in $\mathcal{M}$ between maps $f, g: X \to Y$ (in the sense of [29, §1]), define $Ch = \sum_{i=0}^p (-1)^i h_i$ on the image of $X_{pq}$ in $CX$; then $Ch$ is a homotopy between $Cf$ and $Cg$. Define $Ch + Ch - d = Cf - Cg$. If $U$ is a differential $R$-algebra and if $X \in \mathcal{M}_U$, then the module products $X_{pq} \otimes U \to X_{pq}$ induce a structure of right differential $U$-module on $CX$, and $C$ restricts to a functor $\mathcal{M}_U \to \mathcal{M}_U$. The analogous statement for $\mathcal{M}$ fails by a sign, and there is no choice of signs for $C$ which behaves properly both for left and for right differential $U$-modules.

Before returning to the bar construction, we develop some properties of the functor $C$. Recall that the category $\mathcal{M}$ admits the tensor product defined by $(X \otimes Y)_p = X_p \otimes Y_p$, with $\delta_i = \delta_i \otimes \delta_i$ and $s_i = s_i \otimes s_i$ for $X, Y \in \mathcal{M}$. The standard shuffle and Alexander-Whitney maps generalize to yield comparisons between $C(X \otimes Y)$ and $CX \otimes CY$. 

---

**Differential Torsion Products**

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Proposition A.3. Let \( X, Y \in \mathcal{M} \). Define \( \eta : C(X \otimes Y) \to C(X \otimes Y) \) and \( \xi : C(X \otimes Y) \to C(X \otimes Y) \) by the following formulas:

\[
\eta(x \otimes y) = \sum_{(\mu, \nu)} (-1)^{\sigma(\mu)} x^{s_1 \ldots s_\nu} y^{s_{\mu_1} \ldots s_{\mu_p}}, \quad x \in X_{\mu}, \quad y \in Y_{\nu},
\]

where the sum is taken over all \((p, q)\)-shuffles \((\mu, \nu)\) and

\[
\sigma(\mu) = \sum_{i=1}^{p+1} (\mu_i - i + 1) \text{ is the signature of the corresponding permutation;}
\]

\[
\xi(x \otimes y) = \sum_{i=0}^{p} (-1)^{(p-i)} y \otimes x^{s_0 \ldots s_i}, \quad x \in X_{pq}, \quad y \in Y_{pr},
\]

where \( s_{p-i} \) is the iterated last face operator \( \partial_{i+1} \ldots \partial_p \).

Then \( \eta \) is a commutative and associative natural transformation of functors \( \mathcal{M} \times \mathcal{M} \to \mathcal{M} \), \( \xi \) is an associative natural transformation, the composite \( \xi \eta \) is the identity, and the composite \( \eta \xi \) is naturally homotopic to the identity. Moreover, if \( X, X', Y, Y' \in \mathcal{M} \), then the following diagram is commutative when read with either the solid or the dotted vertical arrows:

\[
\begin{array}{ccc}
C(X \otimes Y) & \rightarrow & C(X \otimes Y) \\
\eta \downarrow & & \downarrow \eta \\
C(X \otimes Y') & \rightarrow & C(X \otimes Y')
\end{array}
\]

\[
\begin{array}{ccc}
\xi \otimes \xi & & \eta \otimes \eta \\
\downarrow & & \downarrow \\
C(I \otimes T \otimes 1) & & C(I \otimes T \otimes 1)
\end{array}
\]

\[
\begin{array}{ccc}
C(X \otimes Y) & \rightarrow & C(X \otimes Y) \\
\eta \downarrow & & \downarrow \eta \\
C(X \otimes Y') & \rightarrow & C(X \otimes Y')
\end{array}
\]

Here \( T(a \otimes b) = (-1)^{\deg a \cdot \deg b} a \otimes b \) on elements of objects \( A, B \in \mathcal{M} \), and \( T_\ast \) is defined by \( T_\ast = T \) on the tensor product of objects of \( \mathcal{M} \). (Note that if \( \eta \) is replaced by \( \xi \) in the horizontal arrows, then the diagram is no longer commutative, but only homotopy commutative.)

Proof. Except for the diagram, the proof is essentially the same as that originally given by Eilenberg and Mac Lane [12, §5 and 13, §2]. With dotted arrows, the commutativity of the diagram follows from the commutativity and associativity of \( \eta \). With solid arrows, the commutativity of the diagram is proven by essentially the same calculation as was used by Eilenberg and Moore [14, §17] to prove Lemma 3.4; indeed, that lemma is a direct consequence of the commutativity of the stated diagram.

Of course, \( \xi \) fails to be commutative; its deviation from commutativity is studied in the following proposition.

Proposition A.4. Let \( X, Y \in \mathcal{M} \). Define \( \xi_1 : C(X \otimes Y) \to C(X \otimes Y) \) by

\[
\xi_1(x \otimes y) = \sum_{1 \leq i \leq p} (-1)^{i+j+1-1} y^{s_j} x^{s_i} \otimes \otimes_{j=0}^{i-1} \otimes_{j=p-1}^{i-1} y, \quad x \in X_{pq}, \quad y \in Y_{pr},
\]

where \( x \in X_{pq}, y \in Y_{pr}, \) and \( s_t = 0 \otimes 1 \otimes 2 \otimes \cdots \otimes t \) for \( s \leq t \).

Then \( \xi_1 \) is a homotopy from \( \xi \) to \( T \xi \circ CT \); that is,

\[
d \xi_1 + \xi_1 d = \xi - T \xi \circ CT.
\]

Moreover, the following formula is satisfied for \( X, Y, Z \in \mathcal{M} \) (where we have written \( \xi(\cdot) \) and \( \xi_1(\cdot) \) to indicate variables when necessary to avoid ambiguity):
\[(\xi \otimes 1)\xi_4(X \otimes Y, Z) - (1 \otimes \xi_4)\xi(X, Y \otimes Z) - (1 \otimes T)(\xi_4 \otimes 1)\xi(X \otimes Y, Z)\]
\[= \delta - \delta + C(1 \otimes T \alpha), \text{ where for } x \in X_p, y \in Y_p, z \in Z_p,\]
\[\delta(x \otimes y \otimes z) = \sum_{1 \leq i \leq j \leq k \leq p} (-1)^{i+j+1}((-1)^{p-q+r} + (p-k)q^j i + p-k)j^i p-k p-j p-k p\]

The proof consists of tedious, but quite straightforward, calculations based on the manipulation of simplicial identities. We fix notations in the following corollaries.

**Corollary A.5.** If \(X\) is a simplicial differential algebra with product \(\mu_x\), then \(CX\) is a differential algebra with product \(\mu = C(1 \otimes \eta).\) If \(X\) is commutative, \(\mu_x = \mu_x \circ T_x\), then \(CX\) is commutative, \(\mu = \mu \circ T\).

**Corollary A.6.** If \(X\) is a simplicial differential coalgebra with coproduct \(\Delta_x\), then \(CX\) is a differential coalgebra with coproduct \(\Delta = \xi \circ C\Delta_x\).

Define \(\Delta_4 = \xi \circ \Delta_x\). If \(X\) is cocommutative, \(\Delta_x = T_x \circ \Delta_x\), then \(CX\) is homotopy cocommutative via \(\Delta_4\).

\[(\xi \otimes 1)\Delta_4 - (1 \otimes \Delta_4)\Delta + (1 \otimes \Delta)(\Delta_4 \otimes 1)\Delta = 0.\]

The dual algebra \(C^*X\) is therefore homotopy commutative via a \(\mu_4\)-product \(\Delta_4^*\) which satisfies the Hirsch formula.

**Proof.** Clearly (4) implies (6), and (5) implies (7) by the naturality of \(\xi\) and \(\xi_4\). Formulas (6) and (7) yield formulas (i) and (ii) of Example 2.2 by dualization.

**Differential Torsion Products**

It is important to observe that the cocommutativity of \(X\) is sufficient, but by no means necessary, for the validity of formulas (6) and (7). The diagrams of Proposition A.3 imply the following result.

**Corollary A.7.** If \(X\) is a simplicial differential Hopf algebra, then \(CX\) is a differential Hopf algebra; if \(X\) and \(Y\) are simplicial differential Hopf algebras, then the shuffle map \(\eta: CX \otimes CY \rightarrow C(X \otimes Y)\) is a morphism of differential Hopf algebras.

Observe that the category of ungraded \(R\)-modules is contained in \(\mathcal{M}\) as the full subcategory of objects concentrated in degree zero (with zero differential) and that \(C\) here restricts to the standard functor from simplicial \(R\)-modules to differential graded \(R\)-modules. If \(K\) is a simplicial set, then the free simplicial \(R\)-module \(F_eK\) generated by \(K\)

\((F_eK)_p = FK_p\) where \(F\) is the free functor from sets to \(R\)-modules) admits the co-product \(\Delta_e\) defined by \(\Delta_e(k) = k \otimes k\) on elements of \(K\). As usual, we write \(C_e(K; R) = CFK\) and \(C^e(K; R) = C(FCF_eK)\) for the chains of \(K\).

Here \(\Delta_4\) on \(C_e(K; R)\) is just the obvious generalization to the simplicial setting of the dual of Steenrod's original \(\cup_4\)-product [38, p. 293], and the computation used to prove (7) is dual to that of Hirsch [18, p. 924]. We have thus recovered the standard \(\cup_4\)-product and verified the Hirsch formula on the cochains of a simplicial set \(K\).

We now return to the bar construction. We define \(B = C \circ B_e: \mathcal{M} \rightarrow \mathcal{M}\)

As usual, we write \(m[u_1|\ldots|u_p]\) for typical elements of \(B(M, U, N)\).
The signs in our definition of $B$ are non-standard; they are dictated both by the present theory and by the geometric bar construction $[31,30].$

**Lemma A.8.** Define $\varepsilon: B(M, U, U) \to M$ and $\eta: M \to B(M, U, U)$ by

$$\varepsilon(m[u_1|\ldots|u_p]u) = 0 \text{ if } p > 0, \quad \varepsilon(m[u]u) = mu, \text{ and } \eta(m) = [\cdot].$$

Define $s: B(M, U, U) \to B(M, U, U),$ of degree one, by

$$s(m[u_1|\ldots|u_{p-1}]u_p) = (-1)^p m[u_1|\ldots|u_{p-1}]u_p.$$ 

Then $ds + sd = 1 - \eta\varepsilon$ and $s^2 = 0$ on $B(M, U, U).$ Therefore $B(M, U, U)$ is a split resolution of $M,$ it is a Künneth resolution of $M$ under appropriate flatness hypotheses.

**Proof.** $s = Ch,$ where $h$ is the simplicial contracting homotopy defined in $[29, 9, 9].$ $B(M, U, U)$ is clearly a split object of $\mathcal{F} M_U$ when filtered by simplicial degree, and $B(M, U, U)$ is a resolution of $M$ in the sense of Definition 1.4 because $s$ raises filtration by at most one. The last part follows, with hypotheses made precise, by Lemma 1.5 and Remark 1.6.

**Theorem 1.7** now has the following immediate corollary.

**Corollary A.9.** There is a natural transformation of functors

$$\gamma: \text{Tor}^U(M, N) \to \text{HB}(M, U, N),$$

and $\gamma$ is an isomorphism under appropriate flatness hypotheses.

The geometric situation is rather different than one would expect from the preceding corollary.

**Remarks A.10.** Under the hypotheses of Theorem 3.3 or of Theorem 3.9, we can define a map $\theta'$ (precisely as the map $\theta$ was defined but with $X$ replaced by the appropriate bar construction) such that the following diagram is commutative:

$$\begin{array}{ccc}
\text{Tor}_{C^s(B)}(C^s_A, C^s_Y) & \xrightarrow{\theta} & \text{Tor}_{C^s(G)}(C^s_A, C^s_Y) \\
\downarrow \gamma & & \downarrow \gamma \\
\text{HB}(C^s_A, C^s_B, C^s_Y) & \xrightarrow{\theta'} & \text{HB}(C^s_A, C^s_G, C^s_Y)
\end{array}$$

The proofs of the cited theorems apply (with a slight added argument to deal with the arrow $1 \otimes \tau_1$ in the diagram used to prove Theorem 3.3) almost verbatim to show that $\theta'$ is an isomorphism. Therefore $\gamma$ is an isomorphism in all situations of geometric interest, regardless of flatness hypotheses. However, the spectral sequence derived from the bar construction need not have a classical torsion product for its $E_2$-term, hence the comparison of spectral sequences need not be an isomorphism.

Observe that the functor $B_{\otimes}$ preserves tensor products in the sense that the evident shuffling isomorphisms on $p$-simplices for each $p$ define a commutative and associative natural isomorphism

$$V: B_{\otimes}(M, U, N) \otimes B_{\otimes}(M', U', N') \to B_{\otimes}(M \otimes M', U \otimes U', N \otimes N').$$

Now let $U$ be augmented via $\xi: U \to R.$ We agree to write $BU = B(R, U, U)$ and $\overline{BU} = B(R, U, R),$ and we define $\tau = C_{\pi_{\otimes}}: BU \to \overline{BU},$ where $\pi_{\otimes} = B(1, 1, 1).$ BU and $\overline{BU}$ are augmented and unital differential
R-modules; their augmentations $B(\varepsilon)$ and $\overline{B}(\varepsilon)$ and units $B(\eta)$ and $\overline{B}(\eta)$ will again be denoted by $\varepsilon$ and $\eta$. By abuse, we shall sometimes regard $\overline{B}U$ as a sub R-module of $BU$. In the following lemmas, we use Corollaries A.5 and A.6 to fix notations and we compare the "simplicial" internal structures on $BU$ and $\overline{B}U$ derived above to the more usual "homological" internal structures. Observe that $\overline{B}U$ is a differential co-algebra with respect to the coproduct $D$ defined by the formula

$$D[u_1|\ldots|u_p] = \sum_{i=0}^{p} (-1)^{(p-i)} q[u_1|\ldots|u_i] \otimes [u_{i+1}|\ldots|u_p] ,$$

$$q_i = \sum_{j=1}^{i} \deg u_j .$$

**Lemma A.11.** Let $U$ be commutative with product $\mu$ and let

$$\phi^* = B_a(\mu) \cdot V : B_a U \otimes B_a U \to B_a U .$$

Then $\phi^*: BU \otimes BU \to BU$ is the unique $\mu$-equivariant morphism of augmented differential R-modules such that $\phi^*(BU \otimes BU) \subseteq \overline{B}U$, namely that defined inductively by $\phi = \phi^* \phi$ on $BU \otimes BU \otimes BU$. If $\phi = B_a(\mu) \ast V$, then $\phi$ is the restriction of $\tau \ast \phi$ to $BU \otimes BU$ and $\phi$ and $D$ give $\overline{B}U$ a structure of commutative differential Hopf algebra.

**Lemma A.12.** Let $U$ be a differential Hopf algebra with coproduct $\Delta$ and let $\phi^* = V^{-1}B_a(\Delta): B_a U \to B_a U \otimes B_a U$. Then $\phi: BU \to BU \otimes BU$ is the unique $\Delta$-equivariant morphism of augmented differential R-modules such that $\phi(BU) \subseteq \overline{B}U \otimes \overline{B}U$, namely that defined inductively by

$$\psi = (s \otimes \eta \varepsilon + 1 \otimes s)\phi .$$

If $\overline{\psi} = V^{-1}B_a(\Delta)$, then $\overline{\psi}$ is the restriction of $(\pi \otimes \pi)\psi$ to $\overline{B}U$, and $\overline{\psi}$ also coincides with the explicit coproduct $D$ of $(\phi)$.

**Lemma A.13.** Again, let $U$ be a differential Hopf algebra. Then the homotopy $\psi_1 = \xi \ast C \psi_1^*$ agrees with that defined inductively by

(i) $\psi_1: BU \to BU \otimes BU$ is $\Delta$-equivariant; and

(ii) $\psi_1(\varepsilon) = 0$ and $\psi_1 s = (s \otimes \eta \varepsilon + \eta \varepsilon \otimes s)(\psi s - T\psi s - \psi_1) .

The homotopy $\overline{\psi}_1 = \xi \ast C \overline{\psi}_1^*$ is the restriction of $(\pi \otimes \pi)\psi_1^*$ to $\overline{B}U$ and is also given by the following explicit formula

$$\overline{\psi}_1[u_1|\ldots|u_p] = \sum_{1 \leq i \leq j \leq p} (-1)^q[u_1|\ldots|u_{i-1}|u_{i+1}|\ldots|u_j|\ldots|u_p] \otimes [u_1^n|\ldots|u_p^n],$$

where $\Delta(u_n) = \sum u_k^n \otimes u_k^n$ (with the index of summation understood) and

$$\alpha = i + (j+1-i)(p-j) + \sum_{a=1}^{p} \deg u_a - \sum_{b=1}^{p} \deg u_b^n + \sum_{b=1}^{p} \deg u_{i-b} + \sum_{c=1}^{p} \deg u_{c-b} + \sum_{d=1}^{p} \deg u_{d+1} .$$

Moreover $\overline{\psi}_1$ and $\psi$ satisfy formulas (6) and (7) on $\overline{B}U$, hence the dual algebra $B^\ast U$ is homotopy commutative via a $\psi_1^*$-product $\psi_1^*$ which satisfies the Hirsch formula.

**Proof.** $\psi_1$ satisfies (i) by a trivial verification. By (4), $\psi_1$ on $BU$ is a homotopy from $\psi$ to $T\psi'$, where $\psi' = \xi \ast C(T\psi')$. It follows by induction on homological degree that $\psi_1 s = S(\psi s - T\psi' s - \psi_1)$, where
these calculations is the role of the augmentation of $U$ in the zeroth and last faces on $\bar{B}U$ and the formula $(\varepsilon \otimes 1)\Delta = 1 = (1 \otimes \varepsilon)\Delta$ in the definition of a Hopf algebra.

We next study the $W$ construction. Our definition differs from that of Mac Lane [21] only in that we have reversed the order of the face and degeneracy operators so as to simplify the comparison with the bar construction (as a right, rather than as a left, module). For an ungraded $R$-module $E$, let $E_0$ denote the simplicial $R$-module such that $E_0 = E$ and each $0_i$ and $1_i$ is the identity map.

**Definition A.14.** Let $A$ be a simplicial augmented $R$-algebra with product $\mu: A \otimes A \to A$, unit $\eta: R_e \to A$, and augmentation $\varepsilon: A \to R_e$.

Define simplicial $R$-modules $W_0A$ and $\bar{W}_0A$ by

(i) $W_0A = A_0 \otimes \cdots \otimes A_p$, with $s_1 = 1 \otimes \eta \otimes s^{p+1-1}$ if $0 \leq i \leq p$;

$$\partial_0 = \varepsilon \otimes \partial_0^R \text{ and } \partial_i = i^{i-1} \otimes \mu_{i-1} (1 \otimes \theta_i) \otimes s^2 \quad \text{if } 0 < i \leq p;$$

(ii) $\bar{W}_0A = R$ and $\bar{W}_0A = A_0 \otimes \cdots \otimes A_{p-1}$ if $p > 0$, with $s_1 = 1 \otimes \eta \otimes s^{p-1}$,

$$\partial_0 = \varepsilon \otimes \partial_0^R \text{ and } \partial_i = i^{i-1} \otimes \mu_{i-1} (1 \otimes \theta_i) \otimes s^2 \quad \text{if } 0 < i \leq p,$$

$$\partial_p = 1 \otimes \varepsilon \otimes s_p.$$  

(The exponents in these formulas all denote iterated tensor products.) The maps $1^p \otimes \mu: W_0A \otimes A_p \to W_0A$ and $1^p \otimes \varepsilon: W_0A \to \bar{W}_0A$ are the components of morphisms $\tilde{\mu}: W_0A \otimes A \to W_0A$ and $\eta: W_0A \to \bar{W}_0A$ of simplicial $R$-modules. By abuse, $\bar{W}_0A$ will sometimes be regarded as a sub-
R-module of \( \mathbb{W}_A \) via \( \beta \otimes \eta \); then \( \mathbb{W}_A = \mathbb{W}_A \otimes A \cdot p \), and all degeneracies and all faces except the last on \( \mathbb{W}_A \) agree with those of the tensor product. With \( f_0 \otimes \cdots \otimes f_{p-1} \) on \( \mathbb{W}_A \) and \( f_0 \otimes \cdots \otimes f_{p-1} \), \( \mathbb{W}_A \) and \( W_A \) define functors from the category of simplicial augmented \( R \)-algebras to the category of simplicial augmented and unital \( R \)-modules. The augmentations \( \mathbb{W}_A \eta_A \) and \( \mathbb{W}_A \eta_A \) and units \( \mathbb{W}_A \eta_A \) and \( \mathbb{W}_A \eta_A \) of \( \mathbb{W}_A \) and \( \mathbb{W}_A \) will be denoted by \( \eta_A \) and \( \eta_A \).

Observe that the functors \( \mathbb{W}_A \) and \( W_A \) preserve tensor products in the sense that the evident shuffling isomorphisms in \( p \)-simplices (which here involve no signs) define commutative and associative natural isomorphisms
\[
V : \mathbb{W}_A \otimes \mathbb{W}_A ^! \to \mathbb{W}_A (A \otimes A) ^!, \quad V : \mathbb{W}_A \otimes \mathbb{W}_A ^! \to \mathbb{W}_A (A \otimes A) ^!.
\]
A comparison of Definitions 2.1 and 2.4 yields the following result.

**Lemma 2.5.** Let \( A = E_A \) be a simplicial augmented \( R \)-algebra, where \( E \) is an augmented \( R \)-algebra. Then \( \mathbb{W}_A E = \mathbb{E}_A \) and \( \mathbb{W}_A A = \mathbb{E}_A \).

In analogy with the definition of the bar construction, we define
\[
\mathbb{W}_A = \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A, \quad \mathbb{W}_A = \mathbb{W}_A \mathbb{W}_A, \quad \mathbb{W}_A = \mathbb{W}_A \mathbb{W}_A.
\]
We write \( \eta \) and \( \xi \) for the augmentations \( \mathbb{W}_A A, \mathbb{W}_A, \) and \( \mathbb{W}_A \) and we write \( \mu \) and \( \eta \) for the product on \( CA \) and for the right action of \( CA \) on \( WA \) induced from \( \mu \) and from \( \eta \). The following lemmas are the analogs for the \( W \) construction of Lemmas 2.8 and 2.11-A, 2.13 for the bar construction. We write typical elements of \( \mathbb{W}_A \) and of \( \mathbb{W}_A \) in the form
\[
\langle a_0, \ldots, a_{p-1} \rangle \circ a_p \quad \text{and} \quad \langle a_0', \ldots, a_{p-1}' \rangle, \quad a_i \in A_i.
\]

**Lemma 2.16.** Let \( \mathbb{W}_A \to \mathbb{W}_A \) be the morphism of \( \mathbb{W}_A \) of degree one defined by \( \tau \langle a_0', \ldots, a_{p-1}' \rangle = (-1)^{p} \langle a_0', \ldots, a_{p-1}' \rangle \). Then
\[
\tau \circ \tau = 1 - \eta \quad \text{and} \quad \tau = 0 \quad \text{on} \quad \mathbb{W}_A.
\]

We use Corollaries 2.5 and 2.6 to fix notations in the following lemmas. Observe that \( \mathbb{W}_A \) is a differential coalgebra with respect to the coproduct \( D \) defined by the formula
\[
D \langle a_0, \ldots, a_{p-1} \rangle = \sum_{i=0}^{p} \langle a_0', \ldots, a_{i-1}', \tau \langle a_i, \ldots, a_{p-1} \rangle \rangle \otimes \tau \langle a_0', \ldots, a_{i-1}', a_i, \ldots, a_{p-1} \rangle.
\]

**Lemma 2.17.** Let \( \mathbb{W}_A \to \mathbb{W}_A \) be commutative and let
\[
\tau := W(\mu) \circ V : \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \to \mathbb{W}_A.
\]

(i) \( \tau : \mathbb{W}_A \mathbb{W}_A \to \mathbb{W}_A \) is \( \mu \)-equivariant; and

(ii) \( \tau (\mathbb{W}_A \mathbb{W}_A) \subseteq \mathbb{W}_A \), hence \( \tau = \tau \circ \tau \) on \( \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \). If \( \tau \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \), then \( \tau \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \) is the restriction of \( \tau \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \) to \( \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \) and \( \tau \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \mathbb{W}_A \).

Proof. (i) follows from the (dotted arrow) diagram of Proposition 2.3. The first half of (ii) is evident from the definition of the shuffle map, and the second half follows by use of Lemma 2.16. The last statement follows from Corollary 2.7. If \( \mathbb{W}_A \) is a simplicial Hopf algebra, and the general case can be deduced from this special case by the argument in the proof of Theorem A.20 below.

**Lemma 2.18.** Let \( \mathbb{W}_A \) be a simplicial Hopf algebra with coproduct \( \Delta \) and let \( \psi : \mathbb{W}_A \to \mathbb{W}_A \) be a Hopf algebra. Then
\[
\tau \tau = \tau \tau \tau.
\]
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(i) \( \psi: WA \to WA \otimes WA \) is \( \Delta \)-equivariant; and

(ii) \( \psi(\overline{WA}) \subset WA \otimes \overline{WA} \), hence \( \psi s = (s \otimes \eta \varepsilon + 1 \otimes s)\psi \).

If \( \overline{\psi} = \psi^{-1} \overline{WA}(\Delta_0) \), then \( \overline{\psi} \) is the restriction of \( (\pi \otimes \pi)\psi \) to \( \overline{WA} \), and \( \overline{\psi} \) also coincides with the explicit coproduct \( D \) of (10).

Proof. (i) follows from the (solid arrow) diagram of Proposition A.3. The first half of (ii) is evident from the definition of the Alexander–Whitney map, and the second half follows by use of Lemma A.16. The last statement is verified by an easy calculation.

Lemma A.19. Again, let \( A \) be a simplicial Hopf algebra. Then the homotopy \( \psi = \xi \circ C(\psi): WA \to WA \otimes WA \) satisfies the formulas

(i) \( \psi_1(xa) = \psi_1(x)\Delta(a) \in \text{Ker} (\pi \otimes \pi) \) for \( x \in WA \) and \( a \in CA \);

(ii) \( \psi_1 > = 0 \) and \( \psi_1 s = (s \otimes 1 + \eta \varepsilon \otimes s)(\psi s - T\psi s - \psi_1) \).

The homotopy \( \overline{\psi} = \xi \circ \overline{\psi} \otimes \overline{\psi} \) is the restriction of \( (\pi \otimes \pi)\psi \) to \( \overline{WA} \). Moreover, \( \overline{\psi} \) and \( \overline{\psi} \) satisfy formulas (6) and (7) on \( \overline{WA} \), hence the dual algebra \( \overline{W}^* A \) is homotopy commutative via a \( \circ \) product which satisfies the Hirsch formula.

Proof. To prove (i), consider the following diagram:

If the two paths from \( WA \otimes CA \) to \( WA \otimes WA \) given by the outer edges of the diagram were equal, then \( \psi_1 \) would be \( \Delta \)-equivariant. This is not the case since the largest left (six vertex) rectangle fails to commute, in contrast to the analogous diagram of Proposition A.3. However, it is trivial to verify that all sub diagrams with five or less vertices commute, and therefore \( \psi_1 \) is \( \Delta \)-equivariant modulo the kernel of \( \pi \otimes \pi \), as claimed in (i). The proofs of (ii) on \( WA \) and of (6) and (7) on \( \overline{WA} \) are precisely analogous to the proofs of the corresponding parts of Lemma A.13. Of course, these results
would be obvious if $A$ were cocommutative, and this will be the case in our applications.

With these preliminaries, we can prove the following theorem.

**Theorem A.20.** Let $A$ be a simplicial augmented $R$-algebra. Then there is a unique morphism $\lambda : BCA \to WA$ of differential right $CA$-modules augmented over $R$ such that $\lambda(BCA) \subset WA$. Let $\bar{\lambda} : \overline{BCA} \to \overline{WA}$ be the re-

(i) $\overline{\lambda}$ is a morphism of coalgebras and induces an isomorphism on homology.

(ii) If $A$ is commutative, then $\lambda$ and $\overline{\lambda}$ are morphisms of algebras.

(iii) If $A$ is a simplicial Hopf algebra, then $\lambda$ and $\overline{\lambda}$ are morphisms of

coalgebras and $\overline{\lambda}$ (but not $\lambda$) commutes with the homotopy $\psi_4$.

**Proof.** $\lambda$ can and must be defined inductively by $\lambda = \text{id}_{\overline{BA}}$ on $\overline{BA} \cap \ker \in \overline{BA}$. Note that $\lambda$s = $\lambda$ follows. Parts (ii) and (iii) are now immediate consequences (by induction on degree) of Lemmas A.11-A.13 and A.17-A.19. For the second half of (iii), one first verifies that

$$ (\overline{\psi} \otimes \psi) (\overline{\lambda} \otimes \lambda) \psi_4 = (\overline{\psi} \otimes \psi) \psi_4 \lambda $$

on $\overline{BCA}$ and then deduces the required formula $\overline{\psi_4} (\overline{\lambda} \otimes \lambda) \psi_4 = \psi_4 \overline{\lambda}$ on $\overline{BCA}$. The first part of (i) can be deduced by naturality from (iii), since $A$ is a quotient algebra of the free associative simplicial algebra $T_{\psi}A$, by the simplicial $R$-module $A$, and $T_{\psi}A$ admits a structure of simplicial Hopf algebra. The second half of (i) is proven by an argument precisely analogous to the proof of Theorem 3.9.

Comparison theorem is applied with respect to the ground ring $H_0(GA)$. Further details may be found in Moore's paper [32, exposé 13].

If $G$ is a simplicial monoid, we can define simplicial sets $WG$ and $W_{\psi}G$ by analogy with Definition A.14. We simply replace tensor products by Cartesian products and $R_{\psi}$ by the point complex $(e)_{\psi}$. (In the context of monoidal categories [29, §10], a single definition would suffice.) We give $F_{\psi}G$ a structure of simplicial Hopf algebra by regarding $F_{\psi}G$, which is the free $R$-module $FG$ generated by $G_p$, as the monoid ring of $G_p$. The functor $F$ converts finite Cartesian products to tensor products and it follows that $F_{\psi}WG$ and $F_{\psi}W_{\psi}G$ may be identified with $W_*F_{\psi}G$ and $W_\psi F_{\psi}G$ as simplicial coalgebras and, if $G$ is Abelian, as simplicial Hopf algebras.

The functors $W$ and $W_{\psi}$ from simplicial monoids to simplicial sets differ from the standard constructions [25, §21] only in that we have reversed the order of faces and degeneracies. Therefore, by the theory of minimal complexes [25, §9] and of simplicial $K(\pi, n)$'s [25, §23], we have the following result.

**Proposition A.21.** Let $\pi$ be an Abelian group. Define $W^0 = \pi_*$ and $W_n = W(W^{n-1} - \pi)$ if $n \geq 1$. Then $W_{\psi}$ is isomorphic to a strong deformation retract of any Kan complex $K$ such that $\pi_n(K) = \pi$ and $\pi_{n+1}(K) = 0$ for $i \not= n$.

Thus we can replace the singular chains and cochains of any geometric $K(\pi, n)$ by the chains and cochains of the simplicial set $W_{\psi}$. Now Theorem A.20 implies the following key result.
Theorem A.22. For an Abelian group $\pi$, define $\overline{B}^n = \overline{B}^{n-1} \otimes \pi$ for $n \geq 1$. Let $\pi_i$ be an Abelian group and let $n_i \geq 1$, $1 \leq i \leq q$. Then there is a morphism
\[ \xi : \overline{B}^q \otimes \bigotimes_{i=1}^q \overline{B}^{n_i-1} \pi_i \rightarrow \overline{B}^q \otimes \bigotimes_{i=1}^q \text{K}(\pi_i, n_i)_1 \otimes \pi \]
of commutative and homotopy cocommutative differential Hopf algebras such that $\xi$ induces an isomorphism on homology.

Proof. \[ \bigotimes_{i=1}^q \text{K}(\pi_i, n_i)_1 = \bigotimes_{i=1}^q \overline{W}^{n_i-1} \pi_i = \overline{W} \bigotimes_{i=1}^q \overline{W}^{n_i-1} \pi_i. \]
Since $F \overline{W} = \overline{W} F$ and $F \pi_i$ converts Cartesian products to tensor products, we have
\[ F \bigotimes_{i=1}^q \text{K}(\pi_i, n_i)_1 = \overline{W} \bigotimes_{i=1}^q \overline{W}^{n_i-1} \pi_i, \]
where $\overline{W} \pi_i = F \pi_i$ and $\overline{W} \pi_i = \overline{W}^{n_i-1}$ for $n_i \geq 1$. $\overline{B}^n = \text{C} \overline{W} \pi$ since $\text{C} \pi_i = F \pi_i$ and $\overline{B}^1 = \text{C} \overline{W} \pi$ by Lemma A.15 since $F \pi_i = (F \pi_i)_0$. We define $\xi = 1$ on $\overline{B}^0 \pi$ and on $\overline{B}^1 \pi$. By induction on $n_i - 1$, assume $\xi$ has been defined when $q = 1$ and $n_i < n_i$ for all $i$. Then define $\xi$ to be the composite:
\[ \overline{B}^q \otimes \bigotimes_{i=1}^q \pi_i \rightarrow \overline{B}^q \otimes \bigotimes_{i=1}^q \text{C} \overline{W} \pi_i \rightarrow \overline{B}^q \otimes \bigotimes_{i=1}^q \overline{W}^{n_i-1} \pi_i \]
\[ \rightarrow \overline{W} \bigotimes_{i=1}^q \overline{W}^{n_i-1} \pi_i. \]
Here $\eta$ is the iterated shuffle map (or the identity if $q = 1$) and is a morphism of differential Hopf algebras by Corollary A.7. If $f : U \rightarrow U'$ is a homology isomorphism of non-negatively graded $R$-flat differential $R$-algebras, then $\overline{B}f$ is also a homology isomorphism since, by the spectral sequences $\overline{B}(BU)$ and $\overline{B}(BU')$ obtained from the filtration by simplicial degree are functors of $HU$ and $HU'$. Thus $\xi$ is a composite of homology isomorphisms which, by Lemma A.13 and Theorem A.20, preserve all requisite structure, including the homotopy for cocommutativity.

It remains to study the homology of the bar constructions of Theorem A.22. We shall need the following general result concerning the classical (bigraded) torsion product of a (graded) augmented algebra without internal differential.

Proposition A.23. Let $\Lambda$ be an augmented algebra and let $K\Lambda = \overline{K}\Lambda \otimes \Lambda$ be a differential right $\Lambda$-module augmented over $R$. Then there is a unique morphism $\theta : K\Lambda \rightarrow B\Lambda$ of right differential $\Lambda$-modules augmented over $R$ such that $\theta(K\Lambda) \subset \overline{B}\Lambda$. Let $\overline{\theta} : \overline{K}\Lambda \rightarrow \overline{B}\Lambda$ be the restriction of $\theta$ to $\overline{K}\Lambda$. Then

(i) if $\Lambda$ and $\overline{K}\Lambda$ are projective $R$-modules and if $K\Lambda$ is a resolution of $R$, then $\overline{\theta}$ induces an isomorphism on homology.

(ii) if $\Lambda$ is commutative with product $\mu$ and if $K\Lambda$ admits a $\mu$-equivariant product $f : K\Lambda \otimes K\Lambda \rightarrow K\Lambda$ such that $f(K\Lambda \times K\Lambda) \subset \overline{K}\Lambda$, then $\theta$ and $\overline{\theta}$ are morphisms of algebras.

(iii) if $\Lambda$ is a Hopf algebra with coproduct $\Delta$ and if $K\Lambda$ admits a $\Delta$-equivariant coproduct $\psi : K\Lambda \rightarrow K\Lambda \otimes K\Lambda$ such that $\Delta(K\Lambda) \subset K\Lambda \otimes K\Lambda$, the $\theta$ and $\overline{\theta}$ are morphisms of coalgebras.

(iv) if, under the hypotheses of (iii), $\Lambda$ and $K\Lambda$ are cocommutative, then the homotopy $\psi_1$ on $B\Lambda$ vanishes on the image of $\theta$. 

Proof. \( \theta \) can and must be defined inductively by \( \theta = s \theta d \) on \( \overline{K} \Lambda \cap \text{Ker} \xi \). Now (i) is obvious and (ii) and (iii) are easily verified by use of the contracting homotopies \( s \) on \( B \Lambda \) and \( s \times \eta \xi + 1 \otimes s \) on \( B \Lambda \otimes B \Lambda \); the product and coproduct on \( B \Lambda \) are those defined in Lemma A.12. To prove (iv), proceed by induction on the homological degree. By Lemma A.13, \( \psi_1^0 = 0 \) provided that \( \psi_1 \theta = 0 \) on \( \overline{K} \Lambda \cap \text{Ker} \xi \) and here, with \( S = s \otimes 1 + \eta \xi \otimes s \) on \( BU \otimes BU \), we find inductively that
\[
\psi_1 \theta = \psi_1 s \theta d = S(\psi s - T \psi s - \psi d) = S(\theta \otimes \theta)(\psi - T \psi) - S \psi_1 \theta d = 0.
\]

In (ii) and (iii), we can always obtain \( \phi \) and \( \psi \) which are appropriately equivariant, and this is enough for the conclusions to hold up to equivariant homotopy, provided that \( \overline{K} \Lambda \) is a \( \Lambda \)-projective resolution of \( R \). However, there may be no choice for \( \phi \) or \( \psi \) which satisfies the hypothesis concerning \( \overline{K} \Lambda \). The following lemma simplifies application of the proposition.

Lemma A.24. Let \( \overline{K} \Lambda_1 = \overline{K} \Lambda_1 \otimes \Lambda_1 \) be a differential right \( \Lambda_1 \)-module augmented over \( R \), where \( \Lambda_1 \) is an augmented \( R \)-algebra and \( 1 \) runs through an ordered indexing set \( I \). Let \( \Lambda = \bigotimes_{i \in I} \Lambda_i \), \( \overline{K} \Lambda = \bigotimes_{i \in I} \overline{K} \Lambda_i \), and \( \overline{K} \Lambda = \overline{K} \Lambda \otimes \Lambda \). Give \( \overline{K} \Lambda \) the differential determined from the differentials on the \( \overline{K} \Lambda_i \) by means of the evident shuffle isomorphism between \( \overline{K} \Lambda \) and \( \bigotimes_{i \in I} \overline{K} \Lambda_i \). Then \( \Lambda \) and \( \overline{K} \Lambda \) satisfy the hypotheses of (i), (ii), (iii), or (iv) of the proposition if \( \Lambda_1 \) and \( \overline{K} \Lambda_i \) satisfy the corresponding hypotheses for all \( i \in I \).

We now give a list of resolutions for monogenic commutative Hopf algebras. In each case, we shall write down an algebra \( \overline{K} \Lambda \) and a differential \( d \) on the generators of \( \overline{K} \Lambda \subseteq \overline{K} \Lambda \otimes \Lambda \). The differential on \( \overline{K} \Lambda \) is then uniquely determined by the requirement that \( \overline{K} \Lambda \) be a differential \( \Lambda \)-algebra, and the hypotheses of (ii) will be satisfied in all cases. We shall also specify \( \theta \) on the generators of \( \overline{K} \Lambda \); for this purpose, we shall write
\[
\gamma(x_1 \cdots x_n) = [x_1 \cdots x_n | x_1 \cdots x_n | x_1 \cdots x_n],
\]
for factors \( [x_1 \cdots x_n] \). Finally, we shall write down a coproduct on \( \overline{K} \Lambda \subseteq \overline{K} \Lambda \); the hypotheses of (iii) will be satisfied in cases I, II, IV, and V, and the hypotheses of (iv) will also be satisfied in cases IV and V. In all cases, \( \overline{K} \Lambda \) will be a differential coalgebra under the resulting equivariant coproduct. The acyclicity of the various \( \overline{K} \Lambda \) can easily be verified by construction of a contracting homotopy. The letters \( E, P, \) and \( \Gamma \) will denote exterior, polynomial, and divided polynomial algebras. Recall that \( \Gamma(y) \) is the Hopf algebra dual to \( P(y^\theta) \); we shall usually write \( \gamma(x) \) instead of the customary \( \gamma(x) \) for the basis element dual to \( (y^\theta)^\Gamma \).

(I) \( \Lambda = F \tau \), where \( \tau \) is the infinite cyclic group with generator \( x \);
\[
\overline{K} \Lambda = E(y), \ deg y = 1; \ d(y) = 1 - x; \ \theta(y) = [x]; \ \psi(y) = y \otimes 1 + x \otimes y.
\]

(II) \( \Lambda = F \tau \), where \( \tau \) is cyclic of order 2 with generator \( x \) and char \( R = 2 \);
\[
\overline{K} \Lambda = \Gamma(y), \ deg y = 1; \ d(y) = y; \ \theta(y) = y[1]; \ \psi(y) = \sum_{i,j=x} \gamma^i \otimes y^j.
\]
coalgebras for any choice of $\psi$ in these cases. If $p = 2$ and $n = 1$, then III and VI reduce to II and V (with $z = y^2/2$), and here $\theta$ and $\bar{\theta}$ clearly are morphisms of coalgebras.

In all cases, $\mathbb{K}A$ is a minimal resolution of $R$ in the sense that the induced differential on $\mathbb{K}A = \mathbb{K}A \otimes_A R$ is zero. Moreover, the induced Hopf algebra structure on $\mathbb{K}A$ is the standard one (provided that we exclude the case $p = 2$ and $n = 1$ from III and VI), so that $\text{Tor}^A(R, R) = \mathbb{K}A$ as a Hopf algebra. Thus if an augmented $R$-algebra $A$ is isomorphic to a tensor product of algebras of types I-VI, then $\text{Tor}^A(R, R)$ is isomorphic as a Hopf algebra to a tensor product of exterior and divided polynomial algebras. For example, by Borel's theorem [5, 6, 1; see also 28], any connected commutative Hopf algebra of finite type over a perfect field $R$ is isomorphic as an algebra to a tensor product of algebras of types IV-VI.

It should be emphasized that the coproduct of $\text{Tor}^A(R, R)$ depends only on $A$ as an algebra. Therefore, if $\mathbb{K}A$ is a minimal resolution of $R$ and if $A$ admits a Hopf algebra structure, then the coproduct on $\mathbb{K}A$ induced from any $\psi$ for any coproduct $\Delta$ must coincide with the coproduct of $\text{Tor}^A(R, R)$. We conclude that $\bar{\psi}: \mathbb{K}A \to \mathbb{K}A$ is a morphism of Hopf algebras if $A$ is isomorphic as an algebra to a tensor product of algebras of types I, II, IV and V, irrespective of any possible given coproduct on $A$. If $A$ is isomorphic as a Hopf algebra to a tensor product of exterior and polynomial algebras, then we conclude further that $\bar{\psi} \bar{\theta} = 0$. 

Of course, IV is the Koszul resolution used in section 2; here $\theta(y) = [x]$ dictates the sign in the definition of $d$. Observe that if $p > 2$ or $n > 1$ in III or VI, then the images of $\theta$ and $\bar{\theta}$ are not closed under the coproducts of $BA$ and $\mathbb{K}A$; thus neither $\theta$ nor $\bar{\theta}$ can be a morphism of
We now use these resolutions, regraded by total degree, to study the homology of $K(n, q)$'s. Recall that the group ring $F(q_1 \times \ldots \times q_k)$ is isomorphic as a Hopf algebra to $F(q_1 \bigotimes \ldots \bigotimes F q_k)$ for Abelian groups $q_i$. Since $\mathbb{K}$ applied to a tensor product of Hopf algebras of type I yields a tensor product of Hopf algebras of type $V$, the arguments above imply the following theorem.

**Theorem A.25.** Let $\pi$ be the direct sum of finitely many infinite cyclic groups and let $R$ be any commutative ring. Define

$$
\overline{g}^1 = \overline{\mathcal{E}}_{\Phi}(K(q, 1); R) = \overline{\mathcal{K}Fq} \rightarrow \overline{\mathcal{E}Fq} = \overline{\mathcal{E}Fq}^1
$$

and let $\overline{g}^2 : H_q(K(q, 2); R) \rightarrow \overline{\mathcal{E}Fq}^2$ be the composite

$$H_q(K(q, 2); R) = \overline{\mathcal{K}H_q}(K(q, 1); R) \xrightarrow{\overline{g}^1} \overline{\mathcal{E}H_q}(K(q, 1); R) \xrightarrow{\overline{b}^1} \overline{\mathcal{E}Fq}^2. $$

Then $\overline{g}^1$ and $\overline{g}^2$ are morphisms of differential Hopf algebras which induce isomorphisms on homology. Moreover, $\psi_1 \overline{g}^2 = 0$.

With $g = (\xi \overline{g}^2)$, Theorem 4.1 follows immediately from Theorems A.22 and A.25 and Proposition A.21. Similarly, the following theorem will imply Theorem 4.2.

**Theorem A.26.** Let $\pi$ be a finitely generated Abelian group and let $n_i \geq 1$, $1 \leq i < q$. Let $p$ be a prime. Then there is a morphism

$$
\chi : H_q(\bigotimes_{i=1}^q K(q_i, n_i); Z_p) \rightarrow \mathcal{E}(\bigotimes_{i=1}^q \mathcal{E}Fq_i) \rightarrow \overline{\mathcal{B}(\bigotimes_{i=1}^q \mathcal{E}Fq_i)} \chi
$$

of commutative differential algebras such that $\chi$ induces an isomorphism on homology. For all $p$, $\overline{\mathcal{B}(\chi)}$ is a morphism of differential Hopf algebras and $\overline{\mathcal{B}(\chi)}$ is a morphism of homotopy commutative differential Hopf algebras. If $p = 2$ and if $\pi$ has no $4$-torsion (elements of order 4) when $n_i = 1$, then $\chi$ is a morphism of differential Hopf algebras and $\overline{\mathcal{B}(\chi)}$ is a morphism of homotopy commutative differential Hopf algebras.

**Proof.** We may assume, without loss of generality, that $\pi$ contains no $p$-torsion prime to $p$. We write $H(\pi, n, p)$ for $H_q(K(\pi, n); Z_p)$. $H(\pi, 0, p)$ is obviously a tensor product of Hopf algebras of types I-III and, by Borel's theorem, $H(\pi, n, p)$ is a priori a tensor product of algebras of types IV-VI if $n \geq 1$ (actually, type IV algebras will not appear). Define $\chi = 1$ on $H(\pi, 0, p) = \mathcal{E}Fq = \mathcal{E}Fq^0$. Proceeding inductively, define $\chi$ to be the composite

$$
\overline{\mathcal{K}(\bigotimes_{i=1}^q H(q_i, n_i-1, p))} \rightarrow \overline{\mathcal{B}(\bigotimes_{i=1}^q H(q_i, n_i-1, p))} \rightarrow \overline{\mathcal{B}(\bigotimes_{i=1}^q \mathcal{E}Fq_i)} = \overline{\mathcal{B}(\chi)}.
$$

By Proposition A.23 and by the algebraic Eilenberg-Moore spectral sequence $q \overline{\mathcal{B}(\chi)}$ induces isomorphism on homology. The right-hand side computes $H_q(\bigotimes_{i=1}^q K(q_i, n_i); Z_p)$ by Theorem A.22 and the left-hand side has zero differential and may thus be identified with this homology. By I-VI, Lemma A.24, and Proposition A.23, $\overline{g}$ is certainly a morphism of algebras.

Observe that, as an algebra, $\Gamma(y) = \bigotimes_{r=0}^\infty P(x_r)/(x_r^p)$ where $x_r = y x_r$. Thus, if $p = 2$, $\Gamma(y) = E_0(x_r)$ and we do not need VI to define $\overline{g}$. Our assumption as to 4-torsion guarantees that $\overline{g}$ is a morphism of Hopf algebras.

Since $\overline{\mathcal{B}}$ is a functor from commutative augmented differential algebras to
commutative differential Hopf algebras by Lemma A.11 and from differential Hopf algebras to homotopy cocommutative differential coalgebras (with specified homotopy) by Lemma A.13, the proof is complete.

Of course, the theorem yields an inductive calculation of the specified homology. We must still indicate precisely how to compute \( \omega \)-products in \( H^*(\bigotimes_{i=1}^q K(n_i, n_1 n_2); Z_2^*) \). Assume for simplicity that \( \gamma \) has no 4-torsion if \( n_1 = 1 \) or \( n_2 = 2 \); then the map \( \overline{B}(\bigotimes_{i=1}^q \chi) \) in the composite definition of \( \chi \) commutes with the homotopy \( \psi \) and the Hopf algebra \( \bigotimes_{i=1}^q H(\pi_i, n_1 n_2, 2) \) is a tensor product of divided polynomial algebras. Thus we need only study the dual map \( \overline{B}^* : B^* \Lambda \rightarrow K^* \Lambda \) where \( \Lambda = \Gamma(x_i) \) as a Hopf algebra over \( Z_2^* \). As an algebra, \( \Lambda = \Sigma(x_i) \), where \( x_i = y_i \otimes 1 \). Therefore \( \overline{K}^* \Lambda = \Gamma(y_i) \), where \( \deg y_i = 1 + \deg x_i \), and \( \overline{K}^* \Lambda = \Sigma(y_i) \). Clearly \( \overline{B}^* \Lambda \) is the free associative algebra generated by \( \{w\} \), where \( w \) runs through the dual of a basis for the monomials in the \( x_i \). \( \overline{B}^* \) is the morphism of algebras determined by \( \overline{B}^*[x_i] = y_i \) and \( \overline{B}^*[w] = 0 \) for all duals \( w \) of decomposable monomials in the \( x_i \). To compute \( \omega \)-products \( \gamma_1 \cdots \gamma_p \) of elements \( \gamma_i \in \overline{B}^* \Lambda \), we choose elements \( \delta_i \in \overline{B}^* \Lambda \) such that \( \overline{B}^*(\delta_i) = \gamma_i \); compute \( \delta_1 \cdots \delta_p \) in \( \overline{B}^* \Lambda \), and apply \( \overline{B}^* \); the result is independent of the choice of the \( \delta_i \) by Lemma 2.4. To compute the \( \omega \)-products in \( \overline{B}^* \Lambda \), we use the following formula of Adams [1], which is obtained by dualization of (9) and applies to any Hopf algebra \( \Lambda \) of finite type over \( Z_2^* \).

\[
\begin{align*}
(11) \quad [\alpha_1, \ldots, \alpha_p] & \omega [\beta_1, \ldots, \beta_p] \\
& = \sum_{r=1}^p \sum_{s=1}^q [\alpha_{r-1}, \alpha_r^{(1)} \beta_1, \ldots, \alpha_r^{(q)} \beta_q] \alpha_{s+1}, \ldots, \alpha_p, \beta_1, \ldots, \beta_p,
\end{align*}
\]

where \( \alpha \), \( \beta \in \Lambda^* \), and the iterated coproduct \( \mu^* : \Lambda^* \rightarrow (\Lambda^*)^q \) is written \( \mu^*(\alpha) = \sum a^{(1)} \otimes \cdots \otimes a^{(q)} \), with the index of summation understood.

When \( \mu^*(\gamma_i) = \gamma_i \otimes 1 + 1 \otimes \gamma_i \) for each \( i \), formula (11) reduces to

\[
(12) \quad [\alpha_1, \ldots, \alpha_p] \omega [\beta_1, \ldots, \beta_p] \\
& = \sum_{r=1}^p \sum_{s=1}^q [\alpha_{r-1}, \alpha_r \beta_1, \ldots, \beta_s, \ldots, \beta_p] \alpha_{s+1}, \ldots, \alpha_p, \beta_1, \ldots, \beta_p.
\]

For example, with \( \Lambda = \Gamma(x_i) \) as above, we have \( \Lambda^* = \Sigma(x_i^*) \), where \( x_i^* = x_{i,0}^* \) and \( (x_i^*)^2 = x_{i+1}^* \) are primitive; thus, in \( \overline{K}^* \Lambda \),

\[
\begin{align*}
[y_{ij} \omega y_{kl}] &= \delta^*([x_{ij}] \omega [x_{kl}]) = \delta^*([x_{ij}] [x_{kl}]) = \begin{cases} 0 & \text{if } i \neq k \text{ or } j \neq l \\ y_{ij,kl} & \text{if } i = k \text{ and } j = l
\end{cases},
\end{align*}
\]

Now let \( \Lambda = H^*(K(n, n-1)) \); here \( \gamma \omega \gamma = Sq^q \gamma \) if \( \gamma \in \overline{K}^* \Lambda = H^{q+1} K(n, n) \) and we conclude inductively that

\[
\gamma^*_{ij} = Sq^j Sq^{i+j} \cdots Sq^q y_{1,0}, \quad \deg x_i = q.
\]

By comparison with Serre's calculations [36, Théorèmes 2-5], it follows that the Cartan basis for \( H^* K(n, n) \), which is obtained by dualization from Theorem A.26, coincides with the Serre basis in terms of admissible monomials in the Steenrod squares.
BIBLIOGRAPHY


Differential Torsion Products


