# OPERADS, ALGEBRAS, MODULES, AND MOTIVES

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Abstract. With motivation from algebraic topology, algebraic geometry, and string theory, we study various topics in differential homological algebra. The work is divided into five largely independent parts:

I Definitions and examples of operads and their actions
II Partial algebraic structures and conversion theorems
III Derived categories from a topological point of view
IV Rational derived categories and mixed Tate motives
V Derived categories of modules over $E_\infty$ algebras

In differential algebra, operads are systems of parameter chain complexes for multiplication on various types of differential graded algebras "up to homotopy", for example commutative algebras, $n$-Lie algebras, $n$-braid algebras, etc. Our primary focus is the development of the concomitant theory of modules up to homotopy and the study of both classical derived categories of modules over DGA's and derived categories of modules up to homotopy over DGA's up to homotopy. Examples of such derived categories provide the appropriate setting for one approach to mixed Tate motives in algebraic geometry, both rational and integral.

## Contents

## Part . Introduction

There are many different types of algebra: associative, associative and commutative, Lie, Poisson, etc., etc. Each comes with an appropriate notion of a module and thus with an associated theory of representations. Moreover, as is becoming more and more important in a variety of fields, including algebraic topology, algebraic geometry, differential geometry, and string theory, it is very often necessary to deal with "algebras up to homotopy" and with "partial algebras". The associated theories of modules have not yet been developed in the published literature, but these notions too are becoming increasingly important. We shall study various aspects of the theory of such generalized algebras and modules in this paper. We shall also develop some related algebra in the classical context of modules over DGA's. While much of our motivation comes from the theory of mixed Tate motives in algebraic geometry, there are pre-existing and potential applications in all of the other fields mentioned above.

The development of abstract frameworks in which to study such algebras has a long history. It now seems to be widely accepted that, for most purposes, the most convenient setting is that given by operads and their actions [46]. While the notion was first written up in a purely topological framework, due in large part to the resistance of topologists to abstract nonsense at that period, it was already understood by 1971 that the basic definitions apply equally well in any underlying symmetric monoidal (= tensor) category [35]. In fact, certain chain level concepts, the PROP's and PACT's of Adams and MacLane [42], were important precursors of operads. From a topological point of view, the switch from algebraic to topological PROP's, which was made by Boardman and Vogt [11], was a major step forwards. Perhaps for this reason, a chain level algebraic version of the definition of an operad did not appear in print until the 1987 paper of Hinich and Schechtman [31]. Applications of such algebraic operads and their actions have appeared in a variety of contexts in other recent papers, for example [27, 28, 29, 32, 34, 33, 56].

In the algebraic setting, an operad $\mathscr{C}$ consists of suitably related chain complexes $\mathscr{C}(j)$ with actions by the symmetric groups $\Sigma_j$. An action of $\mathscr{C}$ on a chain complex $A$ is specified by suitably related $\Sigma_j$-equivariant chain maps

$$\mathscr{C}(j) \otimes A^j \to A,$$

where $A^j$ is the $j$-fold tensor power of $A$. The $\mathscr{C}(j)$ are thought of as parameter complexes for $j$-ary operations. When the differentials on the $\mathscr{C}(j)$ are zero, we think of $\mathscr{C}$ as purely algebraic, and it then determines an appropriate class of (differential) algebras. When the differentials on the $\mathscr{C}(j)$ are non-zero, $\mathscr{C}$ determines a class of (differential) algebras "up to homotopy", where the homotopies are determined by the homological properties of the $\mathscr{C}(j)$. For example, we say that $\mathscr{C}$ is an $E_\infty$ operad if each $\mathscr{C}(j)$ is $\Sigma_j$-free and acyclic, and we then say that $A$ is an $E_\infty$ algebra. An $E_\infty$ algebra $A$ has a product for each degree zero cycle of $\mathscr{C}(2)$. Each such product is unital, associative, and commutative up to all possible coherence homotopies, and all such products are homotopic. There is a long history in topology and category theory that makes precise what these "coherence homotopies" are. However, since the homotopies are all encoded in the operad action, there is no need to be explicit. There is a class of operads that is related to Lie algebras as $E_\infty$ operads are related to commutative algebras, and there is a concomitant notion of a "strong homotopy Lie algebra". In fact, any type of algebra that is defined in terms of suitable identities admits an analogous "strong homotopy" generalization expressed in terms of actions by appropriate operads.

We shall give an exposition of the basic theory of operads and their algebras and modules in Part I. While we shall give many examples, the deeper parts of the theory that are geared towards particular applications will be left to later parts. In view of its importance to string theory and other areas of current interest, we shall illustrate ideas by describing the relationship between the little $n$-cubes operads of iterated loop space theory on the one hand and $n$-Lie algebras and $n$-braid algebras on the other. An operad $\mathscr{S}$ of topological spaces gives rise to an operad $C_{\#}(\mathscr{S})$ of chain complexes by passage to singular chains. On passage to homology with field coefficients, there results a purely algebraic operad $H_*(\mathscr{S})$. There is a particular operad of topological spaces, denoted $\mathscr{C}_n$, that acts naturally on $n$-fold loop spaces. For $n \geq 2$, the algebras defined by $H_*(\mathscr{C}_n;\mathbb{Q})$ are exactly the $(n-1)$-braid algebras. Even before doing any calculation, one sees from a purely homotopical theorem of [46] that, for any path connected space $X$, $H_*(\Omega^n\Sigma^n X;\mathbb{Q})$ is the free $H_*(\mathscr{C}_n; bQ)$-algebra generated by $H_*(X;\mathbb{Q})$. This allows a topological proof, based on the Serre spectral sequence, of the algebraic fact that the free $n$-braid algebra generated by a graded vector space $V$ is the free commutative algebra generated by the free $n$-Lie algebra generated by $V$. Actually, the results just summarized are the easy characteristic zero case of Cohen's much deeper calculations in arbitrary characteristic [15, 16], now over twenty years old.

Operads and their actions are specified in terms of maps that are defined on tensor products of chain complexes. In practice, one often encounters structures that behave much like algebras and modules, except that the relevant maps are only defined on suitable submodules of tensor products. For geometric intuition, think of intersection products that are only defined between elements that are in general position. Such partial algebras have been used in topology since the 1970's, for example in [48] and in unpublished work of Boardman and Segal. In Part II, we shall generalize the notions of algebras over operads and of modules over algebras over operads to the context of partially defined structures. Such partially defined structures are awkward to study algebraically, and it is important to know when they can be replaced by suitably equivalent globally defined structures. We shall show in favorable cases that partial algebras can be replaced by quasi-isomorphic genuine algebras over operads, and similarly for modules. When $k$ is a field of characteristic zero, we shall show further that $E_\infty$ algebras and modules can be replaced by quasi-isomorphic commutative algebras and modules and, similarly, that strong homotopy Lie algebras and modules can be replaced by quasi-isomorphic genuine Lie algebras and modules. The arguments work equally well for other kinds of algebras.

One of the main features of the definition of an operad is that an operad determines an associated monad that has precisely the same algebras. This interpretation is vital to the use of operads in topology. The proofs of the results of Part II are based on this feature. The key tool is the categorical "two-sided monadic bar construction" that was introduced in the same paper that first introduced operads [46]. This construction has also been used to prove topological analogs of many of the present algebraic results, along with various other results that are suggestive of further algebraic analogs [47, 49, 26, 52]. In particular, the proofs in Part II are exactly analogous to a topological comparison between Segal's Γ-spaces [55] and spaces with operad actions that is given in [26].

While these results can be expected to have other applications, the motivation came from algebraic geometry. For a variety $X$, Bloch [7] defined the Chow complex $\mathfrak{Z}(X)$. This is a simplicial abelian group whose homology groups are the Chow groups of $X$. It has a partially defined intersection product, and we show in Part II that it gives rise to a quasi-isomorphic $E_\infty$ algebra, denoted $\mathscr{N}(X)$. After tensoring with the rationals, we obtain a commutative differential graded algebra (DGA) $\mathscr{N}_{\mathbb{Q}}(X)$ that is quasi-isomorphic to $\mathscr{N}(X) \otimes \mathbb{Q}$. The construction of these algebras answers questions of Deligne [20] that were the starting point of the present work. His motivation was the intuition that, when $X = \mathrm{Spec}(F)$ for a field $F$, the associated derived categories of modules ought to be the appropriate homes for categories of integral and rational mixed Tate motives over $F$.

This raises several immediate problems. On the rational level, it is necessary to connect this approach to mixed Tate motives with others. On the integral level, in order to take the intuition seriously, one must first construct the derived category of modules over an $E_\infty$ algebra. As a preliminary to the solution of these problems, in Part III we shall give a new, topologically motivated, treatment of the classical derived category of modules over a DGA. We shall give a theory of "cell modules" that is just like the theory of "CW spectra" in stable homotopy theory, and we shall prove direct algebraic analogs of such standard and elementary topological results as the homotopy extension and lifting property, the Whitehead theorem, and Brown's representability theorem. One point is that there is not the slightest difficulty in handling unbounded algebras and modules: except that the details are far simpler, our substitute for the usual approximation of differential modules by projective resolutions works in exactly the same way as the approximation of arbitrary spectra by (infinite) CW spectra with cells of arbitrarily small dimension, which has long been understood. Similarly, derived tensor products of modules work in the same way as smash products of spectra.

In Part IV, we shall specialize this theory to study the derived category $\mathscr{D}_A$ of cohomologically bounded below $A$-modules, where $A$ is a cohomologically connected commutative DGA over a field of characteristic zero. In the language of [3], we shall give the triangulated category $\mathscr{D}_A$ a $t$-structure. Its heart $\mathscr{H}_A$ will be the Abelian subcategory of modules whose indecomposable elements have homology concentrated in degree zero. In the language of [21], we shall show that the full subcategory $\mathscr{F}\mathscr{H}_A$ of finite dimensional modules in $\mathscr{H}_A$ is a neutral Tannakian category. It is therefore the category of representations of an affine group scheme or, equivalently, of finite dimensional comodules over a Hopf algebra.

In fact, without using Tannakian theory, we shall prove directly that $\mathscr{H}_A$ is equivalent to the category of comodules over the explicit commutative Hopf algebra $\chi_A = H^0\bar{B}(A)$. The "cobracket" associated to the coproduct on $\chi_A$ induces a structure of "co-Lie algebra" on its vector space $\gamma_A$ of indecomposable elements, and we shall see that $\mathscr{H}_A$ is also equivalent to the category of generalized nilpotent representations of the co-Lie algebra $\gamma_A$.

Part IV is really a chapter in rational homotopy theory, and it may well have applications to that subject. As was observed by Sullivan [58], a co-Lie algebra $\gamma$ determines a structure of DGA on the exterior algebra $\wedge(\gamma[-1])$, where $\gamma[-1]$ is a copy of $\gamma$ concentrated in degree one. For a cohomologically connected DGA A,

$\wedge(\gamma_A[-1])$ is the 1-minimal model of $A$. We shall prove the rather surprising result that the derived category of modules over the DGA $\wedge(\gamma_A[-1])$ is equivalent to the derived category of the Abelian category $\mathscr{H}_A$. Curiously, although the theory of minimal rational DGA's has been widely studied since Sullivan's work, the analogous theory of minimal modules does not appear in the literature. That theory will be central to our work in Part IV.

In view of the relationship between Chow groups and $K$-groups, the Beilinson-Soulé conjecture for the field $F$ is equivalent to the assertion that the DGA $\mathscr{N}_{\mathbb{Q}} = \mathscr{N}_{\mathbb{Q}}(\mathrm{Spec}(F))$ is cohomologically connected. When the conjecture holds, the results just summarized apply to $A = \mathscr{N}_{\mathbb{Q}}$. Assuming the Beilinson-Soulé conjecture (and assuming our construction of the DGA $A$), Deligne [20], [17], proposed $\mathscr{F}\mathscr{H}_A$ as a candidate for the Abelian category $\mathscr{M}\mathscr{T}\mathscr{M}(F)$ of mixed Tate motives over $F$. He (in [18]) and Bloch also proposed the category of finite dimensional comodules over $\chi_A$ as a candidate for $\mathscr{M}\mathscr{T}\mathscr{M}(F)$, and [6] proves realization theorems in étale and Hodge theory starting from this definition. Our work shows that these two categories are equivalent, and it gives a fairly concrete and explicit description of them. When $A$ is a $K(\pi,1)$, in the sense that $A$ is quasi-isomorphic to its 1-minimal model, we shall have the relation

$$\mathrm{Ext}^p_{\mathscr{M}\mathscr{T}\mathscr{M}(F)}(\mathbb{Q}, \mathbb{Q}(r)) \cong \mathrm{gr}^r_\gamma K_{2r-p}(F) \otimes \mathbb{Q}$$

between the Abelian category $\mathscr{M}\mathscr{T}\mathscr{M}(F)$ and the algebraic $K$-theory of $F$. (Undefined notations are explained in the introduction to Part IV.)

Finally, in Part V, we shall construct the derived category of modules over an $A_\infty$ or $E_\infty$ $k$-algebra $A$, where $k$ is a commutative ground ring. Here $A_\infty$ algebras are DGA's up to homotopy (without commutativity). There are a number of subtleties. From Part I, we know that $A$-modules are equivalent to modules over an associative, but not commutative, universal enveloping DGA $U(A)$. In particular, $U(k) = \mathscr{C}(1)$. In earlier parts, all $E_\infty$ operads were on the same footing. In Part V, we work with a particular $E_\infty$ operad $\mathscr{C}$ that enjoys special properties, but we show that restriction to this choice results in no loss of generality. Remarkably, with this choice, the category of $E_\infty$ $k$-modules, alias the category of $\mathscr{C}(1)$-modules, admits a commutative and associative "tensor product" $\boxtimes$. This product is not unital on the module level, although there is a natural unit map $k \boxtimes M \to M$ that becomes an isomorphism in the derived category. This fact leads us to introduce certain modified versions of the product $M \boxtimes N$ that are applicable when one or both of $M$ and $N$ is unital, in the sense that it has a given map $k \to M$. The product "$\boxdot$" that applies when both $M$ and $N$ are unital is commutative, associative, and unital up to coherent natural isomorphism; that is, the category of unital $E_\infty$ $k$-modules is symmetric monoidal under $\boxdot$.

Conceptually, we now change ground categories from the category of $k$-modules to the category of $E_\infty$ $k$-modules. It turns out that $A_\infty$ and $E_\infty$ algebras can be described very simply in terms of products $A \boxtimes A \to A$. In fact, an $A_\infty$ $k$-algebra is exactly a monoid in the symmetric monoidal category of unital $E_\infty$ $k$-modules, and an $E_\infty$ $k$-algebra is a commutative monoid. There is a similar conceptual description of modules over $A_\infty$ and $E_\infty$ algebras. From here, the development of the triangulated derived category $\mathscr{D}_A$ of modules over an $A_\infty$ algebra $A$ proceeds exactly as in the case of an actual DGA in Part III. When $A$ is an $E_\infty$ algebra, the category of $A$-modules admits a commutative and associative tensor product $\boxtimes_A$

and a concomitant internal Hom functor $\mathrm{Hom}_A^{\boxtimes}$. Again, there is a natural unit map $A \boxtimes_A M \to M$ that becomes an isomorphism on passage to derived categories. There are Eilenberg-Moore, or hyperhomology, spectral sequences for the computation of the homology of $M \boxtimes_A N$ and $\mathrm{Hom}_A^{\boxtimes}(M, N)$ in terms of the classical Tor and Ext groups

$$\mathrm{Tor}_*^{H^*(A)}(H^*(M), H^*(N)) \quad \text{and} \quad \mathrm{Ext}^*_{H^*(A)}(H^*(M), H^*(N)).$$

Thus our new derived categories of modules over $A_\infty$ and $E_\infty$ algebras enjoy all of the basic properties of the derived categories of modules over DGA's and commutative DGA's.

In view of the unfamiliarity of the constructions in Part V, we should perhaps say something about our philosophy. In algebraic topology, it has long been standard practice to work in the stable homotopy category. This category is hard to construct rigorously, and its objects are hard to think about on the point-set level. (Although the definitional framework in algebraic geometry is notoriously abstract, the objects that algebraic geometers usually deal with are much more concrete than the spectra of algebraic topology.) However, once the machinery is in place, the stable homotopy category gives an enormously powerful framework in which to perform explicit calculations. It may be hoped that our new algebraic derived categories will eventually serve something of the same purpose.

Actually, the analogy with topology is more far-reaching. There are analogs of $E_\infty$ algebras in stable homotopy theory, namely the $E_\infty$ ring spectra that were introduced in [47]. With Elmendorf [25], we have worked out a theory of module spectra over $A_\infty$ and $E_\infty$ ring spectra that is precisely parallel to the algebraic theory of Part V. Although it is much more difficult, its constructive and calculational power are already evident. Basic spectra that previously could only be constructed by the Baas-Sullivan theory of manifolds with singularities are easily obtained from the theory of modules over the $E_\infty$ ring spectrum $MU$ that represents complex cobordism. Spectral sequences that are the precise analogs of the Eilenberg-Moore (or hyperhomology) spectral sequences in Part V include Künneth and universal coefficient spectral sequences that are of clear utility in the study of generalized homology and cohomology. Some other applications were announced in [24], and many more are now in place. An exposition of the analogy between the algebraic and topological theories is given in [51].

Parts II and V constitute a revision and expansion of material in the preprint [37], which had a rather different perspective. That draft was intended to lay foundations for work in both algebra and topology, but it has since become apparent that, despite the remarkably close analogy between the two theories and the resulting expository duplication, the technical differences dictate separate and self-contained treatments. Some of the present results were announced in [38].

Each part has its own introduction, and we have tried to make the parts readable independently of one another. Part III has nothing whatever to do with operads and is wholly independent of Parts I and II. Although the examples that motivated Part IV are constructed by use of Part II, the theory in Part IV also has nothing to do with operads and is independent of Parts I and II. Part V is independent of Part IV and nearly independent of Part II.

A reference of the form "II.m.n" is to statement m.n in Part II; within Part II, the reference would be to "m.n". We shall work over a fixed commutative ground ring $k$. There are no restrictions on $k$ in Parts I, III, and V; $k$ is assumed to be a Dedekind ring in Part II and to be a field of characteristic zero in Part IV.

We wish to thank many people who have taken an interest in this work. Part I can serve as an introduction not only to this paper, but also to the closely related papers of Ginzburg and Kapranov [29], Getzler and Jones [27, 28], and Hinich and Schechtman [31, 32]. Some of the more interesting insights in Part I are due to these authors, and we are grateful to them for sharing their ideas with us. The second author wishes to take this opportunity to offer his belated thanks to Max Kelly and Saunders MacLane for conversations in 1970-71. Discussions then about operads in symmetric monoidal categories are paying off now. We are also very grateful to Jim Stasheff, who alerted us to how seriously operads are being used in mathematical string theory, urged us to give the general exposition of Parts I and II, and offered helpful criticism of preliminary versions. We also thank our colleague Spencer Bloch for detecting an error in the first version of Part II and for ongoing spirited discussions about motives. We are especially grateful to our collaborator Tony Elmendorf; the original version of the theory in Part V was far more complicated, and this material has been reshaped by the insights developed in our parallel topological work with him. It is a pleasure to thank Deligne for his letters that led to this paper and for his suggestions for improving its exposition.

## **Part** I. **Definitions and examples of operads and operad actions**

We define operads in Section 1, algebras over operads in Section 2, and modules over algebras over operads in Section 4, giving a number of variants and examples. The term "operad" is meant to bring to mind suitably compatible collections of $j$-ary product operations. It was coined in order to go well with the older term "monad" (= triple), which specifies a closely related mathematical structure that has a single product. As we explain in Section 3, operads determine associated monads in such a way that an algebra over an operad is the same thing as an algebra over the associated monad. While not at all difficult, this equivalence of definitions is central to the theory and its applications. Section 4 includes a precisely analogous description of modules as algebras over a suitable monad, together with a quite different, and more familiar, description as ordinary modules over universal enveloping algebras. Both points of view are essential.

In Section 5, we discuss the passage from topological operads and monads to algebraic operads and monads via chain complexes and homology. We speculate that similar ideas will have applications to other situations, for example in algebraic geometry, where one may encounter operads in a category that has a suitable homology theory defined on it. In Section 6, we specialize to the little $n$-cubes operads $\mathscr{C}_n$. These arose in iterated loop space theory and are now understood to be relevant to the mathematics of string theory. We show that $H_*(\mathscr{C}_n)$ contains a suboperad which, when translated to degree zero, is isomorphic to the operad that defines Lie algebras, and we observe that work in Cohen's 1972 thesis [15, 16] implies that the full operad $H_*(\mathscr{C}_n)$ defines $n$-braid algebras. While current interest focuses on characteristic zero information, we shall give some indications of the deeper mod $p$ theory. In particular, in Section 7, we shall describe the Dyer-Lashof operations that are present on the mod $p$ homologies of $E_\infty$ algebras. Such operations are central to infinite loop space theory, and our later work will indicate that they are also relevant to the mod $p$ higher Chow groups in algebraic geometry.

## 1. Operads

We work in the tensor category of differential $\mathbb{Z}$-graded modules over our ground ring $k$, with differential decreasing degree by 1. Thus $\otimes$ will always mean $\otimes_k$. Readers who prefer the opposite grading convention may reindex chain complexes $C_*$ by setting $C^n = C_{-n}$. While homological grading is most convenient in Parts I and II, we shall find it convenient to switch to cohomological grading in later parts. We agree to refer to chain complexes over $k$ simply as "$k$-modules". As usual, we consider graded $k$-modules without differential to be differential graded $k$-modules with differential zero, and we view ungraded $k$-modules as graded $k$-modules concentrated in degree 0. These conventions allow us to view the theory of generalized algebras as a special case of the theory of differential graded generalized algebras. The differentials play little role in the theory of the first four sections. As will become relevant in Part II, everything in these sections works just as well in the still more general context of simplicial $k$-modules.

We begin with the definition of an operad of $k$-modules. While there are perhaps more elegant equivalent ways of writing the definition, the original explicit version of [46] still seems to be the most convenient, especially for concrete calculational purposes. Whenever we deal with permutations of $k$-modules, we implicitly use the standard convention that a sign $(-1)^{pq}$ is to be inserted whenever an element of degree $p$ is permuted past an element of degree $q$.

**Definition 1.1.** An operad $\mathscr{C}$ consists of $k$-modules $\mathscr{C}(j)$, $j \geq 0$, together with a unit map $\eta : k \to \mathscr{C}(1)$, a right action by the symmetric group $\Sigma_j$ on $\mathscr{C}(j)$ for each $j$, and maps

$$\gamma : \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \to \mathscr{C}(j)$$

for $k \geq 1$ and $j_s \geq 0$, where $\sum j_s = j$. The $\gamma$ are required to be associative, unital, and equivariant in the following senses.
(a) The following associativity diagrams commute, where $\sum j_s = j$ and $\sum i_t = i$; we set $g_s = j_1 + \cdots + j_s$, and $h_s = i_{g_{s-1}+1} + \cdots + i_{g_s}$ for $1 \leq s \leq k$:

$$\begin{array}{ccc}
\mathscr{C}(k)\otimes\left(\bigotimes_{s=1}^{k}\mathscr{C}(j_s)\right)\otimes\left(\bigotimes_{r=1}^{j}\mathscr{C}(i_r)\right) & \xrightarrow{\gamma\otimes\mathrm{Id}} & \mathscr{C}(j)\otimes\left(\bigotimes_{r=1}^{j}\mathscr{C}(i_r)\right) \\
\Big\downarrow{\scriptstyle \text{shuffle}} & & \Big\downarrow{\scriptstyle\gamma} \\
 & & \mathscr{C}(i) \\
 & & \Big\uparrow{\scriptstyle\gamma} \\
\mathscr{C}(k)\otimes\left(\bigotimes_{s=1}^{k}\left(\mathscr{C}(j_s)\otimes\left(\bigotimes_{q=1}^{j_s}\mathscr{C}(i_{g_{s-1}+q})\right)\right)\right) & \xrightarrow[\mathrm{Id}\otimes(\otimes_s\gamma)]{} & \mathscr{C}(k)\otimes\left(\bigotimes_{s=1}^{k}\mathscr{C}(h_s)\right).
\end{array}$$

(b) The following unit diagrams commute:

$$\begin{array}{ccc} \mathscr{C}(k)\otimes(k)^k & \xrightarrow{\cong} & \mathscr{C}(k) \\ {\scriptstyle \mathrm{Id}\otimes\eta^k}\downarrow & \nearrow_{\gamma} & \\ \mathscr{C}(k)\otimes\mathscr{C}(1)^k & & \end{array} \qquad \begin{array}{ccc} k\otimes\mathscr{C}(j) & \xrightarrow{\cong} & \mathscr{C}(j) \\ {\scriptstyle \eta\otimes\mathrm{Id}}\downarrow & \nearrow_{\gamma} & \\ \mathscr{C}(1)\otimes\mathscr{C}(j) & & \end{array}$$

(c) The following equivariance diagrams commute, where $\sigma \in \Sigma_k, \tau_s \in \Sigma_{j_s}$, $\sigma(j_1, \ldots, j_k) \in \Sigma_k$ permutes $k$ blocks of letter as $\sigma$ permutes $k$ letters, and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_k$ is the block sum:

$$\begin{array}{ccc} \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathscr{C}(k)\otimes\mathscr{C}(j_{\sigma(1)})\otimes\cdots\otimes\mathscr{C}(j_{\sigma(k)}) \\ \gamma\downarrow & & \downarrow\gamma \\ \mathscr{C}(j) & \xrightarrow{\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})} & \mathscr{C}(j) \end{array}$$

and

$$\begin{array}{ccc} \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k) & \xrightarrow{\mathrm{Id}\otimes\tau_1\otimes\cdots\otimes\tau_k} & \mathscr{C}(k)\otimes\mathscr{C}(j_1)\otimes\cdots\otimes\mathscr{C}(j_k) \\ \gamma\downarrow & & \downarrow\gamma \\ \mathscr{C}(j) & \xrightarrow{\tau_1\otimes\cdots\otimes\tau_k} & \mathscr{C}(j) \end{array}$$

The $\mathscr{C}(j)$ are to be thought of as modules of parameters for "$j$-ary operations" that accept $j$ inputs and produce one output. Thinking of elements as operations, we think of $\gamma(c \otimes d_1 \otimes \cdots \otimes d_k)$ as the composite of the operation $c$ with the tensor product of the operations $d_s$. We emphasize that the definition makes sense in any symmetric monoidal ground category, with product $\otimes$ and unit object $k$. In the present algebraic context, the unit map $\eta$ is specified by a degree zero cycle $1 \in \mathscr{C}(1)$. The definition admits several minor variants and particular types. Recall that a map of $k$-modules is said to be a quasi-isomorphism if it induces an isomorphism of homology groups.

**Variants 1.2.** (i) *Non-$\Sigma$ operads.* When modelling non-commutative algebras, it is often useful to omit the permutations from the definition, giving the notion of a non-$\Sigma$ operad. However, one may also keep the permutations in such contexts, using them to record the order in which products are taken. An operad is a non-$\Sigma$ operad by neglect of structure.
(ii) *Unital operads.* By convention, the $0^{\text{th}}$ tensor power of a $k$-module $A$ is interpreted to be $k$ (concentrated in degree 0). The module $\mathscr{C}(0)$ parametrizes "0-ary operations" $k \to A$. In practice, one is most often concerned with unital algebras, and one thinks of the unit element $1 \in A$ as specifying a map $k \to A$. In such contexts, it is sensible to insist that $\mathscr{C}(0) = k$, and we then say that $\mathscr{C}$ is a unital operad. For types of algebras without units, such as Lie algebras, it is natural to set $\mathscr{C}(0) = 0$.
(iii) *Augmentations.* If $\mathscr{C}$ is unital, the $\mathscr{C}(j)$ have the augmentations

$$\epsilon = \gamma : \mathscr{C}(j) \cong \mathscr{C}(j) \otimes \mathscr{C}(0)^j \to \mathscr{C}(0) = k.$$

**Definition 1.3.** Let $\mathscr{C}$ be a unital operad. We say that $\mathscr{C}$ is acyclic if its augmentations are quasi-isomorphisms. We say that $\mathscr{C}$ is $\Sigma$-free (or $\Sigma$-projective) if $\mathscr{C}(j)$ is $k[\Sigma_j]$-free (or $k[\Sigma_j]$-projective) for each $j$. We say that $\mathscr{C}$ is an $E_\infty$ operad if it is both acyclic and $\Sigma$-free; $\mathscr{C}(j)$ is then a $k[\Sigma_j]$-free resolution of $k$.

**Example 1.4.** An explicit example of an $E_\infty$ operad $\mathscr{C}$ can be obtained as follows. There is a standard product-preserving functor $D_*$ from sets to contractible simplicial sets [46, §10]. The set $D_q(X)$ of $q$-simplices of $D_*(X)$ is the $(q+1)$-fold Cartesian power $X^{q+1}$; the faces and degeneracies are given by projections and diagonal maps. For a group $G$, $D_*(G)$ is a free simplicial group, and its normalized $k$-chain complex is the classical homogeneous bar resolution for the group ring $k[G]$ (e.g. [14, p. 190]). Letting $\mathscr{C}(j)$ be the normalized $k$-chain complex of $D_*(\Sigma_j)$, we can use functoriality to construct structural maps $\gamma$ making $\mathscr{C}$ an $E_\infty$ operad.

Passage to normalized singular $k$-chain complexes from $E_\infty$ operads of spaces gives other examples; see Section 5.

## 2. Algebras over operads

Let $X^j$ denote the $j$-fold tensor power of a $k$-module $X$, with $\Sigma_j$ acting on the left. Again, $X^0 = k$. (We shall never use Cartesian powers in the algebraic context.)

**Definition 2.1.** Let $\mathscr{C}$ be an operad. A $\mathscr{C}$-algebra is a $k$-module $A$ together with maps $\theta : \mathscr{C}(j) \otimes A^j \to A$, $j \geq 0$, that are associative, unital, and equivariant in the following senses.
(a) The following associativity diagrams commute, where $j = \sum j_s$:

$$\begin{array}{ccc}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A^j & \xrightarrow{\gamma \otimes \mathrm{Id}} & \mathscr{C}(j) \otimes A^j \\
 & & \downarrow \theta \\
\downarrow \text{shuffle} & & A \\
 & & \uparrow \theta \\
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes A^{j_1} \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A^{j_k} & \xrightarrow{\mathrm{Id} \otimes \theta^k} & \mathscr{C}(k) \otimes A^k
\end{array}$$

(b) The following unit diagram commutes:

$$\begin{array}{ccc}
k \otimes A & \xrightarrow{\cong} & A \\
\downarrow \eta \otimes \mathrm{Id} & \nearrow \theta & \\
\mathscr{C}(1) \otimes A & &
\end{array}$$

(c) The following equivariance diagrams commute, where $\sigma \in \Sigma_j$;

$$\begin{array}{ccc}
\mathscr{C}(j) \otimes A^j & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathscr{C}(j) \otimes A^j \\
 & {\scriptstyle\gamma} \searrow \quad \swarrow {\scriptstyle\gamma} & \\
 & A &
\end{array}$$

One way to motivate the precise data in the definition is to define the endomorphism operad $\mathrm{End}(X)$ of a $k$-module $X$. For $k$-modules $X$ and $Y$, let $\mathrm{Hom}(X,Y)$ be the $k$-module whose elements of degree $n$ are the homomorphisms $f : X \to Y$ of graded $k$-modules (not commuting with differential) that raise degree by $n$. The differential is specified by

$$(df)(x) = d(f(x)) - (-1)^n f(d(x)).$$

If $\mathscr{K}$ denotes the category of $k$-modules and maps of degree 0, then

$$\mathscr{K}(X \otimes Y, Z) \cong \mathscr{K}(X, \mathrm{Hom}(Y,Z)).$$

Now define

$$\mathrm{End}(X)(j) = \mathrm{Hom}(X^j, X).$$

The unit is given by the identity map $X \to X$, the right actions by symmetric groups are given by their left actions on tensor powers, and the maps $\gamma$ are given by the following composites, where $\sum j_s = j$:

$$\begin{array}{c}
\mathrm{Hom}(X^k, X) \otimes \mathrm{Hom}(X^{j_1}, X) \otimes \cdots \otimes \mathrm{Hom}(X^{j_k}, X) \\
\big\downarrow {\scriptstyle \mathrm{Id}\otimes(k\text{-fold tensor product of maps})} \\
\mathrm{Hom}(X^k, X) \otimes \mathrm{Hom}(X^j, X^k) \\
\big\downarrow {\scriptstyle \text{composition}} \\
\mathrm{Hom}(X^j, X)
\end{array}$$

Conditions (a)-(c) of Definition 1.1 are then forced by direct calculation. An action of $\mathscr{C}$ on $A$ can be redefined in adjoint form as a morphism of operads $\mathscr{C} \to \mathrm{End}(A)$, and conditions (a)-(c) of Definition 2.1 are then also forced by direct calculation.

**Examples 2.2.** (i) The unital operad $\mathscr{M}$ has $\mathscr{M}(j) = k[\Sigma_j]$ as a right $k[\Sigma_j]$-module (concentrated in degree 0). The unit map $\eta$ is the identity and the maps $\gamma$ are dictated by the equivariance formulas of Definition 1.1(c). Explicitly, for $\sigma \in \Sigma_k$ and $\tau_s \in \Sigma_{j_s}$,

$$\gamma(\sigma; \tau_1, \ldots, \tau_k) = \sigma(j_1, \ldots, j_k)(\tau_1 \oplus \cdots \oplus \tau_k).$$

An $\mathscr{M}$-algebra $A$ is the same thing as a "DGA", that is, a unital and associative differential graded algebra. The action $\theta$ on a DGA $A$ is given by the explicit formula

$$\theta(\sigma \otimes a_1 \otimes \cdots \otimes a_j) = \pm a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(j)},$$

where $\sigma \in \Sigma_j$ and $a_i \in A$. (The sign is given by our standing convention.)
(ii) The unital operad $\mathscr{N}$ has $\mathscr{N}(j) = k$ for all $j$. The $\Sigma_j$-actions are trivial, the unit map $\eta$ is the identity, and the maps $\gamma$ are the evident identifications. An $\mathscr{N}$-algebra is the same thing as a commutative DGA. If we regard $\mathscr{N}$ as a non-$\Sigma$ operad and delete the equivariance diagram from Definition 2.1, then the resulting notion of an $\mathscr{N}$-algebra is again a not necessarily commutative DGA.
(iii) For a unital operad $\mathscr{C}$, the augmentations $\epsilon : \mathscr{C}(j) \to k$ give a map $\epsilon : \mathscr{C} \to \mathscr{N}$ of operads. Therefore, by pullback along $\epsilon$, an $\mathscr{N}$-algebra may be viewed as a $\mathscr{C}$-algebra.
(iv) We define an $E_\infty$ algebra to be a $\mathscr{C}$-algebra for any $E_\infty$ operad $\mathscr{C}$. We do not insist on a particular choice of $\mathscr{C}$. Hinich and Schechtman [31] studied algebras of this type, which they called "May algebras".

One can treat operads as algebraic systems to which one can apply versions of classical algebraic constructions. An ideal $\mathscr{I}$ in an operad $\mathscr{C}$ consists of a sequence of sub $k[\Sigma_j]$-modules $\mathscr{I}(j)$ of $\mathscr{C}(j)$ such that $\gamma(c \otimes d_1 \otimes \cdots \otimes d_k)$ is in $\mathscr{I}$ if either $c$ or any of the $d_s$ is in $\mathscr{I}$. There is then a quotient operad $\mathscr{C}/\mathscr{I}$ with $j^{\text{th}}$ $k$-module $\mathscr{C}(j)/\mathscr{I}(j)$. As observed by Ginzburg and Kapranov [29], one can adapt work of Boardman and Vogt [11, 11§2] to construct the free operad $\mathscr{F}\mathscr{G}$ generated by any sequence $\mathscr{G} = \{\mathscr{G}(j)\}$ of $k[\Sigma_j]$-modules, and one can then construct an operad that describes a particular type of algebra by quotienting out by the ideal generated by an appropriate sequence $\mathscr{R} = \{\mathscr{R}(j)\}$ of defining relations, where $\mathscr{R}(j)$ is a sub $k[\Sigma_j]$-module of $(\mathscr{F}\mathscr{G})(j)$. Actually, there are two variants of the construction, one unital and one non-unital.

In many familiar examples, called quadratic operads in [29], $\mathscr{G}(j) = 0$ for $j \neq 2$ and $\mathscr{R}(j) = 0$ for $j \neq 3$. Here, if $\mathscr{G}(2)$ is $k[\Sigma_2]$ and $\mathscr{R}(3) = 0$, this reconstructs $\mathscr{M}$. If $\mathscr{G}(2) = k$ with trivial $\Sigma_2$-action and $\mathscr{R}(3) = 0$, this reconstructs $\mathscr{N}$. In these cases, we use the unital variant. If $k$ is a field of characteristic other than 2 or 3, we can use the non-unital variant to construct an operad $\mathscr{L}$ whose algebras are the Lie algebras over $k$. To do this, we take $\mathscr{G}(2) = k$, with the transposition in $\Sigma_2$ acting as $-1$, and take $\mathscr{R}(3)$ to be the space $(\mathscr{F}\mathscr{G})(3)^{\Sigma_3}$ of invariants, which is one dimensional. Basis elements of $\mathscr{G}(2)$ and $\mathscr{R}(3)$ correspond to the bracket operation and the Jacobi identity. As we explain in Section 6, $\mathscr{L}$ can be realized homologically by the topological little $n$-cubes operads for any $n > 1$. Various other examples of quadratic operads are described in [29]. Note that, in these "purely algebraic" examples, all $\mathscr{C}(j)$ are concentrated in degree zero, with zero differential.

The definition of a Lie algebra over a field $k$ requires the additional relations $[x, x] = 0$ if $\mathrm{char}(k) = 2$ and $[x, [x, x]] = 0$ if $\mathrm{char}(k) = 3$. Purely algebraic operads are not well adapted to codify such relations with repeated variables, still less such nonlinear operations as the restriction (or $p^{\text{th}}$ power operation) of restricted Lie algebras in characteristic $p$. The point is simply that the elements of an operad specify operations, and operations by their nature cannot know about special properties (such as repetition) of the variables to which they are applied.

As an aside, since in the absence of diagonals it is unclear that there is a workable algebraic analog, we note that a topological theory of $E_\infty$ ring spaces has been developed in [49]. The sum and product, with the appropriate version of the distributive law, are codified in actions by two suitably interrelated operads.

*Remarks* 2.3. (i) A $k$-module $X$ also has a "co-endomorphism operad" Co-End$(X)$; its $j$th $k$-module is $\mathrm{Hom}(X, X^j)$, and its structural maps are given in an evident way by composition and tensor products. We define a coaction of an operad $\mathscr{C}$ on a $k$-module $X$ to be a map of operads $\mathscr{C} \to$ Co-End$(X)$; such an action is given by suitably interrelated maps $\mathscr{C}(j) \otimes X \to X^j$.
(ii) We have defined operads in terms of maps. If we reverse the direction of every arrow in Definition 1.1, we obtain the dual notion of "co-operad". Similarly, if we reverse the direction of every arrow in Definition 2.1, we obtain the notion of a coalgebra over a co-operad. Again, if we reverse the direction of every arrow in Definition 4.1 below, we obtain the notion of a comodule over such a coalgebra.

## 3. Monadic reinterpretation of algebras

We recall some standard categorical definitions.

**Definition 3.1.** Let $\mathscr{G}$ be any category. A monad in $\mathscr{G}$ is a functor $C : \mathscr{G} \to \mathscr{G}$ together with natural transformations $\mu : CC \to C$ and $\eta : \mathrm{Id} \to C$ such that the following diagrams commute:

$$
\begin{array}{ccccc}
C & \xrightarrow{\eta C} & CC & \xleftarrow{C\eta} & C \\
& {\scriptstyle \mathrm{Id}}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \mathrm{Id}} & \\
& & C & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
CCC & \xrightarrow{C\mu} & CC \\
{\scriptstyle \mu C}\downarrow & & \downarrow{\scriptstyle \mu} \\
CC & \xrightarrow{\mu} & C
\end{array}
$$

A $C$-algebra is an object $A$ of $\mathscr{G}$ together with a map $\xi : CA \to A$ such that the following diagrams commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta} & CA \\
& {\scriptstyle \mathrm{Id}}\searrow & \downarrow{\scriptstyle \xi} \\
& & A
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
CCA & \xrightarrow{C\xi} & CA \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \xi} \\
CA & \xrightarrow{\xi} & A
\end{array}
$$

Taking $\xi = \mu$, we see that $CX$ is a $C$-algebra for any $X \in \mathscr{G}$. It is the free $C$-algebra generated by $X$. That is, for $C$-algebras $A$, restriction along $\eta : X \to CX$ gives an adjunction isomorphism

$$C[\mathscr{G}](CX, A) \cong \mathscr{G}(X, A), \tag{3.2}$$

where $C[\mathscr{G}]$ is the category of $C$-algebras. The inverse isomorphism assigns the composite $\xi \circ Cf : CX \to A$ to a map $f : X \to A$. Formally, we are viewing $C$ as a functor $\mathscr{G} \to C[\mathscr{G}]$, and our original monad is given by its composite with the forgetful functor $C[\mathscr{G}] \to \mathscr{G}$. Thus the monad $C$ is determined by its algebras. Quite generally, every pair $L : \mathscr{G} \to \mathscr{H}$ and $R : \mathscr{H} \to \mathscr{G}$ of left and right adjoints determines a monad $RL$ on $\mathscr{G}$, but many different pairs of adjoint functors can define the same monad.

Returning to the category of $k$-modules, we have the following simple construction of the monad of free algebras over an operad $\mathscr{C}$.

**Definition 3.3.** Define the monad $C$ associated to an operad $\mathscr{C}$ by letting

$$CX = \bigoplus_{j \geq 0} \mathscr{C}(j) \otimes_{k[\Sigma_j]} X^j.$$

The unit $\eta : X \to CX$ is $\eta \otimes \mathrm{Id} : X = k \otimes X \to \mathscr{C}(1) \otimes X$ and the map $\mu : CCX \to CX$ is induced by the maps ($j = \sum j_s$)

$$
\begin{array}{c}
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^{j_k} \\
\downarrow{\scriptstyle \text{shuffle}} \\
\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^j \\
\downarrow{\scriptstyle \gamma \otimes \mathrm{Id}} \\
\mathscr{C}(j) \otimes X^j
\end{array}
$$

**Proposition 3.4.** *A $\mathscr{C}$-algebra structure on a $k$-module $A$ determines and is determined by a $C$-algebra structure on $A$. Formally, the identity functor on the category* *of $k$-modules restricts to give an isomorphism between the categories of $\mathscr{C}$-algebras and of $C$-algebras.*

*Proof.* Maps $\theta_j : \mathscr{C}(j) \otimes_{\Sigma_j} A^j \to A$ that together specify an action of $\mathscr{C}$ on $A$ are the same as a map $\xi : CA \to A$ that specifies an action of $C$ on $A$. □

Not all monads come from operads. Rather, operads single out a particularly convenient, algebraically manageable, collection of monads.

For the operad $\mathscr{M}$, the free algebra $MX$ is just the free associative $k$-algebra generated by $X$, with the differential induced from that of $X$. Similarly, for the operad $\mathscr{N}$, the free algebra $NX$ is the free associative and commutative algebra generated by $X$, with its induced differential. Again, for the operad $\mathscr{L}$, we obtain the free Lie algebra functor $L$. While these observations can be checked by observation, they are also formal consequences of the freeness adjunction (3.2). Some less obvious examples are discussed in Section 6 and are generalized to situations of particular interest in string theory in [27, 28].

In the rest of this section, we suppose that $\mathscr{C}$ is a unital operad. In this case, there is a monad that is different from that defined above but that nevertheless has essentially the same algebras. Since $\mathscr{C}$ is unital, a $\mathscr{C}$-algebra $A$ comes with a unit $\eta \equiv \theta_0 : k \to A$. Thinking of the unit as preassigned, it is natural to change ground categories to the category of unital $k$-modules and unit-preserving maps. Working in this ground category, we obtain a reduced monad $\tilde{C}$. This monad is so defined that the units of algebras that are built in by the $\theta_0$ component of operad actions coincide with the preassigned units $\eta$.

In detail, note that we have "degeneracy maps" $\sigma_i : \mathscr{C}(j) \to \mathscr{C}(j-1)$ specified by

$$\sigma_i(c) = \gamma(c \otimes 1^{i-1} \otimes * \otimes 1^{j-i}) \tag{3.5}$$

for $1 \leq i \leq j$, where 1 denotes $\eta(1)$ in $\mathscr{C}(1)$ and $*$ denotes the identity element in $k = \mathscr{C}(0)$. For a unital $k$-module $X$ with unit 1, define $\tilde{C}X$ to be the quotient of $CX$ obtained by the identifications

$$\begin{aligned} c \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_j = \\ \sigma_i(c) \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_j \end{aligned} \tag{3.6}$$

for $1 \leq i \leq j$. With unit map $\eta$ and product map $\mu$ induced from those of the monad $C$, $\tilde{C}$ is a monad in the category of unital $k$-modules.

**Proposition 3.7.** *Let $\mathscr{C}$ be a unital operad. Then a $\mathscr{C}$-algebra structure satisfying $\eta = \theta_0$ on a unital $k$-module $A$ determines and is determined by a $\tilde{C}$-algebra structure on $A$.*

The proof is immediate from Proposition 3.4 and the definitions. With a slight restriction, the monads $C$ and $\tilde{C}$ determine each other. Define an augmentation of a unital $k$-module $X$ to be a map $\epsilon : X \to k$ whose composite with the unit is the identity.

**Proposition 3.8.** *(i) For a $k$-module $X$, let $X_+$ be the unital $k$-module $X \oplus k$. Then $CX \cong \tilde{C}(X_+)$ as $\mathscr{C}$-algebras.*
*(ii) For an augmented $k$-module $X$, let $\tilde{X}$ be the $K$-module $Ker(\epsilon)$. Then $\tilde{C}X \cong C\tilde{X}$ as $\mathscr{C}$-algebras.*

*Proof.* Part (i) can be viewed as a special case of part (ii). For (ii), the composite of the map $C\tilde{X} \to CX$ induced by the inclusion $\tilde{X} \to X$ and the quotient map $CX \to \tilde{C}X$ gives the required isomorphism, and the following diagrams commute:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ C\tilde{X} & \xrightarrow{\cong} & \tilde{C}X \end{array} \qquad \begin{array}{ccccc} CC\tilde{X} & \xrightarrow{\cong} & C\tilde{C}X & \longrightarrow & \tilde{C}\tilde{C}X \\ {\scriptstyle\mu}\downarrow & & & & \downarrow{\scriptstyle\mu} \\ C\tilde{X} & & \xrightarrow{\cong} & & \tilde{C}X. \end{array} \quad \square$$

There is an obvious analogy with the adjunction of a disjoint basepoint to a space $X$ to obtain a space $X_+$ such that $H_*(X) \cong \tilde{H}_*(X_+)$. In the original topological theory of [46], all operads were unital and the reduced topological monad $\tilde{C}$ associated to an operad $\mathscr{C}$ was denoted $C$. In that context, as we shall recall in Sections 5 and 6, there is a great difference in homotopy types between $\tilde{C}$ and $C$, with $\tilde{C}$ being by far the more interesting construction. While there is an evident topological analog of the first part of the previous proposition, there is no analog of the second part: topologically, the reduced construction is strictly more general. In the preprint version of this paper [37], $\tilde{C}$ was denoted by $C$. We have followed a suggestion of Deligne in placing the emphasis on the simpler construction $C$ in the present algebraic context.

## 4. Modules over $\mathscr{C}$-algebras

Fix an operad $\mathscr{C}$ and a $\mathscr{C}$-algebra $A$.

**Definition 4.1.** An $A$-module is a $k$-module $M$ together with maps $\lambda : \mathscr{C}(j) \otimes A^{j-1} \otimes M \to M$ for $j \geq 1$ that are associative, unital, and equivariant in the following sense.
(a) The following associativity diagrams commute, where $j = \sum j_s$:

$$\begin{array}{ccc} (\mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k} \mathscr{C}(j_s))) \otimes A^{j-1} \otimes M & \xrightarrow{\gamma\otimes\mathrm{Id}} & \mathscr{C}(j) \otimes A^{j-1} \otimes M \\ & & \downarrow{\scriptstyle\lambda} \\ \downarrow{\scriptstyle\text{shuffle}} & & M \\ & & \uparrow{\scriptstyle\lambda} \\ \mathscr{C}(k) \otimes (\bigotimes_{s=1}^{k-1} (\mathscr{C}(j_s) \otimes A^{j_s})) \otimes (\mathscr{C}(j_k) \otimes A^{j_k-1} \otimes M) & \xrightarrow[\mathrm{Id}\otimes\theta^{k-1}\otimes\lambda]{} & \mathscr{C}(k) \otimes A^{k-1} \otimes M \end{array}$$

(b) The following unit diagram commutes:

$$\begin{array}{ccc} k \otimes M & \xrightarrow{\cong} & M \\ {\scriptstyle\eta\otimes\mathrm{Id}}\downarrow & \nearrow{\scriptstyle\lambda} & \\ \mathscr{C}(1) \otimes M & & \end{array}$$

(c) The following equivariance diagram commutes, where $\sigma \in \Sigma_{j-1} \subset \Sigma_j$;

$$\begin{array}{ccc} \mathscr{C}(j) \otimes A^{j-1} \otimes M & \xrightarrow{\sigma \otimes \sigma^{-1} \otimes \mathrm{Id}} & \mathscr{C}(j) \otimes A^{j-1} \otimes M \\ & {\scriptstyle\lambda} \searrow \quad \swarrow {\scriptstyle\lambda} & \\ & M & \end{array}$$

A map $f : M \to N$ of $k$-modules between $A$-modules $M$ and $N$ is a map of $A$-modules if the following diagram commutes for each $j \geq 1$:

$$\begin{array}{ccc} \mathscr{C}(j) \otimes A^{j-1} \otimes M & \longrightarrow & M \\ {\scriptstyle \mathrm{Id} \otimes \mathrm{Id} \otimes f} \downarrow & & \downarrow {\scriptstyle f} \\ \mathscr{C}(j) \otimes A^{j-1} \otimes N & \longrightarrow & N \end{array}$$

We think of these as left modules. However, motivated by the first of the following examples, one can also think of them as bimodules [29].

**Examples 4.2.** (i) For an $\mathscr{M}$-algebra $A$, an $A$-module $M$ in our sense is the same as an $A$-bimodule in the classical sense. Precisely, given the maps $\lambda$, we define $am = \lambda(e \otimes a \otimes m)$ and $ma = \lambda(\sigma \otimes a \otimes m)$, where $e$ and $\sigma$ are the identity and transposition in $\Sigma_2$. Conversely, just as in Example 2.2(i), given an $A$-bimodule $M$, we define

$$\lambda(\sigma \otimes a_1 \otimes \cdots \otimes a_j) = \pm a_{\sigma(1)} \cdots a_{\sigma(j)},$$

where $\sigma \in \Sigma_j$, $a_i \in A$ for $1 \leq i < j$ and $a_j \in M$.
(ii) For an $\mathscr{N}$-algebra $A$, an $A$-module in our sense is the same as an $A$-module in the classical sense. If we use $\mathscr{N}$ regarded as a non-$\Sigma$ operad to define non-commutative algebras and delete part (c) of the definition, then a module over an $\mathscr{N}$-algebra $A$ is a classical left $A$-module.
(iii) For an $\mathscr{L}$-algebra $L$, an $L$-module in our sense is the same as a Lie algebra module in the classical sense.

Just as for algebras, modules admit a monadic reinterpretation.

**Definition 4.3.** For $k$-modules $X$ and $Y$, define

$$C(X;Y) = \bigoplus_{j \geq 1} \mathscr{C}(j) \otimes_{k[\Sigma_{j-1}]} X^{j-1} \otimes Y.$$

Define $\eta : Y \to C(X;Y)$ to be $\eta \otimes \mathrm{Id} : Y = k \otimes Y \to \mathscr{C}(1) \otimes Y$ and define

$$\mu : C(CX; C(X;Y)) \to C(X;Y)$$

to be the map induced by the following composites ($j = \sum j_s$):

$$\begin{array}{c} \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes X^{j_1} \otimes \cdots \otimes \mathscr{C}(j_{k-1}) \otimes X^{j_{k-1}} \otimes \mathscr{C}(j_k) \otimes X^{j_k - 1} \otimes Y \\ \downarrow {\scriptstyle \text{shuffle}} \\ \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes X^{j-1} \otimes Y \\ \downarrow {\scriptstyle \gamma \otimes \mathrm{Id}} \\ \mathscr{C}(j) \otimes X^{j-1} \otimes Y \end{array}$$

Define a monad $C[1]$ in the category of pairs $(X;Y)$ by letting

$$C[1](X;Y) = (CX; C(X;Y)).$$

The unit $\eta$ and product $\mu$ of $C[1]$ are given by the evident pairs $(\eta;\eta)$ and $(\mu;\mu)$.

**Proposition 4.4.** *A $\mathscr{C}$-algebra structure on a $k$-module $A$ together with an $A$-module structure on a $k$-module $M$ determine and are determined by a $C[1]$-algebra structure on the pair $(A;M)$. Formally, the identity functor on the category of pairs of $k$-modules restricts to an isomorphism between the evident category of $\mathscr{C}$-algebras together with modules and the category of $C[1]$-algebras.*

When $\mathscr{C}$ is unital, there is a similar reduced monad $\tilde{C}[1]$ in the category of pairs $(X;Y)$, where $X$ is a unital $k$-module and $Y$ is an arbitrary $k$-module. Explicitly, define $\tilde{C}(X;Y)$ to be the quotient of $C(X;Y)$ obtained by the identifications

$$c \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_j = \sigma_i(c) \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_j \tag{4.5}$$

for $1 \le i < j$, where $x_i \in X$ if $i < j$ and $x_j \in Y$. Then define

$$\tilde{C}[1](X;Y) = (\tilde{C}X; \tilde{C}(X;Y)).$$

The unit map $\eta$ and product map $\mu$ are induced from those of $C[1]$.

**Proposition 4.6.** *Let $\mathscr{C}$ be a unital operad. A $\mathscr{C}$-algebra structure satisfying $\eta = \theta_0$ on a unital $k$-module $A$ together with an $A$-module structure on a $k$-module $M$ determines and is determined by a $\tilde{C}[1]$-algebra structure on the pair $(A;M)$.*

**Proposition 4.7.** *(i) For $k$-modules $X$ and $Y$, $C(X;Y) \cong \tilde{C}(X_+;Y)$.*
*(ii) For an augmented $k$-module $X$ with $\tilde{X} = Ker(\epsilon)$ and a $k$-module $Y$, $\tilde{C}(X;Y) \cong C(\tilde{X};Y)$.*

Observe that free objects in our monadic context are pairs $(CX; C(X;Y))$, where $C(X;Y)$ is a module over $CX$. Formally, we can rewrite the present instance of the freeness adjunction (3.2) in the form

$$C[1][\mathscr{K}[1]]((CX;C(X;Y)),(A;M)) \cong \mathscr{K}[1]((X;Y),(A;M)),$$

where $\mathscr{K}[1]$ denotes the category of pairs of $k$-modules.

Of course, this is quite different from fixing an algebra $A$ and constructing free $A$-modules $FY = F(A;Y)$. Such a free module functor $F$ is characterized by an adjunction

$$\mathrm{Hom}_A(FY,M) \cong \mathrm{Hom}(Y,M)$$

relating maps of $A$-modules and maps of $k$-modules. We shall construct the free $A$-module functor $F(A;?)$ for an algebra $A$ over an operad $\mathscr{C}$ in a moment, and we will then have the following formal comparison of definitions.

**Proposition 4.8.** *For any operad $\mathscr{C}$ and any $k$-modules $X$ and $Y$, $C(X;Y)$ is isomorphic to the free $CX$-module $F(CX;Y)$ generated by $Y$.*

*Proof.* The forgetful functor $C[1][\mathscr{K}[1]] \to \mathscr{K}[1]$ factors through the category of pairs $(A;Y)$, where $A$ is a $\mathscr{C}$-algebra and $Y$ is a $k$-module. That is, we can first forget the module structure on the second coordinate and then forget the algebra structure on the first coordinate. These two forgetful functors have left adjoints $(Id, F(\mathrm{Id},?))$ and $(C,\mathrm{Id})$. Their composite must coincide with $C[1]$ by the uniqueness of adjoints. □

With the morphisms of Definition 4.1, it is clear that the category of $A$-modules is abelian. In fact, as was observed in [29] and [32], it is equivalent to the category of modules over the universal enveloping algebra $U(A)$ of $A$. Of course, at our present level of generality, $U(A)$ must be a DGA. This gives us the free $A$-module functor $F$ just asked for as the ordinary free $U(A)$-module functor. The definition of $U(A)$ is forced by Definition 4.1.

**Definition 4.9.** Let $A$ be a $\mathscr{C}$-algebra. The action maps

$$\lambda : \mathscr{C}(j) \otimes A^{j-1} \otimes M \to M$$

of an $A$-module $M$ together define an action map

$$\lambda : C(A;k) \otimes M = C(A;M) \to M.$$

Thus $C(A;k)$ may be viewed as a $k$-module of operators on $A$-modules. The free DGA $M(C(A;k))$ generated by $C(A;k)$ therefore acts iteratively on all $A$-modules. Define the universal enveloping algebra $U(A)$ to be the quotient of $M(C(A;k))$ by the ideal of universal relations. Explicitly, reading off from Definition 4.1, the element $1 \in \mathscr{C}(1)$ must be identified with the unit element of the algebra and the element

$$\gamma(d \otimes c_1 \otimes \cdots \otimes c_k) \otimes a_1 \otimes \cdots \otimes a_{j-1} \in \mathscr{C}(j) \otimes A^{j-1}$$

must be identified with the product

$$[d \otimes \theta(c_1; b_1) \otimes \cdots \otimes \theta(c_{k-1}; b_{k-1})][c_k \otimes b_k] \in [\mathscr{C}(k) \otimes A^{k-1}][\mathscr{C}(j_k) \otimes A^{j_k - 1}],$$

where $d \in \mathscr{C}(k)$, $c_s \in \mathscr{C}(j_s)$, $a_i \in A$, and $b_s$ is the tensor product of the $s$th block of $a$'s; $b_s$ has $j_s$ tensor factors if $s < k$ and $j_k - 1$ factors if $s = k$. Taking $c_i = 1$ for $i < k$ and changing notation, we obtain the relation

$$[c \otimes a_1 \otimes \cdots \otimes a_j][d \otimes a'_1 \otimes \cdots \otimes a'_k] = \gamma(c \otimes 1^j \otimes d) \otimes a_1 \otimes \cdots \otimes a_k \otimes a'_1 \otimes \cdots \otimes a'_j$$

for $c \in \mathscr{C}(j+1)$ and $d \in \mathscr{C}(k+1)$. Reinterpreting this formula as a product on $C(A;k)$, we see that $U(A)$ can be described more economically as the quotient of the algebra $C(A;k)$ by the relations originally specified.

The following result is immediate from the definition.

**Proposition 4.10.** *The category of $A$-modules is isomorphic to the category of $U(A)$-modules.*

It is an illuminating and not quite trivial exercise to check the first of the following examples from the explicit relations just specified.

**Examples 4.11.** (i) For an $\mathscr{M}$-algebra $A$, $U(A)$ is isomorphic to $A \otimes A^{\mathrm{op}}$.
(ii) For an $\mathscr{N}$-algebra $A$, $U(A)$ is isomorphic to $A$.
(iii) For an $\mathscr{L}$-algebra $L$, $U(L)$ is isomorphic to the classical universal enveloping algebra of $L$.

In Part V, we shall construct a derived tensor product on modules over an $E_\infty$ algebra $A$. From the universal enveloping algebra point of view, this should look most implausible: a $U(A)$-module is just a left module, and, since $U(A)$ is far from being commutative, there is no obvious way to define a tensor product of $A$-modules, let alone a tensor product that is again a module.

## 5. Algebraic operads associated to topological operads

Recall that operads can be defined in any symmetric monoidal category, such as the category of topological spaces under Cartesian product. Thus an operad $\mathscr{S}$ of spaces consists of spaces $\mathscr{S}(j)$ with right actions of $\Sigma_j$, a unit element $1 \in \mathscr{S}(1)$, and maps

$$\gamma : \mathscr{S}(k) \times \mathscr{S}(j_1) \times \cdots \times \mathscr{S}(j_k) \to \mathscr{S}(j)$$

such that associativity, unity, and equivariance diagrams precisely like those in Definition 1.1 commute. For definiteness, we assume that $\mathscr{S}(0)$ is a point.

Via the singular complex functor, an operad of topological spaces gives rise to an operad of simplicial sets. Via the free $k$-module functor, an operad of simplicial sets gives rise to an operad of simplicial ungraded $k$-modules. By passage to normalized chains, which we denote by $C_\#$, an operad of simplicial ungraded $k$-modules gives rise to an operad of $k$-modules in our original differential graded sense. The proof of the last assertion depends on the associativity and commutativity of the standard shuffle quasi-isomorphism (e.g. [44, §29], or [30, Appendix])

$$C_\#(X) \otimes C_\#(Y) \to C_\#(X \times Y).$$

Therefore the normalized singular $k$-chain functor restricts to a functor from operads of spaces to operads of $k$-modules. We write $C_\#(\mathscr{S})$ for the operad of $k$-modules associated to an operad $\mathscr{S}$ of spaces.

The operad $\mathscr{S}$ is said to an $E_\infty$ operad if each space $\mathscr{S}(j)$ is $\Sigma_j$-free and contractible (a universal $\Sigma_j$-bundle), and $C_\#(\mathscr{S})$ is then an $E_\infty$ operad in the sense of Definition 1.3. Similarly, the chain functor $C_\#$ carries $\mathscr{S}$-algebras (= $\mathscr{S}$-spaces) to $C_\#(\mathscr{S})$-algebras and carries modules over an $\mathscr{S}$-algebra to modules over the associated $C_\#(\mathscr{S})$-algebra.

Following [29, 27, 28] and others, we can go further and define homology operads. We take $k$ to be a field in the rest of this section, and all homology groups are to be taken with coefficients in $k$.

**Definition 5.1.** Let $\mathscr{S}$ be an operad of spaces. Define $H_*(\mathscr{S})$ to be the unital operad whose $j$th $k$-module is the graded $k$-module $H_*(\mathscr{S}(j))$, with algebraic structure maps $\gamma$ induced by the topological structure maps. For $n \geq 0$, define $H_n(\mathscr{S})$ to be the suboperad of $H_*(\mathscr{S}(j))$ whose $j$th $k$-module is $H_{n(j-1)}(\mathscr{S}(j))$ for $j \geq 0$; in particular, the 0th $k$-module is zero unless $n = 0$. The degrees are so arranged that the definition makes sense. We retain the grading that comes naturally, so that the $j$th term of $H_n(\mathscr{S})$ is concentrated in degree $n(j-1)$. We obtain a "degree zero translate" operad associated to $H_n(\mathscr{S})$ by regrading so that all terms are concentrated in degree zero.

If the spaces $\mathscr{S}(j)$ are all connected, then $H_0(\mathscr{S}) = \mathscr{N}$ and $H_*(X)$ is a commutative algebra for any $\mathscr{S}$-space $X$. If the spaces $\mathscr{S}(j)$ are all contractible, for example if $\mathscr{S}$ is an $E_\infty$ operad, then $H_*(\mathscr{S}) = \mathscr{N}$. Thus, on passage to homology, $E_\infty$ operads record only the algebra structure on the homology of $\mathscr{S}$-spaces, although the chain level operad action gives rise to the homology operations discussed in Section 7. It is for this reason that topologists did not formally introduce homology operads decades ago.

In fact, there is a sharp dichotomy between the calculational behavior of operads in characteristic zero and in positive characteristic. The depth of the original topological theory lies in positive characteristic, where passage to homology operads jettisons most of the interesting structure. In characteristic zero, in contrast, the homology operads completely determine the homology of the monads $S$ and $\tilde{S}$ associated to an operad $\mathscr{S}$. Here, for a space $X$,

$$SX = \coprod \mathscr{S}(j) \times_{\Sigma_j} X^j.$$

For a based space $X$, $\tilde{S}X$ is the quotient of $SX$ obtained by basepoint identifications, exactly as in (3.6). The space $\tilde{S}X$ has a natural filtration with successive quotients

$$\mathscr{S}(j)_+ \wedge_{\Sigma_j} X^{(j)},$$

where $X^{(j)}$ denotes the $j$-fold smash power of $X$. ($X \wedge Y$ is the quotient of the product $X \times Y$ obtained by identifying the wedge $X \vee Y$ to a point.)

The calculational difference comes from a simple general fact: if a finite group $\pi$ acts on a space $X$, then, with coefficients in a field of characteristic zero, $H_*(X/\pi)$ is naturally isomorphic to $H_*(X)/\pi$. (We are assuming that our spaces are not pathological; for example, they may be $\pi$-CW complexes.) In fact, $H_*(X/\pi)$ is a homology theory on $X$—this being true in any characteristic—and $H_*(x)/\pi$ is a homology theory on $X$ since the functor $M/\pi = M \otimes_{k[\pi]} k$ on $k[\pi]$-modules $M$ is exact (e.g. because $k$ is a direct summand of $k[\pi]$). It is obvious that these theories agree on orbits $\pi/\rho$, and it follows exactly as in nonequivariant algebraic topology that they are isomorphic. In the cases of interest to us, the shuffle map induces a chain map

$$(*) \qquad C_\#(\mathscr{S}(j)) \otimes_{\Sigma_j} C_\#(X)^j \to C_\#(\mathscr{S}(j) \times_{\Sigma_j} X^j),$$

from which we obtain an instance of our general isomorphism on passage to homology, and similarly for $\mathscr{S}(j)_+ \wedge_{\Sigma_j} X^{(j)}$. This leads to the following result.

**Theorem 5.2.** *Let $\mathscr{S}$ be any operad of spaces. Let $S$ denote both the monad in the category of spaces associated to $\mathscr{S}$ and the monad in the category of $k$-modules associated to $H_*(\mathscr{S})$. Similarly, let $\tilde{S}$ denote both the monad in the category of based spaces associated to $\mathscr{S}$ and the monad in the category of unital $k$-modules associated to $H_*(\mathscr{S})$. If $k$ is a field of characteristic zero, then*

$$H_*(SX) \cong SH_*(X) \quad \text{and} \quad H_*(\tilde{S}X) \cong \tilde{S}(H_*(X))$$

*as $H_*(\mathscr{S})$-algebras for all spaces $X$ (based spaces in the reduced case).*

*Proof.* On passage to homology, the unit $X \to SX$ and the action of $\mathscr{S}$ on $SX$ induce the composite map

$$\alpha : S(H_*(X)) \to S(H_*(SX)) \to H_*(SX)$$

of $H_*(\mathscr{S})$-algebras. Similarly, in the reduced case we have a composite

$$\tilde{\alpha} : \tilde{S}(H_*(X)) \to \tilde{S}(H_*(\tilde{S}X)) \to H_*(\tilde{S}X).$$

In the unreduced case, $\alpha$ is the direct sum of isomorphisms induced by the chain maps $(*)$. For the reduced case, observe that if $V$ is an augmented $k$-module, then the $k$-module $\tilde{S}V$ has an evident filtration with successive quotients $H_*(\mathscr{S}(j)) \otimes_{\Sigma_j} \tilde{V}^j$. The map $\tilde{\alpha}$ is filtration-preserving, and its successive quotients are isomorphisms

$$H_*(\mathscr{S}(j)) \otimes_{\Sigma_j} \tilde{H}_*(X)^j \cong H_*(\mathscr{S}(j)_+ \wedge_{\Sigma_j} X^{(j)})$$

induced by the chain maps $(*)$. Therefore $\tilde{\alpha}$ is an isomorphism by induction up the filtration and passage to colimits. $\square$

This allows us to realize free algebras topologically. For example, we have the obvious topological (actually, discrete) versions of the operads $\mathscr{M}$ and $\mathscr{N}$, with $\mathscr{M}(j) = \Sigma_j$ and $\mathscr{N}(j)$ a point. For a based space $X$, $\tilde{M}X$ is the James construction (or free topological monoid) on $X$, and it is homotopy equivalent to $\Omega\Sigma X$ if $X$ is connected. Similarly, $\tilde{N}X$ is the infinite symmetric product (or free commutative topological monoid) on $X$, and it is homotopy equivalent to the product over $n \geq 1$ of the Eilenberg-MacLane spaces $K(H_n(X), n)$ if $X$ is connected. Note that the unreduced constructions $MX$ and $NX$ are just disjoint unions of Cartesian powers and symmetric Cartesian powers and are therefore much less interesting. At least in characteristic zero, we conclude that

$$H_*(\tilde{M}X) \cong \tilde{M}(H_*(X)) \quad \text{and} \quad H_*(\tilde{N}X) \cong \tilde{N}(H_*(X)).$$

By Proposition 3.5, these are the free and free commutative algebras generated by $\tilde{H}_*(X)$. Note that any positively graded $k$-module can be realized as $\tilde{H}_*(X)$ by taking $X$ to be a suitable wedge of spheres.

## 6. Operads, loop spaces, $n$-Lie algebras, and $n$-braid algebras

We here specialize to the operads that come from the study of iterated loop spaces. These operads turn out to encode notions of $n$-Lie algebra and $n$-braid algebra. Implicitly or explicitly, the case $n = 1$ has received a great deal of attention in the recent literature of string theory. See, e.g. [27, 28, 56], and the references therein.

For each $n > 0$, there is a little $n$-cubes operad $\mathscr{C}_n$. It was invented (before the introduction of operads) by Boardman and Vogt [11]; see also [46]. Its $j$th space $\mathscr{C}_n(j)$ consists of $j$-tuples of little $n$-cubes embedded with parallel axes and disjoint interiors in the standard $n$-cube. There is an analogous little $n$-disks operad defined in terms of embeddings of little disks in the unit disk via radial contraction and translation. These are better suited to considerations of group actions and of geometry, but they do not stabilize over $n$. There is a more sophisticated variant, due to Steiner [57], that enjoys the good properties of both the little $n$-cubes and the little $n$-disks operads. Each of these operads comes with a canonical equivalence from its $j$th space to the configuration space $F(\mathbb{R}^n, j)$ of $j$-tuples of distinct points of $\mathbb{R}^n$. The little $n$-cubes operad (and any of its variants) acts naturally on all $n$-fold loop spaces $\Omega^n Y$.

Since $\mathscr{C}_1$ maps by a homotopy equivalence to $\mathscr{M}$, we concentrate on the case $n > 1$. When $k$ is a field of characteristic $p > 0$, the homology of a $\mathscr{C}_n$-space, such as $\Omega^n Y$, has an extremely rich and complicated algebraic structure, carrying Browder operations and some of the Dyer-Lashof operations that are present in the homology of $E_\infty$ algebras (see the next section). For a detailed description, see Cohen, [16, II§1]. (Minor corrections are given in Wellington, [61, I,§1].) We will here describe the characteristic zero information and a portion of the mod $p$ information in Cohen's exhaustive mod $p$ calculations. We take $k$ to be a field throughout this section.

Cohen's calculations have two essential starting points. One is his complete and explicit calculation of the integral homology of $F(\mathbb{R}^n, j)$, with its action by $\Sigma_j$, for all $n$ and $j$ [16, II §§6–7]. He used this to define homology operations. The other is the "approximation theorem" of [46]. It asserts that, for a based space $X$, the reduced free $\mathscr{C}_n$-space $\tilde{\mathscr{C}}_n X$ maps to $\Omega^n\Sigma^n X$ via a natural map of $\mathscr{C}_n$-spaces that is an equivalence when $X$ is connected. This allowed Cohen to combine the homology operations with the Serre spectral sequence to compute simultaneously both $H_*(\tilde{C}_n X)$ and $H_*(\Omega^n\Sigma^n X)$ for any $X$.

In characteristic zero, the calculations simplify drastically since Theorem 5.2 shows that calculation of the homology operads $H_*(\mathscr{C}_n)$ is already enough to determine $H_*(\tilde{C}_n X)$. Cohen showed that each space $F(\mathbb{R}^n, j)$ has the same integral homology as a certain product of wedges of $(n-1)$-spheres. Therefore, with the notations of Definition 5.1, the operad $H_*(\mathscr{C}_n)$ can be written additively as the reduced sum $\mathscr{N}\tilde{\oplus}H_{n-1}(\mathscr{C}_n)$ of its suboperads $\mathscr{N}$ and $H_{n-1}(\mathscr{C}_n)$, where the reduced sum is obtained from the direct sum by identifying the unit elements in $\mathscr{N}(1)$ and $H_0(\mathscr{C}_n(1))$. When $\mathrm{char}(k) = 0$ and $n = 1$, the following result was implicit in [4] and was made explicit by Schechtman and Ginzburg. It was observed by Getzler and Jones [28] that the general case was already implicit in Cohen's thesis [15].

**Theorem 6.1.** *If char$(k) \neq 2$ or $3$, then, for all $n \geq 1$, the degree zero translate of the operad $H_n(\mathscr{C}_{n+1})$ is isomorphic to the operad $\mathscr{L}$ that defines Lie algebras over $k$.*

We are more interested in the algebras defined by the untranslated operads and by the full homology operads. If $\mathrm{char}(k) \neq 2$ or 3, these turn out to be the $n$-Lie algebras and $n$-braid algebras. (A 1-braid algebra is also called a braid algebra or a Gerstenhaber algebra.) Recall our standing convention that $k$-modules are $\mathbb{Z}$-graded and have differentials.

**Definition 6.2.** An $n$-Lie algebra is a $k$-module $L$ together with a map of $k$-modules $[\ ,\ ]_n : L \otimes L \to L$ that raises degrees by $n$ and satisfies the following identities, where $\deg(x) = q - n$, $\deg(y) = r - n$, and $\deg(z) = s - n$.
(i) (Anti-symmetry)

$$[x, y]_n = -(-1)^{qr}[y, x]_n.$$

(ii) (Jacobi identity)

$$(-1)^{qs}[x, [y, z]_n]_n + (-1)^{qr}[y, [z, x]_n]_n + (-1)^{rs}[z, [x, y]_n]_n = 0.$$

(iii) $[x, x]_n = 0$ if $\mathrm{char}(k) = 2$ and $[x, [x, x]_n]_n = 0$ if $\mathrm{char}(k) = 3$.

Of course, a 0-Lie algebra is just a Lie algebra. For a $k$-module $Y$ and an integer $n$, define the $n$-fold suspension $\Sigma^n Y$ by $(\Sigma^n Y)_q = Y_{q-n}$, with differential $(-1)^n d$. (The sign depends on conventions: see III, §1.)

**Proposition 6.3.** *The category of $n$-Lie algebras is isomorphic to the category of Lie algebras. There is an operad $\mathscr{L}_n$ whose algebras are the $n$-Lie algebras, and its degree zero translate is isomorphic to $\mathscr{L}$.*

*Proof.* For an $n$-Lie algebra $L$, $\Sigma^n L$ is a Lie algebra with bracket

$$[\Sigma^n x, \Sigma^n y] = \Sigma^n [x, y]_n.$$

Similarly, for a Lie algebra $L$, $\Sigma^{-n} L$ is an $n$-Lie algebra. This gives the first statement. For the second, $\mathscr{L}_n$ can be constructed by a precisely similar use of suspensions, and the isomorphism with $\mathscr{L}$ is then obvious. □

**Definition 6.4.** An $n$-braid algebra is a $k$-module $A$ that is an $n$-Lie algebra and a commutative DGA such that the bracket and product satisfy the following identity, where $\deg(x) = q - n$ and $\deg(y) = r - n$.
(i) (Poisson formula)

$$[x, yz]_n = [x, y]_n z + (-1)^{q(r-n)} y[x, z]_n.$$

The Poisson formula implies and is implied by the following identities, where $\deg(x) = q - n$, $\deg(y) = r - n$, $\deg(z) = s - n$, and $\deg(w) = t - n$.
(ii) $[1, x]_n = 0$, where 1 is the unit for the product.
(iii) $[xy, zw]_n = x[y, z]_n w + (-1)^{(r-n)s}[x, z]_n w + (-1)^{(q+r-n)(s-n)} zx[y, w]_n$
$+(-1)^{q(s-n)+(r-n)(s+t-n)} z[x, w]_n y.$

The Poisson formula asserts that the map $d_x = [x, ?]_n$ is a graded derivation, in the sense that

$$d_x(yz) = d_x(y)z + (-1)^{\deg(y)\deg(d_x)} y d_x(z).$$

Batalin-Vilkovisky algebras are examples of 1-braid algebras [27], hence the general case, with non-zero differentials, is relevant to string theory. However, our concern here is with structures that have zero differential.

**Theorem 6.5.** *The homology $H_*(X)$ is an $n$-braid algebra for any $\mathscr{C}_{n+1}$-space $X$ and any field of coefficients.*

The $n$-bracket is denoted $\lambda_n$ and called a Browder operation in [45, §6], and [16, II], where the theorem is proven. The first appearance of $\lambda_n$ was in [12], in characteristic 2. We have displayed (iii) since that is the version of the Poisson formula given in [16] (where signs are garbled on page 216 but correct on page 317). Identity (iii) of Definition 6.2 is of conceptual interest: it cannot be visible in the operad $H_n(\mathscr{C}_{n+1})$, but it follows directly from the chain level definition of $\lambda_n$.

For a $k$-module $V$, let $L_n V$ be the free $n$-Lie algebra generated by $V$; as in Proposition 6.3, $L_n V = \Sigma^{-n} L \Sigma^n V$. For the moment, let $\Lambda_n$ denote the monad on $k$-modules associated to the operad $H_n(\mathscr{C}_{n+1})$, and recall the duplicative use of the notation $\tilde{C}_{n+1}$ from Theorem 5.2. For a $\mathscr{C}_{n+1}$-space $X$, the action of $\mathscr{C}_{n+1}$ induces an action of $H_n(\mathscr{C}_{n+1})$ on $H_*(\tilde{C}_{n+1}X)$. It is clear from the decomposition of $H_*(\mathscr{C}_{n+1})$ as a reduced direct sum that all of the iterated $n$-bracket operations must be codified as part of this action.

**Theorem 6.6** (Cohen)**.** *Assume that $char(k) = 0$. For any based space $X$,*

$$\tilde{C}_{n+1} H_*(X) \cong H_*(\tilde{C}_{n+1}X) \cong N L_n \tilde{H}_*(X)$$

*is the free commutative algebra generated by the free $n$-Lie algebra generated by $\tilde{H}_*(X)$. Moreover, the image of $\Lambda_n \tilde{H}_*(X)$ in $H_*(\tilde{C}_{n+1}X)$ under the composite*

$$\Lambda_n \tilde{H}_*(X) \to \Lambda_n \tilde{H}_*(\tilde{C}_{n+1}X)) \to H_*(\tilde{C}_{n+1}X)$$

*induced by the unit $X \to \tilde{C}_{n+1}X$ and the action of $H_n(\mathscr{C}_{n+1})$ coincides with the $n$-Lie algebra $L_n \tilde{H}_*(X)$.*

The first isomorphism is given by Theorem 5.2 and the second by Cohen's calculations. With $\mathrm{char}(k) = 0$, the deduction of Theorem 6.1 from Proposition 6.3 and Theorem 6.6 is a conceptual exercise. The fact that $H_n(\mathscr{C}_{n+1})$ induces the $n$-Lie bracket on the homology of $\mathscr{C}_{n+1}$-spaces implies that there is a map of operads $\mathscr{L}_n \to H_n(\mathscr{C}_{n+1})$. Any positively graded $k$-module $V$ is the homology of some space. Therefore this map of operads induces an isomorphism $L_nV \to \Lambda_nV$ for all such $V$. This is enough to conclude that $\mathscr{L}_n \to H_n(\mathscr{C}_{n+1})$ is an isomorphism. A similar exercise gives the $\mathrm{char}(k) = 0$ case of the following further consequence of Theorem 6.6. The second statement can be proven algebraically, but it is more amusing to deduce it from the topology.

**Theorem 6.7.** *If $\mathrm{char}(k) \neq 2$ or 3, then, for all $n \geq 1$, the algebras over the operad $H_*(\mathscr{C}_{n+1})$ are exactly the $n$-braid algebras. The free $n$-braid algebra generated by a $k$-module $V$ is isomorphic to $NL_nV$.*

It remains to say something about the proofs of Theorem 6.1 and 6.7 in positive characteristic. Here we still have a natural map

$$\tilde{C}_{n+1}H_*(X) \to H_*(\tilde{C}_{n+1}X),$$

but it is no longer an isomorphism. Cohen's complete calculation of the target shows that it contains $NL_n\tilde{H}_*(X)$, and one again sees that all iterated Browder operations are determined by the action of elements of $H_n(\mathscr{C}_{n+1})$. Now the dimension of the $k$-module $\mathscr{L}_n(j)$ is independent of the characteristic by Proposition 6.3 and the corresponding fact for Lie algebras, while the dimension of $H_n(\mathscr{C}_{n+1}(j))$ is independent of the characteristic by Cohen's integral calculations. By the characteristic zero result, these dimensions must be equal for all characteristics. We deduce that the displayed map must be an isomorphism onto $NL_n\tilde{H}_*(X)$, and the rest of the argument goes as before.

## 7. Homology operations in characteristic $p$

When $\mathscr{C}$ is an $E_\infty$ operad, an action of $\mathscr{C}$ on $A$ builds in the kinds of higher homotopies for the multiplication of $A$ that are the source, for example, of the Dyer-Lashof operations in the homology of infinite loop spaces and the Steenrod operations in the cohomology of general spaces. We describe the form that these operations take in the homology of general $E_\infty$ algebras $A$ in this section. When we connect up partial algebras and $E_\infty$ algebras in Part II, this will give new homological invariants on the mod $p$ higher Chow groups. Many other examples are known to topologists, such as the Steenrod operations in the Ext groups of cocommutative Hopf algebras (e.g. [45, §11]) and in the cohomology of simplicial restricted Lie algebras (e.g. [53], [45, §8]).

We begin with the trivial observation that, in characteristic zero, $E_\infty$ operads carry no more homological information than the operad $\mathscr{N}$.

**Lemma 7.1.** *Let $\epsilon : \mathscr{C} \to \mathscr{P}$ be a quasi-isomorphism of operads over a field $k$ of characteristic zero, such as the augmentation $\epsilon : \mathscr{C} \to \mathscr{N}$ of an acyclic operad. Then the maps $CX \to PX$ and $C(X;Y) \to P(X;Y)$ induced by $\epsilon$ are quasi-isomorphisms for all $k$-modules $X$ and $Y$.*

*Proof.* This is an easy consequence of the definitions and the fact that all modules over the group ring $k[G]$ of a finite group $G$ are projective. □

Taking $\mathscr{P} = \mathscr{N}$ and $\mathscr{P} = \mathscr{L}$, we will see in Part II that this leads to a proof that, when $k$ is a field of characteristic zero, $E_\infty$ algebras are quasi-isomorphic to commutative DGA's and strong homotopy Lie algebras are quasi-isomorphic to differential Lie algebras, and similarly for modules.

We take $k = \mathbb{Z}$ and consider algebras $A$ over an integral $E_\infty$ operad $\mathscr{C}$ in the rest of this section. Let $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and consider the mod $p$ homology $H_*(A;\mathbb{Z}_p)$.

**Theorem 7.2.** *For $s \geq 0$, there exist natural homology operations*

$$Q^s : H_q(A;\mathbb{Z}_2) \to H_{q+s}(A;\mathbb{Z}_2)$$

*and*

$$Q^s : H_q(A;\mathbb{Z}_p) \to H_{q+2s(p-1)}(A;\mathbb{Z}_p)$$

*if $p > 2$. These operations satisfy the following properties*

(1) $Q^s(x) = 0$ *if $p = 2$ and $s < q$ or if $p > 2$ and $2s < q$.*
(2) $Q^s(x) = x^p$ *if $p = 2$ and $s = q$ or if $p > 2$ and $2s = q$.*
(3) $Q^s(1) = 0$ *if $s > 0$, where $1 \in H_0(A;\mathbb{Z}_p)$ is the identity element.*
(4) *(Cartan formula)* $Q^s(xy) = \sum Q^t(x)Q^{s-t}(y)$.
(5) *(Adem relations) If $p \geq 2$ and $t > ps$, then*

$$Q^tQ^s = \sum_i (-1)^{t+i}(pi - t, t-(p-1)s-i)Q^{s+t-i-1}Q^i;$$

*if $p > 2$, $t \geq ps$, and $\beta$ denotes the mod $p$ Bockstein, then*

$$(7.3) \quad Q^t\beta Q^s = \sum_i (-1)^{t+i}(pi-t, t-(p-1)s-i)\beta Q^{s+t-i}Q^i - \sum_i (-1)^{t+i}(pi-t-1, t-(p-1)s-i)Q^{s+t-i}\beta Q^i;$$

*here $(i,j) = \dfrac{(i+j)!}{i!j!}$ if $i \geq 0$ and $j \geq 0$ (where $0! = 1$), and $(i,j) = 0$ if $i$ or $j$ is negative; the sums run over $i \geq 0$.*

The proof is the same as in [16, I§1]; as there, one simply checks that one is in the general algebraic framework of [45], which does the relevant homological algebra once and for all. (Actually, [45] should be read as a paper about operad actions. Unfortunately, it was written shortly before operads were invented.) The point is that $\mathscr{C}(p)$ is a $\Sigma_p$-free resolution of $\mathbb{Z}$, so that the homology of $\mathscr{C}(p)\otimes_{\Sigma_p} A^p$ is readily computed, and computation of $\theta_* : H_*(\mathscr{C}(p)\otimes_{\Sigma_p} A^p;\mathbb{Z}_p) \to H_*(A;\mathbb{Z}_p)$ allows one to read off the operations. The Cartan formula and the Adem relations are derived from special cases of the diagrams in Definition 2.1(a) via calculations in the homology of groups.

Notice the grading. The first non-zero operation is the $p$th power, and there can be infinitely many non-zero operations on a given element. This is in marked contrast with Steenrod operations in the cohomology of spaces, where the last non-zero operation is the $p$th power. In fact, Steenrod operations are defined on cohomologically graded $E_\infty$ algebras that are concentrated in positive degrees, where the cochain complexes $\mathscr{C}(j)$ of the relevant $E_\infty$ operad are concentrated in negative degrees. If we systematically regrade homologically, then Dyer-Lashof and Steenrod operations both fit into the general context of the theorem, except that the adjective "Dyer-Lashof" is to be used when the underlying chain complexes are positively graded and the adjective "Steenrod" is to be used when the underlying chain complexes are negatively graded.

## Part II. Partial algebraic structures and conversion theorems

As in Part I, let $\mathscr{C}$ be an operad of $k$-modules, where $k$-modules are understood to be $\mathbb{Z}$-graded and to have differentials. We assume that $k$ is a Dedekind ring in this part. In Part I, we defined $\mathscr{C}$-algebras and modules over $\mathscr{C}$-algebras, and we showed how to interpret these notions in terms of actions of monads associated to $\mathscr{C}$. We here generalize these ideas by specifying partial $\mathscr{C}$-algebras and their modules in Section 2 and then expressing these notions in terms of monads in Section 3. As with the theory of Part I, everything in these sections works equally well in the greater generality of *simplicial* $\mathbb{Z}$-graded differential $k$-modules.

The main point of this part is the conversion of partial algebras and modules to quasi-isomorphic genuine algebras and modules. As we shall see in Part V, we can construct derived categories of modules over $E_\infty$ algebras that enjoy all of the standard properties of derived categories of modules over commutative DGA's. One might instead try to develop a theory of derived categories of partial modules over partial algebras. However, modules over $E_\infty$ algebras are much more tractable for this purpose since they are defined entirely in terms of actual iterated tensor products rather than the tensor products up to quasi-isomorphism that are intrinsic to the definition of partial algebras and modules.

For subtle technical reasons, explained in Section 5, our conversion theorems do not work in the full generality of our definitions. Rather, we must work in the category of "simplicial $k$-modules", where $k$-modules may or may not be graded but do not have differentials. Fortunately, this is the situation that occurs in the motivating examples that arise in algebraic geometry. We discuss these examples briefly in Section 6. We explain the proofs of our conversion theorems in Sections 4 and 5.

## 1. Statements of the conversion theorems

Modulo precise definitions, our conversion theorems read as follows.

**Theorem 1.1.** *Let $\mathscr{C}$ be a $\Sigma$-projective operad of simplicial $k$-modules. Then there is a functor $V$ that assigns a quasi-isomorphic $\mathscr{C}$-algebra $VA$ to a partial $\mathscr{C}$-algebra $A$. There is also a functor $V$ that assigns a quasi-isomorphic $VA$-module $VM$ to a partial $A$-module $M$.*

When $k$ is a field of characteristic zero, every operad $\mathscr{C}$ is $\Sigma$-projective and we have the following complement.

**Theorem 1.2.** *Let $k$ be a field of characteristic zero and let $\epsilon : \mathscr{C} \to \mathscr{P}$ be a quasi-isomorphism of operads of simplicial $k$-modules. Then there is a functor $W$ that assigns a quasi-isomorphic $\mathscr{P}$-algebra $WA$ to a partial $\mathscr{C}$-algebra $A$. There is also a functor $W$ that assigns a quasi-isomorphic $WA$-module $WM$ to a partial $A$-module $M$.*

An acyclic operad $\mathscr{C}$ is one that maps by a quasi-isomorphism to the operad $\mathscr{N}$ that defines commutative simplicial $k$-algebras, hence the following result is a special case.

**Corollary 1.3.** *Let $k$ be a field of characteristic zero and let $\mathscr{C}$ be an acyclic operad of simplicial $k$-modules. Then there is a functor $W$ that assigns a quasi-isomorphic simplicial commutative $k$-algebra $WA$ to a partial $\mathscr{C}$-algebra $A$. There is also a functor $W$ that assigns a quasi-isomorphic $WA$-module $WM$ to a partial $A$-module $M$.*

As usual, we apply the normalized chain complex functor to pass from simplicial $k$-modules to differential graded $k$-modules, and a map of simplicial $k$-modules is said to be a quasi-isomorphism if the associated map of differential graded $k$-modules is a quasi-isomorphism (induces an isomorphism on homology). The passage from simplicial $k$-modules to differential graded $k$-modules carries an operad of simplicial $k$-modules to an operad of differential graded $k$-modules. Similarly, it preserves algebras and modules. However, it does *not* preserve *partial* algebras and modules. For essentially the same technical reason, we do not have an analog of Theorem 1.1 in the category of partial differential graded $k$-modules. Therefore, although our motivation and applications concern chain complexes, we are forced to work on the simplicial level as long as possible, only passing to the differential graded level after the conversion of partial algebras and modules to genuine algebras and modules.

Of course, in view of Theorem 1.1, Theorem 1.2 is only needed when $A$ is already a genuine $\mathscr{C}$-algebra. There is a version of this case of Theorem 1.2 that does work in the differential graded context.

**Theorem 1.4.** *Let $k$ be a field of characteristic zero and let $\epsilon : \mathscr{C} \to \mathscr{P}$ be a quasi-isomorphism of operads of differential graded $k$-modules. Then there is a functor $W$ that assigns a quasi-isomorphic $\mathscr{P}$-algebra $WA$ to a $\mathscr{C}$-algebra $A$. There is also a functor $W$ that assigns a quasi-isomorphic $WA$-module $WM$ to an $A$-module $M$.*

**Corollary 1.5.** *Let $k$ be a field of characteristic zero and let $\mathscr{C}$ be an acyclic operad of differential graded $k$-modules. Then there is a functor $W$ that assigns a quasi-isomorphic commutative DGA $WA$ to a $\mathscr{C}$-algebra $A$. There is also a functor $W$ that assigns a quasi-isomorphic $WA$-module $WM$ to an $A$-module $M$.*

While our original motivation came from algebraic geometry, these results may also be of interest in other subjects. For example, operads of differential graded $k$-modules whose algebras are "strong homotopy Lie algebras" are becoming increasingly important in string theory (see [32, 56] and the references therein). The defining property of such an operad $\mathscr{J}$ is that it must admit a quasi-isomorphism $\epsilon : \mathscr{J} \to \mathscr{L}$, where $\mathscr{L}$ is the operad that defines Lie algebras over $k$. We then say that $\mathscr{J}$ is a strong homotopy Lie operad. Theorem 1.4 applies directly to replace strong homotopy Lie algebras by quasi-isomorphic genuine differential graded Lie algebras. A version of this result is known to the experts, via an entirely different proof, but the corresponding result for modules is new.

**Corollary 1.6.** *Let $k$ be a field of characteristic zero and let $\mathscr{J}$ be a strong homotopy Lie operad of differential graded $k$-modules. Then there is a functor $W$ that assigns a quasi-isomorphic differential graded Lie algebra $WL$ to a $\mathscr{J}$-algebra $L$. There is also a functor $W$ that assigns a quasi-isomorphic $WL$-module $WM$ to an $L$-module $M$.*

Similarly, modulo the appropriate definitions, Theorems 1.1 and 1.2 apply to convert partial simplicial strong homotopy Lie algebras first to genuine simplicial strong homotopy Lie algebras and then, when $k$ is a field of characteristic zero, to simplicial Lie algebras.

## 2. Partial algebras and modules

One often encounters $k$-modules $A$ that come with products that are only defined on appropriate submodules of $A \otimes A$. We first define the ground categories for such partial algebras and their modules, then specify partial commutative DGA's and their partial modules, and finally generalize to define partial structures defined by operad actions for any operad $\mathscr{C}$ of $k$-modules. In the following definition, "domain" should be thought of as shorthand for "domain of definition". As in Part I, we let $X^j$ denote the $j$-fold tensor power of $X$, with $X^0 = k$.

**Definition 2.1.** Let $X$ be a flat $k$-module. A domain $X_*$ in $X$ is a sequence of $\Sigma_j$-invariant submodules $X_j$ of $X^j$ such that the given inclusions $\delta_j : X_j \to X^j$ satisfy the following properties.
(a) $\delta_o$ and $\delta_1$ are identity maps and each $\delta_j$ is a quasi-isomorphism.
(b) For each partition $j = j_1 + \cdots + j_k$, $\delta_j$ factors through $X_{j_1} \otimes \cdots \otimes X_{j_k}$, as indicated in the following commutative diagram:

$$\begin{array}{ccc} X^{j_1} \otimes \cdots \otimes X^{j_k} & \xrightarrow{\cong} & X^j \\ {\scriptstyle \delta_{j_1}\otimes\cdots\otimes\delta_{j_k}}\uparrow & & \uparrow{\scriptstyle \delta_j} \\ X_{j_1} \otimes \cdots \otimes X_{j_k} & \hookleftarrow & X_j. \end{array}$$

Our standing assumption that $k$ is a Dedekind ring and our requirement that $X$ be a flat $k$-module ensure that the tensor product $\delta_{j_1} \otimes \cdots \otimes \delta_{j_k}$ in the diagram just given is both an inclusion and a quasi-isomorphism. This is a consequence of the following lemma and the fact that, over any commutative ring $k$, a tensor product of flat $k$-modules is flat and a tensor product of inclusions of flat modules is an inclusion.

**Lemma 2.2.** *Assume that $k$ is a Dedekind ring. Then submodules of flat $k$-modules are $k$-flat. Let $X$ be a $k$-module and $f : Y \to Y'$ be a quasi-isomorphism of $k$-modules, where either $X$ or both $Y$ and $Y'$ are flat. Then $1 \otimes f : X \otimes Y \to X \otimes Y'$ is a quasi-isomorphism.*

*Proof.* The first part is standard, and it implies that the cycles and boundaries of flat $k$-modules are flat. In turn, this implies that a flat $k$-module $X$ is the union of its bounded below flat $k$-modules $X[n]$, where $X[n]_q = 0$ for $q < n$, $X[n]_n = Z_nX$ (the cycles of degree $n$), and $X[n]_q = X_q$ for $q > n$. For positively graded $k$-modules $X$ and $Y$, one of which is flat, there is a Künneth spectral sequence that converges from $\mathrm{Tor}^k_{*,*}(H_*(X), H_*(Y))$ to $H_*(X \otimes Y)$ [41, XII.12.1]. By generalizing from positive to bounded below $k$-modules and passing to direct limits, we obtain a natural convergent Künneth spectral sequence for any two $k$-modules $X$ and $Y$, one of which is flat. The conclusion follows. □

A map $f_* : X_* \to X'_*$ between domains in $X$ and $X'$ is a sequence of maps $f_j : X_j \to X'_j$ such that $f_j$ is the restriction of $f^j$, where $f = f_1$. The map $f_*$ is said to be a quasi-isomorphism if $f : X \to X'$ is a quasi-isomorphism. It follows from the lemma and the definitions that each $f_j$ is then also a quasi-isomorphism.

Let $\mathscr{K}$ be the category of flat $k$-modules and $\mathscr{D}$ be the category of domains in flat $\mathscr{K}$-modules. Let $L : \mathscr{D} \to \mathscr{K}$ be the functor that sends $X_*$ to $X$ and $R : \mathscr{K} \to \mathscr{D}$ be the functor that sends $X$ to $\{X^j\}$. Then $LR = \mathrm{Id}$ and the inclusions $\delta_j$ define a natural map $\delta : X_* \to RX = RLX_*$ such that $L\delta = \mathrm{Id}$. We therefore have an adjunction

$$\mathscr{K}(LX_*, Y) \cong \mathscr{D}(X_*, RY). \tag{2.3}$$

Informally, given any type of algebraic structure that is defined in terms of maps $A^j \to A$, we define a partial structure on $A$ to be a domain $A_*$ in $A$ together with maps $A_j \to A$ that satisfy the same formal properties as the given type of structure. We shall shortly formalize this with a general definition of a partial operad action.

For motivation, and because it is the type of structure that we are most interested in, we first consider commutative DGA's explicitly. Such an algebra $A$ has a $j$-fold product $\mu_j : A^j \to A$, with $\mu_0 = \eta : k \to A$ and $\mu_1 = \mathrm{Id}$. For $\sigma \in \Sigma_j$, $j \geq 2$, $\mu_j \circ \sigma = \mu_j$. For any partition $j = j_1 + \cdots + j_k$ with $j_i \geq 0$, the following associativity and unity diagram commutes:

$$\begin{CD} A^{j_1} \otimes \cdots \otimes A^{j_k} @>{\cong}>> A^j \\ @V{\mu\otimes\cdots\otimes\mu}VV @VV{\mu}V \\ A^k @>{\mu}>> A \end{CD}$$

Recall that $2x^2 = 0$ if $x$ has odd degree. It is standard in topology to say that a commutative DGA is "strictly commutative" if $x^2 = 0$ when $x$ has odd degree. Unless $A$ is strictly commutative (or the ring $k$ is of characteristic two), $A$ will not be a flat $k$-module.

We have the concomitant notion of a partial commutative DGA $A_*$. The only point that might require clarification is the partial version of the previous diagram, which now takes the form

$$\begin{array}{ccccc} A_{j_1} \otimes \cdots \otimes A_{j_k} & \hookleftarrow & & & A_j \\ \downarrow{\scriptstyle \mu\otimes\cdots\otimes\mu} & & & \swarrow & \downarrow{\scriptstyle \mu} \\ A^k & \hookleftarrow & A_k & \xrightarrow{\mu} & A \end{array}$$

That is, the restriction to $A_j$ of $\mu \otimes \cdots \otimes \mu : A_{j_1} \otimes \cdots \otimes A_{j_k} \to A_k$ factors through $A_k$, and the two resulting maps from $A_j$ to $A$ coincide. More generally, we have the following direct generalization of I.2.1.

**Definition 2.4.** Let $\mathscr{C}$ be an operad. A partial $\mathscr{C}$-algebra is a domain $A_*$ in a flat $k$-module $A$ together with $\Sigma_j$-equivariant maps

$$\theta_j : \mathscr{C}(j) \otimes A_j \to A, \quad j \geq 0,$$

such that
(a) $\theta_1(1 \otimes a) = a$,
(b) the map

$$\mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A_j \to A^k$$

that is obtained by including $A_j$ in $A_{j_1} \otimes \cdots \otimes A_{j_k}$, shuffling, and applying $\theta^k$ factors through $A_k$, where $j = \sum j_s$, and
(c) the following associativity diagrams commute:

$$\begin{array}{ccccc} \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A_j & \xrightarrow{\gamma\otimes\mathrm{Id}} & \mathscr{C}(j) \otimes A_j & & \\ \downarrow{\scriptstyle \subset} & \searrow & & \searrow{\scriptstyle \theta} & \\ \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A_{j_1} \otimes \cdots \otimes A_{j_k} & & \mathscr{C}(k) \otimes A_k & \xrightarrow{\theta} & A \\ \downarrow{\scriptstyle \text{shuffle}} & & \downarrow{\scriptstyle \subset} & & \\ \mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes A_{j_1} \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A_{j_k} & \xrightarrow{\mathrm{Id}\otimes\theta^k} & \mathscr{C}(k) \otimes A^k & & \end{array}$$

Note that $RA = \{A^j\}$ may be viewed as a partial $\mathscr{C}$-algebra if $A$ is a $\mathscr{C}$-algebra. In our formal theory, we generally write $A_*$ for a partial $\mathscr{C}$-algebra. Informally, however, as in the statements of the results in the introduction, we think of the submodules $A_j$ of the $A^j$ as implicitly given and simply write $A$. The following examples generalize I.2.2.

**Examples 2.5.** (i) A partial $\mathscr{M}$-algebra is a partial DGA.
(ii) A partial $\mathscr{N}$-algebra is a partial commutative DGA, as defined above.
(iii) By pullback along $\epsilon : \mathscr{C} \to \mathscr{N}$, a partial commutative DGA is a partial $\mathscr{C}$-algebra for any unital operad $\mathscr{C}$.
(iv) We define a partial $E_\infty$ algebra to be a partial $\mathscr{C}$-algebra, where $\mathscr{C}$ is any $E_\infty$ operad.

*Remark* 2.6. We noted in I.7.2 that the mod $p$ homology of an integral $E_\infty$ algebra has homology operations. The cited result is true precisely as stated with $E_\infty$ algebras replaced by partial $E_\infty$ algebras. In fact, one can construct the operations by passing to mod $p$ homology from the diagram

$$\mathscr{C}(p) \otimes_{\Sigma_p} A^p \longleftarrow \mathscr{C}(p) \otimes_{\Sigma_p} A_p \longrightarrow A.$$

The first arrow is induced by the inclusion $A_p \subset A^p$, and this arrow induces an isomorphism on mod $p$ homology by Lemma 3.1 below. The second arrow is $\theta_p$. If we start with a partial commutative DGA $A_*$, then $\theta_p = \epsilon_p \otimes \mu_p$. Use of the augmentation $\epsilon_p : \mathscr{C}(p) \to \mathbb{Z}$ may make it appear that the resulting operations ought to be trivial. However, as is explained in [26], nontriviality is allowed by the fact that the inclusion $A_p \subset A^p$ need not be a $\Sigma_p$-equivariant homotopy equivalence.

We have a precisely parallel definition of a partial module over a partial algebra.

**Definition 2.7.** Define a domain $(X_*; Y_*)$ in a pair of flat $k$-modules $(X; Y)$ to be a domain $X_*$ in $X$ together with a sequence of $\Sigma_{j-1}$-invariant submodules $Y_j$ of $X^{j-1} \otimes Y$, $j \geq 1$, such that the given inclusions $\delta_j : Y_j \to X^{j-1} \otimes Y$ satisfy the following properties.
(a) $\delta_1 = \mathrm{Id}$ and each $\delta_j$ is a quasi-isomorphism.
(b) For $j = j_1 + \cdots + j_k$, $\delta_j$ factors through $X_{j_1} \otimes \cdots \otimes X_{j_{k-1}} \otimes Y_{j_k}$, as indicated in the following commutative diagram:

$$\begin{array}{ccc} X^{j_1} \otimes \cdots \otimes X^{j_{k-1}} \otimes X^{j_k - 1} \otimes Y & \xrightarrow{\cong} & X^{j-1} \otimes Y \\ \uparrow{\scriptstyle \delta_{j_1} \otimes \cdots \otimes \delta_{j_k}} & & \uparrow{\scriptstyle \delta_j} \\ X_{j_1} \otimes \cdots \otimes X_{j_{k-1}} \otimes Y_{j_k} & \hookleftarrow & Y_j \end{array}$$

A map $(f_*; g_*) : (X_*; Y_*) \to (X'_*; Y'_*)$ consists of a map of domains $f_* : X_* \to X'_*$ and a sequence of maps of $k$-modules $g_j : Y_j \to Y'_j$ such that $g_j$ is the restriction of $f^{j-1} \otimes g$, where $g = g_1$. The map $(f_*; g_*)$ is said to be a quasi-isomorphism if $f : X \to X'$ and $g : Y \to Y'$ are quasi-isomorphisms, and then each $f_j$ and $g_j$ is also a quasi-isomorphism.

Let $\mathscr{K}[1]$ be the category of pairs of flat $k$-modules and $\mathscr{D}[1]$ be the category of domains in such pairs. Let $L : \mathscr{D}[1] \to \mathscr{K}[1]$ send $(X_*; Y_*)$ to $(X; Y)$ and $R : \mathscr{K}[1] \to \mathscr{D}[1]$ send $(X; Y)$ to $(\{X^j\}; \{X^{j-1} \otimes Y\})$. Again, $LR = $Id and the $\delta_j$ specify a natural map $\delta : X_* \to RX = RLX_*$ such that $L\delta = $Id, hence we have an adjunction

$$\mathscr{K}[1](L(X_*;Y_*),(X';Y')) \cong \mathscr{D}[1]((X_*;Y_*),R(X';Y')). \tag{2.8}$$

We shall often abbreviate $(X_*;Y_*)$ to $Y_*$ when $X_*$ is implicit from the context.

Let $A_*$ be a partial commutative DGA. An $A_*$-module (informally, a partial $A$-module) is a domain $M_*$ together with maps of $k$-modules $\lambda_j : M_j \to M$ such that $\lambda_1 =$Id, $\lambda_j \circ \sigma = \lambda_j$ for $\sigma \in \Sigma_{j-1}$, and the following diagrams commute, where $j_i \geq 0$ for $i < k, j_k \geq 1$, and $j = j_1 + \cdots + j_k$;

$$\begin{array}{ccccc}
A_{j_1} \otimes \cdots \otimes A_{j_{k-1}} \otimes M_{j_k} & & \hookleftarrow & & M_j \\
\downarrow{\scriptstyle \mu\otimes\cdots\otimes\mu\otimes\lambda} & & & \swarrow{\scriptstyle \lambda} & \downarrow{\scriptstyle \mu} \\
A^{k-1} \otimes M & \hookleftarrow & M_k & \longrightarrow & M
\end{array}$$

That is, the restriction to $M_j$ of $\mu \otimes \cdots \otimes \mu \otimes \lambda$ factors through $M_k$, and the two resulting maps from $M_j$ to $M$ coincide. This is the special case $\mathscr{C} = \mathscr{N}$ of the following definition, which generalizes I.4.1.

**Definition 2.9.** Let $\mathscr{C}$ be an operad and $A_*$ be a partial $\mathscr{C}$-algebra in $A$. An $A_*$-module $M_*$ in $M$ is a domain $(A_*;M_*)$ in $(A;M)$ together with $\Sigma_{j-1}$-equivariant maps

$$\lambda_j : \mathscr{C}(j) \otimes M_j \to M, \quad j \geq 1,$$

such that
(a) $\lambda_1(1 \otimes m) = m$,
(b) the map

$$\mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes M_j \to M_k$$

that is obtained by including $M_j$ in $A_{j_1} \otimes \cdots \otimes A_{j_{k-1}} \otimes M_{j_k}$, shuffling, and applying $\theta^{k-1} \otimes \lambda$ factors through $A_k$, where $j = \sum j_s$, and
(c) the following associativity diagrams commute:

$$\begin{array}{ccccc}
\mathscr{C}(k) \otimes (\otimes_{s=1}^{k-1}\mathscr{C}(j_s)) \otimes \mathscr{C}(j_k) \otimes M_j & \longrightarrow & \mathscr{C}(j) \otimes M_j & & \\
\downarrow{\scriptstyle \cap} & \searrow & & \searrow{\scriptstyle \lambda} & \\
\mathscr{C}(k) \otimes (\otimes_{s=1}^{k-1}\mathscr{C}(j_s)) \otimes \mathscr{C}(j_k) \otimes (\otimes_{s=1}^{k=1} A_{j_s}) \otimes M_{j_k} & & \mathscr{C}(k) \otimes M_k & \xrightarrow{\lambda} & M \\
\downarrow{\scriptstyle \text{shuffle}} & & \downarrow{\scriptstyle \cap} & & \\
\mathscr{C}(k) \otimes (\otimes_{s=1}^{k-1}\mathscr{C}(j_s) \otimes A_{j_s}) \otimes \mathscr{C}(j_k) \otimes M_{j_k} & \xrightarrow[\mathrm{Id}\otimes\theta^{k-1}\otimes\lambda]{} & \mathscr{C}(k) \otimes A^{k-1} \otimes M & &
\end{array}$$

*Remark* 2.10. There is also a notion of a partial operad $\mathscr{C}$, with structural maps $\gamma$ defined on submodules $\mathscr{C}(k;j_1,\ldots,j_k)$ of $\mathscr{C}(k) \otimes \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k)$. In topology, the little convex bodies partial operads of [47, VII §2], were the first examples, but Steiner [57] later showed how to replace these particular partial operads by equivalent genuine operads with all of the desired properties. Partial operads have arisen more recently, and more substantially, in work of Huang and Lepowski on vertex operator algebras [34, 33]. It is an easy matter to generalize the definitions above to specify partial algebras and modules over partial operads. However, from an algebraic point of view, the resulting concepts are harder to work with since they apparently cannot be described in equivalent monadic terms.

## 3. Monadic reinterpretation of partial algebras and modules

In this section, we assume given a fixed operad $\mathscr{C}$ that is $\Sigma$-projective, in the sense that each $\mathscr{C}(j)$ is a projective $k[\Sigma_j]$-module. This condition holds automatically when $k$ is a field of characteristic zero since every module over the group ring $k[G]$ of a finite group $G$ is then projective. It allows us to make use of the following standard observation, which complements Lemma 2.2.

**Lemma 3.1.** *Let $G$ be a group and $P$ be a projective $k[G]$-module. Then a quasi-isomorphism $X \to X'$ of $k[G]$-modules induces a quasi-isomorphism $P \otimes_{k[G]} X \to P \otimes_{k[G]} X'$ of $k$-modules.*

*Proof.* If we filter $P \otimes_{k[G]} X$ by the degrees in $P$, we obtain a natural spectral sequence that converges from $H_*(P \otimes_{k[G]} H_*(X))$ to $H_*(P \otimes_{k[G]} X)$. □

In I §§3–4, we constructed monads $C$ in $\mathscr{K}$ and $C[1]$ in $\mathscr{K}[1]$ such that a $C$-algebra determines and is determined by a $\mathscr{C}$-algebra and a $C[1]$-algebra determines and is determined by a $\mathscr{C}$-algebra together with a module over it. In this part, $\mathscr{K}$ and $\mathscr{K}[1]$ are restricted to flat $k$-modules and pairs, and our assumption on $\mathscr{C}$ ensures that $C$ and $C[1]$ take flat modules and pairs to flat modules and pairs. We generalize these constructions to the context of partial algebras and modules.

**Definition 3.2.** Define the monad $C_*$ in $\mathscr{D}$ associated to $\mathscr{C}$ as follows. Let $X_*$ be a domain in $X$. Define

$$CX_* = \bigoplus_{j \geq 0} \mathscr{C}(j) \otimes_{k[\Sigma_j]} X_j.$$

For $k \geq 0$, define $C_k X_* \subset (CX_*)^k$ to be the direct sum of the images of the following composites (where $j_s \geq 0$ and $j = \sum j_s$):

$$\begin{array}{c} \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes_{k[\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}]} X_j \\ \big\downarrow \\ \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes_{k[\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}]} X_{j_1} \otimes \cdots \otimes X_{j_k} \\ \big\downarrow \text{shuffle} \\ \mathscr{C}(j_1) \otimes_{k[\Sigma_{j_1}]} X_{j_1} \otimes \cdots \otimes \mathscr{C}(j_k) \otimes_{k[\Sigma_{j_k}]} X_{j_k} \end{array}$$

This inclusion is a quasi-isomorphism by Lemma 3.1, hence the inclusion $C_k X_* \subset (CX)^k$ is a quasi-isomorphism. The action of $\Sigma_k$ on $C_k X_*$ is induced from the action on $(CX)^k$; more explicitly, it is obtained from permutations of the variables $\mathscr{C}(j_s)$ and action of the permutations $\sigma(j_1, \ldots, j_k)$ associated to $\sigma \in \Sigma_k$ on the factors $X_j$ (see I.1.1). Condition (b) of Definition 2.1 is inherited from the corresponding condition for $X_*$. Let $\eta_k : X_k \to C_k X_*$ be induced by the map

$$\eta \otimes \cdots \otimes \eta \otimes \mathrm{Id} : X_k = k \otimes \cdots \otimes k \otimes X_k \to \mathscr{C}(1) \otimes \cdots \otimes \mathscr{C}(1) \otimes X_k.$$

Similarly, let $\mu_k : C_kC_*X_* \to C_kX_*$ be induced by the following maps, where $\sum j_s = j$, $\sum i_t = i$, $g_s = j_1 + \cdots + j_s$, and $h_s = i_{g_{s-1}+1} + \cdots + i_{g_s}$ for $1 \le s \le k$:

$$\begin{array}{c} \mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes \mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_j) \otimes X_i \\ \Big\downarrow \text{shuffle} \\ \bigotimes_s (\mathscr{C}(j_s) \otimes \mathscr{C}(i_{g_{s-1}+1}) \otimes \cdots \otimes \mathscr{C}(i_{g_s})) \otimes X_i \\ \Big\downarrow (\bigotimes_s \gamma) \otimes \mathrm{Id} \\ \mathscr{C}(h_1) \otimes \cdots \otimes \mathscr{C}(h_k) \otimes X_i \end{array}$$

It is easy to check that $(C_*, \mu_*, \eta_*)$ is a monad in $\mathscr{D}$.

The following observation is immediate from Lemma 3.1.

**Lemma 3.3.** *If $f : X_* \to X'_*$ is a quasi-isomorphism of domains, then so is $C_*f : C_*X_* \to C_*X'_*$.*

We have the following generalizations of I.3.4. Recall (2.3).

**Theorem 3.4.** *Let $\mathscr{C}$ be a $\Sigma$-projective operad.*
*(i) A partial $\mathscr{C}$-algebra determines and is determined by a $C_*$-algebra in $\mathscr{D}$. Formally, the identity functor on $\mathscr{D}$ restricts to an isomorphism between the categories of partial $\mathscr{C}$-algebras and of $C_*$-algebras.*
*(ii) $RC = C_*R$, hence $C = LC_*R$, and the unit $\eta$ and product $\mu$ for $C$ are given as follows in terms of the unit $\eta_*$ and product $\mu_*$ of $C_*$:*

$$\eta = L\eta_*R : Id = LR \to LC_*R = C,$$

*and*

$$\mu = L\mu_*R : CC = LC_*RC = LC_*C_*R \to LC_*R = C.$$

*Proof.* If $A_*$ is a partial $\mathscr{C}$-algebra, then the given maps $\theta : \mathscr{C}(j) \otimes A_j \to A$ together induce a map $\xi : CA_* \to A$. For $k \ge 1$, the maps

$$\mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A_j \to A_k$$

that factor the evident map to $A^k$ (as in Definition 2.4) together induce a map $\xi_k : C_kA_* \to A_k$. It is easily checked that $(A_*, \xi_*)$ is a $C_*$-algebra. Conversely, if $(A_*, \xi_*)$ is a $C_*$-algebra, then the evident composites

$$\mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes A_j \to C_kA_* \to A_k, \quad k \ge 1,$$

give $A_*$ a structure of a partial $\mathscr{C}$-algebra. Part (ii) is easily verified by a direct comparison of definitions. □

The theory for partial modules is precisely analogous and generalizes material in I §4. Recall (2.8).

**Definition 3.5.** Define the monad $C_*[1]$ in $\mathscr{D}[1]$ associated to $\mathscr{C}$ as follows. Let $(X_*; Y_*)$ be a domain in $(X; Y)$. Define

$$CY_* = \bigoplus \mathscr{C}(j) \otimes_{k[\Sigma_{j-1}]} \otimes Y_j.$$

For $k \geq 1$, define $C_kY_* \subset (CX_*)^{k-1} \otimes CY_*$ to be the direct sum of the images of the following composites (where $j_s \geq 0$ and $j = \sum j_s$):

$$\begin{array}{c}
\mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes_{k[\Sigma_{j_1} \times \cdots \times \Sigma_{j_{k-1}} \times \Sigma_{j_k - 1}]} Y_j \\
\big\downarrow \\
\mathscr{C}(j_1) \otimes \cdots \otimes \mathscr{C}(j_k) \otimes_{k[\Sigma_{j_1} \times \cdots \times \Sigma_{j_{k-1}} \times \Sigma_{j_k - 1}]} X_{j_1} \otimes \cdots \otimes X_{j_{k-1}} \otimes Y_{j_k} \\
\big\downarrow \text{shuffle} \\
\mathscr{C}(j_1) \otimes_{k[\Sigma_{j_1}]} X_{j_1} \otimes \cdots \otimes \mathscr{C}(j_{k-1}) \otimes_{k[\Sigma_{j_{k-1}}]} X_{j_{k-1}} \otimes \mathscr{C}(j_k) \otimes_{k[\Sigma_{j_k - 1}]} Y_{j_k}
\end{array}$$

This inclusion is a quasi-isomorphism by Lemma 3.1, hence the inclusion $C_kY_* \subset (CX_*)^{k-1} \otimes CY_*$ is a quasi-isomorphism. The action of $\Sigma_{k-1}$ on $C_kY_*$ is induced from the action on $(CX_*)^{k-1} \otimes CY_*$. Condition (b) of Definition 2.7 is inherited from the corresponding condition for $Y_*$. Maps $\eta_k : Y_k \to C_kY_*$ and $\mu_k : C_kC_*Y_* \to C_kY_*$ are defined as in Definition 3.2 and I.4.3. Taking $C[1]_*(X_*; Y_*)$ to be $(C_*X_*; C_*Y_*)$ and using the pairs of maps $(\eta_*; \eta_*)$ and $(\mu_*; \mu_*)$ as the unit and product, we obtain the desired monad in $\mathscr{D}[1]$.

**Lemma 3.6.** *If $(f_*; g_*) : (X_*; Y_*) \to (X'_*; Y'_*)$ is a quasi-isomorphism of domains, then the induced map $C_*(f_*; g_*) : C_*Y_* \to C_*Y'_*$ is a quasi-isomorphism.*

**Theorem 3.7.** *Let $\mathscr{C}$ be a $\Sigma$-projective operad.*
*(i) A $C[1]_*$-algebra structure on a domain $(A_*; M_*)$ determines and is determined by a partial $\mathscr{C}$-algebra structure on $A_*$ together with a partial $A_*$-module structure on $M_*$.*
*(ii) $RC[1] = C[1]_*R$, hence $C[1] = LC[1]_*R$, and the unit $\eta$ and product $\mu$ of $C[1]$ are given in terms of the unit $\eta_*$ and product $\mu_*$ of $C[1]_*$ by $\eta = L\eta_*R$ and $\mu = L\mu_*R$.*

*Remark* 3.8. For a unital operad $\mathscr{C}$, there are generalizations to the partial context of the reduced monads that we constructed in I §§3–4. These were used in the preprint version [37] of this paper. Since the reduced monads are not essential to the theory and the details are fairly technical, we shall omit these constructions in the interests of brevity.

## 4. The two-sided bar construction and the conversion theorems

We begin by recalling some categorical definitions from [46, §§2,9]. Their use to prove the theorems stated in the introduction will follow a conceptual pattern that is explained in detail in [49, §5].

**Definition 4.1.** Let $(C, \mu, \eta)$ be a monad in a category $\mathscr{T}$. A (right) $C$-functor in a category $\mathscr{V}$ is a functor $F : \mathscr{T} \to \mathscr{V}$ together with a natural transformation $\nu : FC \to F$ such that the following diagrams commute:

$$\begin{array}{ccc}
FC & \xleftarrow{F\eta} & F \\
{\scriptstyle\nu}\big\downarrow & \swarrow{\scriptstyle \mathrm{Id}} & \\
F & &
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
FCC & \xrightarrow{\nu C} & FC \\
{\scriptstyle F\mu}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\
FC & \xrightarrow{\nu} & F
\end{array}$$

For a triple $(F, C, A)$ consisting of a monad $C$ in $\mathscr{T}$, a $C$-algebra $A$, and a $C$-functor $F$ in $\mathscr{V}$, define a simplicial object $B_*(F, C, A)$ in $\mathscr{V}$ by letting the $q$-simplices $B_q(F, C, A)$ be $FC^qA$ (where $C^q$ denotes $C$ composed with itself $q$ times); the faces and degeneracies are given by

$$\begin{aligned} \partial_0 &= \nu C^{q-1}, \\ \partial_i &= FC^{i-1}\mu C^{q-i-1} \text{ for } 1 \leq i < q, \\ \partial_q &= FC^{q-1}\epsilon, \text{ and} \\ s_i &= FC^i \eta C^{q-i}. \end{aligned}$$

In an evident sense, $B_*(F, C, A)$ is functorial in all three variables. Given a monad $C'$ in $\mathscr{V}$ and a left action $\lambda : C'F \to F$, we say that $F$ is a $(C', C)$-bifunctor if the following diagram commutes:

$$\begin{array}{ccc} C'FC & \xrightarrow{\lambda C} & FC \\ {\scriptstyle C'\nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ C'F & \xrightarrow{\lambda} & F \end{array}$$

For such an $F$, $B_*(F, C, A)$ is a simplicial $C'$-algebra.

**Example 4.2.** An obvious example of a $(C, C)$-bifunctor is $C$ itself, with both left and right action $\mu$. Thus we may regard $C$ as a functor from $\mathscr{T}$ to the category $C[\mathscr{T}]$ of $C$-algebras in $\mathscr{T}$. This example gives a simplicial $C$-algebra $B_*(C, C, A)$ associated to a $C$-algebra $A$. Let $\underline{A}$ denote $A$ regarded as a constant simplicial object, $\underline{A}_q = A$ for all $q$, with each face and degeneracy the identity map. Iterates of $\mu$ and $\xi$ give a map $\psi_* : B_*(C, C, A) \to \underline{A}$ of simplicial $C$-algebras in $\mathscr{T}$. Similarly, iterates of $\eta$ give a map $\eta_* : \underline{A} \to B_*(C, C, A)$ of simplicial objects in $\mathscr{T}$ (but not in $C[\mathscr{T}]$) such that $\psi_*\eta_* = \mathrm{Id}$. Moreover, there is a simplicial homotopy $\eta_*\psi_* \simeq \mathrm{Id}$ [46, 9.8]. This is a generalized version of the classical bar resolution in homological algebra, and we shall often abbreviate notation by setting

$$B_*(A) = B_*(C, C, A).$$

The following examples should be viewed as formal precursors of Theorems 1.1, 1.2, and 1.4. Fix a $\Sigma$-projective operad $\mathscr{C}$.

**Example 4.3.** (i) As explained in [49, 5.5], part (ii) of Theorem 3.4 implies that $CL : \mathscr{D} \to \mathscr{K}$ is a $(C, C_*)$-bifunctor with $C_*$-action the composite

$$\mu L \circ CLC_*\delta : CLC_* \to CLC_*RL = CCL \to CL$$

and that $C_*\delta : C_* \to C_*RL = RCL$ is a map of $(C_*, C_*)$-bifunctors $\mathscr{D} \to \mathscr{D}$. Since $\delta$ is a natural quasi-isomorphism, Lemma 3.3 allows us to view $C_*\delta$ as inducing a quasi-isomorphism of simplicial $C_*$-algebras

$$\delta_* : B_*(C_*, C_*, A_*) \to B_*(RCL, C_*, A_*) = RB_*(CL, C_*, A_*)$$

for any $C_*$-algebra $A_*$. Introduce the abbreviated notations

$$B_*A_* = B_*(C_*, C_*, A_*) \text{ and } V_*A_* = B_*(CL, C_*, A_*).$$

Then $B_*A_*$ is a simplicial $C_*$-algebra, $V_*A_*$ is a simplicial $C$-algebra, and $\psi_*$ and $\delta_*$ give a natural diagram of simplicial $C_*$-algebras

$$\underline{A}_* \leftarrow B_*A_* \to RV_*A_*.$$

(ii) Similarly, part (ii) of Theorem 3.7 implies that we may replace $C$ by $C[1]$ in (i) and obtain the analogous conclusions: $C[1]L : \mathscr{D}[1] \to \mathscr{K}[1]$ is a $(C[1], C[1]_*)$-bifunctor with $C[1]_*$-action given by $\mu L \circ C[1]LC[1]_*\delta$, and $C[1]_*\delta : C[1]_* \to RC[1]L$ is a map of $(C[1]_*, C[1]_*)$-bifunctors $\mathscr{D}[1] \to \mathscr{D}[1]$. By Lemma 3.6, we may view $C[1]_*\delta$ as inducing a quasi-isomorphism of simplicial $C[1]_*$-algebras

$$\delta_* : B_*(C[1]_*, C[1]_*, (A_*; M_*)) \to RB_*(C[1]L, C[1]_*, (A_*; M_*))$$

for any $C[1]_*$-algebra $(A_*; M_*)$; recall that $A_*$ is a $C_*$-algebra and $M_*$ is an $A_*$-module. We extend the abbreviated notations of (i) by setting

$$(B_*A_*; B_*M_*) = B_*(C[1]_*, C[1]_*, (A_*; M_*))$$

and

$$(V_*A_*; V_*M_*) = B_*(C[1]L, C[1]_*, (A_*; M_*)).$$

Then $B_*M_*$ is a simplicial $B_*A_*$-algebra, $V_*M_*$ is a simplicial $V_*A_*$-algebra, and $\psi_*$ and $\delta_*$ give a natural diagram of simplicial $C[1]_*$-algebras

$$(\underline{A}_*; M_*) \leftarrow (B_*A_*; B_*M_*) \to R(V_*A_*; V_*M_*).$$

**Example 4.4.** Let $\epsilon : \mathscr{C} \to \mathscr{P}$ be a quasi-isomorphism of $\Sigma$-projective operads and let $\epsilon$ also denote the induced maps of monads $C \to P$ and $C[1] \to P[1]$. If $k$ is a field of characteristic zero, then $\epsilon : CX \to PX$ and $\epsilon : C(X; Y) \to P(X; Y)$ are quasi-isomorphisms for all $k$-modules $X$ and $Y$. (This is I.7.1, and it also follows directly from Lemma 3.1.) In this case, the maps $\epsilon_*$ in the rest of this example are all quasi-isomorphisms.

(i) $PL : \mathscr{D} \to \mathscr{K}$ is a $(P, C_*)$-bifunctor with $C_*$-action the composite

$$\mu L \circ P\epsilon L \circ PLC_*\delta : PLC_* \to PLC_*RL = PCL \to PPL \to PL;$$

$\epsilon L : CL \to PL$ is a map of $(C, C_*)$-bifunctors and therefore induces a map of simplicial $C$-algebras

$$\epsilon_* : VA_* = B_*(CL, C_*, A_*) \to B_*(PL, C_*, A_*) \equiv W_*A_*$$

for any $C_*$-algebra $A_*$, where $W_*A_*$ is abbreviated notation for the simplicial $P$-algebra $B_*(PL, C_*, A_*)$.

(ii) $P[1]L : \mathscr{D}[1] \to \mathscr{K}[1]$ is a $(P[1], C[1]_*)$-bifunctor; $\epsilon L : C[1]L \to P[1]L$ is a map of $(C[1], C[1]_*)$-bifunctors and therefore induces a map of simplicial $C[1]$-algebras

$$\epsilon_* : (V_*A_*; V_*M_*) \to (W_*A_*; W_*M_*),$$

where $W_*M_*$ is abbreviated notation for the second coordinate of the simplicial $P[1]$-algebra $B_*(P[1]L, C[1]_*, (A_*; M_*))$.

(iii) For a $C$-algebra $A$, define

$$B_*A = B_*(C, C, A) \text{ and } W_*A = B_*(P, C, A).$$

Then $W_*A$ is a simplicial $P$-algebra and $\epsilon_* : B_*A \to W_*A$ is a map of simplicial $C$-algebras. Similarly, for a $C[1]$-algebra $(A; M)$, define

$$(B_*A; B_*M) = B_*(C[1], C[1], (A; M)), \text{ and}$$

$$(W_*A; W_*M) = B_*(P[1], C[1], (A; M)).$$

Then $W_*M$ is a simplicial $W_*A$-module and

$$\epsilon_* : (B_*A; B_*M) \to (W_*A; W_*M)$$

is a map of simplicial $C[1]$-algebras. That is, $\epsilon_* : B_*M \to W_*M$ is a map of simplicial $B_*A$-modules, where $W_*M$ is a simplicial $B_*A$-module by pullback along $\epsilon_* : B_*A \to W_*A$.

To go from these examples to the theorems of Section 1, we need only transport the information that they give from the category $\mathscr{S}\mathscr{K}$ of simplicial $k$-modules to the category $\mathscr{K}$ of $k$-modules. Recall that, so far, the term "$k$-module" has been used in all-embracing generality, allowing $\mathbb{Z}$-graded differential $k$-modules or even simplicial $\mathbb{Z}$-graded differential $k$-modules. To obtain a sufficiently well-behaved functor $\mathscr{S}\mathscr{K} \to \mathscr{K}$, one that preserves all kinds of algebras and modules in sight, we must shrink $\mathscr{K}$. This is the subject of the next section.

## 5. Totalization and diagonal functors; proofs

We change notational conventions at this point and use the term "$k$-module" in the classical sense of an ungraded $k$-module without differential. We use the term "chain complex" for a differential graded $k$-module. We shall have to consider, and carefully distinguish among, simplicial $k$-modules, simplicial chain complexes, and bisimplicial $k$-modules (which arise naturally as simplicial simplicial $k$-modules).

Recall that the tensor product of simplicial $k$-modules $X$ and $Y$ has $q$-simplices $X_q \otimes Y_q$, with faces and degeneracies $\partial_i \otimes \partial_i$ and $s_i \otimes s_i$. Indeed, this defines the tensor product between simplicial objects in any category with a tensor product.

For the applications of the next section, we must allow simplicial graded $k$-modules $X$. This means that $X = \{X(r) | r \in \mathbb{Z}\}$ is a sequence of simplicial $k$-modules. A map $X \to Y$ is a sequence of maps $X(r) \to Y(r)$ of simplicial $k$-modules. The grading $r$ is vital to the geometric context, where it is closely related to the grading of rational algebraic $K$-theory by the eigenvectors of Adams operations, but it will carry through the theory of this section without introducing substantive complications. We will generally call it the "Adams grading" to avoid confusion with any other grading that we may have. Write $X_p(r)$ for the $k$-module of $p$-simplices in $X(r)$ and define $(X \otimes Y)(r) = \sum X(s) \otimes Y(r-s)$, where the tensor products on the right are as specified above. The unit for the tensor product is the constant simplicial $k$-module $\underline{k}$, thought of as concentrated in Adams grading zero.

We must also allow Adams graded chain complexes $X$. This means that $X = \{X(r) | r \in \mathbb{Z}\}$ is a sequence of chain complexes. For the general theory, the $X(r)$ can be $\mathbb{Z}$-graded, but they will be positively graded in our examples. A map $X \to Y$ is a sequence of chain maps $X(r) \to Y(r)$. Define $(X \otimes Y)(r) = \sum X(s) \otimes Y(r-s)$, where the tensor products on the right are the usual tensor products of chain complexes. The unit is $k$ regarded as a chain complex concentrated in degree zero and Adams grading zero.

We view the Adams grading as if it were concentrated in even degrees: it will not contribute to signs under permutations. While the examples in the next section are concentrated in positive Adams grading, a satisfactory theory of modules requires us to allow negative degrees.

Let $\mathscr{K}$ denote the category of Adams graded $k$-modules and let $\mathscr{C}\mathscr{C}$ denote the category of Adams graded chain complexes. For any category $\mathscr{T}$, let $\mathscr{S}\mathscr{T}$ denote the category of simplicial objects in $\mathscr{T}$. We thus have categories $\mathscr{S}\mathscr{K}$, $\mathscr{S}\mathscr{C}\mathscr{C}$, and $\mathscr{S}\mathscr{S}\mathscr{K}$. The last is the category of bisimplicial Adams graded $k$-modules, and such objects will arise as simplicial bar constructions $B_*(F, C, A)$, where $F$ takes values in $\mathscr{S}\mathscr{K}$. These categories are all symmetric monoidal under their respective tensor products.

To understand our conversion theorems, we must understand the properties of the normalized chain complex functor

$$C_{\#} : \mathscr{S}\mathscr{K} \to \mathscr{C}\mathscr{C},$$

its generalization to the totalization functor

$$C_{\#} : \mathscr{S}\mathscr{C}\mathscr{C} \to \mathscr{C}\mathscr{C},$$

and the diagonal functor

$$\Delta : \mathscr{S}\mathscr{S}\mathscr{K} \to \mathscr{S}\mathscr{K}.$$

(We are using the notation $C_{\#}$ to avoid confusion with our use of the notation $C_*$ for the monads in domains associated to an operad $\mathscr{C}$.)

For a simplicial $k$-module $X$, $C_{\#}(X)$ is just the chain complex $X/D(X)$ with differential $d = \sum(-1)^i \partial_i$, where $D(X)$ is the subcomplex of $X$ generated by the degenerate simplices (which is acyclic [44, 22.3]). Since $D(X)$ is a direct summand of $X$ [44, 22.2], $C_{\#}$ preserves inclusions. If $X$ is Adams graded, we define $(C_{\#}X)(r) = C_{\#}(X(r))$.

The functor $C_{\#}$ preserves algebraic structures that are defined in terms of tensor products, but it does not carry partial algebras to partial algebras in general. To see this, recall that, for simplicial $k$-modules $X$ and $Y$, we have the shuffle map

$$g : C_{\#}(X) \otimes C_{\#}(Y) \to C_{\#}(X \otimes Y)$$

and the Alexander-Whitney map

$$f : C_{\#}(X \otimes Y) \to C_{\#}(X) \otimes C_{\#}(Y).$$

These are inverse chain homotopy equivalences and, because we are working on the normalized level, $f \circ g =\mathrm{Id}$ [44, 29.10]. Thus $g$ is a split inclusion. Moreover, $g$ is commutative, associative, and unital by [44, 29.9] and inspection.

Given any kind of algebraic structure defined in terms of maps $\theta : X_1 \otimes \cdots \otimes X_j \to X$ in $\mathscr{S}\mathscr{K}$, we obtain a similar kind of algebraic structure in $\mathscr{C}\mathscr{C}$ by composing the maps $C_{\#}\theta$ with iterates of $g$. Here, if we start with a structure defined in terms of an operad $\mathscr{C}$ of simplicial $k$-modules, we end with a structure defined in terms of the operad $C_{\#}(\mathscr{C})$ of chain complexes. All of our operads, in both $\mathscr{S}\mathscr{K}$ and $\mathscr{C}\mathscr{C}$, are to be concentrated in Adams grading zero.

If $A_*$ is a partial algebra in a simplicial $k$-module $A$, so that $A_j$ is a simplicial submodule of $A^j$ and the inclusion is a quasi-isomorphism, then the obvious way to try to define a domain $C_{\#}(A_*)_*$ in the chain complex $C_{\#}(A)$ is to set

$$C_{\#}(A_*)_j = g^{-1}(C_{\#}(A_j) \cap g(C_{\#}(A)^j)) = f(C_{\#}(A_j) \cap g(C_{\#}(A)^j)) \subset C_{\#}(A)^j.$$

Thus the following diagram is a pullback, where $g'$ is the restriction of $g$:

$$\begin{array}{ccc} C_{\#}(A_*)_j & \hookrightarrow & C_{\#}(A)^j \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C_{\#}(A_j) & \hookrightarrow & C_{\#}(A^j) \end{array}$$

In general, the top inclusion need not be a quasi-isomorphism. It is a quasi-isomorphism if $f$ restricts to a left inverse $f'$ of $g'$, that is, if

$$f(C_{\#}(A_j)) \subset f(C_{\#}(A_j) \cap g(C_{\#}(A)^j)) = C_{\#}(A)_j,$$

since we then have compatible direct sum decompositions. While one can write down explicit conditions in terms of faces and degeneracies which ensure that these inclusions hold, this approach is not very satisfactory.

Thus we accept that the functor $C_\#$ fails to carry partial algebras of simplicial $k$-modules to partial algebras of chain complexes in general. To get around this, we prove our conversion functors for partial algebras on the simplicial level, as stated in Theorems 1.1 and 1.2, and then apply $C_\#$.

Before getting to this, we briefly consider the generalization of $C_\#$ to a functor $\mathscr{S}\mathscr{C}\mathscr{C} \to \mathscr{C}\mathscr{C}$. This is needed to prove Theorem 1.4 and will also be used in Part V. For a simplicial Adams graded chain complex $X$, let $X_{p,q}(r)$ denote the $k$-module of $p$-simplices of ordinary grading $q$ and Adams grading $r$. Then $C_\# X$ is constructed by letting $(C_\# X)_n(r)$ be the quotient of $\sum X_{p,q}(r)$ by its subgroup of degenerate simplices. The differential on $C_\# X$ is the sum of the simplicial differential $\sum(-1)^i d_i$ and $(-1)^p$ times the internal differential; see [30, pp. 65-68] for details (some of which will be recalled in IV §4).

By [30, A.2], the functor $C_\#$ carries simplicial homotopies of the sort occurring in Example 4.2 to chain homotopies, and a standard spectral sequence argument shows that it carries simplicial quasi-isomorphisms to quasi-isomorphisms. The shuffle product and Alexander-Whitney map are generalized and shown to continue to enjoy all of the properties that we mentioned above in [30, A.3].

Therefore, given any kind of algebraic structure defined in terms of maps $\theta : X_1 \otimes \cdots \otimes X_j \to X$ in $\mathscr{S}\mathscr{C}\mathscr{C}$, we obtain a similar kind of algebraic structure in $\mathscr{C}\mathscr{C}$ by composing the maps $C_\#\theta$ with iterates of $g$. We are interested in simplicial $\mathscr{C}$-algebras and their modules, where $\mathscr{C}$ is an operad of chain complexes. These are the same things as $\underline{\mathscr{C}}$-algebras and their modules, where $\underline{\mathscr{C}}$ is the operad of simplicial chain complexes given by the constant simplicial chain complexes $\underline{\mathscr{C}(j)}$. Clearly, $C_\#(\underline{\mathscr{C}}) = \mathscr{C}$, and it follows that the functor $C_\#$ carries simplicial $\mathscr{C}$-algebras and modules to $\mathscr{C}$-algebras and modules. Of course, this fails on the partial level.

*Proof of Theorem 1.4.* With the hypotheses and notations of Theorem 1.4 and Example 4.4(iii), we define functors $B = C_\# B_*$ and $W = C_\# W_*$ on both $\mathscr{C}$-algebras $A$ and $A$-modules $M$. Noting that $A = C_\# \underline{A}$, we define

$$\psi = C_\#\psi_* : BA \to A \text{ and } \epsilon = C_\#\epsilon_* : BA \to WA.$$

Then $WA$ is a $\mathscr{P}$-algebra and $\psi$ and $\epsilon$ are quasi-isomorphisms and maps of $\mathscr{C}$-algebras. Thus these maps give a natural quasi-isomorphism between the $\mathscr{C}$-algebra $A$ and the $\mathscr{P}$-algebra $WA$. The argument for modules is identical. □

Now consider $\mathscr{S}\mathscr{S}\mathscr{K}$. An object $X = \{X_{p,q}(r)\}$ in this category has a "horizontal" simplicial variable $p$ and a "vertical" simplicial variable $q$, as well as the Adams grading $r$. We again have a total chain complex functor $C_\# : \mathscr{S}\mathscr{S}\mathscr{K} \to \mathscr{C}\mathscr{C}$, and we say that a map $f : X \to Y$ is a quasi-isomorphism if $C_\# f$ is a quasi-isomorphism. More generally, we must consider $\mathscr{S}\mathscr{S}\mathscr{D}$, the category of bisimplicial domains of Adams graded $k$-modules. Such an object $X_*$ consists of inclusions $\delta_j : X_j \to X^j$ of $\Sigma_j$-invariant subobjects, where $\delta_0$ is the identity of $k$, $\delta_1$ is the identity of $X = X_1$, and each of the $\delta_j$ is a quasi-isomorphism.

**Definition 5.1.** The diagonal functor $\Delta : \mathscr{S}\mathscr{S}\mathscr{K} \to \mathscr{S}\mathscr{K}$ sends

$$X = \{(X_{p,q})(r)\} \text{ to } \Delta X = \{(X_{q,q})(r)\},$$

with the diagonal face and degeneracy operations and the obvious Adams grading. Extend $\Delta$ to a functor $\Delta_* : \mathscr{S}\mathscr{S}\mathscr{D} \to \mathscr{S}\mathscr{D}$ by setting $\Delta_j X_* = \Delta X_j$; the required inclusion $\Delta_j X \subset (\Delta X)^j$ is obtained by restriction of the given inclusions $X_j \subset X^j$.

To validate this definition, we need to check that the cited inclusions are equivalences, but this is immediate from the first statement of the following standard result. As usual, the horizontal and vertical simplicial structures of a bisimplicial $k$-module $X$ give rise to corresponding iterated homology groups, and there are spectral sequences that converge from these iterated homology groups to the homology of the total chain complex of $X$.

**Lemma 5.2.** *(*[22, Satz 2.9]*) For a bisimplicial $k$-module $X$, the total chain complex of $X$ is naturally quasi-isomorphic to the chain complex associated to $\Delta X$. Therefore there are spectral sequences converging to $H_*(\Delta X)$ from the vertical homology of the horizontal homology simplicial $k$-module and from the horizontal homology of the vertical homology simplicial $k$-module.*

We are concerned with actions by an operad $\mathscr{C}$ of simplicial $k$-modules. We say that $\mathscr{C}$ is $\Sigma$-free or $\Sigma$-projective if each $\mathscr{C}_q(j)$ is $\Sigma$-free or $\Sigma$-projective, and we say that $\mathscr{C}$ is acyclic if $C_\#\mathscr{C}$ is acyclic. We say that $\mathscr{C}$ is an $E_\infty$ operad if it is $\Sigma$-free and acyclic; $C_\#\mathscr{C}$ is then an $E_\infty$ operad of chain complexes. As observed in I §5, examples arise naturally from operads of topological spaces. We repeat that everything in §§2-4 works precisely as written with "$k$-modules" interpreted as "simplicial $k$-modules".

We think of the given simplicial structure on $\mathscr{C}$ as vertical, and we let $\underline{\mathscr{C}}$ be the associated horizontally constant bisimplicial operad. When the functor $F$ takes values in simplicial $\mathscr{C}$-algebras, $B_*(F, C, A)$ takes values in simplicial simplicial $\mathscr{C}$-algebras, which are the same things as bisimplicial $\underline{\mathscr{C}}$-algebras. Partial actions work similarly. The crucial, if trivial, fact about the functor $\Delta$ is that it manifestly preserves any such operad actions, even partial ones. This makes $\Delta$ a valuable technical substitute for the total chain complex functor.

*Proofs of Theorems 1.1 and 1.2.* With the hypotheses and notations of Theorem 1.1 and Example 4.3, define functors $B = \Delta B_*$ and $V = \Delta V_*$ on both partial $\mathscr{C}$-algebras and their modules. Note that $\Delta \underline{A}_* = A_*$ and define $\psi = \Delta\psi_* : BA_* \to A_*$. While the horizontal homotopy $\eta_*\psi_* \simeq \mathrm{Id}$ does not give rise to a homotopy on application of $\Delta$, it does imply that $\psi_*$ restricts to a horizontal equivalence on each fixed vertical degree, and $\psi$ is therefore a quasi-isomorphism. Define $\delta = \Delta\delta_* : BA_* \to RVA_*$. Since $\delta_*$ restricts to a vertical quasi-isomorphism on each fixed horizontal degree, $\delta$ is a quasi-isomorphism. Since $\psi$ and $\delta$ are maps of partial $\mathscr{C}$-algebras, they define a natural quasi-isomorphism

$$A_* \leftarrow BA_* \to RVA_*$$

between $A_*$ and the genuine $\mathscr{C}$-algebra $VA_*$. Similarly, with the hypotheses and notations of Theorem 1.2 and Example 4.4, define functors $W = \Delta W_*$ on both algebras and modules and define $\epsilon = \Delta\epsilon_* : VA_* \to WA_*$. Since $\epsilon_*$ restricts to a vertical equivalence on each fixed horizontal degree, $\epsilon$ is a quasi-isomorphism. Since $\epsilon$ is a map of $\mathscr{C}$-algebras, it combines with $\psi$ and $\delta$ to define a natural quasi-isomorphism between $A_*$ and the $\mathscr{P}$-algebra $WA_*$. The proofs for modules are identical. $\square$

## 6. Higher Chow complexes

Before getting to our motivating examples, we insert some general remarks about extension of scalars and about the specialization of the arguments just given to partial commutative simplicial $k$-algebras.

*Remark* 6.1. Let $k$ be a subring of $K$ such that $K$ is a flat $k$-module. If $\mathscr{C}$ is an operad over $k$, $A$ is a $\mathscr{C}$-algebra, and $M$ is an $A$-module, then $\mathscr{C}\otimes_k K$ is an operad over $K$, $A\otimes_k K$ is a $(\mathscr{C}\otimes_k K)$-algebra, and $M\otimes_k K$ is an $(A\otimes_k K)$-module. This remains true of partial structures (in view of our flatness hypothesis). All of our monads, hence also our bar constructions, also commute with extension of scalars. Under the varying hypotheses of Theorems 1.1, 1.2, and 1.4, there are natural isomorphisms

$$(BA)\otimes_k K \cong B(A\otimes_k K),$$
$$(VA)\otimes_k K \cong V(A\otimes_k K), \text{ and}$$
$$(WA)\otimes_k K \cong W(A\otimes_k K)$$

that preserve all structure in sight and are compatible with the various natural quasi-isomorphisms that were used in the proofs of the cited results. The same conclusions hold for modules.

*Remark* 6.2. (i) (A shortcut). Consider a partial commutative simplicial $k$-algebra $A_*$, where $k$ is a field of characteristic zero. Then $A_*$ is a partial $\mathscr{N}$-algebra, or, equivalently, an $N_*$-algebra, and we can work directly with $\mathscr{N}$ and its monads to effect the conversion of Theorems 1.1 and 1.2. That is, we set

$$BA_* = \Delta B_*(N_*, N_*, A_*) \text{ and } VA_* = \Delta B_*(NL, N_*, A_*).$$

Then $VA_*$ is a commutative simplicial $k$-algebra, and we have quasi-isomorphisms of $N_*$-algebras

$$\psi = \Delta\psi_* : BA_* \to A_* \text{ and } \delta = \Delta\delta_* : BA_* \to RVA_*.$$

A similar shortcut converts partial $A_*$-modules to genuine $VA_*$-modules. If $\mathscr{C}$ is another acyclic operad, we have compatible quasi-isomorphisms

$$B_*(C_*, C_*, A_*) \to B_*(N_*, N_*, A_*) \text{ and } B_*(NL, C_*, A_*) \to B_*(NL, N_*, A_*).$$

(ii) Now let $A_*$ be a partial commutative simplicial $k$-algebra over a general commutative ring $k$. Here, to effect the conversion of Theorem 1.1, we choose an $E_\infty$ operad $\mathscr{C}$ of simplicial $k$-modules and regard $A_*$ as a partial $\mathscr{C}$-algebra by pullback along the augmentation $\epsilon : \mathscr{C} \to \mathscr{N}$. The point is that $\Sigma$-projectivity is essential to the proof. Thus Theorem 1.1 eliminates the partialness of our structures at the expense of fattening up the operad. When $k = \mathbb{Z}$, we obtain operations on mod $p$ homology by passage to homology from the diagram

$$C_\#(\mathscr{C}(p))\otimes_{\Sigma_p} C_\#(A)^p \to C_\#(\mathscr{C}(p)\otimes_{\Sigma_p} A^p) \leftarrow C_\#(\mathscr{C}(p)\otimes_{\Sigma_p} A_p) \to C_\# A.$$

The left arrow is the shuffle map, and it and the middle arrow induce isomorphisms on mod $p$ homology. The rights arrow is $C_\#(\epsilon\otimes\theta)$. A diagram chase shows that the resulting operations agree with those of the quasi-isomorphic $\mathscr{C}$-algebra $VA_*$, and Theorem I.7.2 is still valid as stated (compare [26]). The only point worth mentioning is that we have added an Adams grading, and the operations $Q^s$ and $\beta Q^s$ carry elements of Adams grading $r$ to elements of Adams grading $pr$.

We now recall the motivating examples, as defined by Bloch [7]. These are partial commutative simplicial rings, with ground ring $k = \mathbb{Z}$.

**Example 6.3.** Let $X$ be a (smooth, quasi-projective) variety over a field $F$. Bloch [7] has defined an Adams graded simplicial Abelian group $\mathfrak{Z}(X)$. Its group $\mathfrak{Z}^r(X,q)$ of $q$-simplices in Adams grading $r$ is free Abelian on the set of those codimension $r$ irreducible subvarieties of $X \times \Delta^q$ which meet all faces properly, where

$$\Delta^q = \mathrm{Spec}(F[t_0, \ldots, t_q]/(\sum t_i - 1)).$$

There is a partially defined intersection product on this graded simplicial Abelian group. In Adams grading $r$ and simplicial degree $q$, the domain $\mathfrak{Z}(X)_j$ of the $j$-fold product is the sum over all partitions $\{r_1, \ldots, r_j\}$ of $r$ of the subgroups of

$$\mathfrak{Z}^{r_1}(X,q) \otimes \cdots \otimes \mathfrak{Z}^{r_j}(X,q)$$

spanned by those $j$-tuples of simplices all intersections of subsets of which meet all faces properly. An "easy" moving lemma of Bloch, implicit in [7] and proven in detail when $X = \mathrm{Spec}(F)$ in [8], gives that the inclusion $\mathfrak{Z}(X)_j \to \mathfrak{Z}(X)^j$ is a quasi-isomorphism. It is evident that the intersection product is commutative, associative, and unital. If $\pi : X \to Y$ is a flat map, we obtain a map $\pi^* : \mathfrak{Z}(Y) \to \mathfrak{Z}(X)$ of Adams graded simplicial Abelian groups by pulling cycles in simplices back along the flat maps $\pi : X \times \Delta^q \to Y \times \Delta^q$. It extends to a map of partial rings. That is, the partial commutative simplicial ring $\mathfrak{Z}(X)_*$ is contravariantly functorial on flat maps. In particular, letting $\mathfrak{Z} = \mathfrak{Z}(\mathrm{Spec}(F))$, we obtain a map $\pi^* : \mathfrak{Z}_* \to \mathfrak{Z}(X)_*$ of partial commutative simplicial rings for any $X$.

The integral higher Chow groups of $X$ are defined by

$$CH^r(X,q) = H_q(\mathfrak{Z}^r(X,*);\mathbb{Z}).$$

By the previous remarks, if we define the mod $p$ Chow groups by taking mod $p$ homology, then these groups admit homology operations just like those familiar in algebraic topology. A harder moving lemma of Bloch, first claimed in [7] and recently proven in [9], implies that

$$CH^r(X,q) \otimes \mathbb{Q} \cong (K_q(X) \otimes \mathbb{Q})^{(r)}.$$

Here the right side is the $n^r$-eigenspace of the Adams operation $\psi^n$ for any $n > 1$ (which is independent of $n$); $K_q(X) \otimes \mathbb{Q}$ is the direct sum of these eigenspaces. Levine [39] has recently given a different proof of this isomorphism that avoids Bloch's hard moving lemma.

As in Remark 6.2(ii), we choose an $E_\infty$ operad $\mathscr{C}$ of simplicial Abelian groups and regard our partial commutative simplicial rings as partial $\mathscr{C}$-algebras. We apply Theorem 1.1 to convert partial $\mathscr{C}$-algebras to quasi-isomorphic genuine $\mathscr{C}$-algebras, still in the category of simplicial Abelian groups. We then apply the functor $C_\#$ to convert to algebras over the associated $E_\infty$ operad $C_\#\mathscr{C}$ of chain complexes. It is these chain complex level structures that really interest us. Recall that we defined an $E_\infty$ algebra to be an algebra over any $E_\infty$ operad of chain complexes, such as $C_\#\mathscr{C}$. Of course, we can proceed in the same way for modules.

**Definition 6.4.** Fix a field $F$ and consider varieties $X$ over $F$.
(i) Let $\mathscr{A}(X)$ be the $E_\infty$ algebra obtained from $\mathfrak{Z}(X)_*$ by applying the functor $C_\# V$. Write $\mathscr{A}$, or $\mathscr{A}/F$, for $\mathscr{A}(\mathrm{Spec}(F))$. Write $\pi^* : \mathscr{A}(Y) \to \mathscr{A}(X)$ for the map of $E_\infty$ algebras induced by a flat map $\pi : X \to Y$; $\mathscr{A}(X)$ is an $\mathscr{A}(Y)$-module via

$\pi^*$. In particular, $\mathscr{A}(X)$ is an $\mathscr{A}$-module for every $X$.
(ii) Let $\mathscr{A}_{\mathbb{Q}}(X)$ be the commutative DGA obtained from $\mathfrak{Z}(X)_*\otimes\mathbb{Q}$ by applying the functor $C_\# W$. Write $\mathscr{A}_{\mathbb{Q}}$, or $\mathscr{A}_{\mathbb{Q}}/F$, for $\mathscr{A}_{\mathbb{Q}}(\mathrm{Spec}(F))$. Write $\pi^* : \mathscr{A}_{\mathbb{Q}}(Y) \to \mathscr{A}_{\mathbb{Q}}(X)$ for the map of DGA's induced by a flat map $\pi : Y \to X$.

**Proposition 6.5.** *For varieties $X$, $\mathscr{A}(X)\otimes\mathbb{Q}$ is an $E_\infty$ algebra, and there is a quasi-isomorphism $\mathscr{A}(X)\otimes\mathbb{Q} \to \mathscr{A}_{\mathbb{Q}}(X)$ of $E_\infty$ algebras.*

*Proof.* By Remark 6.1, $\mathscr{A}(X)\otimes\mathbb{Q} \cong C_\# V(\mathfrak{Z}(X)\otimes\mathbb{Q})$. The map $\epsilon : V \to W$ used in the proof of Theorem 1.2 gives the desired quasi-isomorphism. □

*Remark* 6.6. In order to relate these definitions to the usual cohomology theories in algebraic geometry, it is appropriate to regrade by setting

$$\mathscr{N}^{2r-p}(X)(r) = \mathscr{A}_p(X)(r).$$

With this grading, it is reasonable to define

$$H^i_{\mathrm{Mot}}(X;\mathbb{Q}(r)) \equiv H^i(\mathscr{N}_{\mathbb{Q}}(X))(r). \tag{6.7}$$

Write $\mathscr{N}$, or $\mathscr{N}/F$, for $\mathscr{N}(\mathrm{Spec}(F))$. The $E_\infty$ algebras $\mathscr{N}(X)$ may be viewed as $\mathscr{N}$-modules and thus as objects of the derived category $\mathscr{D}_{\mathscr{N}}$. It is a suggestion of Deligne [20, 17] that this derived category should provide an appropriate site in which to define integral mixed Tate motives. In fact, if one accepts the integral analog of (6.7) as the definition of integral motivic cohomology, one can view $\mathscr{D}_{\mathscr{N}}$ as a "derived category" of integral mixed Tate modules. If there is a good Abelian category of integral mixed Tate motives, it should be an admissible Abelian subcategory [3, 1.2.5] of this triangulated derived category, and it would then necessarily be its heart with respect to a suitable $t$-structure [3, 1.3.13]. However, in order to take this idea seriously, we must first understand such derived categories of modules over $E_\infty$ algebras: that is the subject of Part V.

Most work on mixed Tate motives has concentrated on the rational theory, and our work gives a classical category of derived modules in which to think about the subject. We shall return to consideration of mixed Tate motives in Part IV, after developing a new approach to the study of classical derived categories in Part III.

## Part III. Derived categories from a topological point of view

Let $k$ be a commutative ring and let $A$ be a differential graded associative and unital $k$-algebra (= DGA). As many topologists recognize, there is an extremely close analogy between the derived category $\mathscr{D}_A$ of differential graded $A$-modules and the stable homotopy category of spectra. However, there is no published account of derived categories from this point of view. With the goal of developing an *integral* theory of mixed Tate motives, we shall generalize the derived category $\mathscr{D}_A$ to the case of $E_\infty$ algebras $A$ over $k$ in Part V. Understanding of that more difficult theory requires a prior knowledge of our treatment of the derived categories of ordinary DGA's. This elementary theory is adequate and illuminating for one approach to *rational* mixed Tate motives ([20, 17, 6]), as we shall show in Part IV.

Therefore, with the hope that our way of thinking about derived categories will prove useful to others, we here give a topologically motivated, although purely algebraic, exposition of the classical derived categories of DGA's. These categories admit remarkably simple and explicit descriptions in terms of "cell modules", which are the precise algebraic analogs of cell spectra. Such familiar topological results as

Whitehead's theorem and Brown's representability theorem transcribe directly into algebra. There is also a theory of CW modules, but these are less useful (at least in our motivating examples), due to the limitations of the cellular approximation theorem. Derived tensor products and Hom functors, together with differential Tor and Ext functors and Eilenberg-Moore (or hyperhomology) spectral sequences for their computation, drop out quite easily.

Our methods can be abstracted and applied more generally, and some of what we do can be formalized in Quillen's context of closed model categories [54]. We prefer to be more concrete and less formal. We repeat that many topologists have long known some of this material. For example, although the emphasis is quite different, our work overlaps that of [30] and [2]. On a technical note, we emphasize that, as in [30], $k$ is an arbitrary commutative ring and we nowhere impose boundedness or flatness hypotheses.

## 1. Cell $A$-modules

Motivated by the motivic context, we take all $k$-modules $X$ to be $\mathbb{Z}$-bigraded, with gradings written $X = \{X^q(r)\}$. We call $q$ the ordinary grading or degree and $r$ the Adams grading or degree. We assume given a differential $d : X^q(r) \to X^{q+1}(r)$. Thus $X$ is really an "Adams graded differential graded $k$-module". By convention, the grading does and the Adams grading does not introduce signs under permutations. The reader with other motivations may prefer to forget the Adams grading and to regrade homologically, setting $X_q = X^{-q}$; this makes the analogy with topology far more transparent. Except where otherwise specified, a map $f : X \to Y$ of $k$-modules means a map of bidegree $(0,0)$ that commutes with the differentials; $f$ is a quasi-isomorphism if it induces an isomorphism on homology.

We sometimes write $x \in (q,r)$ to indicate that an element of some module is of bidegree $(q,r)$. We begin with some utterly trivial notions, expressed so as to show the analogy with topology. Let $I$ denote the "unit interval $k$-module". It is free on generators $[0] \in (0,0)$, $[1] \in (0,0)$, and $[I] \in (-1,0)$, with $d[I] = [0] - [1]$. A homotopy is a map $X \otimes I \to Y$, where $\otimes$ means $\otimes_k$. Of course,

$$(X \otimes Y)^q(r) = \bigoplus_{m+n=q,s+t=r} X^m(s) \otimes Y^n(t),$$

with $d(x \otimes y) = dx \otimes y + (-1)^{\deg(x)} x \otimes dy$. The cone $CX$ is the quotient module $X \otimes (I/k[1])$ and the suspension $\Sigma X$ is $X \otimes (I/\partial I)$, where $\partial I$ has basis $[0]$ and $[1]$. Additively, $CX$ is the sum of copies of $X$ and $\Sigma X$, but with differential arranged so that $H^*(CX) = 0$. The usual algebraic notation for the suspension is $\Sigma X = X[1]$, and$(\Sigma X)^q = X^{q+1}$. Since we have tensored the interval coordinate on the right, the differential on $\Sigma X$ is the same as the differential on $X$, without the introduction of a sign.

The cofiber of a map $f : X \to Y$ is the pushout of $f$ along the inclusion $X = X \otimes [0] \to CX$. There results a short exact sequence

$$0 \to Y \to Cf \to \Sigma X \to 0.$$

Up to sign, the connecting homomorphism of the resulting long exact homology sequence is $f^*$. Explicitly, $(Cf)^q = Y^q \otimes X^{q+1}$, with differential

$$d(y,x) = (dy + (-1)^q fx, dx).$$

The sequence

$$X \to Y \to Cf \to \Sigma X$$

is called a cofiber sequence, or an exact triangle.

Now assume given a DGA $A$ over $k$; $A$ is to be associative and unital, but not necessarily commutative, and $A$-modules will usually mean left $A$-modules. If $X$ is a $k$-module and $M$ is an $A$-module, then $M \otimes X$ is an $A$-module, hence the notion of a homotopy between maps of $A$-modules is defined. Since we defined cofiber sequences in terms of tensoring with $k$-modules, the cofiber sequence generated by a map of $A$-modules is clearly a sequence of $A$-modules. Let $\mathscr{M}_A$ denote the category of $A$-modules and $h\mathscr{M}_A$ its homotopy category. Then the derived category $\mathscr{D}_A$ is obtained from $h\mathscr{M}_A$ by adjoining formal inverses to the quasi-isomorphisms of $A$-modules. In Construction 2.7, we shall give an explicit description that makes it clear that there are no set theoretic difficulties. (This point is typically ignored in algebraic geometry and obviated by concrete construction in algebraic topology.)

The sequences isomorphic to cofiber sequences in the respective categories give $h\mathscr{M}_A$ and $\mathscr{D}_A$ classes of exact triangles with respect to which they become triangulated categories in the sense of Verdier [60]. More precisely, they become so after the introduction of graded maps or rather, in our context, bigraded maps. A map of bidegree $(s,t)$ consists of maps $M^q(r) \to N^{q+s}(r+t)$ that commute with the differentials and $A$-actions. Such maps can be thought of as maps $M \to \Sigma^s(t)N$ of bidegree $(0,0)$, where the suspension functor $\Sigma^s(t)$ is specified by

$$(\Sigma^s(t)M)^q(r) = M^{q+s}(r+t),$$

with differential and $A$-action inherited from $M$. Since we have allowed ourselves $\mathbb{Z}$-bigrading, each such functor is an automorphism of $\mathscr{M}_A$, and the introduction of bigraded morphisms is in principle a notational device that can add nothing of substance to the mathematics. It becomes crucial when we define Hom modules of bigraded morphisms, but until then it is convenient to think solely in terms of maps of bidegree $(0,0)$.

It is also convenient to think of the suspension functors in a different way. Let $S^s(t)$ be the free $k$-module generated by a cycle $i^s(t) \in (s,t)$. Then our suspension functors are just

$$\Sigma^s(t)M = M \otimes S^s(t).$$

We think of the $S^s(t)$ as sphere $k$-modules. We let $F^s(t) = A \otimes S^s(t)$ and think of the $F^s(t)$ as sphere $A$-modules; they are free on the generating cycles $i^s(t)$. Since $s$ and $t$ run through $\mathbb{Z}$, the analogy is with stable homotopy theory: that is where negative dimensional spheres live.

In fact, the modern description of the stable homotopy category [40] translates directly into our new description of the derived category. (The preamble of [40] explains the relationship with earlier treatments of the stable homotopy category, which do not have the same flavor.) In brief, one sets up a category of spectra. In that category, one defines a theory of cell and CW spectra that allows negative dimensional spheres. One shows that a weak homotopy equivalence between cell spectra is a homotopy equivalence and that every spectrum is weakly homotopy equivalent to a cell spectrum. The stable homotopy category is obtained from the homotopy category of spectra by formally inverting the weak homotopy equivalences, and it is described more concretely as the homotopy category of cell spectra.

With spectra and weak homotopy equivalences replaced by $A$-modules and quasi-isomorphisms, precisely the same pattern works algebraically—but of course far more simply.

**Definitions 1.1.** (i) A cell $A$-module $M$ is the union of an expanding sequence of sub $A$-modules $M_n$ such that $M_0 = 0$ and $M_{n+1}$ is the cofiber of a map $\phi_n : F_n \to M_n$, where $F_n$ is a direct sum of sphere modules $F^s(t)$ (of varying bidegrees). The restriction of $\phi_n$ to a summand $F^s(t)$ is called an attaching map and is determined by the "attaching cycle" $\phi_n(i^s(t))$. An attaching map $F^s(t) \to M_n$ induces a map

$$CF^s(t) = A \otimes CS^s(t) \to M_{n+1} \subset M,$$

and such a map is called an $(s-1,t)$-cell. Thus $M_{n+1}$ is obtained from $M_n$ by adding a copy of $F^{s-1}(t)$ for each attaching map with domain $F^s(t)$, but giving the new generators $j^{s-1}(t) = i^s(t) \otimes [I]$ the differentials

$$d(j^{s-1}(t)) = (-1)^s \phi_n(i^s(t)).$$

We call such a copy of $F^{s-1}(t)$ in $M$ an open cell; if we ignore the differential, then $M$ is the direct sum of its open cells.
(ii) A map $f : M \to N$ between cell $A$-modules is cellular if $f(M_n) \subset N_n$ for all $n$.
(iii) A submodule $L$ of a cell $A$-module $M$ is a cell submodule if $L$ is a cell $A$-module such that $L_n \subset M_n$ and the composite of each attaching map $F^s(t) \to L_n$ of $L$ with the inclusion $L_n \to M_n$ is an attaching map of $M$. Thus every cell of $L$ is a cell of $M$.

We call $\{M_n\}$ the sequential filtration of $M$. It is essential for inductive arguments, but it should be regarded as flexible and subject to change whenever convenient. It merely records the order in which cells are attached and, as long as the cycles to which attachment are made are already present, it doesn't matter when we attach cells.

**Lemma 1.2.** *Let $f : M \to N$ be an $A$-map between cell $A$-modules. Then $M$ admits a new sequential filtration with respect to which $f$ is cellular.*

*Proof.* Assume inductively that $M_n$ has been filtered as a cell $A$-module $M_n = \cup M'_q$ such that $f(M'_q) \subset N_q$ for all $q$. Let $x \in M_n$ be an attaching cycle for the construction of $M_{n+1}$ from $M_n$ and let $\chi : CF^s(t) \to M_{n+1}$ be the corresponding cell. Let $q$ be minimal such that both $x \in M'_q$ and $f \circ \chi$ has image in $N_{q+1}$. Extend the filtration of $M_n$ to $M_{n+1}$ by taking $x$ to be a typical attaching cycle of a cell $CF^s(t) \to M'_{q+1}$. □

From a topological point of view, our cohomological grading has the effect that we are looking at things upside down: the bottom summand of a cone $CF^s(t)$ is the one that involves the unit interval. That may help explain the intuition behind the following definition.

**Definition 1.3.** The dimension of a cell $CF^s(t) \to M_{n+1}$ is $s-1$. A cell $A$-module $M$ is said to be a CW $A$-module if each cell is attached only to cells of higher dimension, in the sense that the defining cycles $\phi_n(i^s(t))$ are elements in the sum of the images of cells of dimension at least $s$. The $n$-skeleton $M^n$ of a CW $A$-module is the sum of the images of its cells of dimension at least $n$, so that $M^n \subset M^{n-1}$. We require of cellular maps $f : M \to N$ between CW $A$-modules that they be "bicellular", in the sense that both $f(M^n) \subset N^n$ and $f(M_n) \subset N_n$ for all $n$. By

Lemma 1.2, the latter condition can be arranged by changing the order in which the cells of $M$ are attached.

**Definition 1.4.** A cell $A$-module is finite dimensional if it has cells in finitely many dimensions. It is finite if it has finitely many cells.

Just as finite cell spectra are central to the topological theory, so finite cell $A$-modules are central here, especially when we restrict to commutative DGA's and discuss duality. The collection of cell $A$-modules enjoys the following closure properties, which imply many others.

**Proposition 1.5.** *(i) A direct sum of cell $A$-modules is a cell $A$-module.*
*(ii) If $L$ is a cell submodule of a cell $A$-module $M$, $N$ is a cell $A$-module, and $f : L \to M$ is a cellular map, then the pushout $N \cup_f M$ is a cell $A$-module with sequential filtration $\{N_n \cup_f M_n\}$. It contains $N$ as a cell submodule and has one cell for each cell of $M$ not in $L$.*
*(iii) If $L$ is a cell submodule of a cell $A$-module $M$ and $X$ is a cell submodule of a cell $k$-module $Y$, then $M \otimes Y$ is a cell $A$-module with sequential filtration $\{\sum_p (M_p \otimes Y_{n-p})\}$. It contains $L \otimes Y + M \otimes X$ as a cell submodule and has a $(q+s, r+t)$-cell for each pair consisting of a $(q,r)$-cell of $M_p$ and an $(s,t)$-cell of $Y_{n-p}$, $0 \le p \le n$.*
*(iv) The mapping cylinder $Mf = N \cup_f (L \otimes I)$ of $f : L \to N$ is the pushout defined by taking $L = L \otimes k[0] \subset L \otimes I$. If $f$ is a cellular map between cell $A$-modules, then $Mf$ is a cell $A$-module, $L = L \otimes k[1]$ is a cell submodule, the inclusion $N \to Mf$ is a homotopy equivalence, and $Cf = Mf/L$.*

*Proof.* Parts (i) and (ii) are easy and (iv) follows from (ii) and (iii). For (iii), observe that there are evident canonical isomorphisms

$$S^q(r) \otimes S^s(t) \cong S^{q+s}(r+t) \text{ and } F^q(r) \otimes S^s(t) \cong F^{q+s}(r+t).$$

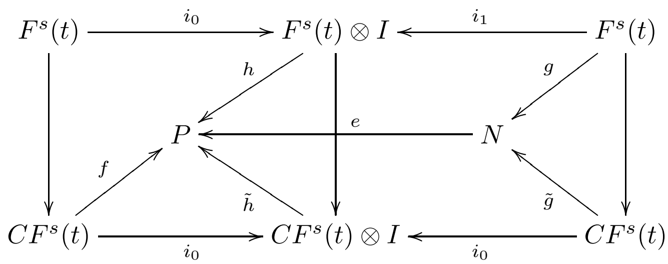
$M \otimes Y$ has an open cell $F^{q+s}(r+t)$ for each open cell $F^q(r)$ of $M$ and $S^s(t)$ of $Y$; the differential on its canonical basis element is the cycle

$$d(j^q(r)) \otimes j^s(t) + (-1)^q (j^q(r)) \otimes d(j^s(t)). \quad \square$$

## 2. Whitehead's theorem and the derived category

A quick space level version of some of the results of this section may be found in [50], and the spectrum level model is given in [40, I §5]. We construct the derived category explicitly in terms of cell modules. As in topology, the "homotopy extension and lifting property" is pivotal. It is a direct consequence of the following trivial observation. Let $i_0$ and $i_1$ be the evident inclusions of $M$ in $M \otimes I$.

**Lemma 2.1.** *Let $e : N \to P$ be a map such that $e^* : H^*(N) \to H^*(P)$ is a monomorphism in degree $s$ and an epimorphism in degree $s-1$. Then, given maps $f$, $g$, and $h$ such that $f|F^s(t) = hi_0$ and $eg = hi_1$ in the following diagram, there* *are maps $\tilde{g}$ and $\tilde{h}$ that make the entire diagram commute.*



*Proof.* Let $i = i^s(t) \otimes [0]$ and $j = i^s(t) \otimes [I]$ be the basis elements of $CF^s(t)$, so that $d(j) = (-1)^s i$. Then $eg(i) = h(i \otimes [1])$ and $f(i) = h(i \otimes [0])$, hence

$$d(h(i \otimes [I]) - f(j)) = (-1)^{s+1} eg(i).$$

Since $eg(i)$ bounds in $P$, $g(i)$ must bound in $N$, say $d(n') = g(i)$. Then

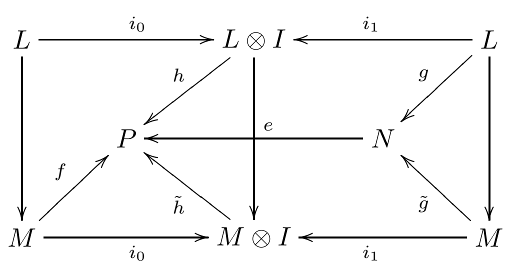
$$p \equiv e(n') + (-1)^s (h(i \otimes [I]) - f(j))$$

is a cycle. There must be a cycle $n \in N$ and a chain $q \in P$ such that

$$d(q) = p - e(n).$$

Define $\tilde{g}(j) = (-1)^s (n' - n)$ and $\tilde{h}(j \otimes [I]) = q$. □

**Theorem 2.2** (HELP)**.** *Let $L$ be a cell submodule of a cell $A$-module $M$ and let $e : N \to P$ be a quasi-isomorphism of $A$-modules. Then, given maps $f : M \to P$, $g : L \to N$, and $h : L \otimes I \to P$ such that $f|L = hi_0$ and $eg = hi_1$ in the following diagram, there are maps $\tilde{g}$ and $\tilde{h}$ that make the entire diagram commute.*



*Proof.* By induction up the filtration $\{M_n\}$ and pullback along cells not in $L$, this quickly reduces to the case $(M, L) = (CF^s(t), F^s(t))$ of the lemma. □

For objects $M$ and $N$ of any category Cat, let $\mathrm{Cat}(M, N)$ denote the set of morphisms in Cat from $M$ to $N$.

**Theorem 2.3** (Whitehead)**.** *If $M$ is a cell $A$-module and $e : N \to P$ is a quasi-isomorphism of $A$-modules, then $e_* : h\mathscr{M}_A(M, N) \to h\mathscr{M}_A(M, P)$ is an isomorphism. Therefore a quasi-isomorphism between cell $A$-modules is a homotopy equivalence.*

*Proof.* Take $L = 0$ in HELP to see the surjectivity. Replace $(M, L)$ by the pair $(M \otimes I, M \otimes (\partial I))$ to see the injectivity. When $N$ and $P$ are cell $A$-modules, we may take $M = P$ and obtain a homotopy inverse $f : P \to N$. □

**Theorem 2.4** (Cellular approximation)**.** *Let $L$ be a cell submodule of a CW $A$-module $M$, let $N$ be a CW $A$-module such that $H^s(N/N^s) = 0$ for all $s$, and let $f : M \to N$ be a map whose restriction to $L$ is cellular. Then $f$ is homotopic relative to $L$ to a cellular map. Therefore any map $M \to N$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

*Proof.* By Lemma 1.2, we may change the sequential filtration of $M$ to one for which $f$ is sequentially cellular. Proceeding by induction up the filtration $\{M_n\}$, we construct compatible cellular maps $g_n : M_n \to N_n$ and a homotopy $h_n : M_n \otimes I \to N_n$ from $f|M_n$ to $g_n$. The result quickly reduces to the case of a single cell of $M$ that is not in $L$ and thus to the case when $(M, L) = (CF^s(t), F^s(t))$. The conclusion follows by application of Lemma 2.1 to the inclusions $e : (N_n)^{s-1} \to N_n$. □

*Remark* 2.5. If $H^s(A) = 0$ for all $s > 0$, then the hypothesis holds for all $N$, and we can work throughout with CW $A$-modules and cellular maps rather than with cell $A$-modules. Of course, if we regrade homologically, then this means that $H_s(A) = 0$ for $s < 0$, which matches the intuition: CW theory works topologically because the homotopy groups of the zero sphere spectrum are zero in negative degrees.

**Theorem 2.6** (Approximation by cell modules)**.** *For any $A$-module $M$, there is a cell $A$-module $N$ and a quasi-isomorphism $e : N \to M$.*

*Proof.* We construct an expanding sequence $N_n$ and compatible maps $e_n : N_n \to M$ inductively. Choose a cycle $\nu \in (q, r)$ in each homology class of $M$, let $N_1$ be the direct sum of $A$-modules $F^q(r)$, one for each $\nu$, and let $e_1 : N_1 \to M$ send the $\nu$th canonical basis element to the cycle $\nu$. Inductively, suppose that $e_n : N_n \to M$ has been constructed. Choose a pair of cycles $(\nu, \nu')$ in each pair of unequal homology classes on $N_n$ that map under $(e_n)^*$ to the same element of $H^*(M)$. Let $N_{n+1}$ be the "homotopy coequalizer" obtained by adjoining a copy of $F^q(r) \otimes I$ to $N_n$ along the evident map $F^q(r) \otimes \partial I \to N_n$ determined by each such pair $(\nu, \nu') \in (q, r)$. Proposition 1.5 implies that $N_{n+1}$ is a cell $A$-module such that $N_n$ is a cell submodule. Any choice of chains $\mu \in M$ such that $d(\mu) = \nu - \nu'$ determines an extension of $e_n : N_n \to M$ to $e_{n+1} : N_{n+1} \to M$. Let $N$ be the direct limit of the $N_n$ and $e : N \to M$ be the resulting map. Clearly, $N$ is a cell module, $e$ induces an epimorphism on homology since $e_1$ does, and $e$ induces a monomorphism on homology by construction. □

**Construction 2.7.** For each $A$-module $M$, choose a cell $A$-module $\Gamma M$ and a quasi-isomorphism $\gamma : \Gamma M \to M$. By the Whitehead theorem, for a map $f : M \to N$, there is a map $\Gamma f : \Gamma M \to \Gamma N$, unique up to homotopy, such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} \Gamma M & \xrightarrow{\Gamma f} & \Gamma N \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ M & \xrightarrow[f]{} & N \end{array}$$

Thus $\Gamma$ is a functor $h\mathscr{M}_A \to h\mathscr{M}_A$, and $\gamma$ is natural. The derived category $\mathscr{D}_A$ can be described as the category whose objects are the $A$-modules and whose morphisms are specified by

$$\mathscr{D}_A(M, N) = h\mathscr{M}_A(\Gamma M, \Gamma N),$$

with the evident composition. When $M$ is a cell $A$-module,

$$\mathscr{D}_A(M,N) \cong h\mathscr{M}_A(M,N).$$

Using the identity function on objects and $\Gamma$ on morphisms, we obtain a functor $i : h\mathscr{M}_A \to \mathscr{D}_A$ that sends quasi-isomorphisms to isomorphisms and is universal with this property. Let $\mathscr{C}_A$ be the full subcategory of $\mathscr{M}_A$ whose objects are the cell $A$-modules. Then the functor $\Gamma$ induces an equivalence of categories $\mathscr{D}_A \to h\mathscr{C}_A$ with inverse the composite of $i$ and the inclusion of $h\mathscr{C}_A$ in $h\mathscr{M}_A$.

Therefore the derived category and the homotopy category of cell modules can be used interchangeably. Homotopy-preserving functors on $A$-modules that do not preserve quasi-isomorphisms are transported to the derived category by first applying $\Gamma$, then the given functor, a point that we return to in Section 4. Much more is made of this simple procedure in the algebraic than the topological literature: topologists routinely transport constructions to the stable category by passing to CW spectra, without change of notation. In fact, while a great deal of modern work depends heavily on having a good underlying category of spectra, earlier constructions of the stable homotopy category did not even allow spectra that were more general than CW spectra. For this and other reasons, topologists are accustomed to work with CW spectra and their cells in a concrete calculational way, not as something esoteric but rather as something much more basic and down to earth than general spectra. An analogous view of differential graded $A$-modules is rather intriguing.

## 3. Brown's representability theorem

Functors of cohomological type on $\mathscr{D}_A$ are of considerable interest, and we here recall a categorical result that characterizes when they can be represented in the form $\mathscr{D}(?,N)$. The topological analogue has long played an important role.

We have said that we think of the $F^q(r)$ as analogs of sphere spectra. Just as maps out of spheres calculate homotopy groups and therefore detect weak equivalences, so maps out of the $F^q(r)$ calculate homology groups and therefore detect quasi-isomorphisms. We display several versions of this fact for later use: for all $A$-modules $N$,

$$\text{(3.1)}\quad H^q(N)(r) \cong h\mathscr{M}_k(k, N\otimes S^q(r)) \cong h\mathscr{M}_k(S^{-q}(-r),N) \cong h\mathscr{M}_A(F^{-q}(-r),N) \cong \mathscr{D}_A(F^{-q}(-r),N).$$

The category $\mathscr{D}_A$ has "homotopy limits and colimits". These are weak limits and colimits in the sense that they satisfy the existence but not the uniqueness property of categorical limits and colimits. For example, the homotopy pushout of maps $f : L \to M$ and $g : L \to N$ is obtained from $M \oplus (L\otimes I) \oplus N$ by identifying $l\otimes[0]$ with $f(l)$ and $l\otimes[1]$ with $g(l)$. More precisely, we first apply cell approximation and then apply the cited construction. We used a similar homotopy coequalizer in the proof of Theorem 2.6. The homotopy colimit, or telescope $\mathrm{Tel}M_i$, of a sequence of maps $f_i : M_i \to M_{i+1}$ is the homotopy coequalizer of Id: $\oplus M_i \to \oplus M_i$ and $\oplus f_i : \oplus M_i \to \oplus M_i$; equivalently, it is the cofiber of $g : \oplus M_i \to \oplus M_i$, where $g(m) = m - f_i(m)$ for $m \in M_i$. We now have enough information to quote the categorical form of Brown's representability theorem given in [13], but we prefer to run through a quick concrete version of the proof.

**Theorem 3.2** (Brown). *A contravariant functor $J : \mathscr{D}_A \to$ Sets is representable in the form $J(M) \cong \mathscr{D}_A(M, N)$ for some $A$-module $N$ if and only if $J$ converts direct sums to direct products and converts homotopy pushouts to weak pullbacks.*

*Proof.* Necessity is obvious. Thus assume given a functor $J$ that satisfies the specified direct sum and Mayer-Vietoris axioms. Since homotopy coequalizers and telescopes can be constructed from sums and homotopy pushouts, $J$ converts homotopy coequalizers to weak equalizers and telescopes to weak limits. Write $f^* = J(f)$ for a map $f$. Consider pairs $(M, \mu)$ where $M$ is an $A$-module and $\mu \in J(M)$.

Starting with an arbitrary pair $(N_0, \nu_0)$, we construct a sequence of pairs $(N_i, \nu_i)$ and maps $f_i : N_i \to N_{i+1}$ such that $f_i^*(\nu_{i+1}) = \nu_i$. Let $N_1 = N_0 \oplus (\oplus F^q(r))$, where there is a copy of $F^q(r)$ for each element $\phi$ of each set $J(F^q(r))$. Let $\nu_1$ have coordinates $\nu$ and the elements $\phi$, and let $f_0 : N_0 \to N_1$ be the inclusion. Inductively, given $(N_i, \nu_i)$, let $L_i$ be the sum of a copy of $F^q(r)$ for each $(q, r)$ and each unequal pair $(x, y)$ of elements of $H^q(N_i)(r)$ such that, when thought of as maps $F^q(r) \to N_i$ in $\mathscr{D}_A$, $x^*(\nu_i) = y^*(\nu_i)$. Let $f_i : N_i \to N_{i+1}$ be the coequalizer of the pair of maps $L_i \to N_i$ given by the $x$'s and the $y$'s. By the weak equalizer property, there is an element $\nu_{i+1} \in J(N_{i+1})$ such that $f_i^*(\nu_{i+1}) = \nu_i$.

Let $N = \text{Tel}\, N_i$. By the weak limit property, there is an element $\nu \in J(N)$ that pulls back to $\nu_i$ for each $i$. For an $A$-module $M$, define $\theta_\nu : \mathscr{D}_A(M, N) \to J(M)$ by $\theta_\nu(f) = f^*(\nu)$. Then, by construction, $\theta_\nu$ is a bijection for all $F^q(r)$. We claim that $\theta_\nu$ is a bijection for all $M$.

Suppose given elements $x, y \in \mathscr{D}_A(M, N)$ such that $\theta_\nu(x) = \theta_\nu(y)$. Replacing $M$ by a cell approximation if necessary, we can assume that $x$ and $y$ are given by maps $M \to N$. Let $c : N \to N_0'$ be the homotopy coequalizer of $x$ and $y$ and choose an element $\nu_0' \in J(N_0')$ such that $c^*(\nu_0') = \nu$. Construct a pair $(N', \nu')$ by repeating the construction above, but starting with the pair $(N_0', \nu_0')$. Let $j : N_0' \to N'$ be the evident map such that $j^*(\nu') = \nu_0'$. Then, since $(jc)^*(\nu') = \nu$ and both $\theta_\nu$ and $\theta_{\nu'}$ are bijections for all $F^q(r)$, $jc : N \to N'$ is an isomorphism in $\mathscr{D}_A$. Since $cx = cy$ by construction, it follows that $x = y$. Therefore $\theta_\nu$ is an injection for all $A$-modules $M$.

Finally, let $\omega \in J(M)$ for any module $M$. Repeat the construction, starting with the zeroth pair $(M \oplus N, (\omega, \nu))$. We obtain a new pair $(N', \nu')$ together with a map $i : M \to N'$ such that $i^*(\nu') = \omega$ and a map $j : N \to N'$ such that $j^*(\nu') = \nu$. Again, $j$ is an isomorphism in $\mathscr{D}_A$ since both $\theta_\nu$ and $\theta_{\nu'}$ are bijections for all $F^q(r)$. Therefore $\omega = (ij^{-1})^*(\nu)$ and $\theta_\nu$ is a surjection for all $A$-modules $M$. □

Observe that we can start with $N_0 = 0$, in which case $N$ can be given the structure of a cell $A$-module. Of course, it is formal that the module $N$ that represents $J$ is unique up to isomorphism in $\mathscr{D}_A$ and that natural transformations between representable functors are represented by maps in $\mathscr{D}_A$.

There is an analog due to Adams that applies when the functor $J$ is only given on finite cell $A$-modules. The proof is a direct translation from topology to algebra of that given in [1] and will be omitted.

**Theorem 3.3** (Adams). *A contravariant group-valued functor $J$ defined on the homotopy category of finite cell $A$-modules is representable in the form $J(M) \cong \mathscr{D}_A(M, N)$ for some cell $A$-module $N$ if and only if $J$ converts finite direct sums to direct products and converts homotopy pushouts to weak pullbacks of underlying sets.*

Here $N$ is usually infinite and is unique only up to non-canonical isomorphism. More precisely, maps $g, g' : N \to N'$ are said to be weakly homotopic if $gf$ is homotopic to $g'f$ for any map $f : M \to N$ defined on a finite cell $A$-module $M$. There is a resulting weak homotopy category of cell $A$-modules, and $N$ is unique up to isomorphism in that category.

## 4. Derived tensor product and Hom functors: Tor and Ext

We first record some elementary facts about tensor products with cell $A$-modules.

**Lemma 4.1.** *Let $N$ be a cell $A$-module. Then the functor $M \otimes_A N$ preserves exact sequences and quasi-isomorphisms in the variable $M$.*

*Proof.* With differential ignored, $N$ is a free $A$-module, and preservation of exact sequences follows. The sequential filtration of $N$ gives short exact sequences of free $A$-modules

$$0 \longrightarrow N_n \longrightarrow N_{n+1} \longrightarrow N_{n+1}/N_n \longrightarrow 0,$$

where the subquotients $N_{n+1}/N_n$ are direct sums of sphere $A$-modules. The preservation of quasi-isomorphisms holds trivially if $N$ is a sphere $A$-module, and the general case follows by passage to direct sums, induction up the filtration, and passage to colimits. □

It is usual to define the derived tensor product, denoted $M \otimes^L_A N$, by replacing the left $A$-module $N$ (or the right $A$-module $M$) by a suitable resolution $P$ and taking the ordinary tensor product $M \otimes_A P$, in line with the standard rubric of derived functors (see e.g. Verdier [60], who restricts to bounded below modules). Our procedure is the same, except that we take approximation by quasi-isomorphic cell $A$-modules as our version of resolution and, following the pedantically imprecise tradition in topology, we prefer not to change notation. That is, in $\mathscr{D}_k$, $M \otimes_A N$ means $M \otimes_A \Gamma N$. The lemma shows that the definition makes sense. We leave it as an exercise to verify that this definition of the derived tensor product agrees with the usual one. (For example, one might use Theorem 4.13 below.) We can also use the lemma to show that the derived category $\mathscr{D}_A$ depends only on the quasi-isomorphism type of $A$.

**Proposition 4.2.** *Let $\phi : A \to A'$ be a quasi-isomorphism of DGA's. Then the pullback functor $\phi^* : \mathscr{D}_{A'} \to \mathscr{D}_A$ is an equivalence of categories with inverse given by the extension of scalars functor $A' \otimes_A (?)$.*

*Proof.* For $M \in \mathscr{M}_A$ and $M' \in \mathscr{M}_{A'}$, we have

$$\mathscr{M}_{A'}(A' \otimes_A M, M') \cong \mathscr{M}_A(M, \phi^* M').$$

The functor $A' \otimes_A (?)$ preserves sphere modules and therefore cell modules. This implies formally that the adjuction passes to derived categories, giving

$$\mathscr{D}_{A'}(A' \otimes_A M, M') \cong \mathscr{M}_A(M, \phi^* M').$$

If $M$ is a cell $A$-module, then

$$\phi \otimes \mathrm{Id} : M \cong A \otimes_A M \longrightarrow \phi^*(A' \otimes_A M)$$

is a quasi-isomorphism of $A$-modules. These maps give the unit of the adjunction. Its counit is given by the maps of $A'$-modules

$$\mathrm{Id} \otimes_\phi \gamma : A' \otimes_A \Gamma M' \longrightarrow A' \otimes_{A'} M' \cong M',$$

where $\Gamma M'$ is a cell $A$-module and $\gamma : \Gamma M' \longrightarrow M'$ is a quasi-isomorphism of $A$-modules. Since the composite of this map with the quasi-isomorphism $\phi \otimes \mathrm{Id}$ for the $A$-module $\Gamma M'$ coincides with $\gamma$, this map too is a quasi-isomorphism. □

For left $A$-modules $M$ and $N$, let $\mathrm{Hom}_A(M,N)^q(r)$ be the $k$-module of homomorphisms of $A$-modules of bidegree $(q,r)$ with the standard differential $(df)(m) = d(f(m)) - (-1)^q f(d(m))$. For $k$-modules $L$,

$$\mathscr{M}_A(L \otimes M, N) \cong \mathscr{M}_k(L, \mathrm{Hom}_A(M,N)), \tag{4.3}$$

where $A$ acts on $L \otimes M$ through its action on $M$ (with the usual sign convention). This isomorphism clearly passes to homotopy categories. Letting $L$ run through the sphere $k$-modules and using (3.1) and the Whitehead theorem, we see that if $M$ is a cell $A$-module then the functor $\mathrm{Hom}_A(M,N)$ preserves quasi-isomorphisms in $N$.

This allows us to define $\mathrm{Hom}_A(M,N)$ in $\mathscr{D}_A$ for arbitrary modules $M$ and $N$ by first replacing $M$ by a cell approximation $\Gamma M$ and then taking $\mathrm{Hom}_A(\Gamma M, N)$ on the level of modules. Thus, in $\mathscr{D}_k$, $\mathrm{Hom}_A(M,N)$ means $\mathrm{Hom}_A(\Gamma M, N)$. This gives a well-defined functor such that

$$\mathscr{D}_A(L \otimes M, N) \cong \mathscr{D}_k(L, \mathrm{Hom}_A(M,N)). \tag{4.4}$$

*Remark* 4.5. The argument we have just run through is a special case of a general one. If $S$ and $T$ are left and right adjoint functors between two categories of the sort that we are considering, then $S$ preserves objects of the homotopy type of cell modules if and only $T$ preserves quasi-isomorphisms, and in that case the resulting induced functors on derived categories are still adjoint. See [40, I.5.13] for a precise categorical statement.

We can now define differential Tor and Ext (or hyperhomology and hypercohomology) groups as follows. We cheerfully ignore questions of justification in terms of standard homological terms: these are of little interest to us, and such language would be unavailable in the precisely analogous $E_\infty$ context of Part V (let alone the topological context of [25]).

**Definition 4.6.** Working in derived categories, define

$$\mathrm{Tor}^*_A(M,N) = H^*(M \otimes_A N) \text{ and } \mathrm{Ext}^*_A(M,N) = H^*(\mathrm{Hom}_A(M,N)).$$

These are Adams graded $k$-modules (with notation for the Adams grading suppressed). However Tor and Ext are defined, the essential point is to have Eilenberg-Moore, or hyperhomology, spectral sequences for their calculation.

**Theorem 4.7.** *There are natural spectral sequences of the form*

$$E_2^{p,q} = \mathit{Tor}^{p,q}_{H^*A}(H^*M, H^*N) \Longrightarrow \mathit{Tor}^{p+q}_A(M,N) \tag{4.8}$$

*and*

$$E_2^{p,q} = \mathit{Ext}^{p,q}_{H^*A}(H^*M, H^*N) \Longrightarrow \mathit{Ext}^{p+q}_A(M,N). \tag{4.9}$$

These are both spectral sequences of cohomological type, with

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}. \tag{4.10}$$

In (4.8), $p$ is the negative of the usual homological degree, the spectral sequence is non-zero only in the left half-plane, and it converges strongly. In (4.9), the spectral sequence is non-zero only in the right half plane, and it converges strongly if, for each fixed $(p, q)$, only finitely many of the differentials (4.10) are non-zero. (The best study of the convergence of spectral sequences, unfortunately still unpublished, is given in [10].)

Our construction of the spectral sequences follows [30], which is a precursor of the present approach to derived categories. Let $\epsilon : P \to N$ be a quasi-isomorphism of left $A$-modules, where $P$ is a cell $A$-module. Refilter $P$ by setting $F^{1-n}P = P_n$. Thus

$$0 = F^1P \subset F^0P \subset F^{-1}P \subset \cdots \subset F^{-n}P \subset \cdots .$$

Suppressing the Adams grading, we see that this filtration gives rise to a spectral sequence that starts from

$$E_0^{p,q}P = (F^pP/F^{p+1}P)^{p+q} \cong A \otimes (\bar{P}^{p,*})^{p+q},$$

where $\bar{P}^{p,*}$ is $k$-free on the canonical basis elements of the open cells of $P_{1-p}$. The definition of a cell module implies that $d_0 = d \otimes 1$. Therefore

$$E_1^{p,*}P \cong H^*(A) \otimes \bar{P}^{p,*}.$$

Thinking of $N$ as filtered with $F^1N = 0$ and $F^pN = N$ for $p \leq 0$, we see that $E_1^{*,*}P$ gives a complex of left $H^*(A)$-modules

$$\cdots \to E_1^{p-1,*}P \to E_1^{p,*}P \to \cdots \to E_1^{0,*}P \to H^*(N) \to 0. \tag{4.11}$$

**Definition 4.12.** Let $P$ be a cell $A$-module. A quasi-isomorphism $\epsilon : P \to N$ is said to be a distinguished resolution of $N$ if the sequence (4.11) is exact, so that $\{E_1^{p,*}P\}$ is a (negatively indexed) free $H^*(A)$-resolution of $H^*(N)$.

Observe that $\epsilon : P \to N$ is necessarily a homotopy equivalence if $N$ is a cell $A$-module, by Whitehead's theorem. The following result of Gugenheim and May [30, 2.1] should be viewed as a greatly sharpened version of Theorem 2.6: it gives cell approximations with precisely prescribed algebraic properties.

**Theorem 4.13** (Gugenheim-May). *For any $A$-module $N$, every free $H^*(A)$-resolution of $H^*(N)$ can be realized as $\{E_1^{p,*}P\}$ for some distinguished resolution $\epsilon : P \to N$.*

A distinguished resolution $\epsilon : P \to N$ of a cell $A$-module $A$-module $N$ induces a homotopy equivalence $M \otimes_A P \to M \otimes_A N$ for any (right) $A$-module $M$. Filtering $M \otimes_A P$ by

$$F^p(M \otimes_A P) = M \otimes_A (F^pP), \quad p \leq 0,$$

we obtain the spectral sequence (4.8).

Similarly, a distinguished resolution $\epsilon : P \to M$ of a cell $A$-module $A$-module $M$ induces a homotopy equivalence $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_A(P, N)$ for any (left) $A$-module $N$, and the filtration

$$F^p\mathrm{Hom}_A(P, N) = \mathrm{Hom}_A(P/F^{1-p}P, N), \quad p \geq 0,$$

gives rise to the spectral sequence (4.9).

In both cases, the identification of $E_2$-terms is immediate from the definition of a distinguished resolution. Details and applications may be found in [30]. A different construction of the spectral sequences can be obtained by specialization of V§7. It will be immediate from the discussion in the next section that, when $A$ is commutative, $\mathrm{Tor}_A^*(M, N)$ and $\mathrm{Ext}_A^*(M, N)$ are $H^*(A)$-modules and the spectral sequences are spectral sequences of differential $H^*(A)$-modules.

## 5. Commutative DGA's and duality

Let $A$ be commutative throughout this section. We give $\mathscr{D}_A$ a structure of a symmetric monoidal category (= tensor category [21, 1.2]) with internal hom objects. We also discuss duality, characterizing the strongly dualizable objects or, in another language, identifying the largest rigid tensored subcategory of $\mathscr{D}_A$. Again, in $\mathscr{D}_A$, $M \otimes_A N$ means $M \otimes_A \Gamma N$. Of course, since $A$ is commutative, this is an $A$-module. From our present point of view, it makes good sense to resolve both variables since we now have the canonical isomorphisms

$$F^q(r) \otimes_A F^s(t) \cong F^{q+s}(r+t).$$

As in Proposition 1.5(iii), this directly implies that tensor products of cell $A$-modules are cell $A$-modules.

**Proposition 5.1.** *If $M$ and $M'$ are cell $A$-modules, then $M \otimes_A M'$ is a cell $A$-module with sequential filtration $\{\sum_p (M_p \otimes_A N_{n-p})\}$. It has a $(q+s, r+t)$-cell for each pair consisting of a $(q,r)$-cell of $M_p$ and an $(s,t)$-cell of $M'_{n-p}$, $0 \leq p \leq n$.*

For $A$-modules $M$ and $N$, $\mathrm{Hom}_A(M,N)$ is an $A$-module such that

$$\mathscr{M}_A(L \otimes_A M, N) \cong \mathscr{M}_A(L, \mathrm{Hom}_A(M,N)). \tag{5.2}$$

In $\mathscr{D}_A$, $\mathrm{Hom}_A(M,N)$ means $\mathrm{Hom}_A(\Gamma M, N)$, and we have an isomorphism

$$\mathscr{D}_A(L \otimes_A M, N) \cong \mathscr{D}_A(L, \mathrm{Hom}_A(M,N)). \tag{5.3}$$

The standard coherence isomorphisms (= associativity and commutativity constraints) on the tensor product pass to the derived category, which is thus a symmetric monoidal closed category in the sense of [43, 36].

There are general accounts of duality theory in such a context in the literature of both algebraic geometry [21, §1], [19] and algebraic topology [23]; we follow [40, III §§1–2]. Observe first that, by an easy direct inspection of definitions, the functor $\mathrm{Hom}_A(M,N)$ preserves cofiber sequences in both variables. (Actually, in the variable $M$, the functor $\mathrm{Hom}_A$ converts an exact triangle into the negative of an exact triangle.)

The dual of an $A$-module $M$, denoted $M^\vee$ or $DM$, is defined to be $\mathrm{Hom}_A(M,A)$. The adjunction (5.2) specializes to give an evaluation map $\epsilon : M^\vee \otimes_A M \to A$ and a map $\eta : A \to \mathrm{Hom}_A(M,M)$. There is a natural map

$$\nu : \mathrm{Hom}_A(L,M) \otimes_A N \to \mathrm{Hom}_A(L, M \otimes_A N), \tag{5.4}$$

which specializes to

$$\nu : M^\vee \otimes_A M \to \mathrm{Hom}_A(M,M). \tag{5.5}$$

$M$ is said to be "finite" or "strongly dualizable" or "rigid" if, in $\mathscr{D}_A$, there is a coevaluation map $\eta : A \to M \otimes_A M^\vee$ such that the following diagram commutes, where $\tau$ is the commutativity isomorphism.

$$\begin{array}{ccc} A & \xrightarrow{\ \eta\ } & M \otimes_A M^\vee \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathrm{Hom}_A(M,M) & \xleftarrow[\ \nu\ ]{} & M^\vee \otimes_A M \end{array} \tag{5.6}$$

The definition has many purely formal implications. The map $\nu$ of (5.4) is an isomorphism (in $\mathscr{D}_A$) if either $L$ or $N$ is finite. The map $\nu$ of (5.5) is an isomorphism if and only if $M$ is finite, and the coevaluation map $\eta$ is then the composite $\gamma\nu^{-1}\eta$ in (5.6). The natural map

$$\rho : M \to M^{\vee\vee}$$

is an isomorphism if $M$ is finite. The natural map

$$\otimes : \mathrm{Hom}_A(M, N) \otimes_A \mathrm{Hom}_A(M', N') \to \mathrm{Hom}_A(M \otimes_A M', N \otimes_A N')$$

is an isomorphism if $M$ and $M'$ are finite or if $M$ is finite and $N = A$.

Say that a cell $A$-module $N$ is a direct summand up to homotopy of a cell $A$-module $M$ if there is a homotopy equivalence of $A$-modules between $M$ and $N \oplus N'$ for some cell $A$-module $N'$.

**Theorem 5.7.** *A cell $A$-module is finite in the sense just defined if and only if it is a direct summand up to homotopy of a finite cell $A$-module.*

*Proof.* Observe first that $F^q(r)$ is finite with dual $F^{-q}(-r)$, hence any finite direct sum of $A$-modules $F^q(r)$ is finite. Observe next that the cofiber of a map between finite $A$-modules is finite. In fact, the evaluation map $\epsilon$ induces a natural map

$$\epsilon_\# : \mathscr{D}_A(L, N \otimes_A M^\vee) \to \mathscr{D}_A(L \otimes_A M, N),$$

and $M$ is finite if and only if $\epsilon_\#$ is an isomorphism for all $L$ and $N$ [40, III.3.6]. Since both sides turn cofiber sequences in the variable $M$ into long exact sequences, the five lemma gives the observation. We conclude by induction on the number of cells that a finite cell $A$-module is finite. It is formal that a direct summand in $\mathscr{D}_A$ of a finite $A$-module is finite. For the converse, let $M$ be a cell $A$-module that is finite with coevaluation map $\eta : A \to M \otimes_A M^\vee$. Clearly $\eta$ factors through $N \otimes_A M^\vee$ for some finite cell subcomplex $N$ of $M$. By [40, III.1.2], the bottom composite in the following commutative diagram is the identity (in $\mathscr{D}_A$):

$$\begin{array}{ccccccc} & \nearrow & N \otimes_A M^\vee \otimes_A M & \xrightarrow{1\wedge\epsilon} & N \otimes_A A & \xrightarrow{\cong} & N \\ & & \downarrow & & \downarrow & & \downarrow \\ M \cong A \otimes_A M & \xrightarrow[\eta\wedge 1]{} & M \otimes_A M^\vee \otimes_A M & \xrightarrow[1\wedge\epsilon]{} & M \otimes_A A & \xrightarrow{\cong} & M \end{array}$$

Therefore $M$ is a retract up to homotopy and thus, by a comparison of exact triangles, a direct summand up to homotopy of $N$. (Retractions split in triangulated categories.) □

Let $\mathscr{F}\mathscr{C}_A$ be the full subcategory of $\mathscr{C}_A$ whose objects are the direct summands up to homotopy of finite cell $A$-modules. In the language of [21, 1.7], the theorem states that the homotopy category $h\mathscr{F}\mathscr{C}_A$ is the largest rigid tensored subcategory of the derived category $\mathscr{D}_A$. Note that the sequential filtration of a finite cell $A$-module can be arranged so that a single cell is attached at each stage. That is, such a module is just a finite sequence of extensions by free modules on a single generator, and each quotient module $M_n/M_{n-1}$ has the form $F^q(r)$ for some $(q, r)$. A direct summand up to homotopy of a finite cell $A$-module, which is the appropriate analog in $\mathscr{D}_A$ of a finitely generated projective $A$-module, need not be an actual direct summand and need not be isomorphic in $\mathscr{D}_A$ to a finite cell $A$-module. The situation demands the introduction and study of the K-theory $K_0(\mathscr{F}\mathscr{C}_A)$, but we shall desist.

## 6. Relative and unital cell $A$-modules

We here revert to a general DGA $A$, not necessarily commutative, and we assume given a fixed $A$-module $K$. There is a theory of cell $A$-modules relative to $K$ that is exactly like the absolute theory, except that we start with $M_0 = K$ rather than $M_0 = 0$ in the definition of a cell module (Definition 1.1(i)). When $A$ is augmented, so that $k$ is an $A$-module, this theory applies with $K = k$ to give a theory of unital cell $A$-modules. It will be needed in Part V.

The relative theory is adapted to the study of the category of $A$-modules under $K$, by which we understand $A$-modules with a given map of $A$-modules $\eta : K \to M$; a map $f : M \to N$ of $A$-modules under $K$ must satisfy $f \circ \eta = \eta$. We let $\mathscr{M}_A^K$ denote the category of $A$-modules under $K$. We observe an obvious difficulty: the sum of maps under $K$ is not a map under $K$, hence $\mathscr{M}_A^K$ is certainly not an additive category. We say that two maps under $K$ are homotopic (or homotopic rel K) if they are homotopic via a chain homotopy $h$ of $A$-modules such that $h \circ \eta = 0$. That is, if we regard $h$ as a map $M \otimes I \to N$, then $h(\eta(x) \otimes [I]) = 0$ for all $x \in K$. The notion of quasi-isomorphism is unchanged: a map under $K$ is a quasi-isomorphism if it induces an isomorphism on homology. We have the homotopy category $h\mathscr{M}_A^K$ of $A$-modules under $K$, and we construct the derived category $\mathscr{D}_A^K$ from the homotopy category by formally inverting the quasi-isomorphisms.

The theory of relative cell $A$-modules makes this definition rigorous. In fact, if $K \subset L \subset M$, where $L$ is a relative cell submodule of the relative cell $A$-module $M$, then HELP (Theorem 2.2) applies verbatim, by the same proof. The relative Whitehead theorem reads as follows.

**Theorem 6.1.** *If $M$ is a relative cell $A$-module and $e : N \to P$ is a quasi-isomorphism under $K$, then $e_* : h\mathscr{M}_A^K(M, N) \to h\mathscr{M}_A^K(M, P)$ is an isomorphism. Therefore a quasi-isomorphism under $K$ of relative cell $A$-modules is a homotopy equivalence under $K$.*

*Proof.* We see that $e_*$ is surjective by applying HELP to the pair $(M, K)$, taking $g = \eta$, $h$ to be the evident homotopy $\eta \simeq \eta$ rel $K$, and $f : M \to P$ to be any given map under $K$. Injectivity is shown as in the proof of Theorem 2.3. □

Approximation by relative cell $A$-modules works exactly as in Theorem 2.6.

**Theorem 6.2.** *For any $A$-module $M$ under $K$, there is a relative cell $A$ module $N$ and a quasi-isomorphism $\epsilon : N \to M$ under $K$.*

Now Construction 2.7 applies verbatim to the category of $A$-modules under $K$.

**Corollary 6.3.** *The category $\mathscr{D}_A^K$ is equivalent to the homotopy category of relative cell $A$-modules.*

The forgetful functor $\mathscr{M}_A^K \to \mathscr{M}_A$ obviously preserves quasi-isomorphisms and so induces a functor $\mathscr{D}_A^K \to \mathscr{D}_A$. However, this functor fails to take relative cell $A$-modules to $A$-modules of the homotopy type of cell $A$-modules unless $K$ itself is of the homotopy type of a cell $A$-module, which is generally not the case in the applications.

# Part IV. Rational derived categories and mixed Tate motives

We shall do some rational differential homological algebra—alias rational homotopy theory—and use it to prove that two proposed definitions of rational mixed

Tate motives agree [19, 18, 6]. One of them has been proved to admit Hodge and étale realizations [6], but is intrinsically restricted to the rational world. The other can be linked up to a proposed definition of integral mixed Tate motives (or modules). We describe our results in Section 1, which recapitulates much of our announcement [38], and prove them in the remaining sections. We refer the reader to [38] for a number of related conjectures and speculations.

## 1. Statements of results

Let $A$ be a commutative, differential graded, and Adams graded $k$-algebra, abbreviated DGA, where $k$ is a field of characteristic zero. Thus $A$ is bigraded via $k$-modules $A^q(r)$, where $q \in \mathbb{Z}$ and $r \geq 0$. We assume that $A^q(r) = 0$ unless $2r \geq q$. The differential and product behave as follows with respect to the gradings:

$$d : A^q(r) \to A^{q+1}(r) \text{ and } A^q(r) \otimes A^s(t) \to A^{q+s}(r+t).$$

We assume that $A$ has an augmentation $\epsilon : A \to K$. Write $H^q(A)(r)$ for the cohomology of $A$ in bidegree $(q, r)$. In the following three theorems, we assume that $A$ is "cohomologically connected" in the sense that

$$H^q(A)(r) = 0 \text{ if } q < 0, H^0(A)(r) = 0 \text{ if } r > 0,$$

and $\epsilon$ induces an isomorphism $H^0(A)(0) \to k$. While the Adams grading is present and important in our motivating examples, all of our results apply verbatim to DGA's without Adams grading.

Let $\mathscr{D}_A$ be the derived category of cohomologically bounded below $A$-modules. Its objects are differential bigraded $A$-modules $M$, where $M^q(r)$ may be non-zero for any pair of integers $(q, r)$, such that $H^q(M)(r) = 0$ for all sufficiently small $q$. All of our $A$-modules are to satisfy this cohomological condition. We agree to write $\otimes$ for the derived tensor product in $\mathscr{D}_A$. With this convention, we define the "indecomposable elements $QM$" by setting $QM = k \otimes_A M$. Thus $QM$ is a bigraded differential $k$-module. Recall the notion of a $t$-structure and its heart from [3, §1.3], and recall the definition of a neutral Tannakian category from [21, 2.19]. We shall prove the following result in Section 4, after reviewing the theory of minimal DGA's in Section 2 and developing the theory of minimal modules over DGA's in Section 3. Let $\mathscr{H}_A$ be the full subcategory of $\mathscr{D}_A$ consisting of those $M$ such that $H^q(QM) = 0$ for $q \neq 0$. Let $\mathscr{F}\mathscr{H}_A$ be the full subcategory of $\mathscr{H}_A$ consisting of the modules $M$ such that $H^0(QM)$ is finite dimensional and define $\omega(M) = H^0(QM)$.

**Theorem 1.1.** *The triangulated category $\mathscr{D}_A$ admits a $t$-structure whose heart is $\mathscr{H}_A$. In particular, $\mathscr{H}_A$ is Abelian. Moreover, $\mathscr{F}\mathscr{H}_A$ is a (graded) neutral Tannakian category over $k$ with fiber functor $\omega$.*

When $A$ is a polynomial algebra on finitely many generators of even positive degree, most of this is proven in [5, pp.93-101]. It follows from [21, 2.11] that $\mathscr{F}\mathscr{H}_A$ is equivalent (in possibly many ways) to the category of finite dimensional representations of an affine group scheme. What amounts to the same thing [21, 2.2], $\mathscr{F}\mathscr{H}_A$ is equivalent to the category of finite dimensional comodules over a Hopf algebra (= bialgebra). We next specify an explicit such Hopf algebra.

The algebra $A$ has a bar construction $\bar{B}(A)$. Let $IA$ denote the augmentation ideal of $A$. Then $\bar{B}^q(A)(r)$ is the direct sum over $p \geq 0$ of the submodules of the $p$-fold tensor power of $IA$ in bigrading $(q + p, r)$. As we recall in Section 2, we can arrange without loss of generality that $A$ is connected, so that $A^q = 0$ for

$q < 0$ and $A^0 \cong k$. In that case, $\bar{B}^0(A)$ is additively isomorphic to the tensor algebra on the (Adams graded) $k$-module $A^1$. Following [6], let $\chi_A = H^0\bar{B}(A)$. This is a commutative Hopf algebra, and it turns out to be a polynomial algebra. Its $k$-module of indecomposable elements is a co-Lie algebra, which we denote by $\gamma_A$. (The notation $\mathscr{M}_A$ was used in [6], but this conflicts with our notation for the category of $A$-modules.) We think of $\chi_A$ as a kind of universal enveloping Hopf algebra of $\gamma_A$. We shall prove the following theorem in Section 5, where its undefined terms are specified. It gives a concrete and explicit description of the categories $\mathscr{H}_A$ and $\mathscr{F}\mathscr{H}_A$.

**Theorem 1.2.** *Let $A$ be a connected DGA. Then the following categories are equivalent.*
*(i) The heart $\mathscr{H}_A$ of $\mathscr{D}_A$.*
*(ii) The category of generalized nilpotent representations of the co-Lie algebra $\gamma_A$.*
*(iii) The category of comodules over the Hopf algebra $\chi_A$.*
*(iv) The category $\mathscr{T}_A$ of generalized nilpotent twisting matrices in $A$.*
*The full subcategories of finite dimensional objects in the categories (i), (ii), and (iii) and of finite matrices in the category (iv) are also equivalent.*

The hypothesis that $A$ be connected and not just cohomologically connected is needed to allow use of the category $\mathscr{T}_A$. The other three categories are invariant under quasi-isomorphisms of cohomologically connected DGA's. The DGA $A$ has a "1-minimal model" $\iota : A\langle 1\rangle \to A$. The map $\iota$ induces an isomorphism on $H^1$ and a monomorphism on $H^2$. A quick construction, explained in Section 2 and justified in Section 6, is to let $A\langle 1\rangle = \wedge(\gamma_A[-1])$, with differential induced by the cobracket on $\gamma_A$, where $\gamma_A[-1]$ denotes a copy of $\gamma_A$ concentrated in degree one. We say that "$A$ is a $K(\pi,1)$" if $\iota$ is a quasi-isomorphism. It is apparent from the equivalence of (i) and (ii) in Theorem 1.2 that the Abelian category $\mathscr{H}_A$ depends only on $A\langle 1\rangle$. We shall prove the following result in Section 7.

**Theorem 1.3.** *The derived category of bounded below chain complexes in $\mathscr{H}_A$ is equivalent to the derived category $\mathscr{D}_{A\langle 1\rangle}$.*

Let $k(r)$ be a copy of $k$ concentrated in bidegree $(0,r)$ and regarded as a representation of $\gamma_A$ in the evident way.

**Corollary 1.4.** *If $A$ is a $K(\pi,1)$, then*

$$Ext^q_{\mathscr{H}_A}(k, k(r)) \cong H^q(A)(r).$$

While the results above are statements in differential homological algebra, we formulated them as general results that would have to be true if two seemingly different definitions of mixed Tate motives were to agree. We briefly explain the relevance. Let $X$ be a (smooth, quasi-projective) variety over a field $F$. As we recalled in II§6, Bloch [7] defined an Adams graded simplicial Abelian group $\mathfrak{Z}(X)$ whose homology groups are the Chow groups of $X$:

$$CH^r(X,q) = H_q(\mathfrak{Z}(X))(r). \tag{1.5}$$

Bloch [7, 9] (see also Levine [39]) proved that

$$CH^r(X,q)\otimes\mathbb{Q} \cong (K_q(X)\otimes\mathbb{Q})^{(r)}, \tag{1.6}$$

where the right side is the $n^r$-eigenspace of the Adams operation $\psi^n$ (for any $n > 1$), and $K_q(X)\otimes\mathbb{Q}$ is the direct sum of these eigenspaces.

The simplicial Abelian group $\mathfrak{Z}(X)$ has a partially defined product. In II§6, we constructed an $E_\infty$ algebra $\mathscr{A}(X)$ quasi-isomorphic to the associated chain complex of $\mathfrak{Z}(X)$. We also constructed a commutative DGA $\mathscr{A}_{\mathbb{Q}}(X)$ and a quasi-isomorphism of $E_\infty$ algebras $\mathscr{A}(X)\otimes\mathbb{Q}\to\mathscr{A}_{\mathbb{Q}}(X)$. These objects are graded homologically. Cohomological considerations dictate the regrading

$$\mathscr{N}^{2r-p}(X)(r)=\mathscr{A}_p(X)(r) \text{ and } \mathscr{N}_{\mathbb{Q}}^{2r-p}(X)(r)=(\mathscr{A}_{\mathbb{Q}})_p(X)(r). \tag{1.7}$$

Since $\mathscr{A}_p(X)=0$ if $p<0$, $\mathscr{N}^q(X)(r)=0$ unless $2r\geq q$. Thinking of the eigenspaces on the right side of (1.6) as successive terms of the associated grading with respect to the $\gamma$-filtration, we may rewrite (1.6) in the form

$$H^q(\mathscr{N}_{\mathbb{Q}}(X))(r)=\mathrm{gr}^r_\gamma(K_{2r-q}(X)\otimes\mathbb{Q}). \tag{1.8}$$

The "Beilinson-Soulé conjecture for $X$" asserts that these groups are zero if $q<0$ or if $q=0$ and $r\neq 0$, and that the group in bidegree $(0,0)$ is $\mathbb{Q}$. That is, the Beilinson-Soulé conjecture is that $\mathscr{N}_{\mathbb{Q}}(X)$ is cohomologically connected. When it holds, our general results above apply to $\mathscr{N}_{\mathbb{Q}}(X)$.

Specializing to $X=\mathrm{Spec}(F)$, let $\mathscr{N}$ denote the $E_\infty$ algebra $\mathscr{N}(\mathrm{Spec}(F))$ and let $\mathscr{N}_{\mathbb{Q}}$ denote the commutative DGA $\mathscr{N}_{\mathbb{Q}}(\mathrm{Spec}(F))$. Even without the Beilinson-Soulé conjecture, [6] proposed the following definition.

**Definition 1.9.** Let $\chi_{mot}$ denote the Hopf algebra $\chi_{\mathscr{N}_{\mathbb{Q}}}=H^0\bar{B}(\mathscr{N}_{\mathbb{Q}})$. Define the category of (rational) mixed Tate motives of the field $F$, denoted $\mathscr{M}\mathscr{T}\mathscr{M}(F)$, to be the category of finite dimensional comodules over $\chi_{mot}$.

Such a definition had been proposed in general terms by Deligne [18]. Actually, since the equivalence between categories (ii) and (iii) in Theorem 1.2 was not yet understood, the preprint version of [6] confused this category with the category of all finite dimensional representations of $\gamma_{\mathscr{N}_{\mathbb{Q}}}$. Technically, [6] worked with the rationalization of a cubical version of the Chow complex $\mathfrak{Z}(\mathrm{Spec}(F))$. The simplicial version is known to be quasi-isomorphic to the cubical one ([6, 39]). Lack of commutativity makes the cubical version ill-suited to an integral theory, although it is conceivable that a suitable $E_\infty$ operad acts on it—we have not explored this possibility. Theorem 1.2 specializes to give the following equivalence of categories.

**Theorem 1.10.** *If the Beilinson-Soulé conjecture holds for Spec$(F)$, then the category $\mathscr{M}\mathscr{T}\mathscr{M}(F)$ is equivalent to the category $\mathscr{F}\mathscr{H}_{\mathscr{N}_{\mathbb{Q}}}$.*

Deligne [20] first suggested that, if a suitable commutative DGA $\mathscr{N}_{\mathbb{Q}}$ could in fact be constructed, then $\mathscr{F}\mathscr{H}_{\mathscr{N}_{\mathbb{Q}}}$ should give an appropriate definition of $\mathscr{M}\mathscr{T}\mathscr{M}(F)$ when the Beilinson-Soulé conjecture holds for Spec$(F)$. Thus Theorem 1.10 is the promised equivalence of two approaches to mixed Tate motives. In view of (1.8), Corollary 1.4 has the following immediate consequence.

**Theorem 1.11.** *If $\mathscr{N}_{\mathbb{Q}}$ is a $K(\pi,1)$, then*

$$Ext^p_{\mathscr{M}\mathscr{T}\mathscr{M}(F)}(\mathbb{Q},\mathbb{Q}(r))\cong gr^r_\gamma(K_{2r-p}(F)\otimes\mathbb{Q}).$$

This verifies one of the key properties desired of a category of mixed Tate motives. The results of [6] start from Definition 1.9 and give realization functors from $\mathscr{M}\mathscr{T}\mathscr{M}(F)$ to the category of mixed Tate $l$-adic representations in étale theory and to the category of mixed Tate Hodge structures in Hodge theory.

The reinterpretation Definition 1.9 given by Theorem 1.10 leads to a category of integral mixed Tate motives that is related to $\mathscr{M}\mathscr{T}\mathscr{M}(F)$ by extension of scalars.

In fact, in Part V, we shall construct the derived category $\mathscr{D}_A$ associated to an $E_\infty$ algebra $A$. Just as if $A$ were a DGA, $\mathscr{D}_A$ is a triangulated tensor category satisfying all of the usual properties. Like the derived category of modules over a DGA, $\mathscr{D}_A$ can be described as a homotopy category of cell modules. The convenience and workability of such a description will become apparent in our proofs of Theorems 1.1–1.3.

Deligne [20], [17, §3] proposed the resulting derived category $\mathscr{D}_{\mathscr{N}}$ as an integral "catégorie triangulée motivique $\mathscr{D}(F)$", and he gave speculations about its motivic role. One can define Adams graded Ext groups

$$\mathrm{Ext}^q_{\mathscr{N}}(M,N) = \mathscr{D}(F)(M,N[q])$$

for modules $M$ and $N$. These agree with the cohomology groups of the right derived module $\mathrm{Hom}_{\mathscr{N}}(M,N)$ that we shall define in Part V, and we shall there construct a spectral sequence that converges from

$$\mathrm{Ext}^{*;*}_{H^*(\mathscr{N})}(H^*(M),H^*(N))$$

to $\mathrm{Ext}^*_{\mathscr{N}}(M,N)$. Here $H^*(\mathscr{N})$ is the integral Chow ring of $\mathrm{Spec}(F)$, regraded as dictated by (1.7) and (1.8).

Little is known about the integral Chow groups and there is only speculation as to their relationship to the higher algebraic K-groups of $F$. However, our results on derived categories work equally well if we reduce mod $n$, and Suslin [59] has recently proven that if $F$ is an algebraically closed field of characteristic prime to $n$ and $X$ is a smooth affine variety over $F$, then, for $r \geq dim(X)$,

$$CH^r(X,q;\mathbb{Z}/n) \cong H^{2r-q}_{\acute{e}t}(X,\mathbb{Z}/n\,(r)).$$

## 2. Minimal algebras, 1-minimal models, and co-Lie algebras

In the interests of intelligibility, we first review some basic rational homotopy theory, working over our given field $k$ of characteristic zero. We assume once and for all that all DGA's in this part are commutative.

**Definition 2.1.** A connected DGA $A$ is said to be minimal if it is a free commutative algebra with decomposable differential: $d(A) \subset (IA)^2$.

**Definition 2.2.** Let $A$ be a connected DGA and define sub DGA's $A\langle n\rangle$ and $A\langle n,q\rangle$ as follows.
(i) For $n \geq 0$, let $A\langle n\rangle$ be the subalgebra generated by the elements of degree $\leq n$ and their differentials; note that $A\langle 0\rangle = k$.
(ii) For $n \geq 1$, let $A\langle n,0\rangle = A\langle n-1\rangle$ and let $A\langle n,q+1\rangle, q \geq 0$ be the subalgebra generated by

$$A\langle n,q\rangle \cup \{a | a \in A^n \text{ and } d(a) \in A\langle n,q\rangle\}.$$

Say that $A$ is generalized nilpotent if it is free commutative as an algebra and if $A\langle n\rangle = \cup A\langle n,q\rangle$ for each $n \geq 1$. This means that every element of $A^n$ is in some $A\langle n,q\rangle$. Say that $A$ is nilpotent if, for each $n \geq 1$, there is a $q_n$ such that $A\langle n\rangle = A\langle n,q_n\rangle$.

**Proposition 2.3.** *A connected DGA (with Adams grading) is minimal if and only if it is generalized nilpotent.*

*Proof.* If $A$ is generalized nilpotent, then $d(A) \subset (IA)^2$ by an easy double induction on $n$ and $q$ (e.g. [4, 7.3]). Assume that $A$ is minimal. Suppose for a contradiction that $A$ is not generalized nilpotent and let $n$ be minimal such that there is an element of $A^n$ not in any $A\langle n, q\rangle$. Let $a$ be such an element of minimal Adams degree and consider a typical summand $a'a''$ of the decomposable element $d(a)$. We may assume that $0 < \deg(a') \leq \deg(a'')$, and $a'a'' \in A\langle n-1\rangle$ unless $\deg(a') = 1$. Since $A^q(r) = 0$ unless $2r \geq q$, $a'$ and $a''$ have strictly lower Adams grading than $a$. By the assumed minimality, both $a'$ and $a''$ are in some $A\langle n, q\rangle$. Therefore $d(a)$ is in some $A\langle n, q\rangle$, hence so is $a$. □

Except when $A$ is simply connected, the "only if" part would be false without the Adams grading, and we shall not use this implication. Without the Adams grading, the useful notion is that of a generalized nilpotent DGA (hence [4] redefined "minimal" to mean generalized nilpotent). The following result is standard: see [58, §5], or [4, 7.7 and 7.8]. Its proof is just like that of Theorem 3.7 below, except that one adjoins generators of algebras rather than generators of free modules.

**Theorem 2.4.** *If $B$ is a cohomologically connected DGA, then there is a quasi-isomorphism $\phi : A \to B$, where A is generalized nilpotent. If $\phi' : A' \to B$ is another such quasi-isomorphism, then there is an isomorphism $\xi : A \to A'$ such that $\phi'\xi$ is homotopic to $\phi$.*

**Definition 2.5.** An $n$-minimal model of $B$ is a composite map of DGA's

$$A\langle n\rangle \subset A \to B,$$

where $A$ is generalized nilpotent and $A \to B$ is a quasi-isomorphism.

The 1-minimal model admits a canonical description in terms of co-Lie algebras, as we recall next. Here and later, we write $X^\vee = \mathrm{Hom}(X, k)$ and we regard the dual of a map $X \to Y \otimes Z$ of $k$-modules to be the evident composite

$$Y^\vee \otimes Z^\vee \to (Y \otimes Z)^\vee \to X^\vee$$

.

**Definition 2.6.** A co-Lie algebra is a $k$-module $\gamma$ together with a cobracket map $\gamma \to \gamma \otimes \gamma$ such that the dual $\gamma^\vee$ is a Lie algebra via the dual homomorphism. Here $\gamma$ is concentrated in ordinary grading zero; its Adams grading (if it has one), is concentrated in positive degrees.

It is natural to think of the bracket of a Lie algebra $L$ as defined on the subspace of invariants with respect to the involution $x \otimes y \to -y \otimes x$ in $L \otimes L$. The sign suggests that one should think of elements of $L$ as having degree 1. Dually, it is natural to think of the cobracket operation of a co-Lie algebra $\gamma$ as a $k$-linear map $d : \gamma[-1] \to \wedge^2(\gamma[-1])$, where $\gamma[-1]$ is a copy of $\gamma$ concentrated in degree 1 and $\wedge^2(\gamma[-1])$ is the second exterior power. Sullivan observed the following fact, [58, p. 279].

**Lemma 2.7.** *A co-Lie algebra $\gamma$ determines and is determined by a structure of DGA on $\wedge(\gamma[-1])$.*

That is, the (dual) Jacobi identity is equivalent to the assertion that $d$ induces a differential on the exterior algebra $\wedge(\gamma[-1])$. Explicitly, if $\{a_r\}$ is an ordered basis for $\gamma[-1]$ and if

$$d(a_r) = \sum_{p<q} k^r_{p,q} a_p \wedge a_q, \tag{2.8}$$

then, with $k^r_{p,p} = 0$ and $k^r_{q,p} = -k^r_{p,q}$, the $k^r_{p,q}$ are the structural constants of a Lie algebra structure on $\gamma^\vee$ if and only if $d^2 = 0$. We say that $\gamma$ is generalized nilpotent if $\wedge(\gamma[-1])$ is generalized nilpotent. By Proposition 2.3, this always holds when $\gamma$ is suitably Adams graded.

Recall that a Hopf algebra $\chi$ has a sub Lie algebra of primitive elements. Dually, it also has a quotient co-Lie algebra of indecomposable elements. Explicitly, let $I\chi = \mathrm{Ker}(\epsilon)$ be the augmentation ideal and note that the coproduct $\psi$ satisfies

$$\psi(x) \equiv x \otimes 1 + 1 \otimes x \bmod I\chi \otimes I\chi \text{ for } x \in I\chi.$$

We have the cobracket $\psi - \tau\psi$ on $\chi$, where $\tau : \chi \otimes \chi \to \chi \otimes \chi$ is the transposition. If $\gamma = I\chi/(I\chi)^2$ denotes the $k$-module of indecomposable elements, then $\psi - \tau\psi$ induces a cobracket on $\gamma$ such that the quotient map $I\chi \to \gamma$ is a map of co-Lie algebras.

**Definition 2.9.** For a DGA $A$, let $\chi_A$ be the Hopf algebra $H^0\bar{B}(A)$ and let $\gamma_A$ be its co-Lie algebra of indecomposable elements.

We shall recall the definition of the bar construction and prove the following result in Section 6; much of it is implicit or explicit in [6].

**Theorem 2.10.** *Let $A$ be a cohomologically connected DGA.*
*(i) The 1-minimal model $A\langle 1\rangle$ of $A$ is isomorphic to $\wedge(\gamma_A[-1])$.*
*(ii) The Hopf algebras $\chi_{A\langle 1\rangle}$ and $\chi_A$ are isomorphic, hence the co-Lie algebras $\gamma_{A\langle 1\rangle}$ and $\gamma_A$ are isomorphic.*

## 3. Minimal $A$-modules

We assume familiarity with the cell theory of Part III. As explained there, the derived category $\mathscr{D}_A$ is equivalent to the homotopy category $h\mathscr{C}_A$ of cell $A$-modules. Remember that we require all modules to be cohomologically bounded below. By III.3.4, we have the following invariance statement; an $E_\infty$ generalization will be proven in V§4.

**Proposition 3.1.** *If $\phi : A \to A'$ is a quasi-isomorphism of cohomologically connected DGA's, then $\phi$ induces an equivalence $\phi^* : \mathscr{D}_{A'} \to \mathscr{D}_A$ of triangulated tensor categories.*

In particular, by Theorem 2.4, we can and will assume that our given DGA $A$ is connected. Let $M$ be a cell $A$-module. Then $QM$ is the ordinary tensor product $k \otimes_A M$. Ignoring the differential, $M$ is $A$-free on the canonical basis elements $\langle j\rangle$ of its open cells, and this basis projects to a canonical basis of $QM$. We write

$$d\langle j\rangle = \sum a_{i,j}\langle i\rangle,$$

where $\langle i\rangle$ runs through the basis elements of the open cells. Define $M^{\leq n} \subset M$ to be the sum of those open cells with basis elements in (ordinary) degree $\leq n$. Note that $M^{\leq n}$ is not necessarily closed under the differential.

**Definition 3.2.** A bounded below cell $A$-module $M$ is minimal if it is $A$-free and has decomposable differential: $d(M) \subset (IA)M$.

**Proposition 3.3.** *The following conditions on a bounded below cell $A$-module $M$ are equivalent.*
*(i) $M$ is minimal.*
*(ii) $QM = H^0(QM)$; that is, $d = 0$ on $QM$.*
*(iii) All coefficients $a_{i,j}$ have positive degree.*
*(iv) Each $M^{\leq n}$ is closed under $d$ and is thus a cell submodule of $M$.*
*If $f : M \to N$ is a quasi-isomorphism between minimal $A$-modules, then $f$ is an isomorphism.*

*Proof.* Since $A^q = 0$ for $q < 0$ and $\epsilon : A^0 \to k$ is an isomorphism, the equivalence of (i)–(iv) is immediate by inspection of definitions. A quasi-isomorphism $f : M \to N$ induces a quasi-isomorphism $Qf : QM \to QN$ and, if $M$ and $N$ are minimal, $Qf$ itself is then an isomorphism. Thus the last statement follows by Nakayama's lemma: a map $f$ of bounded below free $A$-modules is an isomorphism if and only if $Qf$ is an isomorphism. □

There is an equivalent condition in terms of generalized nilpotency.

**Definition 3.4.** Let $M$ be a bounded below $A$-module (not a priori a cell module) and define sub $A$-modules $M\langle n\rangle$ and $M\langle n,q\rangle$ as follows.
(i) Let $M\langle n\rangle$ be the sub $A$-module generated by the elements of degree $\leq n$ and their differentials; note that $M\langle n\rangle = 0$ for $n$ sufficiently small.
(ii) Let $M\langle n,0\rangle = M\langle n-1\rangle$ and let $M\langle n,q+1\rangle, q \geq 0$, be the sub $A$-module generated by

$$M\langle n,q\rangle \cup \{m | m \in M^n \text{ and } d(m) \in M\langle n,q\rangle\}.$$

(iii) Define the "nilpotent filtration" $\{F_tM\}$ by letting $F_0M = 0$ and, inductively, letting $F_tM$ be the sub $A$-module generated by

$$F_{t-1}M \cup \{m | d(m) \in F_{t-1}M\}.$$

Say that $M$ is generalized nilpotent if it is free as an $A$-module and if

$$M\langle n\rangle = \cup M\langle n,q\rangle$$

for each $n$. This means that every element of $M^n$ is in some $M\langle n,q\rangle$. Say that $M$ is nilpotent if, for each $n$, there is a $q_n$ such that $M\langle n\rangle = M\langle n,q_n\rangle$.

In marked contrast with the case of algebras, the following result for modules is true regardless of whether or not there is an Adams grading.

**Proposition 3.5.** *A bounded below $A$-module $M$ is generalized nilpotent if and only if it is a minimal cell $A$-module, and then $\{F_tM\}$ specifies a canonical choice of sequential filtration for the cell structure on $M$.*

*Proof.* Suppose that $M$ is generalized nilpotent. Then $d(M) \subset (IA)M$ since an $A$-basis element in degree $n$ must have differential in the sub $A$-module generated by the $M^j$ for $j \leq n$. We claim that $M$ is a cell $A$-module with $\{F_tM\}$ as sequential filtration. Certainly $M$ is the union of the $F_tM$ since, if not, there would be a minimal pair $(n,q)$ in the lexicographic ordering such that $M\langle n,q\rangle$ was not contained in the cited union and this would contradict the generalized nilpotency. Assuming inductively that $F_{t-1}M$ is $A$-free, we easily check that $F_tM$ is $A$-free with basis obtained by extending a basis for $F_{t-1}M$. Conversely, assume that $M$ is a minimal cell $A$-module. Suppose for a contradiction that $M$ is not generalized nilpotent and let $n$ be minimal such that there is an element of $M^n$ that is not in any $M\langle n,q\rangle$.

Let $m$ be such an element of minimal sequential filtration. By the definition of a cell $A$-module, $d(m)$ has lower sequential filtration than $m$. But then $d(m)$ is in some $M\langle n,q\rangle$ and $m$ is in $M\langle n,q+1\rangle$. This proves the result. □

*Remark* 3.6. A minimal $A$-module $M$ need not have bounded below Adams grading, as we see by considering infinite direct sums. However, if $M$ has bounded below Adams grading, then it admits a second canonical sequential filtration $\{F_t^{\mathrm{Ad}}M\}$. Precisely, let $\{r_t | t \geq 1\}$ be the ordered set of integers for which the free $A$-module $M$ has a basis element of Adams grading $r_t$. Then $F_0^{\mathrm{Ad}}M = 0$ and $F_t^{\mathrm{Ad}}$ is the sub $A$-module spanned by the basis elements of Adams grading at most $r_t$. Clearly $F_t^{\mathrm{Ad}}M \subset F_tM$, and the inclusion can be proper.

**Theorem 3.7.** *Let $N$ be an $A$-module. Then there is a quasi-isomorphism $e : M \to N$, where $M$ is a minimal $A$-module. If $e' : M' \to N$ is another such quasi-isomorphism, then there is an isomorphism $f : M \to M'$ such that $e'f$ is homotopic to $e$.*

*Proof.* Let $n_0$ be sufficiently small that $H^q(N) = 0$ for $q < n_0$ and let $M[n_0, 0] = 0$. Assume inductively that an $A$-map $e : M[n,0] \to N$ has been constructed such that $e^*$ is an isomorphism on $H^i$ for $i < n$ and a monomorphism on $H^n$. Then, proceeding by induction on $q$, construct $A$-maps $e : M[n,q] \to N$ for $q \geq 0$ as follows. If $q = 0$, choose a set $\{n_s\}$ of representative cycles in $N$ for a basis of

$$\mathrm{Coker}(H^n(M[n,q]) \to H^n(N)).$$

If $q \geq 0$, choose a set $\{m_r\}$ of representative cycles in $H^{n+1}(M[n,q])$ for a basis of

$$\mathrm{Ker}(H^n(M[n,q]) \to H^n(N))$$

and choose elements $n_r$ in $N$ such that $d(n_r) = (-1)^{n+1}e(m_r)$. Then construct $M[n,q+1]$ from $M[n,q]$ by attaching $n$-cells $j_s$ (if $q = 0$) and $i_r$ via attaching cycles 0 and $m_r$; thus the basis elements of the adjoined open $n$-cells satisfy

$$d\langle j_s\rangle = 0 \text{ and } d\langle i_r\rangle = (-1)^{n+1}m_r.$$

Extend $e$ to $M[n,q+1]$ by setting $e\langle j_s\rangle = n_s$ and $e\langle i_r\rangle = n_r$. An easy colimit argument shows that if we define

$$M[n+1,0] = \cup M[n,q]$$

and let $e : M[n+1,0] \to M$ be the induced map, then $e^*$ is an isomorphism on $H^i$ for $i \leq n$ and a monomorphism on $H^{n+1}$. Define $M = \cup M[n,0]$. Then the induced map $e : M \to N$ is a quasi-isomorphism, and $M$ is minimal since it is generalized nilpotent with $M\langle n,q\rangle = M[n,q]$. For the last statement, the Whitehead theorem (III.2.3) gives a map $f : M \to M'$ such that $e'f$ is homotopic to $f$. Obviously $f$ is a quasi-isomorphism, and it is therefore an isomorphism by Proposition 2.3. □

## 4. The $t$-structure on $\mathscr{D}_A$

We here prove Theorem 1.1. Let $A$ be a cohomologically connected DGA. We agree to abbreviate notation by writing $\mathscr{D} = \mathscr{D}_A$, and similarly for other categories that depend on $A$.

**Definition 4.1.** Define full subcategories $\mathscr{D}^{\leq n}$ and $\mathscr{D}^{\geq n}$ of $\mathscr{D}$ by

$$\mathscr{D}^{\leq n} = \mathscr{D}^{\leq 0}[-n] = \{M | H^q(QM) = 0 \text{ for } q > n\}$$

and

$$\mathscr{D}^{\geq n} = \mathscr{D}^{\geq 0}[-n] = \{M | H^q(QM) = 0 \text{ for } q < n\}.$$

Observe that $\mathscr{D}^{\leq 0} \subset \mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 0} \supset \mathscr{D}^{\geq 1}$. Define

$$\mathscr{H} = \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0} = \{M | H^q(QM) = 0 \text{ for } q \neq 0\}.$$

The following result is a more explicit statement of the first part of Theorem 1.1.

**Theorem 4.2.** *Definition 4.1 specifies a t-structure on $\mathscr{D}$.*

Proposition 3.1 implies that the result will be true for $A$ if it is true for a DGA quasi-isomorphic to $A$. Therefore, by Theorem 2.4, we may as well assume that $A$ is connected. This allows us to use the theory of minimal $A$-modules. Taken together, the following two lemmas constitute a restatement of Theorem 4.2.

**Lemma 4.3.** *For $M \in \mathscr{D}$, there is an exact triangle $M^{\leq 0} \to M \to M/M^{\leq 0}$ in $\mathscr{D}$ with $M^{\leq 0}$ in $\mathscr{D}^{\leq 0}$ and $M/M^{\leq 0}$ in $\mathscr{D}^{\geq 1}$.*

*Proof.* It suffices to assume that $M$ is minimal, in which case the conclusion is immediate from Proposition 3.3(iv). □

**Lemma 4.4.** *If $M$ is in $\mathscr{D}^{\leq 0}$ and $N$ is in $\mathscr{D}^{\geq 1}$, then $\mathscr{D}(M, N) = 0$.*

*Proof.* It suffices to assume that $M$ and $N$ are minimal. In that case, $(QM)^q = 0$ for $q > 0$ and $N^q = 0$ for $q \leq 0$, hence there are no non-zero maps of $A$-modules $M \to N$. □

*Remark* 4.5. Theorem 1.1 would be false without the restriction to cohomologically bounded below $A$-modules. An unbounded $A$-module $M$ can have non-zero cohomology and yet satisfy $H^*Q(M) = 0$. For example, if $\alpha \in H^n(A)$ is represented by a cycle $a$ and $M$ is the telescope of the sequence of $A$-maps

$$a : A[-qn] \to A[-(q+1)n],$$

then $H^*(M)$ is the localization $H^*(A)[\alpha^{-1}]$ and $H^*(QM) = 0$. Clearly Lemma 4.4 will usually fail in this situation since $H^q(M)(r) \cong \mathscr{D}(F, M)$, where $F$ is free on one generator of bidegree $(-q, -r)$.

The following lemma implies that $\mathscr{F}\mathscr{H}$ is a rigid Abelian tensor category.

**Lemma 4.6.** *The subcategory $\mathscr{F}\mathscr{H}$ is closed under passage to tensor products and duals in $\mathscr{D}$.*

*Proof.* By III.5.1, if $M$ and $N$ are (finite) cell $A$-modules, then the derived tensor product $M \otimes_A N$ is given by the ordinary tensor product and is a (finite) cell $A$-module such that

$$(*) \qquad Q(M \otimes_A N) \cong QM \otimes QN.$$

If $M$ and $N$ are minimal, then, by Proposition 3.3(ii), so is $M \otimes_A N$. If the indecomposable elements of minimal $A$-modules $M$ and $N$ are concentrated in degree zero, then so are the indecomposable elements of $M \otimes_A N$, and this proves closure under tensor products. For duals, it is easy to check that the $k$-modules $Q(M^\vee)$ and $(QM)^\vee$ are isomorphic when $M$ is a finite cell $A$-module. □

The following lemma completes the proof of the last statement of Theorem 1.1.

**Lemma 4.7.** *$\omega = H^0Q : \mathscr{F}\mathscr{H} \to \mathbb{Q}\mathscr{M}$ is a faithful exact tensor functor.*

*Proof.* An easy formal elaboration of (∗) shows that $\omega$ is a tensor functor. The functor $Q$ is exact since we have restricted to cell $A$-modules. Therefore $H^0Q$ is exact on $\mathscr{H}$ by virtue of the long exact sequences associated to short exact sequences obtained by applying $Q$ to short exact sequences

$$0 \to M' \to M \to M'' \to 0$$

of cell $A$-modules. Alternatively, we can check that it suffices to restrict attention to short exact sequences of minimal $A$-modules. Finally, $\omega$ is faithful since two $A$-maps between minimal $A$-modules in $\mathscr{H}$ are equal if they are equal on indecomposable elements. Note that there is no room for homotopies since there are no elements of degree -1: a map in $\mathscr{H}$ between minimal $A$-modules is just a map of $A$-modules. □

## 5. Twisting matrices and representations of co-Lie algebras

We here prove Theorem 1.2. We begin by describing $\mathscr{H}_A$ in terms of matrices. We then show that representations of co-Lie algebras admit a precisely similar description. We tie in comodules at the end. In view of Theorem 2.4, Proposition 3.1, and the quasi-isomorphism invariance of the homology of the bar construction, we may as well assume that $A$ is generalized nilpotent.

Let $M$ be a minimal $A$-module in $\mathscr{H}_A$. Then $M$ is $A$-free on basis elements $\langle j\rangle$ of degree zero and Adams degree $r(j)$. Here the nilpotent filtration of Definition 3.4 is given by $F_tM = M\langle 0,t\rangle$. Each $\langle j\rangle$ lies in $F_tM - F_{t-1}M$ for some positive integer $t$, which we denote by $t(j)$ and think of as the order of nilpotency. The differential is given by

$$d\langle j\rangle = \sum a_{i,j}\langle i\rangle,$$

where $a_{i,j}$ has degree one and Adams degree $r(j) - r(i)$; in particular, $a_{i,j} = 0$ if $r(j) \le r(i)$. For each $\langle j\rangle$, only finitely many of the $a_{i,j}$ are non-zero, and $a_{i,j} = 0$ if $t(i) \ge t(j)$. Order the basis and write $\mathbf{a} = (a_{i,j})$ and $d\mathbf{a} = (d(a_{i,j}))$. Then the condition $dd = 0$ is easily seen to take the form of the matrix equation $d\mathbf{a} = -\mathbf{aa}$, and this makes sense even when $M$ is infinite dimensional. Note in particular that each $d(a_{i,j})$ must be a decomposable element of the algebra $A$.

Now consider a map $f : M \to N$ of minimal $A$-modules, where the differentials on $M$ and $N$ are given by the matrices $\mathbf{a}$ and $\mathbf{b}$ in $A^1$. Let $f\langle i\rangle = \sum k_{j,i}\langle j\rangle$, where $\langle j\rangle$ runs through the canonical basis of $N^0$ and the $k_{j,i}$ are elements of the ground field. Here $k_{j,i} = 0$ unless $\langle i\rangle$ and $\langle j\rangle$ have the same Adams degree. Moreover, since $f$ preserves the nilpotent filtration, $k_{j,i} = 0$ if $t(j) > t(i)$. Write $\mathbf{k} = (k_{j,i})$. Then the condition $df = fd$ is easily seen to take the form of the matrix equation $\mathbf{bk} = \mathbf{ka}$. These observations lead to the following definition (compare Sullivan [58, §1]) and proposition. By an "initial segment of the positive integers", we understand either the set of all positive integers or the set $\{1, 2, \ldots, n\}$ for some finite $n$.

**Definition 5.1.** A "twisting matrix" in $A$ is an ordered set $I$, a function $r : I \to \mathbb{Z}$, and a row finite $(I\times I)$-matrix $\mathbf{a} = (a_{i,j})$ with entries in $A^1$ such that $a_{i,j}$ has Adams degree $r(j) - r(i)$ and $d\mathbf{a} = -\mathbf{aa}$. We say that $\mathbf{a}$ is indexed on $r$. A twisting matrix $\mathbf{a}$ is generalized nilpotent if there is a surjection $t$ from $I$ to an initial segment of the positive integers such that $a_{i,j} = 0$ if $t(i) \ge t(j)$. A morphism from a twisting matrix $\mathbf{a}$ indexed on $r : I \to \mathbb{Z}$ to a twisting matrix $\mathbf{b}$ indexed on $s : J \to \mathbb{Z}$ is a row finite $(J \times I)$-matrix $\mathbf{k} = (k_{j,i})$ with entries in the ground field such that $k_{j,i} = 0$ if $r(i) \ne s(j)$ and $\mathbf{bk} = \mathbf{ka}$. If $\mathbf{a}$ and $\mathbf{b}$ are generalized nilpotent (with nilpotency functions both denoted $t$), then we require morphisms to satisfy $k_{j,i} = 0$

if $t(i) > t(j)$. With composition specified by the usual product of matrices, there results a category $\mathscr{T}_A$ of generalized nilpotent twisting matrices in $A$.

**Proposition 5.2.** *The category $\mathscr{H}_A$ is equivalent to the category $\mathscr{T}_A$.*

*Proof.* The category $\mathscr{H}_A$ is equivalent to its full subcategory of minimal $A$-modules, maps in $\mathscr{H}_A$ between minimal $A$-modules are just maps of modules, and the discussion above gives the conclusion. □

We next recall the notion of a representation of a co-Lie algebra $\gamma$. Recall Definition 2.6 and Lemma 2.7.

**Definition 5.3.** A representation of a co-Lie algebra $\gamma$ is a $k$-module $V$ together with a coaction map $\nu : V \to \gamma \otimes V$ such that the dual $V^\vee$ is a module over the Lie algebra $\gamma^\vee$ via the dual homomorphism. Here $V$ is concentrated in ordinary grading zero; its Adams grading (if it has one) is unrestricted.

Dualizing and reinterpreting, we see that a representation on $V$ can equally well be viewed as a $k$-linear map $\nu : V \to \gamma[-1] \otimes V$ such that $(d \otimes 1)\nu$ coincides with the map obtained by passage to coinvariants from the composite $(1 \otimes \nu)\nu$; that is

$$(1 \otimes \nu)\nu = (d \otimes 1)\nu : V \to \wedge^2\gamma[-1] \otimes V. \tag{5.4}$$

However, we do not want to allow all such representations.

**Definition 5.5.** Let $V$ be a representation of a co-Lie algebra $\gamma$. Define the nilpotent filtration $\{F_tV\}$ by letting $F_0V = 0$ and letting $F_tV$ be the subspace generated by the union of $F_{t-1}V$ and $\{v|\nu(v) \in \gamma[-1] \otimes F_{t-1}V\}$. Say that $V$ is generalized nilpotent if it is the union of the $F_tV$. Say that $V$ is nilpotent if $V = F_tV$ for some finite $t$.

*Remark* 5.6. A generalized nilpotent representation $V$ need not have bounded below Adams grading. If a representation $V$ has bounded below Adams grading, then it is generalized nilpotent and has the Adams filtration $\{F_t^{\mathrm{Ad}}V\}$ specified by letting $F_0^{\mathrm{Ad}}V = 0$ and letting $F_t^{\mathrm{Ad}}V$ be the subspace of elements with Adams grading at most $r_t$, where $\{r_t|t \geq 1\}$ is the ordered set of integers for which $V$ has an element of Adams grading $r_t$. As in Remark 3.6, $F_t^{\mathrm{Ad}}V \subset F_tV$, and the inclusion can be proper.

Let $V$ be a generalized nilpotent representation of $\gamma$. Fix a basis $\{v_i\}$ for $V$ indexed on an ordered set $I$. Define $r : I \to \mathbb{Z}$ by letting $r(i)$ be the Adams degree of $v_i$ and define a surjection from $I$ to an initial segment of the positive integers by letting $t(i)$ be minimal such that $v_i \in F_{t(i)}V$. Let $\nu(v_j) = \sum a_{i,j} \otimes v_i$. Then $a_{i,j}$ has Adams degree $r(j) - r(i)$ and $a_{i,j} = 0$ if $t(i) \geq t(j)$. We again write $\mathbf{a} = (a_{i,j})$ and $d\mathbf{a} = (d(a_{i,j}))$. Then (5.4) takes the form of the matrix identity $d\mathbf{a} = -\mathbf{aa}$.

Similarly, let $f : V \to W$ be a map of generalized nilpotent representations of $\gamma$ and write $f(v_i) = \sum k_{j,i}(w_j)$, where $w_j$ is the chosen basis of $W$. Then $k_{j,i} = 0$ if $r(i) \neq r(j)$ or if $t(j) > t(i)$. Write $\mathbf{k} = (k_{j,i})$. Then the identity $(1 \otimes f)\nu = \omega f$ takes the form of the matrix identity $\mathbf{bk} = \mathbf{ka}$, where $\omega : W \to \gamma[-1] \otimes W$ is specified by the matrix $\mathbf{b}$. These observations imply the following result.

**Proposition 5.7.** *The category of generalized nilpotent representations of a co-Lie algebra $\gamma$ is equivalent to the category $\mathscr{T}_A$, where $A = \wedge(\gamma[-1])$.*

**Corollary 5.8.** *The categories (i), (ii), and (iv) of Theorem 1.2 are equivalent.*

*Proof.* This is immediate from Proposition 5.2 and Proposition 5.7, applied to the co-Lie algebra $\gamma_A$ of Theorem 2.10. Note that $\mathscr{T}_A$ is equivalent to $\mathscr{T}_{A\langle 1\rangle}$ since $\gamma_A$ is isomorphic to $\gamma_{A\langle 1\rangle}$. $\square$

To complete the proof of Theorem 1.2, we must connect up the category of comodules over $\chi_A$. First recall exactly how a module $V$ over a Lie algebra $L$ determines a module over its universal enveloping algebra $U(L)$: the given action map $L \otimes V \to V$ induces an action $T(L) \otimes V \to V$ of the tensor algebra $T(L)$, by iteration, and this map factors through the quotient map $T(L) \otimes V \to U(L) \otimes V$ to induce the required action $U(L) \otimes V \to V$. We shall dualize this description.

**Definition 5.9.** Define the universal enveloping Hopf algebra $\chi(\gamma)$ of a co-Lie algebra $\gamma$ to be $\chi_A$, where $A = \wedge(\gamma[-1])$.

Let $T(\gamma)$ be the tensor coalgebra of $\gamma$. Additively, it is the same as the tensor algebra, and it has the coproduct $\psi$ given by

$$\psi(c_1 \otimes \cdots \otimes c_n) = \sum_{i+j=n} (c_1 \otimes \cdots \otimes c_i) \otimes (c_{i+1} \otimes \cdots \otimes c_n).$$

We shall prove the following result in the next section. Recall that $\gamma$ is said to be generalized nilpotent if $\wedge(\gamma[-1])$ is generalized nilpotent and that this always holds when $\wedge(\gamma[-1])$ is Adams graded.

**Proposition 5.10.** *Let $\gamma$ be a generalized nilpotent co-Lie algebra. Then there is a canonical commutative diagram of algebras*

$$\begin{array}{ccc} T(\gamma^\vee) & \longrightarrow & T(\gamma)^\vee \\ \big\downarrow & & \big\downarrow \\ U(\gamma^\vee) & \longrightarrow & \chi(\gamma)^\vee. \end{array}$$

Here $T(\gamma^\vee) \to U(\gamma^\vee)$ is the obvious quotient map, $U(\gamma^\vee) \to \chi(\gamma)^\vee$ is the map of algebras induced by the inclusion of Lie algebras dual to the quotient map of co-Lie algebras $\chi(\gamma) \to \gamma$, $T(\gamma^\vee) \to T(\gamma)^\vee$ is the map of algebras induced by the dual of the evident quotient map of $k$-modules $T(\gamma) \to \gamma$, and $T(\gamma)^\vee \to \chi(\gamma)^\vee$ is dual to a canonical embedding of $\chi(\gamma)$ as a subcoalgebra of $T(\gamma)$ that will be explained in the next section.

The dual of a $\chi(\gamma)$-comodule $V$ is a $\chi(\gamma)^\vee$-module and therefore a $U(\gamma^\vee)$-module. Equivalently, it is a $\gamma^\vee$-module, and of course the action of $\gamma^\vee$ is the restriction of the action of $U(\gamma^\vee)$. If the coaction of $\chi(\gamma)$ is given by $\mu : V \to \chi(\gamma) \otimes V$, then we obtain an induced coaction of $\gamma$ by composing with the projection $\chi(\gamma) \otimes V \to \gamma \otimes V$. Conversely, let $V$ be a representation of $\gamma$ with coaction $\nu : V \to \gamma \otimes V$. Then the dual $V^\vee$ is a $\gamma^\vee$-module under the dual of $\nu$. Equivalently, $V^\vee$ is a $U(\gamma^\vee)$-module.

We ask when this action results from dualization of a coaction by $\chi(\gamma)$. By iteration, $\nu$ induces a map $\nu_n : V \to \gamma^n \otimes V$ for each $n \geq 0$, where $\gamma^n$ denotes the $n$-fold tensor power of $\gamma$ and $\nu_0$ is understood to be the identity map of $V$. Under the proviso that, for each $v \in V$, $\nu_n(v) = 0$ for all sufficiently large $n$, the sum $\mu : V \to T(\gamma) \otimes V$ of the maps $\nu_n$ makes sense. It must take values in $\chi(\gamma) \otimes V$ and specify a structure of $\chi(\gamma)$-comodule on $V$, by consideration of the dual situation. A moment's reflection on Definition 5.5 will convince the reader that the proviso holds if and only if $V$ is generalized nilpotent. Again, reflection on the dual situation shows that if we start with a coaction $\mu$ of $\chi(\gamma)$ on $V$, project to obtain a coaction $\nu$ of $\gamma$ on $V$, and then take the sum of the iterates $\nu_n$, we must get back $\nu$. Since $\mu$ is defined by finite sums, this means that $V$ is generalized nilpotent. These arguments, which can be carried out less intuitively and more precisely without use of dualization, lead to the following conclusion, which completes the proof of Theorem 1.2.

**Proposition 5.11.** *The category of generalized nilpotent representations of a generalized nilpotent co-Lie algebra $\gamma$ is equivalent to the category of comodules over $\chi(\gamma)$.*

*Remark* 5.12. When $\gamma$ is not generalized nilpotent, the co-Lie algebra of indecomposable elements of $\chi(\gamma)$ specifies the "generalized nilpotent completion" of $\gamma$. Equivalently, for a minimal DGA $A$ with degree one indecomposable elements, the DGA $\wedge(\gamma_A[-1])$ specifies the "generalized nilpotent completion" of $A$. This corresponds topologically to generalized nilpotent completion of rational $K(\pi,1)$'s.

## 6. The bar construction and the Hopf algebra $\chi_A$

We prove Theorem 2.10 and Proposition 5.10 here, and we also develop preliminaries that will be needed in the proof of Theorem 1.3.

We first recall the basic facts about the bar construction (e.g. from [30, Appendix], or [6]). We shall use the sign conventions of [30]. The two-sided bar construction $B(M,A,N)$ is defined for $A$-modules $M$ and $N$. Even though $A$ is commutative, we think of $M$ as a right and $N$ as a left $A$-module to keep track of signs. As a chain complex, $B(M,A,N)$ is obtained by totalization (as in II§5) of the usual simplicial chain complex $B_*(M,A,N)$ with

$$B_p(M,A,N) = M\otimes A^p\otimes N.$$

Since our totalization includes normalization, $B(M,A,N)$ is additively the direct sum of the vector spaces $M\otimes(IA)^p\otimes N$. (Logically, the cokernel of the unit $k\to A$ should appear in place of the isomorphic $k$-module $IA$.) We grade $B(M,A,N)$ so that the homological degree is negative. Thus elements of $M\otimes(IA)^p\otimes N$ have degree their internal degree minus $p$; the (total) differential on such elements is given by the map

$$(-1)^p d + \sum(-1)^i d_i,$$

where $d$ is the internal differential on the tensor product $M\otimes(IA)^p\otimes N$.

With the evident right action by $A$, $B(M,A,N)$ is a differential $A$-module and

$$B(M,A,N) = B(M,A,A)\otimes_A N.$$

We may think of $B(M,A,N)$ as an explicit model for the derived tensor product of $M$ and $N$, and, as in III§4, we have an Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{H^*(A)}(H^*(M),H^*(N)) \Longrightarrow H^*(B(M,A,N)).$$

Therefore quasi-isomorphisms of its variables induce quasi-isomorphisms of the bar construction. The following minor technical point will become relevant in the next section.

*Remark* 6.1. While $B(M,A,A)$ is a right differential $A$-module with the evident right action, there is no choice of signs in $B(M,A,N)$ for which both this and its analog for $B(A,A,N)$ are true. To make $B(A,A,N)$ a left differential $A$-module, one must modify the obvious action by a sign, defining a new action of $A$ by $a \cdot x = (-1)^{p\deg(a)} ax$, where $x$ has homological degree $p$. The required formula $d(a \cdot x) = d(a) \cdot x + (-1)^{\deg(a)} a \cdot d(x)$ is easily checked.

As usual we abbreviate $\bar{B}(A) = B(k, A, k)$. The product $\phi$ on $\bar{B}A$ is the shuffle product

$$\phi([a_1|\dots|a_r] \otimes [a_{r+1}|\dots|a_s]) = \sum (-1)^{\sigma(\mu)} [a_{\mu(1)}|\dots|a_{\mu(r+s)}],$$

where the sum runs over the $(r,s)$-shuffles $\mu$ in the symmetric group $\Sigma_{r+s}$; $\sigma(\mu)$ is the sum over $(i,j)$ such that $1 \leq i \leq r$, $r < j \leq r+s$, and $\mu(j) < \mu(i)$ of $\deg(a_{\mu(i)})\deg(a_{\mu(j)})$. The coproduct $\psi$ on $\bar{B}(A)$ is

$$\psi([a_1|\dots|a_p]) = \sum (-1)^{\tau(i)} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_p],$$

where the sum runs over $0 \leq i \leq p$ and $\tau(i) = (p-i)(\deg(a_1) + \cdots + \deg(a_i))$.

*Remark* 6.2. This coalgebra structure on the tensor algebra $T(IA)$ is isomorphic to the usual one. In fact, the isomorphism specified by

$$[a_1|\dots|a_p] \to (-1)^{\nu(p)} [a_1|\dots|a_p],$$

where $\nu(p) = p\deg(a_1) + (p-1)\deg(a_2) + \cdots + \deg(a_p)$ throws the coproduct defined with signs onto the coproduct defined without signs.

To prove Theorem 2.10, we may assume without loss of generality that $A$ is generalized nilpotent. Since $A$ is connected, there are no non-zero elements of negative degree in $\bar{B}(A)$. Thus there are no degree zero boundaries and $\chi_A = H^0\bar{B}(A)$ embeds in $\bar{B}(A)$ as its $k$-module of cycles of degree zero. Since $\chi_A$ inherits its Hopf algebra structure from the Hopf algebra structure on $\bar{B}(A)$, this embedding must be a map of Hopf algebras. Note however that, even in simple cases, it is not obvious how to identify cycles explicitly. The elements of degree zero in $\bar{B}(A)$ are the elements of the $(A^1)^p$, so that $\chi_A$ depends only on the elements of $A^1$ and their differentials. When $A$ is generalized nilpotent, this means that $\chi_A = \chi_{A\langle 1\rangle}$, and this already implies Theorem 2.10(ii). The last part of the following calculational description of $\chi_A$ is Theorem 2.10(i).

**Theorem 6.3.** *Let $A = A\langle 1\rangle$. Then the following conclusions hold.*
*(i) The embedding $\chi_A \to \bar{B}(A)$ is a quasi-isomorphism.*
*(ii) $\chi_A$ is isomorphic to the polynomial algebra generated by a copy of $A^1$, translated to lie in degree zero.*
*(iii) There is a degree 1 $k$-map $q : \chi_A \to A^1$ which is the composite of the quotient homomorphism $\chi_A \to \gamma_A$ and an isomorphism $\gamma_A \to A^1$ and which makes the following diagram commute, where $\phi$ is the multiplication of $A$ and $\psi$ is the comultiplication of $\chi_A$:*

$$\begin{array}{ccccc} \chi_A & & \xrightarrow{\quad\psi\quad} & & \chi_A \otimes \chi_A \\ \Big\downarrow{\scriptstyle q} & & & & \Big\downarrow{\scriptstyle q\otimes q} \\ A^1 & \xrightarrow[d]{} & A^1 & \xleftarrow[\phi]{} & A^1 \otimes A^1 \end{array}$$

*(iv) $A$ can be identified with the DGA $\wedge(\gamma_A[-1])$.*

*Proof.* Let $A^{\#}$ be the underlying algebra of $A$, with differential zero. Filtering $\bar{B}(A)$ by homological degree, we obtain a spectral sequence that converges from the homology of $\bar{B}(A^{\#})$, which is $\mathrm{Tor}_{A^{\#}}(k,k)$, to the homology of $\bar{B}(A)$. Here the convergence of the spectral sequence follows by induction and passage to colimits from the generalized nilpotency of $A$. Since $A^{\#}$ is the exterior algebra generated by $A^1$, $\mathrm{Tor}_{A^{\#}}(k,k)$ is the divided polynomial algebra generated by a copy of $A^1$ concentrated in bidegree $(-1,1)$. Since $\mathrm{char}(k)=0$, a divided polynomial algebra is isomorphic to a polynomial algebra. The generators are permanent cycles, by obvious degree considerations, hence $E_2 = E_\infty$ . Thus the homology of $\bar{B}(A)$ is a polynomial algebra concentrated in degree zero since its associated graded algebra is a polynomial algebra with generators of bidegree $(-1,1)$. This proves (i) and (ii). We see from this argument that the elements of $A^1$, thought of as elements $[a]$ in $\bar{B}A$, extend to cycles by addition of summands of lower homological degree. The map $q$ sends a generating cycle "$[a]+$ lower terms" to $a$. That is, $q$ is induced from the homomorphism $\bar{B}A \to A^1$ that is the identity on $A^1$ and is zero on all elements other than those of degree 1 and homological degree -1. To compute the coproduct on generators of $\chi_A$, we must compute the coproduct on generating cycles. Observe that $q\otimes q$ annihilates all summands not of the form $[a']\otimes[a'']$ with $a',a''\in A^1$. With the notation of (2.8), the definition of the differential on $\bar{B}(A)$ forces our basic cycles to have the form

$$[a_r] - \sum_{p<q} k^r_{p,q}[a_p|a_q] + \text{ terms of lower homological degree.}$$

With a cancelling of signs, the form of the coproduct on $\bar{B}(A)$ implies the commutativity of the diagram in (iii), and part (iv) is now immediate by comparison with Lemma 2.7 and the details in the paragraph following it. □

*Proof of Proposition 5.10.* We have that $\chi(\gamma)$ is a sub Hopf algebra of $\bar{B}(\wedge(\gamma[-1]))$, and it lies in the subspace of elements of total degree zero. Using Remark 6.2, we see that this subspace may be identified with the tensor coalgebra $T(\gamma)$. Now all maps in the diagram of Proposition 5.9 are defined. By the universal property of tensor algebras, to show that the diagram commutes we need only show that it commutes when restricted to $\gamma^\vee$, and this is an easy verification from the definitions. □

## 7. The derived category of the heart and the 1-minimal model

We now turn to the proof of Theorem 1.3. Abbreviate $\mathscr{H} = \mathscr{H}_A$ and $\mathscr{D} = \mathscr{D}_A$. We must prove that $\mathscr{D}_{\mathscr{H}}$ is equivalent to $\mathscr{D}$ when $A = A\langle 1\rangle$. Let us first observe that Corollary 1.4 is an immediate consequence.

*Proof of Corollary 1.4.* By III.3.1, $H^q(A)(r) \cong \mathscr{D}(A, F^q(r))$, where $F^q(r)$ is the free $A$-module on one generator of bidegree $(q,r)$. By Theorem 1.3, this is isomorphic to $\mathscr{D}_{\mathscr{H}}(k, k(r)[-q])$, where $k$ and $k(r)$ are regarded as chain complexes concentrated in degree zero, and this is $\mathrm{Ext}^q_{\mathscr{H}}(k, k(r))$. □

To begin the proof of Theorem 1.3, we construct a functor $S : \mathscr{D}_{\mathscr{H}} \to \mathscr{D}$. For this, we need only assume that $A$ is connected. Consider a bounded below chain complex

$$M^* = \{M^n, \delta : M^n \to M^{n+1}\}$$

in $\mathscr{H}$. Since $\mathscr{H} \subset \mathscr{D}$, each $M^n$ is an $A$-module (with differential $d$) and each $\delta$ is a map of $A$-modules. Any such chain complex $M^*$ is quasi-isomorphic to a chain complex of minimal $A$-modules in $\mathscr{H}$, by Theorem 3.7 and the Whitehead theorem (III.2.3), hence we may assume without loss of generality that each $M^n$ is minimal. Then the differential on $M^n$ is specified by a generalized nilpotent twisting matrix $\mathbf{a^n}$ and $\delta$ is specified by matrices $\mathbf{k^n}$ such that $\mathbf{a^{n+1}k^n} = \mathbf{k^n a^n}$. We define a cell $A$-module $SM^*$, called the summation of $M^*$, with one $n$-cell for each 0-cell of $M^n$. We specify the differential on the canonical basis element $\langle j\rangle$ of an open $n$-cell by

$$d\langle j\rangle = \sum_{|i|=n+1} k^n_{i,j}\langle i\rangle + \sum_{|i|=n} a^n_{i,j}\langle i\rangle \tag{7.1}$$

where $|i|$ denotes the degree of a canonical basis element $\langle i\rangle$. If $N^*$ is a chain complex specified by matrices $\mathbf{b^n}$ and $\mathbf{l^n}$ and $f^* : M^* \to N^*$ is a chain map, then $f^*$ is given by matrices $\boldsymbol{\phi}^{\mathbf{n}}$ with entries in $k$ such that $\boldsymbol{\phi}^{\mathbf{n+1}}\mathbf{k^n} = \mathbf{l^n}\boldsymbol{\phi}^{\mathbf{n}}$. We define $Sf^* : SM^* \to SN^*$ by letting $Sf^*$ be prescribed by the matrix $\boldsymbol{\phi}^{\mathbf{n}}$ on the canonical basis for the open $n$-cells. If $f^*$ is a quasi-isomorphism of chain complexes, then $Sf^*$ is a quasi-isomorphism of $A$-modules by a little spectral sequence argument.

Now consider a general cell $A$-module $M$. If $|j| = n$, we can write

$$d\langle j\rangle = \sum_{|i|=n+1} k^n_{i,j}\langle i\rangle + \sum_{|i|=n} a^n_{i,j}\langle i\rangle + \sum_{|i|<n} b^n_{i,j}\langle i\rangle. \tag{7.2}$$

Note that $M$ is minimal if and only if all $k^n_{i,j} = 0$. On the other hand, the functor $S$ takes values in the subcategory of $\mathscr{D}$ consisting of those $M$ such that all $b^n_{i,j} = 0$. The following result is just an observation.

**Lemma 7.3.** *Let $M$ be an $A$-module with differential given by (6.2). If $b^n_{i,j} = 0$ for all $\langle i\rangle$ and $\langle j\rangle$, then $M$ is isomorphic to $SM^*$, where $M^n$ is the $A$-module in $\mathscr{H}$ specified by the twisting matrix $\mathbf{a^n}$ and where the differential $\delta^n : M^n \to M^{n+1}$ is specified by the matrix $\mathbf{k^n}$.*

A map of $A$-modules $g : SM^* \to SN^*$ is given on a canonical basis element $\langle i\rangle$ of $M$ of degree $n$ by

$$g\langle i\rangle = \sum_{|j|=n} \kappa^n_{j,i}\langle j\rangle + \sum_{|j|<n} \alpha^n_{j,i}\langle j\rangle. \tag{7.4}$$

Such a map is of the form $Sf^*$ if and only if all $\alpha^n_{j,i} = 0$.

To prove Theorem 1.3, we must show that, when $A = A\langle 1\rangle$, any cell $A$-module, with differential of the form (7.2), is quasi-isomorphic to some cell $A$-module with differential of the form (7.1). We shall exploit the following conceptual procedure for constructing $A$-modules of the form $SM^*$ out of general $A$-modules.

**Construction 7.5.** Let $\chi$ be a coalgebra (Adams graded, but concentrated in degree zero with respect to the ordinary grading) and suppose given a map $q : \chi \to A^1 \subset A$ of Adams graded $k$-modules such that the following diagram commutes, where $\phi$ is the multiplication of $A$ and $\psi$ is the comultiplication of $\chi$:

$$\begin{array}{ccccc} \chi & & \xrightarrow{\ \psi\ } & & \chi\otimes\chi \\ \downarrow{\scriptstyle q} & & & & \downarrow{\scriptstyle q\otimes q} \\ A & \xrightarrow{\ d\ } & A & \xleftarrow{\ \phi\ } & A\otimes A \end{array}$$

Let $M$ be an $A$-module. Define a new $A$-module $\beta(A,\chi,M)$ by letting $\beta(A,\chi,M)$ be $A\otimes\chi\otimes M$ as an $A$-module, with differential

$$d\otimes 1\otimes 1+(\phi(1\otimes q)\otimes 1\otimes 1)(1\otimes\psi\otimes 1)-(1\otimes 1\otimes\mu(q\otimes 1))(1\otimes\psi\otimes 1)+1\otimes 1\otimes d,$$

where $\mu : A\otimes M\to M$ is the action of $A$ on $M$.

A lengthy but purely formal diagram chase shows that $d^2=0$. The standard sign convention on tensor products of morphisms,

$$(f\otimes g)(x\otimes y)=(-1)^{\deg(g)\deg(x)}f(x)\otimes g(y),$$

is used; observe that this implies, for example, that $(1\otimes d)(d\otimes 1)=-d\otimes d$.

The following special case will lead to the proof of Theorem 1.3. We assume that $A=A\langle 1\rangle$ in the rest of this section.

**Definition 7.6.** As in Theorem 6.2(iii), let $q:\chi_A\to A$ be the composite of the quotient map from $\chi_A$ to $\gamma_A$ and the evident identification of $\gamma_A$ with $\gamma_A[-1]$. Define a functor $R$ from $A$-modules to $A$-modules by

$$R(M)=\beta(A,\chi_A,M).$$

Another little spectral sequence argument shows that the functor $R$ preserves quasi-isomorphisms, and this will also follow from Proposition 7.8 below.

**Proposition 7.7.** *Let $A=A\langle 1\rangle$ and let $M$ be a cell $A$-module. Then $R(M)$ is a cell $A$-module whose differential is given by formula (7.1). If $f:M\to N$ is a map of cell $A$-modules, then $g=R(f)$ is a map of cell $A$-modules such that the coefficients $\alpha^n_{j,i}$ in (7.4) are zero. Therefore $R$ induces a functor $R':\mathscr{D}\to\mathscr{D}_{\mathscr{H}}$ such that $R=SR'$.*

*Proof.* We must specify a sequential filtration $\{F_t\beta(A,\chi_A,M)\}$. We are given a sequential filtration $\{F_tM\}$ of $M$. Let $J$ be the subspace of $M$ spanned by the basis elements of its open cells, so that $M=A\otimes J$ as an $A$-module. We have an induced filtration $\{F_tJ\}$ such that $F_tM=A\otimes F_tJ$. We also have the nilpotent filtration $\{F_tA\}$ of $A$, namely $F_tA=A\langle 1,t\rangle$ in the notation of Definition 2.2. Via the tensor product filtration of the summands $(IA)^p$, there results a filtration of the bar construction $\bar{B}(A)$ and thus a filtration of its subspace $\chi_A$; here $F_0\chi_A=k$. The filtration of $A$ has the property that, for any element $a$, $d(a)=\sum a'a''$ with each $a'$ and $a''$ of strictly lower filtration than $a$. The filtration of $\chi_A$ has the property that, for any element $x$, $\psi(x)=x\otimes 1+1\otimes x+\sum x'\otimes x''$ with each $x'$ and $x''$ of strictly lower filtration than $x$. We define

$$F_t\beta(A,\chi_A,M)=A\otimes F_t(\chi_A\otimes A\otimes J),$$

where the filtration $\{F_t(\chi_A\otimes A\otimes J)\}$ must still be specified. Note first that the tensor product of the three filtrations just specified does not work because, in the differential (7.2) on $M$, we have no control on the filtrations of the $a^n_{i,j}$ and $b^n_{i,j}$. Rather, thinking of the filtration as given by a partial ordering of basis elements, we define a lexicographic filtration by first taking the filtration on $J$, next the filtration on $\chi_A$, and last the filtration on $A$. Formally, this involves an arbitrary choice of total ordering of the lexicographically ordered set of triples $(q,r,s)$ of non-negative integers. The elements of filtration $t$ are linear combinations of elements $x\otimes a\otimes\langle j\rangle$ such that if $t$ corresponds to $(q,r,s)$, then $\langle j\rangle\in F_qJ$, $x\in F_r\chi_A$, and $a\in F_sA$. We obtain a basis for the open cells of filtration $t$ by extending a basis of $F_{t-1}(\chi_A\otimes A\otimes J)$ to a basis of $F_t(\chi_A\otimes A\otimes J)$. In Construction 7.5, of the four summands of the differential, the first is just the differential on $A$ in the free $A$-module structure, the second gives the decomposable summands $a^n_{i,j}\langle i\rangle$ in (7.1), and the third and fourth both give indecomposable summands $k^n_{i,j}\langle i\rangle$. The statement about maps is clear and the last statement follows from Lemma 7.3. □

The following two results complete the proof of Theorem 1.3 by showing that the functors $R : \mathscr{D} \to \mathscr{D}_{\mathscr{H}}$ and $S : \mathscr{D}_{\mathscr{H}} \to \mathscr{D}$ are inverse equivalences of categories.

**Proposition 7.8.** *Let $M$ be a cell $A$-module. Then there is a natural quasi-isomorphism $RM = SR'M \to M$.*

*Proof.* With the signs given in Remark 6.1, we have an $A$-module $B(A, A, M)$. As noted in II.4.2, there is a natural map of $A$-modules $\epsilon : B(A, A, M) \to M$ that is a chain homotopy equivalence and thus a quasi-isomorphism. It suffices to construct a quasi-isomorphism

$$\iota : R(M) = \beta(A, \chi_A, M) \to B(A, A, M).$$

Additively, $B(A, A, M) = A \otimes \bar{B}(A) \otimes M$, and $\chi_A$ is contained in $\bar{B}(A)$ as its sub Hopf algebra of cycles of total degree zero. The resulting inclusion

$$\chi_A \otimes M \to \bar{B}(A) \otimes M$$

extends to the desired map $\iota$ of $A$-modules (but the extension involves insertion of the sign dictated by Remark 6.1).

We must show that $\iota$ commutes with the differentials. In homological degree $p$, the differential on the subspace $\bar{B}(A) \otimes M$ of $B(A, A, M)$ can be written as the sum of the following four terms:
(i) The zeroth face operator $d_0$.
(ii) The last face operator $(-1)^p d_p$.
(iii) $(-1)^p(1 \otimes d)$, where $d$ is the differential on $M$.
(iv) $d \otimes 1$, where $d$ is the differential on the chain complex $\bar{B}(A)$.
Observe that, in $\bar{B}(A)$ itself, the zeroth and last face operators are zero. When we restrict to $\chi(A) \otimes M$, the term (iv) is zero. An inspection of definitions shows that the remaining three terms sum to the differential on the subspace $\chi_A \otimes M$ of $\beta(A, \chi_A, M)$, the essential point being that the zeroth and last faces in the bar construction can be written in terms of its coproduct in the fashion given in Construction 7.5. The rest of the verification that $\iota$ commutes with differentials is just a check of signs.

Finally, we must prove that $\iota$ is a quasi-isomorphism. Filter the source and target of $\iota$ by the sum of the degrees of the first coordinate $A$ and last coordinate $M$; that is, $a \otimes x \otimes m$ is in $F^t$ if $\deg(a) + \deg(m) \geq t$. Then the differential on the $E_1$-term of the resulting spectral sequence for $\beta(A, \chi_A, M)$ is zero, while the differential on the $E_1$-term for $B(A, A, M)$ is induced by term (iv) above. Therefore the induced map of $E_2$-terms is an isomorphism by Theorem 6.3(i). □

**Proposition 7.9.** *Let $M^*$ be a chain complex of minimal $A$-modules in $\mathscr{H}$. Then there is a natural quasi-isomorphism $M^* \to R'SM^*$ of chain complexes in $\mathscr{H}$.*

*Proof.* We change our point of view. Let $V^*$ be the chain complex of $\chi_A$-modules that corresponds to $M^*$ under the equivalence of categories given in Theorem 1.2 and let $\nu : V^n \to \chi_A \otimes V^n$ be the coaction. Observe that, as an $A$-module,

$M^n = A \otimes V^n$. Let $\omega^*$ be the composite

$$\omega^* : V^* \xrightarrow{\nu} \chi_A \otimes V^* = \chi_A \otimes k \otimes V^* \xrightarrow{1\otimes\eta\otimes 1} \chi_A \otimes A \otimes V^*,$$

where $\eta$ is the unit of $A$. We claim that $\omega^*$ is a quasi-isomorphism from $V^*$ to the chain complex of $\chi_A$-comodules that corresponds to $R'SM^*$. On translation back to $\mathscr{H}$, this will imply the result.

We must first show that $\omega^*$ is a map of $\chi_A$-comodules. Clearly $\chi_A \otimes A \otimes V^n$ may be identified with the $k$-module of indecomposable elements of $R'SM^n$. The coaction of $\chi_A$ arises in the manner described above Proposition 5.11 from the coaction of $\gamma_A$, and this arises from the decomposable portion of the differential (7.1). This portion comes from the second term of the differential in Construction 7.5, which reduces on $\chi_A \otimes M$ to $(q\otimes 1\otimes 1)(\psi\otimes 1)$. This implies that the coaction on $\chi_A \otimes A \otimes V^n$ is the obvious one induced by the diagonal map on $\chi_A$. It is now clear from the relation $(\psi\otimes 1)\nu = (1\otimes\nu)\nu$ that $\omega^*$ is a map of $\chi_A$-comodules.

We must next show that $\omega^*$ is a map of chain complexes. The differential on the chain complex of $\chi_A$-comodules that corresponds to $R'SM^*$ is given by the indecomposable portion of the differential (7.1), applied to $RSM^*$. This portion comes from the last two terms of the differential in Construction 7.5, which reduce on $\chi_A \otimes M$ to the sum of

(i) $-(1\otimes\mu(q\otimes 1))(\psi\otimes 1)$

and $1\otimes d$. With $M$ replaced by $A\otimes V^n$, regarded as part of $SM^*$, the factor $d : A\otimes V^n \to A\otimes V^n$ in the summand $1\otimes d$ is itself the sum of the following three terms:

(ii) $d\otimes 1$, where $d$ is the differential on $A$.

(iii) The decomposable part of the differential (7.1) on $V^n \subset M^n$, which is given by the coaction of $\gamma_A$ on $V^n$.

(iv) The indecomposable part of the differential (7.1) on $M^n$, which is given by the differential $V^n \to V^{n+1}$.

On the image of $\omega^*$, the term (ii) obviously vanishes, the term (i) reduces to $-(1\otimes q\otimes 1)(\psi\otimes 1)$, and the sum of the terms (i) and (iii) is zero by a little diagram chase based on the identity $(\psi\otimes 1)\nu = (1\otimes\nu)\nu$. Thus the differential on the image of $\omega^*$ is given by (iv), and it follows that $\omega^*$ is a map of chain complexes.

It remains to prove that $\omega^*$ is a quasi-isomorphism. To see this, assume first that the coaction of $\gamma_A$ on each $V^n$ is zero, so that term (iii) vanishes. The inclusion of $\chi_A$ in $\bar{B}(A)$ induces an inclusion

$$\iota^n : \chi_A \otimes A \otimes V^n \to \bar{B}(A)\otimes A\otimes V^n = B(k, A, A\otimes V^n).$$

The differential on the target is the sum of three terms: the differential on $\bar{B}(A)$, the part of the differential coming from the last face operator, and the differential on the factor $A$ of $A\otimes V^n$. The first of these is zero on $\chi_A$, and the second and third agree under the inclusion with the terms (i) and (ii). Thus $\iota$ is a map of chain complexes. Filtering by degrees in $A\otimes V^n$, we see by a little spectral sequence argument that $\iota$ is a quasi-isomorphism because $\chi_A \to \bar{B}(A)$ is a quasi-isomorphism. Since $V^n$ is just a $k$-module, we have the standard quasi-isomorphism

$$\epsilon^n : B(k, A, A\otimes V^n) \to V^n.$$

The composite $\epsilon^n\iota^n\omega^n : V^n \to V^n$ is the identity map. So far we have ignored the differential $V^n \to V^{n+1}$, but if we filter $\chi_A\otimes A\otimes V^*$ and $\bar{B}(A)\otimes A\otimes V^*$ by degrees in $V^*$, then the differential on the resulting $E_1$-terms is that obtained by ignoring the differential in $V^*$ and, on $E_2$-terms, we obtain copies of the chain complex $V^*$. Thus $\epsilon^*$, $\iota^*$, and $\omega^*$ are quasi-isomorphisms when the $V^n$ are trivial representations of $\gamma_A$.

Finally, we must take account of the coaction of $\gamma_A$. The $V^n$ are generalized nilpotent representations of $\gamma_A$. Since the nilpotent filtration of Definition 5.5 is natural, $V^*$ is the union of its subcomplexes $F_tV^*$, and the quotients $F_tV^*/F_{t-1}V^*$ are complexes of trivial representations. Therefore $\omega^*$ is a quasi-isomorphism in general. □

## **Part** V. **Derived categories of modules over $E_\infty$ algebras**

Let $k$ be a commutative ring and let $\mathscr{C}$ be an $E_\infty$ operad of differential graded $k$-modules. We defined $\mathscr{C}$-algebras and modules over $\mathscr{C}$-algebras in Part I, and we showed how to convert partial $\mathscr{C}$-algebras and modules into genuine $\mathscr{C}$-algebras and modules in Part II. Interesting examples arise in both topology and algebraic geometry.

In this part, we will demonstrate that the derived category of modules over a $\mathscr{C}$-algebra has the same kind of structure as the derived category of modules over a commutative DGA. The essential point is that there is a derived tensor product that satisfies all of the usual properties, but even the rigorous construction of the derived category will require a little work. The standard tools of projective resolutions and derived functors are not present here, and our theory is based on the non-standard approach to the classical derived categories of DGA's that we presented in Part III.

As discussed in Part IV, our original motivation came from Deligne's suggestion [20] that the derived category of modules over the $E_\infty$ algebra $\mathscr{N}(Spec(F))$ that we associated to Bloch's higher Chow complex $\mathfrak{Z}(Spec(F))$ in II§5 is an appropriate derived category of integral mixed Tate motives of $F$. We shall say nothing more about that here. We are confident that the present theory will have other applications. It has been developed in parallel with a precisely analogous, but more difficult, theory of derived categories of modules over $E_\infty$ ring spectra in algebraic topology [25], and that theory has already had very substantial applications.

As in Part I, $k$-modules will mean differential $\mathbb{Z}$-graded $k$-modules, except that, as in Parts III and IV, our $k$-modules will have a second "Adams" grading (which will never introduce signs) and will be graded cohomologically. We let $\mathscr{M}_k$ denote the category of such $k$-modules. In contrast with Part II, it is essential not to restrict attention to flat $k$-modules in this part. Each of the $k$-modules $\mathscr{C}(j)$ of an operad $\mathscr{C}$ is concentrated in Adams grading zero. In view of our cohomological grading, the ordinary grading of the $\mathscr{C}(j)$ is concentrated in negative degrees. When $\mathscr{C}$ is an $E_\infty$ operad, $\mathscr{C}(j)$ is a free $k[\Sigma_j]$-resolution of $k$.

Although our interest is in modules over general $E_\infty$ algebras, we shall first concentrate on the study of "$E_\infty$ modules" over the ground ring $k$. In fact, the theory of this part is based on the idea of changing underlying ground categories from the category of ordinary $k$-modules to that of $E_\infty$ $k$-modules.

For a given operad $\mathscr{C}$, we agree to write $\mathbb{C} = \mathscr{C}(1)$ for brevity. Clearly $\mathbb{C}$ is a differential graded $k$-algebra via $\gamma : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$; it is usually not commutative, but it is homotopy commutative when $\mathscr{C}$ is an $E_\infty$ operad. For a unital operad $\mathscr{C}$, it is easy to see that the category of $\mathbb{C}$-modules can be identified with the category of operadic $k$-modules of I.4.1, where we regard $k$ as a $\mathscr{C}$-algebra via the augmentation

$\mathscr{C} \to \mathscr{N}$ of I.2.2(iii). In fact, what is equivalent, $\mathbb{C}$ coincides with the universal enveloping algebra of $k$ as defined in I.4.9.

Recall that the derived category $\mathscr{D}_k$ of (differential graded) $k$-modules is obtained from the homotopy category of $k$-modules by adjoining formal inverses to the quasi-isomorphisms. Similarly, we have the derived category $\mathscr{D}_{\mathbb{C}}$ of $\mathbb{C}$-modules. If $\mathbb{C} \to k$ is a quasi-isomorphism, then, by III.4.2, the categories $\mathscr{D}_k$ and $\mathscr{D}_{\mathbb{C}}$ are equivalent. When $\mathscr{C}$ is an $E_\infty$ operad, we think of $\mathbb{C}$-modules as $E_\infty$ $k$-modules.

As we explain in Section 1, there is a particularly convenient choice of an $E_\infty$ operad $\mathscr{C}$, and there is no loss of generality if we restrict attention to that choice. The proofs of these claims are deferred until Section 9. We agree to work with this particular $E_\infty$ operad $\mathscr{C}$ throughout the rest of the part. With this choice, we find that the category of $\mathbb{C}$-modules admits an associative and commutative "tensor product", which we denote by $\boxtimes$ to distinguish it from $\otimes = \otimes_k$. Since $\mathbb{C}$ is not commutative, the existence of the operation $\boxtimes$ is a remarkable phenomenon. Under the equivalence between $\mathscr{D}_{\mathbb{C}}$ and $\mathscr{D}_k$, the new derived tensor product $\boxtimes$ agrees with the derived tensor product $\otimes$. Similarly, there is an internal Hom functor on the category of $\mathbb{C}$-modules that agrees with the usual Hom under the equivalence between $\mathscr{D}_{\mathbb{C}}$ and $\mathscr{D}_k$.

In Section 2, we study phenomena connected with the fact that $k$ is not a unit for $\boxtimes$, although there is a natural unit map $\lambda : k \boxtimes M \to M$. We define certain variants of the new tensor product of $\mathbb{C}$-modules that apply when one or both of the given $\mathbb{C}$-modules $M$ is unital, in the sense that it comes with a prescribed map $k \to M$. We write $M \lhd N$ and $N \rhd M$ for the new tensor products of a unital $\mathbb{C}$-module $M$ and a non-unital $\mathbb{C}$-module $N$, and we write $M \boxdot N$ for the new tensor product defined when both $M$ and $N$ are unital. The product $\boxdot$ is associative, commutative, and unital up to coherent natural isomorphism. Thus we have a symmetric monoidal category of unital $\mathbb{C}$-modules, which we denote by $\mathscr{M}^u_{\mathbb{C}}$.

In Section 3, we prove that $A_\infty$ and $E_\infty$ algebras, defined with respect to the particular $E_\infty$ operad $\mathscr{C}$, are exactly the monoids and commutative monoids in the symmetric monoidal category $\mathscr{M}^u_{\mathbb{C}}$. This drastically simplifies the study of these algebraic structures. We also give an appropriate analog for modules over such algebras.

With these preliminaries, we can proceed in precise analogy with the theory of Part III. In fact, we find in Section 4 that the theory of cell modules over a DGA generalizes verbatim to give a theory of cell modules over an $A_\infty$ algebra $A$. The only change is that the free functor from $k$-modules to $A$-modules has a different description. We define and study the tensor product of modules over $A$ in Section 5. We define and study the concomitant Hom functor in Section 6. We also describe the variants of the tensor product for unital $A$-modules and prove that quasi-isomorphic $A_\infty$ algebras have equivalent derived categories there. In Section 7, we define generalized Tor and Ext groups as the homology groups of derived tensor product and Hom modules, and we construct Eilenberg-Moore spectral sequences for their calculation in terms of ordinary Tor and Ext groups. The conclusions are precisely the same as if $A$ were a DGA. In Section 8, we specialize to $E_\infty$ algebras. Here our tensor product of $A$-modules is again an $A$-module, and similarly for Hom. The discussion of duality in Part III carries over directly to the $E_\infty$ context.

## 1. The category of $\mathbb{C}$-modules and the product $\boxtimes$

For the moment, let $\mathscr{C}$ be any operad. Since $\mathbb{C} = \mathscr{C}(1)$ is a DGA, the theory of Part III applies to it. The free functor $F$ from $k$-modules to $\mathbb{C}$-modules is given by $FM = \mathbb{C} \otimes M$, and the free $\mathbb{C}$-modules generated by suspensions of $k$ play the role of sphere $\mathbb{C}$-modules. The derived category $\mathscr{D}_{\mathbb{C}}$ is equivalent to the homotopy category of cell $\mathbb{C}$-modules. When $\mathscr{C}$ is unital and the augmentation $\epsilon : \mathbb{C} \to k$ is a quasi-isomorphism, the derived categories $\mathscr{D}_k$ and $\mathscr{D}_{\mathbb{C}}$ are equivalent. A key point is that the action $\mathbb{C} \otimes M \to M$ is then a quasi-isomorphism for any $\mathbb{C}$-module $M$.

Via instances of the structural maps $\gamma$, we have a left action of $\mathbb{C}$ and a right action of $\mathbb{C}\otimes\mathbb{C}$ on $\mathscr{C}(2)$, and these actions commute with each other. Thus we have a bimodule structure on $\mathscr{C}(2)$. Let $M$ and $N$ be left $\mathbb{C}$-modules. Clearly $M \otimes N$ is a left $\mathbb{C} \otimes \mathbb{C}$-module via the given actions. This makes sense of the following definition.

**Definition 1.1.** For $\mathbb{C}$-modules $M$ and $N$, define $M \boxtimes N$ to be the $\mathbb{C}$-module

$$M \boxtimes N = \mathscr{C}(2) \otimes_{\mathbb{C}\otimes\mathbb{C}} M \otimes N.$$

We have a Hom functor on $\mathbb{C}$-modules that is related to the tensor product $\boxtimes$ by an adjunction of the usual form. In fact, the desired adjunction dictates the definition.

**Definition 1.2.** Let $M$ and $N$ be (left) $\mathbb{C}$-modules. Define

$$\mathrm{Hom}^{\boxtimes}(M, N) = \mathrm{Hom}_{\mathbb{C}}(\mathscr{C}(2) \otimes_{\mathbb{C}} M, N).$$

Here, when forming $\mathscr{C}(2)\otimes_{\mathbb{C}} M$, $\mathbb{C}$ acts on $\mathscr{C}(2)$ through $\eta\otimes\mathrm{Id} : \mathbb{C} = k\otimes\mathbb{C} \to \mathbb{C}\otimes\mathbb{C}$; when forming $\mathrm{Hom}_{\mathbb{C}}$, $\mathbb{C}$ acts on $\mathscr{C}(2) \otimes_{\mathbb{C}} M$ via its left action on $\mathscr{C}(2)$. The right action of $\mathbb{C}$ on $\mathscr{C}(2)$ through $\mathrm{Id} \otimes \eta : \mathbb{C} = \mathbb{C} \otimes k \to \mathbb{C} \otimes \mathbb{C}$ induces a left action of $\mathbb{C}$ on $\mathrm{Hom}^{\boxtimes}(M, N)$.

**Lemma 1.3.** *There is a natural adjunction isomorphism*

$$\mathscr{M}_{\mathbb{C}}(L \boxtimes M, N) \cong \mathscr{M}_{\mathbb{C}}(L, \mathrm{Hom}^{\boxtimes}(M, N)).$$

We must consider the commutativity, associativity, and unity properties of the product $\boxtimes$.

**Lemma 1.4.** *There is a canonical commutativity isomorphism of $\mathbb{C}$-modules*

$$\tau : M \boxtimes N \longrightarrow N \boxtimes M.$$

*Proof.* Use the action of the transposition $\sigma \in \Sigma_2$ on $\mathscr{C}(2)$ together with the transposition isomorphisms $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ and $M \otimes N \to N \otimes M$. □

The following result is fundamental to our work. It comes from our parallel topological work with Elmendorf [25]. We defer the proof to Section 9. Note that $k$ is a $\mathbb{C}$-module via the augmentation $\mathbb{C} \to k$.

**Theorem 1.5.** *There is an $E_\infty$ operad $\mathscr{C}$, called the "linear isometries operad", for which there is a canonical associativity isomorphism of $\mathbb{C}$-modules*

$$(L \boxtimes M) \boxtimes N \cong L \boxtimes (M \boxtimes N).$$

*In fact, for any $j$-tuple $M_1, \ldots, M_j$ of $\mathbb{C}$-modules, there is a canonical isomorphism*

$$M_1 \boxtimes \cdots \boxtimes M_j \cong \mathscr{C}(j) \otimes_{\mathbb{C}^j} (M_1 \otimes \cdots M_j),$$

*where the iterated product on the left is associated in any fashion. Moreover, for $j \geq 2$, the $j$-fold $\boxtimes$-power $\mathbb{C}^{\boxtimes j}$ is isomorphic to $\mathscr{C}(j)$ as a $(\mathbb{C}, \mathbb{C}^j)$-bimodule, and $\mathscr{C}(j)$ is isomorphic to $\mathbb{C}$ as a left $\mathbb{C}$-module.*

**Lemma 1.6.** *There is a natural map of $\mathbb{C}$-modules $\lambda : k \boxtimes N \to N$. The symmetrically defined map $M \boxtimes k \to M$ coincides with the composite $\lambda\tau$. Moreover, under the associativity isomorphism,*

$$\lambda\tau \boxtimes \mathrm{Id} = \mathrm{Id} \boxtimes \lambda : M \boxtimes k \boxtimes N \longrightarrow M \boxtimes N.$$

*Proof.* The degeneracy map $\sigma_1 : \mathscr{C}(2) \to \mathbb{C}$ of I.3.5, $\epsilon \otimes Id : \mathbb{C} \otimes \mathbb{C} \to k \otimes \mathbb{C} \cong \mathbb{C}$, and the isomorphism $k \otimes N \cong N$ together give the required map $\lambda : k \boxtimes N \to \mathbb{C} \otimes_{\mathbb{C}} N \cong N$. The symmetry is clear. Under the isomorphisms of their domains with $\mathscr{C}(3) \otimes M \otimes k \otimes N$, both $\lambda\tau \boxtimes \mathrm{Id}$ and $\mathrm{Id} \boxtimes \lambda$ agree with the tensor product over $\mathrm{Id} \otimes \varepsilon \otimes \mathrm{Id}$ of $\sigma_2 : \mathscr{C}(3) \longrightarrow \mathscr{C}(2)$ and the isomorphism $M \otimes k \otimes N \cong M \otimes N$. □

In our motivating examples from algebraic geometry, we started with partial algebras and converted them to $\mathscr{C}$-algebras, where $\mathscr{C}$ was an arbitrarily chosen $E_\infty$ operad. Clearly, we may as well choose $\mathscr{C}$ to be the linear isometries operad. However, we have the following result. Its proof is a bar construction argument similar to those used in Part II; we defer it to Section 9.

**Theorem 1.7.** *Let $\mathscr{C}$ and $\mathscr{C}'$ be any two $E_\infty$ operads. There is a functor $V$ that assigns a quasi-isomorphic $\mathscr{C}'$-algebra $VA$ to a $\mathscr{C}$-algebra $A$. There is also a functor $V$ that assigns a quasi-isomorphic $VA$-module $VM$ to an $A$-module $M$.*

We construct the derived category of $A$-modules from the homotopy category of $A$-modules by adjoining formal inverses to the quasi-isomorphisms, where a map of $A$-modules is a quasi-isomorophism if it induces an isomorphism on homology, that is, if it is a quasi-isomorphism when regarded as a map of $k$-modules. The theorem can be elaborated to give an equivalence of the derived category of $A$-modules with the derived category of $VA$-modules.

Thus there is no loss of generality if we restrict attention to the linear isometries $E_\infty$ operad $\mathscr{C}$, and we do so throughout the rest of the part. We repeat that $\mathbb{C}$ is an abbreviated notation for $\mathscr{C}(1)$. By use of cell approximations of $\mathbb{C}$-modules, the product $\boxtimes$ induces a derived tensor product, again denoted $\boxtimes$, on $\mathscr{D}_{\mathbb{C}}$. We have the following important consistency statement.

**Proposition 1.8.** *Let $N$ be a cell $\mathbb{C}$-module.*

(i) *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of $\mathbb{C}$-modules, where $M''$ is a cell $\mathbb{C}$-module, then*

$$0 \longrightarrow M' \boxtimes N \longrightarrow M \boxtimes N \longrightarrow M'' \boxtimes N \longrightarrow 0$$

*is an exact sequence of $\mathbb{C}$-modules.*

(ii) *If $f : M \to M'$ is a quasi-isomorphism of $\mathbb{C}$-modules, then*

$$f \boxtimes \mathrm{Id} : M \boxtimes N \longrightarrow M' \boxtimes N$$

*is a quasi-isomorphism of $\mathbb{C}$-modules.*

(iii) *$M \boxtimes N$ is quasi-isomorphic as a $k$-module to $M \otimes N$.*

*Therefore the equivalence of derived categories $\mathscr{D}_{\mathbb{C}} \to \mathscr{D}_k$ that is induced by the forgetful functor from $\mathbb{C}$-modules to $k$-modules carries $\boxtimes$ to $\otimes$.*

*Proof.* The proof is a bit devious, and steps must be taken in the right order. For (i), observe that, with differential ignored, $M''$ is a free $\mathbb{C}$-module and the given sequence is therefore split exact. Upon tensoring with $N$ we obtain an algebraically split exact sequence of $\mathbb{C}^2$-modules and the conclusion follows. We next start on (iii). Choose a degree zero cycle $x \in \mathscr{C}(2)$ that augments to $1 \in \mathbb{C}$ and note that $x$ cannot be a boundary. Clearly $x$ determines a homotopy equivalence $k \to \mathscr{C}(2)$ with homotopy inverse given by the augmentation $\mathscr{C}(2) \to k$. This equivalence and the definition of $\boxtimes$ give us the two maps of $k$-modules

$$M \otimes N \longrightarrow \mathscr{C}(2) \otimes M \otimes N \longrightarrow M \boxtimes N,$$

the first of which is a homotopy equivalence. Assume to begin with that $M$ as well as $N$ is a cell $\mathbb{C}$-module. The displayed functors commute with colimits, and we see by induction up the product of the sequential filtrations and passage to colimits that the second arrow will be a quasi-isomorphism in this case if it is a quasi-isomorphism for all sphere $\mathbb{C}$-modules $M$ and $N$. However, if $M = FK = \mathbb{C} \otimes K$ and $N = \mathbb{C} \otimes L$ for $k$-modules $K$ and $L$, then

$$M \boxtimes N \cong \mathscr{C}(2) \otimes K \otimes L$$

and the second arrow reduces to the homotopy equivalence

$$\mathscr{C}(2) \otimes \mathscr{C}(1) \otimes K \otimes \mathscr{C}(1) \otimes L \longrightarrow \mathscr{C}(2) \otimes K \otimes L$$

induced by the homotopy equivalence $\gamma : \mathscr{C}(2) \otimes \mathscr{C}(1) \otimes \mathscr{C}(1) \longrightarrow \mathscr{C}(2)$. In Section 7, part (i) and the special case of (iii) just proven will be used to construct a spectral sequence that converges from $\mathrm{Tor}_k^{*,*}(H^*(M), H^*(N))$ to $H^*(M \boxtimes N)$, where $N$ but not necessarily $M$ is a cell $\mathbb{C}$-module. The spectral sequence directly implies (ii), and the special case of (iii) already proven now implies the general case of (iii) by cellular approximation of $M$. The naturality of the isomorphism obtained on passage to derived categories is clear from the proof. □

Although the unit map $\lambda : k \boxtimes N \to N$ is not an isomorphism in general, it induces a natural isomorphism on the level of derived categories.

**Corollary 1.9.** *If $N$ is a cell $\mathbb{C}$-module, then the unit map $\lambda : k \boxtimes N \to N$ is a quasi-isomorphism. Therefore $\lambda$ induces a natural isomorphism $k \boxtimes N \to N$ of functors on the derived category $\mathscr{D}_{\mathbb{C}}$.*

*Proof.* Consider the following commutative diagram:

$$\begin{array}{ccccc}
k \otimes \mathbb{C} \otimes N & \xleftarrow{\epsilon\otimes \mathrm{Id}} & \mathscr{C}(2) \otimes \mathbb{C} \otimes N & \longrightarrow & \mathbb{C} \boxtimes N \\
\downarrow{\scriptstyle \mathrm{Id}\otimes\epsilon\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}\otimes\epsilon\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \epsilon\boxtimes\mathrm{Id}} \\
k \otimes N & \xleftarrow{\epsilon\otimes \mathrm{Id}} & \mathscr{C}(2) \otimes k \otimes N & \longrightarrow & k \boxtimes N \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \sigma_1\otimes\mathrm{Id}} & & \downarrow{\scriptstyle \lambda} \\
N & \xleftarrow[\epsilon\otimes \mathrm{Id}]{} & \mathscr{C}(1) \otimes N & \xrightarrow[\nu]{} & N.
\end{array}$$

Here $\nu$ is the action of $\mathscr{C}(1)$ on $N$. By Proposition 1.8 and its proof, all arrows except $\lambda$ are quasi-isomorphisms, hence so is $\lambda$. □

We need a lemma to obtain the analog of Proposition 1.8 for $\mathrm{Hom}^{\boxtimes}$.

**Lemma 1.10.** *For $K$-modules $K$ and $L$, there are isomorphisms of $\mathbb{C}$-modules*

$$FK \boxtimes FL \cong F(K \otimes L) \quad \text{and} \quad \operatorname{Hom}(K,L) \cong \operatorname{Hom}^{\boxtimes}(FK,L).$$

*For cell $\mathbb{C}$-modules $M$ and $N$, $M \boxtimes N$ is a cell $\mathbb{C}$-module.*

*Proof.* The first isomorphism is immediate from the isomorphism $\mathscr{C}(2) \cong \mathscr{C}(1)$ given by the last statement of Theorem 1.5. The second follows in view of the chain of natural isomorphisms

$$\begin{aligned}\mathscr{M}_k(K', \operatorname{Hom}^{\boxtimes}(FK,L)) &\cong \mathscr{M}_{\mathbb{C}}(FK', \operatorname{Hom}^{\boxtimes}(FK,L))\\ &\cong \mathscr{M}_{\mathbb{C}}(FK' \boxtimes FK, L) \cong \mathscr{M}_{\mathbb{C}}((F(K' \otimes K), L)\\ &\cong \mathscr{M}_k(K' \otimes K, L) \cong \mathscr{M}_k(K', \operatorname{Hom}(K,L)).\end{aligned}$$

As in III.1.5(iii) or III.5.1, the last statement follows from the first isomorphism. □

**Proposition 1.11.** *Let $N$ be an arbitrary $\mathbb{C}$-module.*

(i) *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of $\mathbb{C}$-modules, where $M''$ is a cell $\mathbb{C}$-module, then*

$$0 \longrightarrow \operatorname{Hom}^{\boxtimes}(M'',N) \longrightarrow \operatorname{Hom}^{\boxtimes}(M,N) \longrightarrow \operatorname{Hom}^{\boxtimes}(M',N) \longrightarrow 0$$

*is an exact sequence of $\mathbb{C}$-modules.*

(ii) *If $M$ is a cell $\mathbb{C}$-module and $f : N \to N'$ is a quasi-isomorphism of $\mathbb{C}$-modules, then*

$$\operatorname{Hom}^{\boxtimes}(\mathrm{Id}, f) : \operatorname{Hom}^{\boxtimes}(M,N) \longrightarrow \operatorname{Hom}^{\boxtimes}(M,N')$$

*is a quasi-isomorphism of $\mathbb{C}$-modules.*

(iii) *There is an induced adjunction isomorphism*

$$\mathscr{D}_{\mathbb{C}}(L \boxtimes M, N) \cong \mathscr{D}_{\mathbb{C}}(L, \operatorname{Hom}^{\boxtimes}(M,N)).$$

(iv) *If $M$ is a cell $\mathbb{C}$-module, then $\operatorname{Hom}^{\boxtimes}(M,N)$ is quasi-isomorphic as a $k$-module to $\operatorname{Hom}(M,N)$.*

*Therefore the equivalence of derived categories $\mathscr{D}_{\mathbb{C}} \to \mathscr{D}_k$ that is induced by the forgetful functor from $\mathbb{C}$-modules to $k$-modules carries $\operatorname{Hom}^{\boxtimes}$ to $\operatorname{Hom}$.*

*Proof.* Part (i) is clear since the given sequence splits as a sequence of $\mathbb{C}$-modules with differential ignored. Parts (ii) and (iii) follow formally from the lemma; see III.4.5. The functor $\operatorname{Hom}^{\boxtimes}(M,N)$ does not preserve quasi-isomorphisms in $M$, and, in the derived category $\mathscr{D}_{\mathbb{C}}$, $\operatorname{Hom}^{\boxtimes}(M,N)$ means $\operatorname{Hom}^{\boxtimes}(\Gamma M,N)$ where $\Gamma M$ is a cell approximation to $M$. If $M$ is a cell $\mathbb{C}$-module and $K$ is a $k$-module, the quasi-isomorphism $FK \otimes M \longrightarrow FK \boxtimes M$ of Proposition 1.8 and the natural quasi-isomorphism $K \longrightarrow FK$ give rise to the composite

$$\begin{aligned}\mathscr{M}_k(K, \operatorname{Hom}^{\boxtimes}(M,N)) &\cong \mathscr{M}_{\mathbb{C}}(FK, \operatorname{Hom}^{\boxtimes}(M,N))\\ &\cong \mathscr{M}_{\mathbb{C}}(FK \boxtimes M, N) \to \mathscr{M}_k(FK \boxtimes M, N)\\ &\cong \mathscr{M}_k(FK \otimes M, N) \cong \mathscr{M}_k(FK, \operatorname{Hom}(M,N))\\ &\to \mathscr{M}_k(K, \operatorname{Hom}(M,N)).\end{aligned}$$

Since the two arrows are induced by quasi-isomorphisms, the composite induces a natural isomorphism on passage to derived categories, and the image of the identity map is a natural quasi-isomorphism of $k$-modules

$$\operatorname{Hom}^{\boxtimes}(M,N) \longrightarrow \operatorname{Hom}(M,N). \quad \square$$

**Corollary 1.12.** *There is a natural isomorphism* $N \longrightarrow \mathrm{Hom}^{\boxtimes}(k, N)$ *in the derived category* $\mathscr{D}_{\mathbb{C}}$.

*Proof.* This is immediate from the natural isomorphisms

$$\mathscr{D}_{\mathbb{C}}(M, N) \cong \mathscr{D}_{\mathbb{C}}(M \boxtimes k, N) \cong \mathscr{D}_{\mathbb{C}}(M, \mathrm{Hom}^{\boxtimes}(k, N)). \quad \square$$

*Remark* 1.13. It would be of interest to construct an $E_\infty$ operad with the properties of Theorem 1.5 by purely algebraic methods. There are defects to the present construction. For example, we do not know that $\lambda : k \boxtimes k \to k$ is a quasi-isomorphism. It would be desirable to have an operad with the additional property that $\mathscr{C}(2)$ is chain homotopy equivalent to $\mathbb{C}^2$ as a right $\mathbb{C}^2$-module (of course, not $\Sigma_2$-equivariantly). This property would ensure that $M \boxtimes N$ is quasi-isomorphic to $M \otimes N$ for all $\mathbb{C}$-modules $M$ and $N$.

## 2. Unital $\mathbb{C}$-modules and the products $\lhd$, $\rhd$, and $\boxdot$

The fact that $\lambda$ is not an isomorphism before passage to the derived category leads us to introduce some further products. By a unital $\mathbb{C}$-module $M$, we understand a $\mathbb{C}$-module $M$ together with a map of $\mathbb{C}$-modules $\eta : k \to M$. We regard $k$ itself as a unital $\mathbb{C}$-module via the identity map $k \to k$. An augmentation of a unital $\mathbb{C}$-module $M$ is a map $\varepsilon : M \to k$ of unital $k$-modules, so that $\varepsilon\eta = \mathrm{Id}$. For a non-unital $k$-module $M$, we let $M_+$ denote the unital $\mathbb{C}$-module $M \oplus k$. Clearly an augmented $\mathbb{C}$-module $M$ is isomorphic to $(\mathrm{Ker}\,\varepsilon)_+$ as a unital $\mathbb{C}$-module. Our formal arguments will apply to arbitrary unital $\mathbb{C}$-modules, but some of our arguments about quasi-isomorphisms will apply only to augmented $\mathbb{C}$-modules. It is possible to generalize these arguments, but the extra verbiage does not seem to be warranted since the applications we envisage are to augmented $\mathscr{C}$-algebras and since Theorem 2.9 will give a way around such difficulties. A cell theory adapted to unital $\mathbb{C}$-modules is given in III§6 and is used to construct a derived category of unital $\mathbb{C}$-modules. Given this, our results on quasi-isomorphisms lead to conclusions about derived categories. We shall leave the formulation of these interpretations to the reader.

**Definition 2.1.** Let $M$ be a unital $\mathbb{C}$-module and let $N$ be any $\mathbb{C}$-module. Define $M \lhd N$ to be the pushout displayed in the following diagram of $\mathbb{C}$-modules:

$$\begin{array}{ccc} k \boxtimes N & \xrightarrow{\eta \boxtimes \mathrm{Id}} & M \boxtimes N \\ {\scriptstyle\lambda}\downarrow & & \downarrow \\ N & \longrightarrow & M \lhd N. \end{array}$$

Define $N \rhd M$ by symmetry.

**Proposition 2.2.** *Let* $M$ *and* $N$ *be* $\mathbb{C}$*-modules. Then*

$$M_+ \lhd N \cong N \oplus (M \boxtimes N).$$

*If* $N$ *is a cell* $\mathbb{C}$*-module, then the canonical map*

$$M_+ \boxtimes N \longrightarrow M \lhd N$$

*is a quasi-isomorphism.*

*Proof.* The first statement is clear, and the cited canonical map reduces to

$$\mathrm{Id} \oplus \lambda : (M \boxtimes N) \oplus (k \boxtimes N) \longrightarrow (M \boxtimes N) \oplus N.$$

Thus Corollary 1.9 gives the second statement. □

The commutativity and associativity of $\boxtimes$ imply the following commutativity and associativity isomorphisms relating $\boxtimes$ and $\lhd$; these isomorphisms imply various others.

**Lemma 2.3.** *Let $L$ and $M$ be unital $\mathbb{C}$-modules and let $N$ and $P$ be any $\mathbb{C}$-modules. Then there are natural isomorphisms*

$$M \lhd N \cong N \rhd M,$$

$$M \lhd (N \boxtimes P) \cong (M \lhd N) \boxtimes P,$$

*and*

$$L \lhd (N \rhd M) \cong (L \lhd N) \rhd M.$$

We have a Hom functor and a suitable adjunction.

**Definition 2.4.** Let $M$ be a unital $\mathbb{C}$-module and let $N$ be any $\mathbb{C}$-module. Define $\mathrm{Hom}^{\lhd}(M,N)$ to be the $\mathbb{C}$-module displayed in the following pullback diagram:

$$\begin{array}{ccc} \mathrm{Hom}^{\lhd}(M,N) & \longrightarrow & \mathrm{Hom}^{\boxtimes}(M,N) \\ \downarrow & & \downarrow{\scriptstyle \eta^*} \\ N & \longrightarrow & \mathrm{Hom}^{\boxtimes}(k,N); \end{array}$$

here the bottom arrow is adjoint to the unit map $\lambda\tau : N \boxtimes k \cong k \boxtimes N \to N$.

**Lemma 2.5.** *For a unital $\mathbb{C}$-module $M$ and any $\mathbb{C}$-modules $L$ and $N$, there is a natural adjunction isomorphism*

$$\mathscr{M}_{\mathbb{C}}(L \rhd M, N) \cong \mathscr{M}_{\mathbb{C}}(L, \mathrm{Hom}^{\lhd}(M,N)).$$

**Definition 2.6.** Let $M$ and $N$ be unital $\mathbb{C}$-modules. The coproduct of $M$ and $N$ in the category of unital $\mathbb{C}$-modules is the pushout $M \cup_k N$. There is an analogous pushout $(M \boxtimes k) \cup_{k \boxtimes k} (k \boxtimes N)$, and the unit maps $\lambda$ determine a natural map of $\mathbb{C}$-modules

$$\lambda : (M \boxtimes k) \cup_{k \boxtimes k} (k \boxtimes N) \to M \cup_k N.$$

The restrictions to $k \boxtimes k$ of the maps

$$\mathrm{Id} \boxtimes \eta : M \boxtimes k \to M \boxtimes N \quad \text{and} \quad \eta \boxtimes \mathrm{Id} : k \boxtimes N \to M \boxtimes N$$

coincide, hence these maps determine a map

$$\theta : (M \boxtimes k) \cup_{k \boxtimes k} (k \boxtimes N) \longrightarrow M \boxtimes N.$$

Define $M \boxdot N$ to be the pushout displayed in the following diagram of $\mathbb{C}$-modules:

$$\begin{array}{ccc} (M \boxtimes k) \cup_{k \boxtimes k} (k \boxtimes N) & \xrightarrow{\lambda} & M \cup_k N \\ {\scriptstyle \theta}\downarrow & & \downarrow \\ M \boxtimes N & \longrightarrow & M \boxdot N. \end{array}$$

Then $M \boxdot N$ is a unital $\mathbb{C}$-module with unit the composite of the unit $k \to M \cup_k N$ and the displayed canonical map $M \cup_k N \to M \boxdot N$.

**Lemma 2.7.** *Let $M$ and $N$ be $\mathbb{C}$-modules. Then*

$$(M_+) \boxdot (N_+) \cong (M \boxtimes N) \oplus M \oplus N \oplus k.$$

*Remark* 2.8. We have a compatible decomposition

$$(M_+) \boxtimes (N_+) \cong (M \boxtimes N) \oplus (M \boxtimes k) \oplus (k \boxtimes N) \oplus (k \boxtimes k).$$

If we knew that $\lambda : k \boxtimes k \to k$ were a quasi-isomorphism, it would follow from Corollary 1.9 that the canonical map

$$M \boxtimes N \longrightarrow M \boxdot N$$

is a quasi-isomorphism when $M$ and $N$ are cell $\mathbb{C}$-modules. However, such a result would be of limited utility since the "augmentation ideals" of augmented $A_\infty$ or $E_\infty$ algebras are unlikely to be of the homotopy types of cell $\mathbb{C}$-modules.

In the applications of the analogous topological theory, it is vital to overcome the problem pointed out in the previous remark. The way to do this is to to approximate a given $A_\infty$ or $E_\infty$ algebra $A$ by its monadic bar construction $BA$ of II.4.2, which is quasi-isomorphic to $A$ and therefore has an equivalent derived category. We shall be more explicit about the definitions and shall prove the following result in Section 9.

**Theorem 2.9.** *For an $A_\infty$ or $E_\infty$ algebra $A$, there is an $A_\infty$ or $E_\infty$ algebra $BA$ and a natural quasi-isomorphism $\varepsilon : BA \to A$. For an $A$-module $M$, there is a $BA$-module $BM$ and a natural quasi-isomorphism of $BA$-modules $\varepsilon : BM \to M$. If $A$ and $A'$ are augmented $A_\infty$ or $E_\infty$ algebras, there are natural quasi-isomorphisms of $k$-modules*

$$BA \otimes BA' \longrightarrow BA \boxdot BA' \quad \textit{and} \quad BA \otimes BM \longrightarrow BA \lhd BM.$$

The purpose of introducing the products $\lhd$ and $\boxdot$ is to obtain good algebraic properties on the domains of definition of the multiplications on $A_\infty$ and $E_\infty$ algebras and of their actions on modules. The theorem shows that, by use of bar construction approximations, we can obtain such algebraic control without changing the underlying quasi-isomorphism type. The following algebraic properties of $\boxdot$ are easily derived from the associativity and commutativity of $\boxtimes$ together with formal arguments from the definition.

**Lemma 2.10.** *The following associativity relation holds, where $M$ and $N$ are unital $\mathbb{C}$-modules and $P$ is any $\mathbb{C}$-module:*

$$(M \boxdot N) \lhd P \cong M \lhd (N \lhd P).$$

**Proposition 2.11.** *The category of unital $\mathbb{C}$-modules is symmetric monoidal under the product $\boxdot$; that is, $\boxdot$ is associative, commutative, and unital up to coherent natural isomorphism.*

## 3. A new description of $A_\infty$ and $E_\infty$ algebras and modules

Let $\mathscr{C}$ be the linear isometries operad. Recall from I.2.1 that a $\mathscr{C}$-algebra $A$ is a $k$-module together with an associative, unital, and equivariant system of action maps

$$\theta : \mathscr{C}(j) \otimes A^j \to A.$$

Recall from I.4.1 that an $A$-module $M$ is a $k$-module together with an associative, unital, and equivariant system of action maps

$$\lambda : \mathscr{C}(j) \otimes A^{j-1} \otimes M \to M.$$

By Theorem 1.7, up to quasi-isomorphism, all $E_\infty$ algebras and modules are $\mathscr{C}$-algebras and modules. Similarly, if we drop the equivariance conditions, then, up to quasi-isomorphism, all $A_\infty$ algebras and modules are of this form. We agree to refer to $\mathscr{C}$-algebras and modules, with and without equivariance, as $E_\infty$ and $A_\infty$ algebras and modules in the rest of this part.

Restricting the action to $j = 0$ and $j = 1$, we see that an $A_\infty$ algebra is a unital $\mathbb{C}$-module with additional structure. The category $\mathscr{M}^u_{\mathbb{C}}$ of unital $\mathbb{C}$-modules is symmetric monoidal under the product $\boxdot$. As with any symmetric monoidal category, we define a monoid in $\mathscr{M}^u_{\mathbb{C}}$ to be an object $A$ with an associative and unital product $\phi : A \boxdot A \to A$; $A$ is commutative if $\phi\tau = \phi$. The following result is the precise analog of a theorem first discovered in the deeper topological context of [25].

**Theorem 3.1.** *An $A_\infty$ algebra $A$ determines and is determined by a monoid structure on its underlying unital $\mathbb{C}$-module; $A$ is an $E_\infty$ algebra if and only if it is a commutative monoid in $\mathscr{M}^u_{\mathbb{C}}$.*

While this is the most elegant form of the theorem, it is more convenient to prove it in an equivalent form expressed in terms of the $\boxtimes$ product. In fact, the following result is immediate from the description of $\boxdot$ in terms of $\boxtimes$ and $\lambda$.

**Lemma 3.2.** *A monoid structure on a unital $\mathbb{C}$-module $A$ determines and is determined by a product $\phi : A \boxtimes A \to A$ such that the following diagrams commute:*

$$\xymatrix{ k \boxtimes A \ar[r]^{\eta\boxtimes Id} \ar[dr]_{\lambda} & A \boxtimes A \ar[d]^{\phi} & A \boxtimes k \ar[l]_{Id\boxtimes\eta} \ar[dl]^{\lambda} \\ & A & }
\qquad \text{and} \qquad
\xymatrix{ A \boxtimes A \boxtimes A \ar[r]^-{Id\boxtimes\phi} \ar[d]_{\phi\boxtimes Id} & A \boxtimes A \ar[d]^{\phi} \\ A \boxtimes A \ar[r]^{\phi} & A; }$$

*$A$ is commutative if the following diagram commutes:*

$$\xymatrix{ A \boxtimes A \ar[rr]^{\tau} \ar[dr]_{\phi} & & A \boxtimes A \ar[dl]^{\phi} \\ & A. & }$$

The analog of Theorem 3.1 for modules reads as follows; we incorporate the analog of Lemma 3.2 in the statement.

**Theorem 3.3.** *Let $A$ be an $A_\infty$ or $E_\infty$ algebra with product $\phi : A \boxdot A \to A$. An $A$-module is a $\mathbb{C}$-module $M$ together with a map $\mu : A \lhd M \to M$ such that the following diagrams commute, where the second diagram implicitly uses the isomorphism $(A \boxdot A) \lhd M \cong A \lhd (A \lhd M)$:*

$$\xymatrix{ k \lhd M \ar[r]^{\eta\lhd Id} \ar[dr]_{\cong} & A \lhd M \ar[d]^{\mu} \\ & M }
\qquad \text{and} \qquad
\xymatrix{ A \boxdot A \lhd M \ar[r]^-{Id\lhd\mu} \ar[d]_{\phi\lhd Id} & A \lhd M \ar[d]^{\mu} \\ A \lhd M \ar[r]^{\mu} & M. }$$

*Equivalently, an $A$-module is a $\mathbb{C}$-module $M$ together with a map $\mu : A \boxtimes M \to M$ such that the following diagrams commute:*

$$\begin{array}{ccc} k \boxtimes M & \xrightarrow{\eta\boxtimes Id} & A \boxtimes M \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle\mu} \\ & & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A \boxtimes A \boxtimes M & \xrightarrow{Id\boxtimes\mu} & A \boxtimes M \\ {\scriptstyle\phi\boxtimes Id}\downarrow & & \downarrow{\scriptstyle\mu} \\ A \boxtimes M & \xrightarrow{\mu} & M. \end{array}$$

We illustrate the force of these results by giving some formal consequences. Recall that the tensor product of commutative DGA's is their coproduct in the category of commutative DGA's. The proof consists of categorical diagram chases that now carry over to our more general context.

**Corollary 3.4.** *Let $A$ and $B$ be $A_\infty$ algebras. Then $A \boxdot B$ is an $A_\infty$ algebra. If $M$ is an $A$-module and $N$ is a $B$-module, then $M \boxtimes N$ is an $A \boxdot B$-module. If $A$ and $B$ are $E_\infty$ algebras, then $A \boxdot B$ is an $E_\infty$ algebra and is the coproduct of $A$ and $B$ in the category of $E_\infty$ algebras.*

The following corollary will become important in Section 5. We first recall a standard categorical definition [43, VI.6].

**Definition 3.5.** Working in an arbitrary category, suppose given a diagram

$$A \underset{f}{\overset{e}{\rightrightarrows}} B \xrightarrow{g} C$$

in which $ge = gf$. The diagram is called a split coequalizer if there are maps

$$h : C \to B \quad \text{and} \quad k : B \to A$$

such that $gh = \mathrm{Id}_C$, $fk = \mathrm{Id}_B$, and $ek = hg$. It follows that $g$ is the coequalizer of $e$ and $f$.

Observe that, while covariant functors need not preserve coequalizers in general, they clearly do preserve split coequalizers.

**Corollary 3.6.** *Let $A$ be an $A_\infty$ algebra. Then the following diagram of unital $k$-modules is a split coequalizer:*

$$A \boxdot A \boxdot A \underset{\mathrm{Id}\,\boxdot\phi}{\overset{\phi\boxdot\mathrm{Id}}{\rightrightarrows}} A \boxdot A \xrightarrow{\phi} A.$$

*If $M$ is a left $A$-module, then the following diagram of $k$-modules is also a split coequalizer:*

$$(A \boxdot A) \lhd M \cong A \lhd (A \lhd M) \underset{\mathrm{Id}\lhd\mu}{\overset{\phi\boxdot\mathrm{Id}}{\rightrightarrows}} A \lhd M \xrightarrow{\mu} M.$$

*Proof.* The first statement is true for monoids in any symmetric monoidal category. The required maps $h$ and $k$ are $\eta \boxdot \mathrm{Id}$ and $\eta \boxdot \mathrm{Id} \boxdot \mathrm{Id}$. The second statement is equally trivial. □

*Remark* 3.7. In $\mathscr{M}^u_{\mathbb{C}}$, as in any symmetric monoidal category, we have operads $\mathscr{M}$ and $\mathscr{N}$ such that an $\mathscr{M}$-algebra is a monoid and an $\mathscr{N}$-algebra is a commutative monoid; compare I.2.2. Thus these operads define $A_\infty$ and $E_\infty$ algebras. There result monads $M$ and $N$ in $\mathscr{M}^u_{\mathbb{C}}$ which define the free $A_\infty$ and $E_\infty$ algebras. We can start with a $k$-module $K$ and form the free unital $\mathbb{C}$-module $(\mathbb{C}\otimes K)_+ = (\mathbb{C}\otimes K)\oplus k$. The free $A_\infty$ algebra it generates must be the free $A_\infty$ algebra generated by $K$. That is,

$$M((\mathbb{C}\otimes K)_+) \cong C(K) = \sum \mathscr{C}(j)\otimes K^j.$$

Similarly, reinterpreting $C$ in the $E_\infty$ sense,

$$N((\mathbb{C}\otimes K)_+) \cong C(K) = \sum \mathscr{C}(j)\otimes_{\Sigma_j} K^j.$$

The following two lemmas give the proof of Theorem 3.1; the proof of Theorem 3.3 is precisely analogous and will be left to the reader.

**Lemma 3.8.** *Let $A$ be an $A_\infty$ algebra. Then $\theta : \mathscr{C}(2)\otimes A\otimes A \longrightarrow A$ induces a product*

$$\phi : A\boxtimes A \equiv \mathscr{C}(2)\otimes_{\mathbb{C}\otimes\mathbb{C}} A\otimes A \longrightarrow A$$

*such that the first two diagrams of Lemma 3.2 commute. If $A$ is an $E_\infty$ algebra, then the third diagram also commutes.*

*Proof.* That $\theta$ factors through the tensor product $\boxtimes$ is immediate from the associativity diagram in the definition, I.1.1, of an operad action. Since $\eta : k \to A$ is taken to be $\theta : \mathscr{C}(0)\to A$, it is easy to check the commutativity of the unit diagrams from Lemma 1.6 and I.1.1. The associativity diagram is more interesting and depends on the proof of Theorem 1.5. In fact, the two squares in the following diagram commute by I.1.1:

$$\begin{array}{ccc}
\mathscr{C}(2)\otimes\mathscr{C}(1)\otimes\mathscr{C}(2)\otimes A\otimes A^2 & \xrightarrow{\mathrm{Id}\otimes\theta\otimes\theta} & \mathscr{C}(2)\otimes A^2 \\
\big\downarrow{\scriptstyle \gamma\otimes\mathrm{Id}} & & \big\downarrow{\scriptstyle \theta} \\
\mathscr{C}(3)\otimes A^3 & \xrightarrow{\qquad\theta\qquad} & A \\
\big\uparrow{\scriptstyle \gamma\otimes\mathrm{Id}} & & \big\uparrow{\scriptstyle \theta} \\
\mathscr{C}(2)\otimes\mathscr{C}(2)\otimes\mathscr{C}(1)\otimes A^2\otimes A & \xrightarrow{\mathrm{Id}\otimes\theta\otimes\theta} & \mathscr{C}(2)\otimes A^2.
\end{array}$$

The horizontal arrows factor through tensor products over $\mathbb{C}^3$ in the terms in the left column and through tensor products over $\mathbb{C}^2$ in the terms at the top and bottom right corners, and the diagram then becomes

$$\begin{array}{ccc}
A\boxtimes(A\boxtimes A) & \xrightarrow{\mathrm{Id}\boxtimes\phi} & A\boxtimes A \\
\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle \phi} \\
\mathscr{C}(3)\otimes_{\mathbb{C}^3} A^3 & \xrightarrow{\quad\theta\quad} & A \\
\big\uparrow{\scriptstyle \cong} & & \big\uparrow{\scriptstyle \phi} \\
(A\boxtimes A)\boxtimes A & \xrightarrow{\phi\boxtimes\mathrm{Id}} & A\boxtimes A.
\end{array}$$

The two arrows labelled $\cong$ are isomorphisms by the proof of Theorem 1.5 in Section 9, and they give the associativity isomorphism that is implicit in the claim that the associativity diagram of Lemma 3.2 commutes. If $A$ is an $E_\infty$ algebra, then $\theta : \mathscr{C}(2)\otimes A\otimes A \to A$ is $\Sigma_2$-equivariant, and the commutativity of the last diagram of Lemma 3.2 follows. □

**Lemma 3.9.** *Let $A$ be a monoid in the category of unital $\mathbb{C}$-modules. Its monoid structure is uniquely determined by an $A_\infty$ algebra structure, and $A$ is commutative if and only if the $A_\infty$ structure is an $E_\infty$ structure.*

*Proof.* The unit and $\mathbb{C}$-action of $A$ give $\theta : \mathscr{C}(0) \to A$ and $\theta : \mathscr{C}(1) \otimes A \to A$. The product $\phi : A \boxtimes A \to A$ induces $\theta : \mathscr{C}(2) \otimes A^2 \to A$. The associativity of $\phi$ shows that it defines an unambigous map $A^{\boxtimes j} \to A$, where $A^{\boxtimes j}$ denotes the $j$-fold $\boxtimes$-power of $A$. Since

$$A^{\boxtimes j} \cong \mathscr{C}(j) \otimes_{\mathbb{C}^j} A^j,$$

$\phi$ induces a map $\theta : \mathscr{C}(j) \otimes A^j \to A$ for each $j \geq 2$. The verification that the associativity and unity diagrams of I.1.1 commute are laborious diagram chases from the definition and the arguments of Section 9. It is not hard to see that the equivariance diagrams commute if $A$ is commutative. It is clear that the resulting $A_\infty$ algebra structure determines the given monoid structure. Conversely, by the associativity diagrams, the higher maps $\theta$ of an $A_\infty$ algebra structure are determined by the second map, so that the $A_\infty$ structure is uniquely determined by the monoid structure. □

## 4. Cell $A$-modules and the derived category of $A$-modules

Fix an $A_\infty$ algebra $A$. We first observe that the category of $A$-modules is closed under various constructions in the underlying categories of $k$-modules and $\mathbb{C}$-modules. Modules mean left modules unless otherwise specified; right modules are defined by symmetry in terms of action maps $M \boxtimes A \to M$, or $M \rhd A \to M$.

**Proposition 4.1.** *Let $M$ and $N$ be $A$-modules, let $L$ be a $\mathbb{C}$-module, and let $K$ be a $k$-module.*

(i) *Any categorical colimit or limit (in the category of $k$-modules) of a diagram of $A$-modules is an $A$-module.*

(ii) *$M \otimes K$ and $\mathrm{Hom}(K, M)$ are $A$-modules and*

$$\mathscr{M}_A(M \otimes K, N) \cong \mathscr{M}_A(M, \mathrm{Hom}(K, N)).$$

(iii) *$M \boxtimes L$ and $\mathrm{Hom}^{\boxtimes}(L, M)$ are $A$-modules and*

$$\mathscr{M}_A(M \boxtimes L, N) \cong \mathscr{M}_A(M, \mathrm{Hom}^{\boxtimes}(L, N)).$$

(iv) *$Hom^{\boxtimes}(M, L)$ is a right $A$-module.*

(v) *The cofiber of a map of $A$-modules is an $A$-module.*

*Proof.* (i) Note first that colimits and limits of $\mathbb{C}$-modules are calculated as colimits and limits of underlying $k$-modules. The functor $\boxtimes$ commutes with colimits in each of its variables. Thus, for a direct system $M_i$ of $\mathbb{C}$-modules,

$$A \boxtimes (\mathrm{Colim} M_i) \cong \mathrm{Colim}(A \boxtimes M_i).$$

If the $M_i$ are $A$-modules, their structure maps induce a structure map for $\mathrm{Colim} M_i$. For an inverse system $M_i$, canonical projections give a map

$$A \boxtimes (\mathrm{Lim} M_i) \longrightarrow \mathrm{Lim}(A \boxtimes M_i).$$

This implies the analogous conclusion for limits, and this conclusion also follows from the fact, recalled below, that the forgetful functor from $A$-modules to $k$-modules is a right adjoint.

(ii) Certainly $M \otimes K$ is a $\mathbb{C}$-module, and

$$A \boxtimes (M \otimes K) \cong (A \boxtimes M) \otimes K.$$

By applying $A \boxtimes (?)$ to the evaluation map $\mathrm{Hom}(K, M) \otimes K \to M$ and taking the adjoint of the resulting map, we obtain a map of $\mathbb{C}$-modules

$$A \boxtimes \mathrm{Hom}(K, M) \longrightarrow \mathrm{Hom}(K, A \boxtimes M).$$

Therefore, by composition, the structure map for $M$ induces structure maps for $M \otimes K$ and $\mathrm{Hom}(K, M)$. The adjunction is a formal verification.
(iii) The argument is just like the proof of (ii).
(iv) The action of $A$ on $\mathrm{Hom}^{\boxtimes}(M, L)$ is the adjoint of the composite

$$\mathrm{Hom}^{\boxtimes}(M, L) \boxtimes A \boxtimes M \xrightarrow{\mathrm{Id} \boxtimes \mu} \mathrm{Hom}^{\boxtimes}(M, L) \boxtimes M \xrightarrow{\epsilon} L,$$

where $\mu$ is the action of $A$ on $M$ and $\varepsilon$ is the evaluation map. The last statement of Lemma 1.6 is needed for the verification of the unit property.
(v) This follows easily from (i) and (ii). □

To develop the cell theory of $A$-modules, we need a free functor from $k$-modules to $A$-modules, and we already have a free functor from $k$-modules to $\mathbb{C}$-modules, namely $K \to \mathbb{C} \otimes K$. The following observation shows that $A \lhd L$ is the free functor from $\mathbb{C}$-modules to $A$-modules.

**Lemma 4.2.** *For $\mathbb{C}$-modules $L$ and $A$-modules $M$,*

$$\mathscr{M}_A(A \lhd L, M) \cong \mathscr{M}_{\mathbb{C}}(L, M).$$

*Proof.* The unit $k \to A$ induces a $\mathbb{C}$-map $\eta : L \cong k \lhd L \to A \lhd L$, the product on $A$ induces a structure of $A$-module on $A \lhd L$, and an $A$-module structure on $M$ is given by an $A$-map $\mu : A \lhd M \to M$. Therefore an $A$-map $g : A \lhd L \to M$ induces the $\mathbb{C}$-map $g \circ \eta : L \to M$ and a $\mathbb{C}$-map $f : L \to M$ induces the $A$-map $\mu \circ (\mathrm{Id} \lhd f) : A \lhd L \to M$. These are inverse correspondences. □

**Definition 4.3.** For a $k$-module $K$, define an $A$-module $FK$ by

$$FK = A \lhd (\mathbb{C} \otimes K).$$

**Lemma 4.4.** *For $k$-modules $K$ and $A$-modules $M$,*

$$\mathscr{M}_A(FK, M) \cong \mathscr{M}_k(K, M).$$

At this point, we recall that we have already constructed the free $A$-module functor in I.4.9 and I.4.10: since the category of $A$-modules is isomorphic to the category of $U(A)$-modules, the free $A$-module generated by a $k$-module $K$ must be $U(A) \otimes K$. We are entitled to the following consequence, which is special to the linear isometries operad. Note that the unit of $U(A)$ determines a natural $k$-map $K \to U(A) \otimes K$.

**Proposition 4.5.** *For $k$-modules $K$, the natural map $FK \to U(A) \otimes K$ is an isomorphism of $A$-modules.*

In particular, $Fk$ is isomorphic to $U(A)$; we can read off the resulting product on $Fk$ by comparison with I.4.9. The following basic result is intuitively obvious, but we assume that $A$ is augmented in order to obtain a quick proof.

**Proposition 4.6.** *Assume that $A$ is augmented. If $K$ is a cell $k$-module, then the $A$-map $\alpha : FK \to A \otimes K$ induced by the canonical $k$-map $K \to A \otimes K$ is a quasi-isomorphism. If $K$ is a free $k$-module with zero differential, then $H^*(FK)$ is the free $H^*(A)$-module generated by $K$.*

*Proof.* By inspection of definitions, $FK \cong Fk \otimes K$. Thus the result will hold in general if it holds when $K = k$. We have the following commutative diagram of maps of $k$-modules:

$$\begin{array}{ccccc} A \otimes \mathbb{C} & \xrightarrow{\chi} & \mathscr{C}(2) \otimes A \otimes \mathbb{C} & \longrightarrow & A \boxtimes \mathbb{C} \\ {\scriptstyle\iota}\uparrow & & \downarrow{\scriptstyle\beta} & & \downarrow \\ A & \xrightarrow[\mathrm{Id}]{} & A & \xleftarrow[\alpha]{} & A \lhd \mathbb{C}. \end{array}$$

Here $\iota$ is the canonical inclusion, $\chi$ is determined by a chosen degree zero cycle $x \in \mathscr{C}(2)$ that augments to $1 \in k$, and $\beta$ is given by

$$\beta(d \otimes a \otimes c) = \theta(d \otimes a \otimes \theta(c \otimes 1))$$

for $d \in \mathscr{C}(2)$, $a \in A$, and $c \in \mathbb{C}$. The composite $\beta\chi\iota$ is multiplication by the unit $1 \in A$ under the product determined by $x$, and the very definition of an $E_\infty$ operad action implies that $\beta\chi\iota \simeq \mathrm{Id}$. Clearly $\iota$ and $\chi$ are quasi-isomorphisms, hence so is $\beta$. The unlabelled arrows in the right-hand square are quasi-isomorphisms by the proof of Proposition 1.8 and by Proposition 2.2, hence $\alpha$ is also a quasi-isomorphism. The second statement follows. □

At this point, we can simply parrot the theory of Part III in our more general context, replacing the free functor $A \otimes (?)$ used there with the free functor $F(?) = A \lhd (\mathbb{C} \otimes (?))$. To begin with, we define "sphere $A$-modules" $F^s(t)$ by

$$F^s(t) = F(S^s(t)),$$

and we observe that the cones on spheres satisfy

$$CF^s(t) \cong F(CS^s(t)).$$

Part III has been written with this generalization in mind, and we reach the following conclusion.

**Theorem 4.7.** *Without exception, every statement and proof in Sections 1, 2, 3, and 6 of Part III applies verbatim to modules over $A_\infty$ algebras.*

Of course, for an actual DGA $A$, we now have two categories of $A$-modules in sight, namely ordinary ones and $A_\infty$ ones. The latter are the same as $U(A)$-modules, and we have the following expected consistency statement.

**Proposition 4.8.** *If $A$ is a DGA, then the map $\alpha : U(A) \cong Fk \to A$ is a map of DGA's. It induces an equivalence of categories between the ordinary derived category $\mathscr{D}_A$ and the $E_\infty$ derived category $\mathscr{D}_{U(A)}$.*

*Proof.* The first statement is an immediate verification since $\mathscr{C}$ acts on $A$ through the augmentation $\mathscr{C} \to \mathscr{N}$. Since $\alpha$ is a quasi-isomorphism, the second statement follows from III.4.2. □

## 5. The tensor product of $A$-modules

We have not yet defined tensor products of modules over $A_\infty$ algebras. We can mimic classical algebra.

**Definition 5.1.** Let $A$ be an $A_\infty$ algebra and let $M$ be a right and $N$ be a left $A$-module. Define $M \boxtimes_A N$ to be the coequalizer (or difference cokernel) displayed in the following diagram of $\mathbb{C}$-modules:

$$(M \rhd A) \boxtimes N \cong M \boxtimes (A \lhd N) \underset{\mathrm{Id}\boxtimes\nu}{\overset{\mu\boxtimes\mathrm{Id}}{\rightrightarrows}} M \boxtimes N \longrightarrow M \boxtimes_A N,$$

where $\mu$ and $\nu$ are the given actions of $A$ on $M$ and $N$; the canonical isomorphism of the terms on the left is implied by Lemma 2.3.

*Remark* 5.2. We have given the definition in the form most convenient for our later proofs. However, it is equivalent to define $M \boxtimes_A N$ more intuitively as the coequalizer in the following diagram:

$$M \boxtimes A \boxtimes N \underset{\mathrm{Id}\boxtimes\nu}{\overset{\mu\boxtimes\mathrm{Id}}{\rightrightarrows}} M \boxtimes N \longrightarrow M \boxtimes_A N.$$

In fact, by the definitions of our products, there is a natural epimorphism

$$\pi : (M \boxtimes A \boxtimes N) \oplus (M \boxtimes N) \longrightarrow (M \rhd A) \boxtimes N \cong M \boxtimes (A \lhd N).$$

The composites $(\mu\boxtimes\mathrm{Id})\circ\pi$ and $(\mathrm{Id}\boxtimes\nu)\circ\pi$ restrict to $\mu\boxtimes\mathrm{Id}$ and $\mathrm{Id}\boxtimes\nu$ on $M\boxtimes A\boxtimes N$, and both composites restrict to the identity on $M \boxtimes N$.

We have used the notation $\boxtimes_A$ to avoid confusion with $\otimes_A$ in the case of a DGA $A$ regarded as an $A_\infty$ algbra. We have the following consistency statement.

*Remarks* 5.3. When $A = k$, $M \rhd k = M$, $k \lhd N = N$, and we are coequalizing two identity maps. Therefore our new $M\boxtimes_k N$ coincides with $M\boxtimes N$. When $A$ is a DGA and $M$ and $N$ are $A$-modules regarded as $E_\infty$ $A$-algebras, the quasi-isomorphisms constructed in Proposition 1.8 can be elaborated to obtain comparisons of coequalizer diagrams that show that the new derived tensor product of $M$ and $N$ regarded as $E_\infty$ modules is isomorphic in the derived category $\mathscr{D}_k$ to the classical derived tensor product $M \otimes_A N$.

An $A_\infty$ algebra $A$ with product $\phi : A \boxtimes A \to A$ has an opposite algebra $A^{op}$ with product $\phi \circ \tau$, and a left $A$-module with action $\mu$ is a right $A^{op}$-algebra with action $\mu \circ \tau$. A simple comparison of coequalizer diagrams gives the following commutativity isomorphism.

**Lemma 5.4.** *For a right $A$-module $M$ and left $A$-module $N$,*

$$M \boxtimes_A N \cong N \boxtimes_{A^{op}} M.$$

**Lemma 5.5.** *For a $\mathbb{C}$-module $L$,*

$$L \boxtimes (M \boxtimes_A N) \cong (L \boxtimes M) \boxtimes_A N \quad \textit{and} \quad (M \boxtimes_A N) \boxtimes L \cong M \boxtimes_A (N \boxtimes L).$$

For $A_\infty$ algebras $A$ and $B$, we define an $(A, B)$-bimodule to be a left $A$ and right $B$-module $M$ such that the following diagram commutes:

$$\begin{array}{ccc} A \boxtimes M \boxtimes B & \longrightarrow & M \boxtimes B \\ \downarrow & & \downarrow \\ A \boxtimes M & \longrightarrow & M. \end{array}$$

The previous lemma and comparisons of coequalizer diagrams give the following associativity isomorphism and unit map.

**Lemma 5.6.** *Let $L$ be an $(A,B)$-bimodule, $M$ be a $(B,C)$-bimodule, and $N$ be a $(C,D)$-bimodule. Then $L \boxtimes_B M$ is an $(A,C)$-bimodule and*

$$(L \boxtimes_B M) \boxtimes_C N \cong L \boxtimes_B (M \boxtimes_C N)$$

*as $(A,D)$-bimodules.*

**Lemma 5.7.** *The action $\nu : A \boxtimes N \longrightarrow N$ of a left $A$-module $N$ factors through a map of $A$-modules $\lambda : A \boxtimes_A N \longrightarrow N$.*

Observe that, for a $\mathbb{C}$-module $L$, $L \rhd A \cong A \lhd L$ is an $(A,A)$-bimodule. In particular, this applies to the free left $A$-module $FK = A \lhd (\mathbb{C} \otimes K)$ generated by a $k$-module $K$, which may be identified with the free right $A$-module generated by $K$. The following result and its corollary will be used in conjunction with the quasi-isomorphism of Proposition 1.8 relating the $\boxtimes$-product of $\mathbb{C}$-modules with their ordinary tensor product as $k$-modules. Recall from Theorem 1.5 that $\mathbb{C} \boxtimes \mathbb{C} \cong \mathbb{C}$ as a left $\mathbb{C}$-module.

**Proposition 5.8.** *Let $L$ and $L'$ be $\mathbb{C}$-modules and let $N$ be an $A$-module. There is a natural isomorphism of $A$-modules*

$$(L \rhd A) \boxtimes_A N \cong L \boxtimes N.$$

*There is also a natural isomorphism of $(A,A)$-bimodules*

$$(L \rhd A) \boxtimes_A (A \lhd L') \cong A \lhd (L \boxtimes L').$$

*Proof.* Applying the functor $L \boxtimes (?)$ to the representation of $N$ as a split coequalizer in Lemma 3.6 and using isomorphisms from Lemmas 2.3 and 2.10, we find that the resulting split coequalizer diagram is isomorphic to the diagram that defines $FL \boxtimes_A N$. Similarly, we obtain the second isomorphism by applying the functor $(?) \lhd (L \boxtimes L')$ to the representation of $A$ as a split coequalizer in Lemma 3.6. □

**Corollary 5.9.** *Let $K$ and $K'$ be $k$-modules and let $N$ be an $A$-module. There is a natural isomorphism of $A$-modules*

$$FK \boxtimes_A N \cong (\mathbb{C} \otimes K) \boxtimes N.$$

*There is also a natural isomorphism of $(A,A)$-bimodules*

$$FK \boxtimes_A FK' \cong F(K \otimes K').$$

To go further, we must consider the behavior of $\boxtimes_A$ on cell $A$-modules $N$. The sequential filtration of $N$ gives short exact sequences

$$0 \longrightarrow N_n \longrightarrow N_{n+1} \longrightarrow N_{n+1}/N_n \longrightarrow 0,$$

where the quotient is a direct sum of sphere $A$-modules $FS^q(r)$. Just as for DGA's, the sequence is algebraically split when we ignore the differentials, and this implies that the sequence is still exact when we apply the functor $M \boxtimes_A (?)$ for any $M$. This allows us to reduce proofs for general $N$ to the case $N = Fk$, using commutation with suspension to handle sphere modules, commutation with direct sums to handle filtration subquotients, induction and five lemma arguments to handle the $N_n$, and passage to colimits to complete the proof. For example, we have the following result, which is just III.4.1 restated in our new context.

**Lemma 5.10.** *Let $N$ be a cell $A$-module. Then the functor $M \boxtimes_A N$ preserves exact sequences and quasi-isomorphisms in the variable $M$.*

*Proof.* Both statements are clear from Corollary 5.9 and Proposition 1.8 if $N$ is a sphere $A$-module. The general case follows by passage to direct sums, induction up the filtration, and passage to colimits. For the exactness, one uses a $3 \times 3$ lemma to prove the inductive step. □

We construct $\boxtimes_A$ as a functor

$$r\mathscr{D}_A \times \ell\mathscr{D}_A \to \mathscr{D}_k$$

by approximating one of the variables by a cell $A$-module; here "$r$" and "$\ell$" indicate right and left $A$-modules. That is, the derived tensor product of $M$ and $N$ is $M \boxtimes_A (\Gamma N)$. It is unital by the following result.

**Corollary 5.11.** *If $A$ is augmented and $N$ is a cell $A$-module, then the unit map $\lambda : A \boxtimes_A N \to N$ is a quasi-isomorphism.*

*Proof.* It suffices to prove this for the sphere $N = Fk = A \lhd \mathbb{C}$, and we have $A \boxtimes_A Fk \cong A \boxtimes \mathbb{C}$. Comparing the coequalizer diagram that defines $A \boxtimes_A N$ with the coequalizer representation of $N$ in Corollary 3.6 and using Definition 2.1, we see that $\lambda$ coincides with the canonical map $A \boxtimes \mathbb{C} \longrightarrow A \lhd \mathbb{C}$. This is a quasi-isomorphism by Proposition 2.2. □

The following result will be the starting point for the construction of a spectral sequence for the computation of $H^*(M \boxtimes_A N)$.

**Corollary 5.12.** *Let $K$ be a free $k$-module with zero differential and let $N$ be a cell $A$-module. Then there is an isomorphism*

$$H^*(FK \boxtimes_A N) \cong (H^*(A) \otimes K) \otimes_{H^*(A)} H^*(N)$$

*that is natural in the $A$-modules $FK$ and $N$.*

*Proof.* The subtle point is that naturality in $FK$ and not just $K$ will be essential in Section 7. Recall that $\otimes_A$ is defined by a coequalizer diagram like that used to define $\boxtimes_A$. Recall too that Proposition 4.6 gives a quasi-isomorphism of $A$-modules $FK \to A \otimes K$ and that the functor $(?) \boxtimes_A N$ preserves quasi-isomorphisms. We obtain a commutative diagram

$$\begin{array}{ccc} H^*(FK) \otimes_{H^*(A)} H^*(N) & \longrightarrow & H^*(FK \boxtimes_A N) \\ \downarrow & & \downarrow \\ H^*(A \otimes K) \otimes_{H^*(A)} H^*(N) & \longrightarrow & H^*((A \otimes K) \boxtimes_A N) \end{array}$$

in which the vertical arrows are isomorphisms. We see by Corollary 5.9 that the top arrow is an isomorphism when $N$ is a sphere $A$-module, and it follows by our usual induction and passage to limits that it is an isomorphism for any $N$. □

## 6. The Hom functor on $A$-modules; unital $A$-modules

We have a Hom functor to go with our new tensor product. Its definition is dictated by the desired adjunction. Let $A$ be an $A_\infty$ algebra.

**Definition 6.1.** Let $M$ and $N$ be left $A$-modules. Define $\mathrm{Hom}^{\boxtimes}_A(M,N)$ to be the equalizer displayed in the following diagram of $\mathbb{C}$-modules:

$$\mathrm{Hom}^{\boxtimes}_A(M,N) \longrightarrow Hom^{\boxtimes}(M,N) \underset{\omega}{\overset{\mu^*}{\rightrightarrows}} \mathrm{Hom}^{\boxtimes}(A \lhd M, N).$$

Here $\mu^* = \mathrm{Hom}^{\boxtimes}(\mu, \mathrm{Id})$ and $\omega$ is the adjoint of the composite

$$A \lhd (M \boxtimes \mathrm{Hom}^{\boxtimes}_A(M,N)) \xrightarrow{\mathrm{Id} \lhd \varepsilon} A \lhd N \xrightarrow{\ \nu\ } N,$$

where $\varepsilon$ is the evaluation map of the adjunction in Lemma 1.3.

*Remark* 6.2. If $A = k$, and $M$ and $N$ are $E_\infty$ $k$-modules, then our new $\mathrm{Hom}^{\boxtimes}_k(M,N)$ is identical to $\mathrm{Hom}^{\boxtimes}(M,N)$.

**Lemma 6.3.** *For $\mathbb{C}$-modules $L$ and left $A$-modules $M$ and $N$, there is a natural adjunction isomorphism*

$$\mathscr{M}_A(L \boxtimes M, N) \cong \mathscr{M}_{\mathbb{C}}(L, \mathrm{Hom}^{\boxtimes}_A(M,N)).$$

Just as in ordinary module theory, we have the following complementary adjunction.

**Lemma 6.4.** *For $\mathbb{C}$-modules $L$, right $A$-modules $M$, and left $A$-modules $N$, there is a natural adjunction isomorphism*

$$\mathscr{M}_{\mathbb{C}}(M \boxtimes_A N, L) \cong \mathscr{M}_A(M, Hom^{\boxtimes}(N,L)).$$

Proposition 5.8 and Corollary 5.9 imply the following results.

**Proposition 6.5.** *Let $L$ be a $\mathbb{C}$-module and $N$ be an $A$-module. There is a natural isomorphism of $A$-modules*

$$\mathrm{Hom}^{\boxtimes}_A(A \lhd L, N) \cong \mathrm{Hom}^{\boxtimes}(L,N).$$

*Proof.* This is immediate from the following composite of isomorphisms of represented functors, in which $L'$ is a $\mathbb{C}$-module:

$$\begin{aligned}\mathscr{M}_{\mathbb{C}}(L', \mathrm{Hom}^{\boxtimes}_A(A \lhd L, N)) &\cong \mathscr{M}_A(L' \boxtimes (A \lhd L), N)\\ &\cong \mathscr{M}_A((L' \rhd A) \boxtimes_A (A \lhd L), N) \cong \mathscr{M}_A(A \lhd (L' \boxtimes L), N)\\ &\cong \mathscr{M}_{\mathbb{C}}(L' \boxtimes L, N) \cong \mathscr{M}_{\mathbb{C}}(L', \mathrm{Hom}^{\boxtimes}(L,N)). \quad \square\end{aligned}$$

**Corollary 6.6.** *Let $K$ be a $k$-module and $N$ be an $A$-module. There is a natural isomorphism of $A$-modules*

$$\mathrm{Hom}^{\boxtimes}_A(FK, N) \cong \mathrm{Hom}^{\boxtimes}(\mathbb{C} \otimes K, N).$$

Arguing as in III§5, we obtain the following analog of Lemma 5.10.

**Lemma 6.7.** *Let $M$ be a cell $A$-module. Then the functor $\mathrm{Hom}^{\boxtimes}_A(M,N)$ preserves exact sequences and quasi-isomorphisms in the variable $N$. It also preserves exact sequences of cell $A$-modules in the variable $M$.*

In the derived category $\mathscr{D}_A$, $\mathrm{Hom}^{\boxtimes}_A(M,N)$ means $\mathrm{Hom}^{\boxtimes}_A(\Gamma M, N)$, where $\Gamma M$ is a cell approximation of $A$. As in III§5, we are entitled to conclude that

(6.8) $$\mathscr{D}_A(L \boxtimes M, N) \cong \mathscr{D}_{\mathbb{C}}(L, \mathrm{Hom}^{\boxtimes}_A(M,N)).$$

Now Corollary 5.11 has the following formal consequence; compare Corollary 1.11.

**Corollary 6.9.** *The adjoint* $N \to \mathrm{Hom}_A^{\boxtimes}(A, N)$ *of* $\lambda\tau : N \boxtimes_A A \cong A \boxtimes_A N \to N$ *induces a natural isomorphism of functors on the derived category* $\mathscr{D}_A$*.*

Again, as in Corollary 5.12, we can use Proposition 4.6 to deduce the following calculational consequence of Corollary 6.6. It will be needed in the next section.

**Corollary 6.10.** *Let* $K$ *be a free* $k$*-module with zero differential and let* $N$ *be a cell* $A$*-module. Then there is an isomorphism*

$$H^*(\mathrm{Hom}_A^{\boxtimes}(FK, N)) \cong Hom_{H^*(A)}(H^*(A) \otimes K, H^*(N))$$

*that is natural in the* $A$*-modules* $FK$ *and* $N$*.*

We briefly indicate some further developments of the theory, including the important invariance result parallel to III.4.2. By a unital $A$-module, we understand an $A$-module $M$ together with a map $\eta : A \longrightarrow M$ of $A$-modules.

**Definition 6.11.** Define the mixed tensor product $M \lhd_A N$ of a unital right $A$-module $M$ and a left $A$-module $N$ by replacing $k$ by $A$ and $\boxtimes$ by $\boxtimes_A$ in Definition 2.1. Define $\rhd_A$ by symmetry. Define the unital tensor product $M \boxdot_A N$ of a unital right $A$-module $M$ and a unital left $A$-module $N$ by replacing $k$ by $A$ and $\boxtimes$ by $\boxtimes_A$ in Definition 2.6.

When $A$ is an $E_\infty$ algebra, it will follow from the discussion in Section 8 that these products all take values in $A$-modules. The properties of $\lhd$, $\rhd$, and $\boxdot$ listed in Section 2 generalize in the expected fashion. Moreover, the new products admit alternative descriptions in terms of coequalizer diagrams like that which defines $\boxtimes_A$.

**Lemma 6.12.** *For a unital right* $A$*-module* $M$ *and a left* $A$*-module* $N$*,* $M \lhd_A N$ *can be identified with the coequalizer displayed in the diagram*

$$(M \lhd A) \lhd N \cong M \lhd (A \lhd N) \underset{\mathrm{Id} \lhd \nu}{\overset{\mu \lhd \mathrm{Id}}{\rightrightarrows}} M \lhd N \longrightarrow M \lhd_A N.$$

*For a unital right* $A$*-module* $M$ *and a unital left* $A$*-module* $N$*,* $M \boxdot_A N$ *can be identified with the coequalizer displayed in the diagram*

$$(M \boxdot A) \boxdot N \cong M \boxdot (A \boxdot N) \underset{\mathrm{Id} \boxdot \nu}{\overset{\mu \boxdot \mathrm{Id}}{\rightrightarrows}} M \boxdot N \longrightarrow M \boxdot_A N.$$

*Proof.* It is easy to check this on augmented $A$-modules $M = M' \oplus A$, and the general case follows by a formal argument; compare [25, III§3]. □

**Proposition 6.13.** *Let* $\phi : A \to A'$ *be a quasi-isomorphism of augmented DGA's. Then the pullback functor* $\phi^* : \mathscr{D}_{A'} \to \mathscr{D}_A$ *is an equivalence of categories with inverse given by the extension of scalars functor* $A' \lhd_A (?)$*.*

*Proof.* We regard $\phi$ as a map of $A$-modules in forming $A' \lhd_A M$. With this modification of $A' \boxtimes_A (?)$, we have an adjunction isomorphism

$$\mathscr{M}_{A'}(A' \lhd_A M, M') \cong \mathscr{M}_A(M, \phi^* M')$$

for $M \in \mathscr{M}_A$ and $M' \in \mathscr{M}_{A'}$. For a $\mathbb{C}$-module $L$, a formal argument (compare [25, III§3]) gives a natural isomorphism

$$A' \lhd_A (A \lhd L) \cong A' \lhd L.$$

Thus the functor $A' \lhd_A (?)$ preserves sphere modules and therefore cell modules. This implies that the adjuction passes to derived categories. The essential point is that

$$\phi \lhd_A \mathrm{Id} : M \cong A \lhd_A M \longrightarrow A' \lhd_A M$$

is a quasi-isomorphism when $M$ is a cell $A$-module. This will hold in general if it holds when $M$ is a sphere $A$-module. However, when $M = A \lhd L$ for a $\mathbb{C}$-module $L$, $\phi \lhd_A \mathrm{Id}$ reduces to

$$\phi \lhd \mathrm{Id} : A \lhd L \longrightarrow A' \lhd L.$$

For a cell $\mathbb{C}$-module $L$, $\phi \lhd \mathrm{Id}$ is a quasi-isomorphism by Propositions 1.8 and 2.2; the latter result applies in view of our assumption that $A$ and $A'$ are augmented. The rest of the argument is the same as in III.4.2. □

**Definition 6.14.** Let $A$ be an $E_\infty$ algebra. Define an $A$-algebra $B$ and a $B$-module $M$ by replacing $\boxtimes$, $\lhd$ and $\boxdot$ by the corresponding products over $A$ in the diagrammatic descriptions of $A_\infty$ $k$-algebras and their modules given in Theorem 3.1, Lemma 3.2, and Theorem 3.3.

We can carry out homological algebra in this more general context, as suggested by the results of the next section. For example, we can construct the Hochschild homology of $A_\infty$ algebras by mimicking the definition of the standard complex for its computation. We refer the interested reader to our topological paper [25]. It carries the parallel theory considerably further, and its arguments can be transcribed directly into the present algebraic context.

## 7. Generalized Eilenberg-Moore spectral sequences

Fix an $A_\infty$ algebra $A$. Since our derived tensor product and Hom functors generalize those of DGA's, the following definition generalizes III.4.4.

**Definition 7.1.** Working in derived categories, define

$$\mathrm{Tor}^*_A(M,N) = H^*(M \boxtimes_A N) \text{ and } \mathrm{Ext}^*_A(M,N) = H^*(\mathrm{Hom}^{\boxtimes}_A(M,N)).$$

These functors enjoy the same general properties as in the case of DGA's: exact triangles in either variable induce long exact sequences on passage to Tor or Ext, Tor preserves direct sums in either variable, and Ext converts direct sums in $M$ to direct products and preserves direct products in $N$. The behavior on free modules is

$$\mathrm{Tor}^*_A(M, FK) \cong H^*(M \otimes K) \text{ and } \mathrm{Ext}^*_A(FK, N) \cong H^*(\mathrm{Hom}(K,N)). \tag{7.2}$$

The crucial point of our general definition of Tor and Ext is that we still have Eilenberg-Moore spectral sequences for their calculation. Write $M^* = H^*(M)$ for brevity of notation.

**Theorem 7.3.** *There are natural spectral sequences of the form*

$$E_2^{p,q} = Tor^{p,q}_{A^*}(M^*, N^*) \Longrightarrow Tor^{p+q}_A(M,N) \tag{7.4}$$

*and*

$$E_2^{p,q} = Ext^{p,q}_{A^*}(M^*, N^*) \Longrightarrow Ext^{p+q}_A(M,N). \tag{7.5}$$

These are both spectral sequences of cohomological type, with

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}. \tag{7.6}$$

In (7.4), $p$ is the negative of the usual homological degree, the spectral sequence is non-zero only in the left half-plane, and it converges strongly. In (7.5), the spectral sequence is non-zero only in the right half plane, and it converges strongly if, for each fixed $(p, q)$, only finitely many of the differentials (7.6) are non-zero. (See [10] for a general discussion of convergence.) The rest of this section will be devoted to the proof of Theorem 7.3. The starting point is the following construction.

**Construction 7.7.** Let $M$ be an $A$-module and let $Q$ be a submodule of $M^*$ with generating set $\{y_i\}$. If $y_i \in (q_i, r_i)$, we may think of $y_i$ as a map of $k$-modules $S^{q_i}(r_i) \to M$. Let $K = \oplus S^{q_i}(r_i)$, let $f : K \to M$ be the sum of the $y_i$, and let $\tilde{f} : FK \to M$ be the induced map of $A$-modules. Then $(FK)^*$ is the free $A^*$-module on generators $y_i \in (q_i, r_i)$, and the induced homomorphism $f^* : (FK)^* \to M^*$ is a map of $A^*$-modules that sends $x_i$ to $y_i$. Clearly $\mathrm{Im}\tilde{f}^* = Q$.

For a right $A$-module $M$, we choose an $A^*$-free resolution

$$\cdots \longrightarrow F_p \xrightarrow{d_p} F_{p-1} \longrightarrow \cdots F_0 \xrightarrow{\varepsilon} M^* \longrightarrow 0 \tag{7.8}$$

and regrade it cohomologically, setting $F^p = F_{-p}$. Each $F^p$ is bigraded, via degree and Adams degree. We shall pay little attention to the Adams degree since the only complications that it introduces are notational.

Let $Q^0 = \mathrm{Ker}\,\varepsilon$ and $Q^p = \mathrm{Ker}\,d^p$ for $p \leq -1$, so that $d^p$ defines an epimorphism $F^p \to Q^{p+1}$. For $p \leq 0$, let $K^p$ be the sum of a copy of the sphere $k$-module $\Sigma^{-p}k^s(t) = k^{s-p}(t)$ for each basis element of $F^p$ of bidegree $(s, t)$ and let $M^0 = M$. Using Construction 7.7 inductively, we can construct cofiber sequences of right $A$-modules

$$FK^p \xrightarrow{k^p} M^p \xrightarrow{i^p} M^{p-1} \xrightarrow{j^{p-1}} \Sigma FK^p \tag{7.9}$$

for $p \leq 0$ that satisfy the following properties:

(i) $k^0$ realizes $\varepsilon$ on $H^*$.
(ii) $H^*(M^p) = \Sigma^{-p}Q^{p+1}$ for $p \leq -1$.
(iii) $k^p$ realizes $\Sigma^{-p}d^p : \Sigma^{-p}F^p \longrightarrow \Sigma^{-p}Q^{p+1}$ on $H^*$ for $p \leq -1$.
(iv) $i^p$ induces the zero homomorphism on $H^*$ for $p \leq 0$.
(v) $j^{p-1}$ realizes the inclusion $\Sigma^{1-p}Q^p \longrightarrow \Sigma^{1-p}F^p$ on $H^*$ for $p \leq 0$.

Observe that (iii) implies the case $p-1$ of (ii) together with (iv) and (v). We are actually constructing a cell $A$-module relative to $M$, in the sense of III§6.

To obtain the spectral sequence (7.4), we assume that $N$ is a cell $A$-module and we define

$$D_1^{p,q} = H^{p+q-1}(M^{p-1} \boxtimes_A N) \quad \text{and} \quad E_1^{p,q} = H^{p+q}(FK^p \boxtimes_A N), \tag{7.10}$$

where we have ignored the Adams grading. The maps displayed in (6.9) give maps

$$i \equiv (i^{p-1})^* : D_1^{p,q} \longrightarrow D_1^{p-1,q+1}$$

$$j \equiv (j^{p-1})^* : D_1^{p,q} \longrightarrow E_1^{p,q}$$

$$k \equiv (k^p)^* : E_1^{p,q} \longrightarrow D_1^{p+1,q}.$$

By Lemma 5.10, these display an exact couple in standard cohomological form. We see from Corollary 5.12 that $E_1^{p,q} \cong (F^p \otimes_{A^*} N^*)^q$ and that $d_1$ agrees under the isomorphism with $d \otimes 1$. This proves that

$$E_2^{p,q} = \mathrm{Tor}_{A^*}^{p,q}(M^*, N^*).$$

Observe that $k : E_1^{0,q} \longrightarrow D_1^{1,q}$ can and must be interpreted as

$$H^q(FK^0 \boxtimes_A N) \longrightarrow H^q(M \boxtimes_A N).$$

On passage to $E_2$, it induces the edge homomorphism

$$E_2^{0,q} = M^* \otimes_{A^*} N^* \longrightarrow H^*(M \boxtimes_A N). \tag{7.11}$$

The convergence is standard, although it appears a bit differently than in most spectral sequences in current use. Write $i^{0,p}$ for the evident composite $M \to M^p$ and also for its tensor product with $N$. Filter $H^*(M \boxtimes_A N)$ by letting $F^pH^*(M \boxtimes_A N)$ be the kernel of

$$(i^{0,p-1})^* : H^*(M \boxtimes_A N) \longrightarrow H^*(M^{p-1} \boxtimes_A N).$$

By (iv) above, we see that the telescope (= homotopy colimit) $\mathrm{Tel}\, M^p$ has zero homology. Since the functor $(?) \boxtimes_A N$ commutes with telescopes, $\mathrm{Tel}(M^p \boxtimes_A N)$ also has zero homology, as we see by a standard inductive argument using the cell structure on $N$. This implies that the filtration is exhaustive. Consider the $(p, q)$th term of the associated bigraded group of the filtration. It is defined as usual by

$$E_0^{p,q} H^*(M \boxtimes_A N) = F^p H^{p+q}(M \boxtimes_A N)/F^{p+1} H^{p+q}(M \boxtimes_A N),$$

and the definition of the filtration immediately implies that this group is isomorphic to the image of

$$(i^{0,p})^* : H^{p+q}(M \boxtimes_A N) \longrightarrow H^{p+q}(M^p \boxtimes_A N).$$

The target of $(i^{0,p})^*$ is $D_1^{p+1,q}$, and of course $E_1^{p,q} = H^{p+q}(FK^p \boxtimes_A N)$ also maps into $D_1^{p+1,q}$, via $k$. It is a routine exercise in the definition of a spectral sequence to check that $k$ induces an isomorphism

$$E_\infty^{p,q} \longrightarrow \mathrm{Im}(i^{0,p})^*.$$

(We know of no published source, but this verification is given in [10, §6].)

To see the functoriality of the spectral sequence, suppose given a map $f : M \to M'$ of $A$-modules and apply the constructions above to $M'$ (writing $F'^p$, etc). Construct a sequence of maps of $A^*$-modules $f^p : F^p \to F'^p$ that give a map of resolutions. We can realize the maps $f^p$ on homology groups by $A$-module maps $\tilde{f}^p : FK^p \to FK'^p$. Starting with $f = f^0$ and proceeding inductively, a standard exact triangle argument allows us to construct a map $f^p : M^{p-1} \to M'^{p-1}$ such that the following diagram of $A$-modules commutes up to homotopy:

$$\begin{array}{ccccccc}
FK^p & \longrightarrow & M^p & \longrightarrow & M^{p-1} & \longrightarrow & \Sigma FK^p \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
FK'^p & \longrightarrow & M^p & \longrightarrow & M'^{p-1} & \longrightarrow & \Sigma FK'^p.
\end{array}$$

There results a map of spectral sequences that realizes the induced map

$$\mathrm{Tor}_{A^*}^{*,*}(M^*, N^*) \longrightarrow \mathrm{Tor}_{A^*}^{*,*}(M'^*, N'^*)$$

on $E_2$ and converges to $(f \boxtimes_A \mathrm{Id})^*$. Functoriality in $N$ is obvious.

To obtain the analogous Ext spectral sequence, we switch from right to left modules in our resolution (7.8) of $M^*$ and its realization by $A$-modules. It is convenient to work with the homological grading displayed in (7.8) and to regrade (7.9) correspondingly. We obtain a cofiber sequence

$$FK_p \xrightarrow{k_p} M_p \xrightarrow{i_p} M_{p+1} \xrightarrow{j_{p+1}} \Sigma FK_p. \tag{7.12}$$

With this grading, we define

$$D_1^{p,q} = H^{p+q}(\mathrm{Hom}_A^{\boxtimes}(M_p, N)) \ \text{ and } \ E_1^{p,q} = H^{p+q}(\mathrm{Hom}_A^{\boxtimes}(FK_p, N)). \tag{7.13}$$

The maps displayed in (6.12) give maps

$$i \equiv (i_{p-1})^* : D_1^{p,q} \longrightarrow D_1^{p-1,q+1}$$
$$j \equiv (k_p)^* : D_1^{p,q} \longrightarrow E_1^{p,q}$$
$$k \equiv (j_{p+1})^* : E_1^{p,q} \longrightarrow D_1^{p+1,q}.$$

These display an exact couple in standard cohomological form. We see by Corollary 6.10 that $E_1^{p,q} \cong \mathrm{Hom}^q_{A^*}(F_p, N^*)$ and that $d_1$ agrees with $\mathrm{Hom}(d,1)$ under this isomorphism. This proves that

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{A^*}(M^*, N^*).$$

Observe that $j : D_1^{0,q} \to E_1^{0,q}$ can and must be interpreted as

$$H^q(\mathrm{Hom}_A^{\boxtimes}(M, N)) \longrightarrow H^q(\mathrm{Hom}_A^{\boxtimes}(FK^0, N)).$$

On passage to $E_2$, it induces the edge homomorphism

$$H^q(\mathrm{Hom}_A^{\boxtimes}(M, N)) \longrightarrow \mathrm{Hom}^q_{A^*}(M^*, N^*) = E_2^{0,q}. \tag{7.14}$$

To see the convergence, let

$$\iota^{0,p} : \mathrm{Hom}_A^{\boxtimes}(M_p, N) \longrightarrow \mathrm{Hom}_A^{\boxtimes}(M, N)$$

be the map induced by the evident interate $M \to M^p$ and filter $H^*(\mathrm{Hom}_A^{\boxtimes}(M, N))$ by letting $F^pH^*(\mathrm{Hom}_A^{\boxtimes}(M, N))$ be the image of

$$(\iota^{0,p})^* : H^*(\mathrm{Hom}_A^{\boxtimes}(M_p, N)) \longrightarrow H^*(\mathrm{Hom}_A^{\boxtimes}(M, N)).$$

The $(p,q)$th term of the associated bigraded group of the filtration is

$$E_0^{p,q}H^*(\mathrm{Hom}_A^{\boxtimes}(M, N)) = F^pH^{p+q}(\mathrm{Hom}_A^{\boxtimes}(M, N))/F^{p+1}H^{p+q}(\mathrm{Hom}_A^{\boxtimes}(M, N)).$$

The group $E_\infty^{p,q}$ is defined as the subquotient $Z_\infty^{p,q}/B_\infty^{p,q}$ of $E_1^{p,q}$, where

$$B_\infty^{p,q} = j(\mathrm{Ker}(\iota^{0,p})^*),$$

and a routine exercise in the definition of a spectral sequence shows that the additive relation $(\iota^{0,p})^* \circ j^{-1}$ induces an isomorphism

$$E_\infty^{p,q} \cong E_0^{p,q}H^*(\mathrm{Hom}_A^{\boxtimes}(M, N)).$$

Since $\mathrm{Tel}\, M_p$ has zero homology, so does the homotopy limit, or "Microscope",

$$\mathrm{Mic}\,\mathrm{Hom}_A^{\boxtimes}(M_p, N) \cong \mathrm{Hom}_A^{\boxtimes}(\mathrm{Tel}\, M_p, N).$$

As usual for a countable inverse system, there is a $\mathrm{Lim}^1$ exact sequence for the computation of $H^*(\mathrm{Mic}\,\mathrm{Hom}_A^{\boxtimes}(M_p, N))$, and we conclude that

$$\mathrm{Lim}\, H^*(\mathrm{Hom}_A^{\boxtimes}(M_p, N)) = 0 \ \text{ and } \ \mathrm{Lim}^1\, H^*(\mathrm{Hom}_A^{\boxtimes}(M_p, N)) = 0.$$

In the language of [10], this means that the spectral sequence $\{E_r^{p,q}\}$ is conditionally convergent, and [10] shows that strong convergence follows if, for each pair $(p,q)$, only finitely many of the differentials with source $E_r^{p,q}$ are non-zero. The functoriality of the spectral sequence is clear from the argument for torsion products given above.

## 8. $E_\infty$ ALGEBRAS AND DUALITY

We assume that $A$ is an $E_\infty$ algebra in this section, and we show that the study of $E_\infty$ modules works exactly the same way as the study of modules over commutative DGA's. In particular, we discuss composition and Yoneda products and duality. Observe that, although it is not at all obvious from the original definitions of I§4, their reinterpretation in Section 3 implies that we obtain the same $A$-modules whether we regard $A$ as an $E_\infty$ algebra or, by neglect of structure, as an $A_\infty$ module.

If $\mu : A \boxtimes M \to M$ gives $M$ a left $A$-module structure, then $\mu \circ \tau : M \boxtimes A \to M$ gives $M$ a right $A$-module structure such that $M$ is an $(A,A)$-bimodule. Just as in the study of modules over commutative DGA's (where the argument is too trivial to need such a pedantic formalization), this leads to the following important conclusion.

**Theorem 8.1.** *If $M$ and $N$ are $A$-modules, then $M \boxtimes_A N$ and $\mathrm{Hom}_A^{\boxtimes}(M,N)$ have canonical $A$-module structures deduced from the $A$-module structure of $M$ or, equivalently, $N$. The tensor product over $A$ is associative and commutative, and the unit maps $A \boxtimes_A M \to M$ and $A \to \mathrm{Hom}_A^{\boxtimes}(A,N)$ are maps of $A$-modules. There is a natural adjunction isomorphism*

$$\mathscr{M}_A(L \boxtimes_A M, N) \cong \mathscr{M}_A(L, Hom_A^{\boxtimes}(M,N)). \tag{8.2}$$

*The derived category $\mathscr{D}_A$ is symmetric monoidal under the product derived from $\boxtimes_A$, and the adjunction passes to the derived category.*

The analog of III.5.1 is immediate from Corollary 5.9.

**Proposition 8.3.** *If $M$ and $M'$ are cell $A$-modules, then $M \otimes_A M'$ is a cell $A$-module with sequential filtration $\{\sum_p (M_p \otimes_A N_{n-p})\}$.*

As in the previous section, write $A^* = H^*(A)$; it is an associative and (graded) commutative algebra.

**Corollary 8.4.** $\mathrm{Tor}_A^*(M,N)$ *and* $\mathrm{Ext}_A^*(M,N)$ *are $A^*$-modules, and there are natural commutativity and associativity isomorphisms of $A^*$-modules*

$$\mathrm{Tor}_A^*(M,N) \cong \mathrm{Tor}_A^*(N,M) \tag{8.5}$$

*and*

$$\mathrm{Tor}_A^*(L \boxtimes_A M, N) \cong \mathrm{Tor}_A^*(L, M \boxtimes_A N). \tag{8.6}$$

*The spectral sequences of the previous section are spectral sequences of differential $A^*$-modules.*

The formal properties of Theorem 8.1 imply many others. For example,

$$\mathrm{Hom}_A^{\boxtimes}(M \boxtimes_A L, N) \cong \mathrm{Hom}_A^{\boxtimes}(M, \mathrm{Hom}_A^{\boxtimes}(L,N)) \tag{8.7}$$

because the two sides represent isomorphic functors on modules. Using this, we see that the evaluation map

$$\varepsilon : \mathrm{Hom}_A^{\boxtimes}(L, M) \boxtimes_A L \longrightarrow M$$

induces a map

$$\begin{aligned} &\mathscr{M}_A(\mathrm{Hom}_A^{\boxtimes}(M,N), \mathrm{Hom}_A^{\boxtimes}(M,N)) \\ &\quad \to \mathscr{M}_A(\mathrm{Hom}_A^{\boxtimes}(M,N), \mathrm{Hom}_A^{\boxtimes}(\mathrm{Hom}_A^{\boxtimes}(L,M) \boxtimes_A L, N)) \\ &\quad \cong \mathscr{M}_A(\mathrm{Hom}_A^{\boxtimes}(M,N), \mathrm{Hom}_A^{\boxtimes}(\mathrm{Hom}_A^{\boxtimes}(L,M), \mathrm{Hom}_A^{\boxtimes}(L,N))) \\ &\quad \cong \mathscr{M}_A(\mathrm{Hom}_A^{\boxtimes}(M,N) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,M), \mathrm{Hom}_A^{\boxtimes}(L,N)). \end{aligned}$$

The image of the identity map of $\mathrm{Hom}_A^{\boxtimes}(M,N)$ gives a composition pairing

$$\pi : \mathrm{Hom}_A^{\boxtimes}(M,N) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,M) \longrightarrow \mathrm{Hom}_A^{\boxtimes}(L,N). \tag{8.8}$$

This pairing is associative and commutative in the sense that the following diagrams commute; note that the unit of the adjunction (8.2) specializes to give a map $\eta : A \to \mathrm{Hom}_A^{\boxtimes}(M,M)$:

$$\begin{array}{ccc} \mathrm{Hom}_A^{\boxtimes}(M,N) \boxtimes_A A & & \\ \downarrow{\scriptstyle \mathrm{Id} \boxtimes \eta} & \searrow{\scriptstyle \lambda\tau} & \\ \mathrm{Hom}_A^{\boxtimes}(M,N) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(M,M) & \xrightarrow{\pi} & \mathrm{Hom}_A^{\boxtimes}(M,N), \end{array}$$

$$\begin{array}{ccc} A \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,M) & & \\ \downarrow{\scriptstyle \eta \boxtimes \mathrm{Id}} & \searrow{\scriptstyle \lambda} & \\ \mathrm{Hom}_A^{\boxtimes}(M,M) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,M) & \xrightarrow{\pi} & \mathrm{Hom}_A^{\boxtimes}(L,M), \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_A^{\boxtimes}(N,P) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(M,N) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,M) & \xrightarrow{\mathrm{Id} \boxtimes \pi} & \mathrm{Hom}_A^{\boxtimes}(N,P) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,N) \\ \downarrow{\scriptstyle \pi \boxtimes \mathrm{Id}} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Hom}_A^{\boxtimes}(M,P) \boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,M) & \xrightarrow{\pi} & \mathrm{Hom}_A^{\boxtimes}(L,P). \end{array}$$

On passage to homology, the pairing (8.8) induces a Yoneda product on Ext.

**Proposition 8.9.** *There is a natural, associative, and unital system of pairings*

$$\pi_* : \mathit{Ext}_A^*(M,N) \otimes_{A^*} \mathit{Ext}_A^*(L,M) \longrightarrow \mathit{Ext}_A^*(L,N).$$

*Proof.* We have an associative and unital system of isomorphisms in $\mathscr{D}_A$

$$FS^q(r) \boxtimes_A FS^s(t) \cong FS^{q+s}(r+t).$$

Since $H^q(M) \cong \mathscr{D}_A(FS^{-q}(-r), M)$ for an $A$-module $M$, the result follows directly from the pairings. □

These pairings also imply pairings of spectral sequences. We content ourselves with a brief indication of the proof.

**Proposition 8.10.** *The pairing* $\mathrm{Hom}_A^{\boxtimes}(M,N)\boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L,M)\to \mathrm{Hom}_A^{\boxtimes}(L,N)$ *induces a pairing of spectral sequences that coincides with the algebraic Yoneda pairing on the $E_2$-level and converges to the induced pairing of Ext groups.*

*Proof.* Construct a sequence $\{L_p\}$ as in (7.12). Then the maps $M\to M_p$ induce a compatible system of pairings

$$\begin{array}{c} \mathrm{Hom}_A^{\boxtimes}(M_p,N)\boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L_{p'},M) \\ \big\downarrow \\ \mathrm{Hom}_A^{\boxtimes}(M,N)\boxtimes_A \mathrm{Hom}_A^{\boxtimes}(L_{p'},M) \\ \big\downarrow \\ \mathrm{Hom}_A^{\boxtimes}(L_{p'},N). \end{array}$$

These induce the required pairing of spectral sequences. The convergence is clear, and the behavior on $E_2$ terms is correct by comparison with the axioms or by comparison with the usual construction of Yoneda products. □

Modulo the obvious changes of notation, the formal duality theory that we explained in III§5 applies verbatim to the present more general context. Working in $\mathscr{D}_A$, we define $M^\vee=\mathrm{Hom}_A^{\boxtimes}(M,A)$, and we say that $M$ is "finite" if it has a coevaluation map $\eta: A\longrightarrow M\boxtimes M^\vee$ such that the analog of diagram III.5.6 commutes. When $M$ is a finite $A$-module, various natural maps such as

$$\rho: M\longrightarrow M^{\vee\vee}$$

and

$$\nu: M^\vee\boxtimes_A N\longrightarrow \mathrm{Hom}_A^{\boxtimes}(M,N)$$

are isomorphisms in $\mathscr{D}_A$, exactly as if $A$ were a classical $k$-algebra, without differential, and $M$ were a finitely generated projective $A$-module. The last isomorphism has the following implication.

**Proposition 8.11.** *For a finite $A$-module $M$ and any $A$-module $N$,*

$$\mathrm{Tor}_A^n(M^\vee,N)\cong \mathrm{Ext}_A^n(M,N).$$

Although the relation may be obscured by the grading, this is an algebraic counterpart of Spanier-Whitehead duality in algebraic topology. We call particular attention to III.5.7, which we repeat for emphasis.

**Theorem 8.12.** *A cell $A$-module is finite in the sense just defined if and only if it is a direct summand up to homotopy of a finite cell $A$-module.*

## 9. The linear isometries operad; change of operads

We prove Theorems 1.5, 1.7, and 2.9 here. We first define an $E_\infty$ operad $\mathscr{L}$ of topological spaces. The algebraic $E_\infty$ operad $\mathscr{C}$ of Theorem 1.5 is obtained by applying the normalized singular chain complex functor $C_\#$ to $\mathscr{L}$, as discussed at the start of I§5.

Let $U\cong\mathbb{R}^\infty$ be a countably infinite dimensional real inner product space, topologized as the union of its finite dimensional subspaces. Let $U^j$ be the direct sum of

$j$ copies of $U$. Define $\mathscr{L}(j)$ to be the set of linear isometries $U^j \to U$ with the function space topology, that is, the compact-open topology made compactly generated. Note that a linear isometry is an injection but not necessarily an isomorphism. The space $\mathscr{L}(0)$ is the point $i$, $i : 0 \to U$, and $\mathscr{L}(1)$ contains the identity $1 : U \to U$. The left action of $\Sigma_j$ on $U^j$ by permutations induces a free right action of $\Sigma_j$ on $\mathscr{L}(j)$. The structure maps

$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \longrightarrow \mathscr{L}(j_1 + \cdots + j_k)$$

are defined by

$$\gamma(g; f_1, \ldots, f_k) = g \circ (f_1 \oplus \cdots \oplus f_k).$$

The associativity property of Theorem 1.5 stems from a special associativity property of $\mathscr{L}$ that was first observed by Hopkins. Observe that $\mathscr{L}(1)$ acts from the left on any $\mathscr{L}(i)$, via $\gamma$, hence $\mathscr{L}(1) \times \mathscr{L}(1)$ acts from the left on $\mathscr{L}(i) \times \mathscr{L}(j)$. Note too that $\mathscr{L}(1) \times \mathscr{L}(1)$ acts from the right on $\mathscr{L}(2)$. Let us denote these actions by $\nu$ and $\mu$, respectively.

**Lemma 9.1** (Hopkins). *For $i \geq 1$ and $j \geq 1$, the diagram*

$$\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1) \times \mathscr{L}(i) \times \mathscr{L}(j) \underset{\mathrm{Id} \times \nu}{\overset{\mu \times \mathrm{Id}}{\rightrightarrows}} \mathscr{L}(2) \times \mathscr{L}(i) \times \mathscr{L}(j) \xrightarrow{\ \gamma\ } \mathscr{L}(i+j)$$

*is a split coequalizer of spaces.*

*Proof.* Choose isomorphisms $s : U^i \to U$ and $t : U^j \to U$ and define

$$h(f) = (f \circ (s \oplus t)^{-1}, s, t)$$

and

$$k(f; g, g') = (f; g \circ s^{-1}, g' \circ t^{-1}; s, t).$$

It is trivial to check the identities of Definition 3.5. □

**Proposition 9.2.** *Let $i \geq 1$ and $j \geq 1$. Then the structural map $\gamma$ of the operad $\mathscr{C} = C_{\#}(\mathscr{L})$ induces an isomorphism*

$$\mathscr{C}(2) \otimes_{\mathbb{C} \otimes \mathbb{C}} \mathscr{C}(i) \otimes \mathscr{C}(j) \longrightarrow \mathscr{C}(i+j).$$

*Proof.* As in I§5, let $g$ denote the shuffle map

$$C_{\#}(X) \otimes C_{\#}(Y) \longrightarrow C_{\#}(X \times Y)$$

and recall that it is a monomorphism naturally split by the Alexander-Whitney map $f$; $g$ is associative and we continue to write $g$ for maps obtained from it by iteration. The covariant functor $C_{\#}$ preserves split coequalizers, and the map of the statement factors as the composite

$$\mathscr{C}(2) \otimes_{\mathbb{C} \otimes \mathbb{C}} \mathscr{C}(i) \otimes \mathscr{C}(j) \xrightarrow{\ g\ } C_{\#}(\mathscr{L}(2) \times_{\mathscr{L}(1) \times \mathscr{L}(1)} \mathscr{L}(i) \times \mathscr{L}(j) \xrightarrow{\ \gamma_{\#}\ } \mathscr{C}(i+j),$$

where $\gamma_{\#}$ is an isomorphism. We see that $g$ is a split monomorphism by a comparison of coequalizer diagrams, and we must check that $g$ is an epimorphism. Think of isomorphisms $s : U^i \to U$ and $t : U^j \to U$ as singular zero simplices of the spaces $\mathscr{L}(i)$ and $\mathscr{L}(j)$. A singular $n$-simplex $x : \Delta_n \to \mathscr{L}(i+j)$ determines a singular $n$-simplex $y$ of $\mathscr{L}(2)$ by precomposition with $s^{-1} \oplus t^{-1}$. When all but one variable is a zero simplex, the shuffle map takes an obvious form from which it is trivial to check that $(\gamma_{\#} \circ g)(y \otimes s \otimes t) = x$. □

*Proof of Theorem 1.5.* We must construct a natural isomorphism

$$(L \boxtimes M) \boxtimes N \cong L \boxtimes (M \boxtimes N),$$

and we claim that both sides are naturally isomorphic to

$$\mathscr{C}(3) \otimes_{\mathbb{C}^3} L \otimes M \otimes N.$$

Note that $N \cong \mathbb{C} \otimes_{\mathbb{C}} N$. We have the isomorphisms

$$\begin{aligned}(L \boxtimes M) \boxtimes N &\cong \mathscr{C}(2) \otimes_{\mathbb{C}^2} (\mathscr{C}(2) \otimes_{\mathbb{C}^2} L \otimes M) \otimes (\mathbb{C} \otimes_{\mathbb{C}} N)\\ &\cong (\mathscr{C}(2) \otimes_{\mathbb{C}^2} \mathscr{C}(2) \otimes \mathscr{C}(1)) \otimes_{\mathbb{C}^3} (L \otimes M \otimes N)\\ &\cong \mathscr{C}(3) \otimes_{\mathbb{C}^3} (L \otimes M \otimes N).\end{aligned}$$

The symmetric argument shows that this is also isomorphic to $L \boxtimes (M \boxtimes N)$. In view of the generality of Proposition 8.3, the argument iterates to prove that all $j$-fold iterated $\boxtimes$ products are canonically isomorphic to

$$\mathscr{C}(j) \otimes_{\mathbb{C}^j} M_1 \otimes \cdots \otimes M_j.$$

When all $M_i = \mathbb{C}$, this gives an isomorphism $\mathbb{C}^{\boxtimes j} \cong \mathscr{C}(j)$ of $(\mathbb{C}, \mathbb{C}^j)$-bimodules. Finally, if $t : U^j \to U$ is an isomorphism, then composition with $t$ and $t^{-1}$ give inverse homeomorphisms of left $\mathscr{L}(1)$-spaces between $\mathscr{L}(j)$ and $\mathscr{L}(1)$. On passage to chains, these give rise to an isomorphism of left $\mathbb{C}$-modules between $\mathscr{C}(j)$ and $\mathbb{C}$. □

*Proof of Theorem 1.7.* The argument is the exact algebraic analog of one first used in topology in [46]. It is similar to, but simpler than, the arguments of II§§4,5. We assume given any two $E_\infty$ operads $\mathscr{C}$ and $\mathscr{C}'$, and we must construct a $\mathscr{C}'$-algebra from a $\mathscr{C}$-algebra. The argument works equally well for $A_\infty$ and $E_\infty$ algebras. There is an evident notion of the tensor product of operads, with

$$(\mathscr{C} \otimes \mathscr{C}')(j) = \mathscr{C}(j) \otimes \mathscr{C}(j').$$

We abbreviate $\mathscr{C}'' = \mathscr{C} \otimes \mathscr{C}'$. The augmentations of $\mathscr{C}$ and $\mathscr{C}'$ induce maps of operads $\mathscr{C}'' \to \mathscr{C}$ and $\mathscr{C}'' \to \mathscr{C}'$, and these in turn induce maps of monads $C'' \to C$ and $C'' \to C'$. The maps $C''K \to CK$ and $C''K \to C'K$ are homotopy equivalences for all $k$-modules $K$ since all three operads are $E_\infty$ operads. Moreover, the composite of $CC'' \to CC$ and the product of $C$ is a right action of the monad $C''$ on the functor $C$, and $C$ is a $(C, C'')$-bifunctor in the sense of II.4.1. Similarly, if $A$ is a $C$-algebra, then $A$ is a $C''$-algebra by pullback along $C'' \to C$. Now recall the two-sided bar construction

$$B(F, C, A) = C_{\#} B_*(F, C, A)$$

from II.4.1. Here $C_{\#}$ is the totalization functor from simplicial $k$-modules to $k$-modules discussed in II§5. By II.4.2 and the naturality properties of this construction, for a $C$-algebra $A$ we have evident natural maps of $C''$-algebras

$$A \longleftarrow B(C, C, A) \longleftarrow B(C'', C'', A) \longrightarrow B(C', C'', A),$$

all of which are quasi-isomorphisms. We let $VA$ be the $C'$-algebra $B(C', C'', A)$ and have the conclusion of Theorem 1.7 on the algebra level. The argument on the module level is the same, using the monads of I§4. □

Finally, we return to the linear isometries operad and prove Theorem 2.9.

*Proof of Theorem 2.9.* For definiteness, we work with $E_\infty$ algebras. The proof for $A_\infty$ algebras is similar but simpler. We abbreviate $BA = B(C, C, A)$, and we have a natural map of $E_\infty$ algebras $\varepsilon : BA \to A$ that is a homotopy equivalence of $k$-algebras. We also have the monad $C[1]$ of I.4.3 such that a $C[1]$-algebra is a $C$-algebra $A$ together with an $A$-module $M$. We write

$$(BA; BM) = B(C[1], C[1], (A; M)).$$

More explicitly, we apply the totalization functor $C_\#$ to both coordinates of the pair of simplicial $k$-modules $B_*(C[1], C[1], (A; M))$; the first coordinate is $BA$ and we call the second coordinate $BM$. Then $BM$ is a $BA$-module, and we have a map of $BA$-modules $\varepsilon : BM \to M$ that is a homotopy equivalence of $k$-modules. We must construct quasi-isomorphisms of $k$-modules

$$BA \otimes BA' \longrightarrow BA \boxdot BA' \quad \text{and} \quad BA \otimes BM \longrightarrow BA \lhd BM.$$

We give the argument for the first of these; the argument for the second is precisely similar. Clearly $BA \otimes BA'$ is constructed from constituent $k$-modules

$$(\mathscr{C}(i) \otimes (C^p A)^i) \otimes (\mathscr{C}(j) \otimes (C^q A')^j) \cong (\mathscr{C}(i) \otimes \mathscr{C}(j)) \otimes ((C^p A)^i) \otimes (C^q A')^j)$$

by passing to orbits over the action of $\Sigma_i \times \Sigma_j$, passing to direct sums over $i \geq 0$ and $j \geq 0$, and then totalizing over $p$, $q$, and the internal degree; $BA \boxdot BA'$ is constructed similarly from constituent $k$-modules

$$(\mathscr{C}(2) \otimes_{\mathbb{C}^2} (\mathscr{C}(i) \otimes (C^p A)^i) \otimes (\mathscr{C}(j) \otimes (C^q A')^j) \cong \mathscr{C}(i+j) \otimes ((C^p A)^i) \otimes (C^q A')^j).$$

Here we have used Proposition 8.2 when $i \geq 1$ and $j \geq 1$; Lemma 2.7 and the convention $\mathscr{C}(0) \otimes X^0 = k$ give the summands with $i = 0$ or $j = 0$. By choosing a degree cycle $x \in \mathscr{C}(2)$ such that $\varepsilon(x) = 1$ and using the operad structural maps $\gamma$, we obtain a composite $(\Sigma_i \times \Sigma_j)$-map

$$\mathscr{C}(i) \otimes \mathscr{C}(j) \longrightarrow \mathscr{C}(2) \otimes \mathscr{C}(i) \otimes \mathscr{C}(j) \longrightarrow \mathscr{C}(i+j)$$

for each $i$ and $j$. Since $\mathscr{C}$ is an $E_\infty$ operad, this is a map between free $(\Sigma_i \times \Sigma_j)$-resolutions of $k$ and is thus a $(\Sigma_i \times \Sigma_j)$-equivariant homotopy equivalence. Upon tensoring over $\Sigma_i \times \Sigma_j$ with $(C^p A)^i \otimes (C^q A')^j$ and totalizing, these maps induce a well-defined map $\kappa : BA \otimes BA' \longrightarrow BA \boxdot BA'$. We may filter both sides so that the resulting differential on $E_1$ comes from the differentials on our resolutions. The resulting map of $E_2$-terms is an isomorphism, and $\kappa$ is therefore a quasi-isomorphism. □

## References


[1] J. F. Adams. A variant of E. H. Brown's representability theorem. *Topology*, 10:185–198, 1971.

[2] L. Avramov and S. Halperin. Through the looking glass: A dictionary between rational homotopy theory and local algebra, in volume 1183 of *Springer Lecture Notes in Mathematics*. Springer, 1986.

[3] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. *Astérisque*, 100:5–171, 1982.

[4] A. Beilinson and V. Ginzburg. Infinitesimal structures of moduli space of g-bundles. *Duke Math. J.*, 66:63–74, 1992.

[5] J. Bernstein and V. Lunts. Equivariant sheaves and functors. Preprint, 1992.

[6] S. Bloch and I. Kriz. Mixed Tate motives. Annals of Math. To appear.

[7] S. Bloch. Algebraic cycles and higher K-theory. *Adv. Math.*, 61:267–304, 1986.

[8] S. Bloch. Letter to Kriz and May, June 1992.

[9] S. Bloch. The moving lemma for higher Chow groups. Preprint, 1993.

[10] J. M. Boardman. Conditionally convergent spectral sequences. Johns Hopkins University. Preprint, 1982.
[11] J. M. Boardman and R. M. Vogt. *Homotopy invariant structures on topological spaces*, volume 347 of *Springer Lecture Notes in Mathematics*. Springer, 1973.
[12] W. Browder. Homology operations and loop spaces. *Illinois J. Math.*, 4:347–357, 1960.
[13] E. H. Brown, Jr. Abstract homotopy theory. *Trans. Amer. Math. Soc.*, 119:79–85, 1965.
[14] H. Cartan and S. Eilenberg. *Homological Algebra*. Princeton University Press, 1956.
[15] F. R. Cohen. PhD thesis, University of Chicago, 1972.
[16] F. R. Cohen, T.J. Lada, and J. P. May. *The Homology of Iterated Loop Spaces*, volume 533 of *Springer Lecture Notes in Mathematics*. Springer, 1976.
[17] P. Deligne. A quoi servent les motifs? In *Proceedings of the Summer, 1991 Conference on motives*. University of Washington. To appear.
[18] P. Deligne. Letter to Beilinson dated December 6, 1989.
[19] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift*, volume 2, pages 111–195. Birkhauser, 1990.
[20] P. Deligne. Letters to Bloch and May dated March 19, March 20, April 25, and May 2, 1990.
[21] P. Deligne and J. S. Milne. *Tannakian categories*, volume 900 of *Springer Lecture Notes in Mathematics*. Springer, 1982.
[22] A. Dold and D. Puppe. Homologie nicht-additiver Functoren. *Ann. Inst. Fourier*, 11:201–312, 1961.
[23] A. Dold and D. Puppe. Duality, trace, and transfer. In *Proc. International Conference on Geometric Toplogy*, pages 81–102. PWN – Polish Scientific Publishers, 1980.
[24] A. Elmendorf, J. P. C. Greenlees, I. Kriz, and J. P. May. Commutative algebra in stable homotopy theory and a completion theorem. *Mathematical Research Letters* 1, 1994, to appear.
[25] A. Elmendorf, I. Kriz, and J. P. May. Commutative algebra in stable homotopy theory. Monograph in preparation. Preprint, 1993.
[26] Z. Fiedorowicz and J. P. May. Homology operations revisited. *Canadian J. Math.*, 3:700–717, 1982.
[27] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. Preprint, 1992.
[28] E. Getzler and J. D. S. Jones. n-Algebras and Batalin-Vilkovisky algebras. In preparation (undistributed preprint), 1992.
[29] V. A. Ginzburg and M. M. Kapranov. Koszul duality for operads. In preparation (undistributed preprint), 1992.
[30] V. K. A. M. Gugenheim and J. P. May. *On the theory and applications of differential torsion products*, volume 142 of *Memoirs A.M.S.* A.M.S., 1974.
[31] V. A. Hinich and V. V. Schechtman. *On homotopy limit of homotopy algebras*, volume 1289 of *Springer Lecture Notes in Mathematics*. Springer, 1987.
[32] V. A. Hinich and V. V. Schechtman. Homotopy Lie algebras. *Advances in Soviet Mathematics*, 16:1–27, 1993.
[33] Y-Z Huang and J. Lepowsky. Operadic formulation of the notion of vertex operator algebra. Preprint, 1992.
[34] Y-Z Huang and J. Lepowsky. Vertex operator algebras and operads. Preprint, 1992.
[35] G. M. Kelly. On the operads of J. P. May. Preprint, January 1972.
[36] G. M. Kelly and S. MacLane. Coherence in closed categories. *J. Pure and Applied Algebra*, 1:97–140, 1971.
[37] I. Kriz and J. P. May. Differential graded algebras up to homotopy and their derived categories. Preprint, 1992.
[38] I. Kriz and J. P. May. Derived categories and motives. *Mathematical Research Letters*, 1:87–94, 1994.
[39] M. Levine. Bloch's higher Chow groups revisited. Preprint, 1993.
[40] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). *Equivariant stable homotopy theory*, volume 1213 of *Springer Lecture Notes in Mathematics*. Springer, 1986.
[41] S. MacLane. *Homology*. Springer-Verlag, 1963.
[42] S. MacLane. Categorical algebra. *Bull. Amer. Math. Soc.*, 75:40–106, 1965.
[43] S. MacLane. *Categories for the Working Mathematician*. Springer-Verlag, 1976.

[44] J. P. May. Simplicial objects in algebraic toplogy, 1967. Reprinted by the University of Chicago Press, 1982 and 1992.
[45] J. P. May. A general approach to Steenrod operations, volume 168 of *Springer Lecture Notes in Mathematics*. Springer, 1970.
[46] J. P. May. *The Geometry of Iterated Loop Spaces*, volume 271 of *Springer Lecture Notes in Mathematics*. Springer, 1972.
[47] J. P. May. *$E_\infty$ ring spaces and $E_\infty$ ring spectra*, volume 577 of *Springer Lecture Notes in Mathematics*. Springer, 1977.
[48] J. P. May. The spectra associated to $\mathscr{I}$-monoids. *Math. Proc. Camb. Phil. Soc.*, 84:313–322, 1978.
[49] J. P. May. Multiplicative ininite loop space theory. *J. Pure and Applied Algebra*, 26:1–69, 1982.
[50] J. P. May. The dual Whitehead theorems. *London Math. Soc. Lecture Note Series*, 86:46–54, 1983.
[51] J. P. May Derived categories in algebra and topology. In *Proceedings of the Eleventh International Conference on Topology*, Rendiconti dell Istituto Matematico dell Università di Trieste, to appear.
[52] J. P. May and R. W. Thomason. The uniqueness of infinite loop space machines. *Topology*, 17:205–224, 1978.
[53] S. Priddy. Primary cohomology operations for simpicial Lie algebras. *Illinois J. Math.*, 14:585–612, 1970.
[54] D. G. Quillen. *Homotopical algebra*, volume 43 of *Springer Lecture Notes in Mathematics*. Springer, 1967.
[55] G. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.
[56] J. Stasheff. Closed string field theory, strong homotopy Lie algebras, and the operad actions of moduli spaces. Preprints, 1992–93.
[57] R. Steiner. A canonical operad pair. *Math. Proc. Camb. Phil. Soc.*, 86:443–449, 1979.
[58] D. Sullivan. Infinitesimal computations in topology. *Publ. Math. IHES*, 47:269–331, 1978.
[59] A. Suslin. Higher Chow groups of affine varieties and étale cohomology. Preprint, 1993.
[60] J. L. Verdier. *Catégories dérivées*, volume 569 of *Springer Lecture Notes in Mathematics*. Springer, 1971.
[61] R. J. Wellington. The unstable Adams spectral sequence for free iterated loop spaces. *Memoirs of the Amer. Math. Soc.*, 258, 1982.

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