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The Homology of
Iterated Loop Spaces



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Preface

This volume is a collection of five papers (to be referred to as I-V). The first four together give a thorough treatment of homology operations and of their application to the calculation of, and analysis of internal structure in, the homologies of various spaces of interest. The last studies an up to homotopy notion of an algebra over a monad and the role of this notion in the theory of iterated loop spaces. I have established the algebraic preliminaries necessary to the first four papers and the geometric preliminaries necessary for all of the papers in the following references, which shall be referred to by the specified letters throughout the volume.

[A]. A general algebraic approach to Steenrod operations. Springer Lecture Notes in Mathematics Vol. 168, 1970, 153-231.

[G]. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics Vol. 271, 1972.

[G']. E_{∞} spaces, group completions, and permutative categories. London Math. Soc. Lecture Note Series Vol. 11, 1974, 61-93.

In addition, the paper II here is a companion piece to my book (contributed to by F. Quinn, N. Ray, and J. Tornehave)

[R]. E_{∞} Ring Spaces and E_{∞} Ring Spectra.

With these papers, this volume completes the development of a comprehensive theory of the geometry and homology of iterated loop spaces. There are no known results in or applications of this area of topology which do not fit naturally into the framework thus established. However, there are several papers by other authors which seem to me to add significantly to the theory developed in [G]. The relevant references will be incorporated in the list of errata and addenda to [A], [G], and [G'] which concludes this volume.

The geometric theory of [G] was incomplete in two essential respects. First, it worked well only for connected spaces (see [G, p. 156-158]). It was the primary purpose of [G'] to generalize the theory to non-connected spaces. In particular, this allowed it to be applied to the classifying spaces of permutative categories and thus to algebraic K-theory. More profoundly, the ring theory of [R] and II was thereby made possible.

Second, the theory of [G] circumvented analysis of homotopy invariance (see [G, p. 158-160]). It is the purpose of Lada's paper V to generalize the theory of [G] to one based on homotopy invariant structures on topological spaces in the sense of Boardman and Vogt [Springer Lecture Notes in Mathematics, Vol. 347]¹. In Boardman and

¹Incidentally, the claim there (p. VII) that [G] failed to apply to non Σ -free operads is based on a misreading; see [G, p. 22].

Vogt's work, an action up to homotopy by an operad (or PROP) on a space was essentially an action by a larger, but equivalent, operad on the same space. In Lada's work, an action up to homotopy is essentially an action by the given operad on a larger, but equivalent, space. In both cases, the expansion makes room for higher homotopies. While these need not be made explicit in the first approach, it seems to me that the second approach is nevertheless technically and conceptually simpler (although still quite complicated in detail) since the expansion construction is much less intricate and since the problem of composing higher homotopies largely evaporates.

We have attempted to make the homological results of this volume accessible to the reader unfamiliar with the geometric theory in the papers cited above. In I, I set up the theory of homology operations on infinite loop spaces. This is based on actions by E_∞ operads \mathcal{C} on spaces and is used to compute $H_*(CX; Z_p)$ and $H_*(QX; Z_p)$ as Hopf algebras over the Dyer-Lashof and Steenrod algebras, where CX and QX are the free \mathcal{C} -space and free infinite loop space generated by a space X . The structure of the Dyer-Lashof algebra is also analyzed. In II, I set up the theory of homology operations on E_∞ ring spaces, which are spaces with two suitably interrelated E_∞ space structures. In particular, the mixed Cartan formula and mixed Adem relations are proven and are

shown to determine the multiplicative homology operations of the free E_∞ ring space $C(X^\dagger)$ and the free E_∞ ring infinite loop space $Q(X^\dagger)$ generated by an E_∞ space X . In the second half of II, homology operations on E_∞ ring spaces associated to matrix groups are analyzed and an exhaustive study is made of the homology of BSF and of such related classifying spaces as $BTop$ (at $p > 2$) and $BCoker J$. Perhaps the most interesting feature of these calculations is the precise homological analysis of the infinite loop splitting $BSF = BCoker J \times BJ$ at odd primes and of the infinite loop fibration $BCoker J \rightarrow BSF \rightarrow BJ$ at $p = 2$.

In III, Cohen sets up the theory of homology operations on n -fold loop spaces for $n < \infty$. This is based on actions by the little cubes operad \mathcal{C}_n and is used to compute $H_*(C_n X; Z_p)$ and $H_*(\Omega^n \Sigma^n X; Z_p)$ as Hopf algebras over the Steenrod algebra with three types of homology operations. While the first four sections of III are precisely parallel to sections 1, 2, 4, and 5 of I, the construction of the unstable operations (for odd p) and the proofs of all requisite commutation formulas between them (which occupies the rest of III) is several orders of magnitude more difficult than the analogous work of I (most of which is already contained in [A]). The basic ingredient is a homological analysis of configuration spaces, which should be of independent interest. In IV, Cohen computes

$H_*(SF(n); Z_p)$ as an algebra for p odd and n even, the remaining cases being determined by the stable calculations of II. Again, the calculation is considerably more difficult than in the stable case, the key fact being that $H_*(SF(n); Z_p)$ is commutative even though $SF(n)$ is not homotopy commutative. Due to the lack of internal structure on $BSF(n)$, the calculation of $H_*(BSF(n); Z_p)$ is not yet complete.

In addition to their original material, I and III properly contain all work related to homology operations which antedates 1970, while II contains either complete information on or at least an introduction to most subsequent work in this area, the one major exception being that nothing will be said about $BTop$ and BPL at the prime 2. Up to minor variants, all work since 1970 has been expressed in the language and notations established in I §1-§2 and II §1.

Our thanks to Maija May for preparing the index.

J. P. May
August 20, 1975

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J. P. May

Homology operations on iterated loop spaces were first introduced, mod 2, by Araki and Kudo [1] in 1956; their work was clarified and extended by Browder [2] in 1960. Homology operations mod p , $p > 2$, were first introduced by Dyer and Lashof [6] in 1962. The work of Araki and Kudo proceeded in analogy with Steenrod's construction of the Sq^n in terms of U_i -products, whereas that of Browder and of Dyer and Lashof proceeded in analogy with Steenrod's later construction of the P^n in terms of the cohomology of the symmetric group Σ_p . The analogy was closest in the case of infinite loop spaces and, in [A], I reformulated the algebra behind Steenrod's work in a sufficiently general context that it could be applied equally well to the homology of infinite loop spaces and to the cohomology of spaces. Later, in [G], I introduced the notions of E_∞ operad and E_∞ space. Their use greatly simplifies the geometry required for the construction and analysis of the homology operations and, in the non-connected case, yields operations on a wider class of spaces than infinite loop spaces. These operations, and further operations on the homology of infinite loop spaces given by the elements of $H_* \tilde{F}$, will be analyzed in section 1.

Historically, the obvious next step after introduction of the homology operations should have been the introduction of the Hopf algebra of all homology operations and the analysis of geometrically allowable modules

(and more complicated structures) over this algebra, in analogy with the definitions in cohomology given by Steenrod [22] in 1961. However, this step seems not to have been taken until lectures of mine in 1968-69. The requisite definitions will be given in section 2. Since the idea that homology operations should satisfy Adem relations first appears in [6] (although these relations were not formulated or proven there), we call the resulting algebra of operations the Dyer-Lashof algebra; we denote it by R . The main point of section 2 is the explicit construction of free allowable structures over R .

During my 1968-69 lectures, Madsen raised and solved at the prime 2 the problem of carrying out for R the analog of Milnor's calculation of the dual of the Steenrod algebra A . His solution appears in [8]. Shortly after, I solved the problem at odd primes, where the structure of R^* turned out to be surprisingly complicated. The details of this computation ($p = 2$ included) will be given in section 3.

In section 4, we reformulate (and extend to general non-connected spaces X) the calculation of $H_* QX$, $QX = \varinjlim \Omega^n \Sigma^n X$, given by Dyer and Lashof [6]. Indeed, the definitions in section 2 allow us to describe $H_* QX$ as the free allowable Hopf algebra with conjugation over R and A . With the passage of time, it has become possible to give considerably simpler details of proof than were available in 1962. We also compute the Bockstein spectral sequence of QX (for each prime) in terms of that of X .

Just as QX is the free infinite loop space generated by a space X , so CX , as constructed in [G, §2], is the free \mathcal{C} -space generated by X (where \mathcal{C} is an E_∞ operad). In section 5, we prove that $H_* CX$ is the

free allowable Hopf algebra (without conjugation) over R and A . The proof is quite simple, especially since the geometry of the situation makes half of the calculation an immediate consequence of the calculation of $H_* QX$. Although the result here seems to be new, in this generality, special cases have long been known. When X is connected, CX is weakly equivalent to QX by [G, 6.3]. When $X = S^0$, $CX = \coprod K(\Sigma_j, 1)$ and the result thus contains Nakaoka's calculations [16, 17, 18] of the homology of symmetric groups. We end section 5 with a generalization (from S^0 to arbitrary spaces X) of Priddy's homology equivalence $B\Sigma_\infty \rightarrow \Omega_0 S^0$ [20].

In section 6, we describe how the iterated homology operations of an infinite loop space appear successively in the stages of its Postnikov decomposition.

In section 7, we construct and analyze homology operations analogous to the Pontryagin p^{th} powers in the cohomology of spaces. When $p = 2$, these operations were first introduced by Madsen [9].

Most of the material of sections 1-4 dates from my 1968-69 lectures at Chicago and was summarized in [12]. The material of section 5 dates from my 1971-72 lectures at Cambridge. The long delay in publication, for which I must apologize, was caused by problems with the sequel II (to be explained in its introduction).

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§1. Homology operations

We first define and develop the properties of homology operations on E_{∞} spaces. We then specialize to obtain further properties of the resulting operations on infinite loop spaces. In fact, the requisite geometry has been developed in [G, §1,4,5, and 8] and the requisite algebra has been developed in [A, §1-4 and 9]. The proofs in this section merely describe the transition from the geometry to the algebra.

All spaces are to be compactly generated and weakly Hausdorff; \mathcal{T} denotes the category of spaces with non-degenerate base-point [G, p.1]. All homology is to be taken with coefficients in \mathbb{Z}_p for an arbitrary prime p ; the modifications of statements required in the case $p=2$ are indicated inside square brackets.

We require some recollections from [G] in order to make sense of the following theorem. Recall that an E_{∞} space (X, θ) is a \mathcal{C} -space over any E_{∞} operad \mathcal{C} [G, Definitions 1.1, 1.2, and 3.5]; θ determines an H-space structure on X with the base-point $* \in X$ as identity element and with $\theta_2(c): X \times X \rightarrow X$ as product for any $c \in \mathcal{C}(2)$ [G, p.4]. Recall too that the category $\mathcal{C}[\mathcal{T}]$ of \mathcal{C} -spaces is closed under formation of loop and path spaces [G, Lemma 1.5] and has products and fibred products [G, Lemma 1.7].

Theorem 1.1. Let \mathcal{C} be an E_{∞} operad and let (X, θ) be a \mathcal{C} -space. Then there exist homomorphisms $Q^s: H_*X \rightarrow H_*X$, $s \geq 0$, which satisfy the following properties:

- (1) The Q^s are natural with respect to maps of \mathcal{C} -spaces.
- (2) Q^s raises degrees by $2s(p-1)$ [by s].
- (3) $Q^s x = 0$ if $2s < \text{degree}(x)$ [if $s < \text{degree}(x)$], $x \in H_*X$
- (4) $Q^s x = x^p$ if $2s = \text{degree}(x)$ [if $s = \text{degree}(x)$], $x \in H_*X$
- (5) $Q^s \phi = 0$ if $s > 0$, where $\phi \in H_0(X)$ is the identity element.

(6) The external, internal, and diagonal Cartan formulas hold:

$$Q^S(x \otimes y) = \sum_{i+j=S} Q^i x \otimes Q^j y \text{ if } x \otimes y \in H_*(X \times Y), (Y, \theta') \in \mathcal{C}[\mathcal{T}];$$

$$Q^S(xy) = \sum_{i+j=S} (Q^i x)(Q^j y) \text{ if } x, y \in H_* X; \text{ and}$$

$$\psi(Q^S x) = \sum_{i+j=S} \Sigma Q^i x' \otimes Q^j x'' \text{ if } \psi x = \Sigma x' \otimes x'', x \in H_* X.$$

(7) The Q^S are stable and the Kudo transgression theorem holds:

$Q^S \sigma_* = \sigma_* Q^S$, where $\sigma_*: \hat{H}_* \Omega X \rightarrow H_* X$ is the homology suspension; if X is simply connected and if $x \in H_q X$ transgresses to $y \in H_{q-1} \Omega X$ in the Serre spectral sequence of the path space fibration $\pi: PX \rightarrow X$, then $Q^S x$ and $\beta Q^S x$ transgress to $Q^S y$ and $-\beta Q^S y$ and, if $p > 2$ and $q = 2s$, $x^{p-1} \otimes y$ "transgresses" to $-\beta Q^S y$, $d^q(p-1)(x^{p-1} \otimes y) = -\beta Q^S y$.

(8) The Adem relations hold: If $p \geq 2$ and $r > ps$, then

$$Q^r Q^s = \sum_i (-1)^{r+i} (pi-r, r-(p-1)s-i-1) Q^{r+s-i} Q^i;$$

if $p > 2$, $r \geq ps$, and β denotes the mod p Bockstein, then

$$Q^r \beta Q^s = \sum_i (-1)^{r+i} (pi-r, r-(p-1)s-i) \beta Q^{r+s-i} Q^i \\ - \sum_i (-1)^{r+i} (pi-r-1, r-(p-1)s-i) Q^{r+s-i} \beta Q^i$$

(9) The Nishida relations hold: Let $P_*^r: H_* X \rightarrow H_* X$ be dual to P^r where $P^r = Sq^r$ if $p = 2$ (thus $P^r = (P_*^r)^*$ on $H^* X = (H_* X)^*$). Then

$$P_*^r Q^s = \sum_i (-1)^{r+i} (r-pi, s(p-1)-pr+pi) Q^{s-r+i} P_*^i;$$

in particular, $\beta Q^s = (s-1)Q^{s-1}$ if $p=2$; if $p > 2$,

$$P_*^r \beta Q^s = \sum_i (-1)^{r+i} (r-pi, s(p-1)-pr+pi-1) \beta Q^{s-r+i} P_*^i$$

$$+ \sum_i (-1)^{r+i} (r-pi-1, s(p-1)-pr+pi) Q^{s-r+i} P_*^i \beta.$$

(In (8) and (9), $(i, j) = (i+j)!/i!j!$ if $i > 0$ and $j > 0$, $(i, 0) = 1 = (0, i)$ if $i \geq 0$, and $(i, j) = 0$ if $i < 0$ or $j < 0$; the sums run over the integers.)

Proof: The symmetric group Σ_p acts freely on the contractible space $\mathcal{C}(p)$, hence the normalized singular chain complex $C_* \mathcal{C}(p)$ is a Σ_p -free resolution of Z_p [7, IV 11]. Let W be the standard π -free resolution of Z_p [A, Definition 1.2], where π is cyclic of order p , and let $j: W \rightarrow C_* \mathcal{C}(p)$ be a morphism of π -complexes over Z_p . Let $(C_* X)^P$ denote the p -fold tensor product. We are given a Σ_p -equivariant map $\theta_p: \mathcal{C}(p) \times X^P \rightarrow X$, and we define $\theta_*: W \otimes (C_* X)^P \rightarrow C_* X$ to be the following composite morphism of π -complexes:

$$W \otimes (C_* X)^P \xrightarrow{j \otimes \eta} C_* \mathcal{C}(p) \otimes C_* (X^P) \xrightarrow{\eta} C_* (\mathcal{C}(p) \times X^P) \xrightarrow{C_* \theta_p} C_* X.$$

Here η is the shuffle map; for diagram chases, it should be recalled that $\eta: C_* X \otimes C_* Y \rightarrow C_* (X \times Y)$ is a commutative and associative natural transformation which is chain homotopy inverse to the Alexander-Whitney map ξ . In view of [G, Lemmas 1.6 and 1.9 (i)], $(C_* X, \theta_*)$ is a unital and mod p reduced object of the category $\mathcal{C}(p, \infty)$ defined in [A, Definitions 2.1]. Moreover, $(X, \theta) \rightarrow (C_* X, \theta_*)$ is clearly the object map of a functor from $\mathcal{C}[\mathcal{T}]$ to the subcategory $\mathcal{P}(p, \infty)$ of $\mathcal{C}(p, \infty)$ defined in [A, Definitions 2.1]. Let $x \in H_q X$. As in [A, Definitions 2.2], we define

$$(i) \quad Q_i(x) = \theta_*(e_i \otimes x^P), \quad \theta_*: H(W \otimes_{\pi} (H_* X)^P) \cong H(W \otimes_{\pi} (C_* X)^P) \rightarrow H_* X,$$

and we define the desired operations Q^S by the formulas

$$(ii) \quad p = 2: \quad Q^S x = 0 \text{ if } s < q \text{ and } Q^S x = Q_{s-q}(x) \text{ if } s \geq q; \text{ and}$$

$$(iii) \quad p > 2: \quad Q^S x = 0 \text{ if } 2s < q \text{ and } Q^S x = (-1)^{Sv(q)} Q_{(2s-q)(p-1)}(x)$$

$$\text{if } 2s > q, \text{ where } v(q) = (-1)^{q(q-1)m/2} (m!)^q, \text{ with } m = \frac{1}{2}(p-1).$$

The Q^S are homomorphisms which satisfy (1) through (5) by [A, Proposition 2.3 and Corollary 2.4]. Note that [A, Proposition 2.3] also implies that if $p > 2$, then $\beta Q^S x = (-1)^S v(q) Q_{(2s-q)(p-1)-1}(x)$ and the Q^S and βQ^S account for all non-trivial operations Q_i . For (6), recall that the product of \mathcal{C} -spaces (X, θ) and (Y, θ') is $(X \times Y, \tilde{\theta})$, where $\tilde{\theta}_p$ is the composite

$$\mathcal{C}(p) \times (X \times Y)^p \xrightarrow{\Delta \times u} \mathcal{C}(p) \times \mathcal{C}(p) \times X^p \xrightarrow{1 \times t \times 1} \mathcal{C}(p) \times X^p \times \mathcal{C}(p) \times Y^p \xrightarrow{\theta \times \theta'} X \times Y$$

(Here Δ, u , and t are the diagonal and the evident shuffle and interchange maps.)

Similarly, the tensor product of objects (K, θ_*) and (L, θ'_*) in $\mathcal{C}(p, \infty)$ is $(K \otimes L, \tilde{\theta}_*)$, where $\tilde{\theta}_*$ is the composite

$$W \otimes (K \otimes L)^p \xrightarrow{\psi \otimes U} W \otimes W \otimes K^p \otimes L^p \xrightarrow{1 \otimes T \otimes 1} W \otimes K^p \otimes W \otimes L^p \xrightarrow{\theta_* \otimes \theta'_*} K \otimes L.$$

(Here ψ, U , and T are the coproduct on W and the evident shuffle and interchange homomorphisms.) Since $(j \otimes j)\psi$ is π -homotopic to $(\xi \circ C_* \Delta)j$, an easy diagram chase demonstrates that $\eta: C_* X \otimes C_* Y \rightarrow C_*(X \times Y)$ is a morphism in the category $\mathcal{C}(p, \infty)$. The external Cartan formula now follows from [A, Corollary 2.7]. By [G, Lemmas 1.7 and 1.9 (ii)], $\Delta: X \rightarrow X \times X$ is a map of \mathcal{C} -spaces and $(C_* X, \theta_*)$ is a Cartan object of $\mathcal{C}(p, \infty)$; the diagonal and internal Cartan formulas follow by naturality. Part (7) is an immediate consequence of [G, Lemma 1.5] and [A, Theorems 3.3 and 3.4]; the simple connectivity of X serves to ensure that $E^2 = H_* X \otimes H_* \Omega X$ in the Serre spectral sequence of $\pi: PX \rightarrow X$. For (8), note that the following diagram is commutative by [G, Lemma 1.4]:

$$\begin{array}{ccc} \mathcal{C}(p) \times \mathcal{C}(p)^p \times X^{p^2} & \xrightarrow{\gamma \times 1} & \mathcal{C}(p^2) \times X^{p^2} \\ \downarrow 1 \times u & & \searrow \theta_{p^2} \\ \mathcal{C}(p) \times [\mathcal{C}(p) \times X^p]^p & \xrightarrow{1 \times (\theta_p)^p} & \mathcal{C}(p) \times X^p \\ & & \nearrow \theta_p \\ & & X \end{array}$$

An easy diagram chase demonstrates that $(C_* X, \theta_*)$ is an Adem object, in the sense of [A, Definition 4.1], and (8) follows by [A, Theorem 4.7]. Part (9) follows by the naturality of the Steenrod operations from [A, Theorem 9.4], which computes the Steenrod operations in $H_*(\mathcal{C}(p) \times_{\pi} X^p)$. As explained in [A, p.209], our formula differs by a sign from that obtained by Nishida [19].

Let \mathcal{L}_{∞} be the category of infinite loop sequences. Recall that an object $Y = \{Y_i\}$ in \mathcal{L}_{∞} is a sequence of spaces with $Y_i = \Omega Y_{i+1}$ and a morphism $g = \{g_i\}$ in \mathcal{L}_{∞} is a sequence of maps with $g_i = \Omega g_{i+1}$. Y_0 is said to be an infinite loop space, g_0 an infinite loop map. By the results of [10], these notions are equivalent for the purposes of homotopy theory to the more usual ones in which equalities are replaced by homotopies. By [G, Theorem 5.1], there is a functor $W_{\infty}: \mathcal{L}_{\infty} \rightarrow \mathcal{C}_{\infty}[\mathcal{T}]$, with $W_{\infty} Y = (Y_0, \theta_{\infty})$ and $W_{\infty} g = g_0$, where \mathcal{C}_{∞} is the infinite little cubes operad of [G, Definition 4.1]. The previous theorem therefore applies to $H_* Y_0$; the resulting operations Q^S will be referred to as the loop operations. The relevant Pontryagin product is that induced by the loop product on $Y_0 = \Omega Y_1$. Note that there are two different actions of \mathcal{C}_{∞} on ΩY_0 , one coming from [G, Lemma 1.5] and the other from the fact that ΩY_0 is again an infinite loop space; by [G, Lemma 5.6], these two actions are equivariantly homotopic, hence part (7) of the theorem does apply to the loop operations. Similarly, part (6) applies to the loop operations

since, by [G, Lemma 5.7], the two evident actions of \mathcal{L}_∞ on the product of two infinite loop spaces are in fact the same.

The recognition theorem [G, Theorem 14.4; G'] gives a weak homotopy equivalence between any given grouplike E_∞ space X and an infinite loop space B_0X ; moreover, as explained in [G, p.153-155], the homology operations on H_*X coming via Theorem 1.1 from the given E_∞ space structure agree under the equivalence with the loop operations on H_*B_0X . Thus, in principle, it is only for non grouplike E_∞ spaces that the operations of Theorem 1.1 are more general than loop operations. In practice, the theorem gives considerable geometric freedom in the construction of the operations, and this freedom is often essential to the calculations.

The following additional property of the loop operations, which is implied by [G, Remarks 5.8], will be important in the study of non-connected infinite loop spaces. Recall that the conjugation χ on a Hopf algebra, if present, is related to the unit η , augmentation ϵ , product ϕ , and coproduct ψ by the formula $\eta\epsilon = \phi(1 \times \chi)\psi$.

Lemma 1.2. For $Y \in \mathcal{L}_\infty$, $Q^S \chi = \chi Q^S$ on H_*Y_0 , where the conjugation is induced from the inverse map on $Y_0 = \Omega Y_1$.

In the next two sections, we will define and study the global algebraic structures which are naturally suggested by the results above. We make a preliminary definition here.

Definition 1.3. Let A be a Hopf algebra. Let A act on Z_p through its augmentation, $a \cdot 1 = \epsilon(a)$, and let A act on the tensor product $M \otimes N$ of two left A -modules through its coproduct,

$$a(m \otimes n) = \sum (-1)^{\deg a} \deg m_a' m \otimes a'n \text{ if } \psi a = \sum a' \otimes a''.$$

A -left or right structure (algebra, coalgebra, Hopf algebra, Hopf algebra with conjugation, etc.) over A is a left or right

A -module and a structure of the specified type such that all of the structure maps are morphisms of A -modules.

We shall define a Hopf algebra R of homology operations in the next section and, if $Y \in \mathcal{L}_\infty$, H_*Y_0 will be a left Hopf algebra with conjugation over R . For any space X , H_*X is a left coalgebra over the opposite algebra A^0 of the Steenrod algebra; here the opposite algebra enters because dualization is contravariant. Henceforward, although we shall continue to write the Steenrod operations P_*^r on the left, we shall speak of right A -modules rather than of left A^0 -modules. Thus H_*Y_0 is a right Hopf algebra with conjugation over A , and the Nishida relations give commutation formulas between the A and R operations on H_*Y_0 .

There is yet another Hopf algebra which acts naturally and stably on H_*Y_0 namely $H_*\hat{F}$ where \hat{F} is the monoid (under composition) of based maps of spheres. The precise definition of \hat{F} is given in [G, p.74], and it is shown there that composition of maps defines a natural action $c_\infty: Y_0 \times \hat{F} \rightarrow Y_0$ of \hat{F} on infinite loop spaces. The following theorem gives the basic properties of the induced action of $H_*\hat{F}$ on H_*Y_0 .

Theorem 1.4. For $Y \in \mathcal{L}_\infty$, $c_{\infty*}: H_*Y_0 \otimes H_*\hat{F} \rightarrow H_*Y_0$ gives H_*Y_0 a structure of right Hopf algebra with conjugation over $H_*\hat{F}$. Moreover, $c_{\infty*}$ satisfies the following properties, where $c_{\infty*}(x \otimes f) = xf$:

- (1) $c_{\infty*}$ is natural with respect to maps in \mathcal{L}_∞ .
- (2) $\sigma_*(xf) = (\sigma_*x)f$, where $\sigma_*: H_*\Omega Y_0 \rightarrow H_*Y_0$ is the suspension.
- (3) $P_*^r(xf) = \sum_{i+j=r} (P_*^i x)(P_*^j f)$ and $\beta(xf) = (\beta x)f + (-1)^{\deg x} x(\beta f)$.
- (4) $(Q^S x)f = \sum_i Q^{S+i}(xP_*^i f)$ and, if $p > 2$,
 $(\beta Q^S x)f = \sum_i \beta Q^{S+i}(xP_*^i f) - \sum_i (-1)^{\deg x} Q^{S+i}(xP_*^i \beta f)$.

duces $c_{\infty*}$, and the map induced on π -equivariant homology by $d = \Phi(1 \otimes \Delta)$ is explicitly computed in [A, Proposition 9.1]. Of course, it is the presence of d in the diagram which leads to the appearance of Steenrod operations in formula (4). The verification of this formula is now an easy direct calculation from the definition of the operations Q^S .

The essential part of the previous diagram is of course the geometric bottom left square. Henceforward, we shall omit the pedantic details in the passage from geometric diagrams to algebraic formulas.

We evaluate one obvious example of the operations on H_*Y_0 given by right multiplication by elements of $H_*\tilde{F}$.

Lemma 1.5. Let $[i] \in H_0\tilde{F}$ be the class represented by a map of degree i . Then $x[-1] = \chi x$ for all $x \in H_*Y_0$, $Y \in \mathcal{L}_\infty$.

Proof: Define $f: S^n \rightarrow S^n$ by $f(s_1, \dots, s_n) = (1-s_1, s_2, \dots, s_n)$, where $S^n = I^n/\partial I^n$. For any X , the inverse map on $\Omega^n X$ is given by $g \rightarrow g \circ f$, $g: S^n \rightarrow X$, and similarly on Y_0 by passage to limits via [G, p.74]. The result follows.

Recall that $QX = \varinjlim \Omega^n \Sigma^n X$ and $QX = \Omega Q\Sigma X$ [G, p.42].

As we shall see in II § 5, application of the results

above to $Y_0 = QS^0$, where c_∞ reduces to the product on \tilde{F} , completely determines the composition product on $H_*\tilde{F}$.

Remarks 1.6. A functorial definition of a smash product between objects of \mathcal{L}_∞ is given in [13], in which a new construction of the stable homotopy category is given. (In the language of [13], \mathcal{L}_∞ is a category of coordinatized spectra; the smash product is constructed by passing to the category \mathcal{L} of coordinate-free spectra, applying the smash product there, and then returning to \mathcal{L}_∞ .) For objects $Y, Z \in \mathcal{L}_\infty$ and elements

$x, y \in H_*Y_0$ and $z \in H_*Z_0$, [G, Lemma 8.1] and a similar lemma for the inverse map imply the formulas

$$(x * y) \wedge z = \sum (-1)^{\deg y \deg z'} (x \wedge z') * (y \wedge z) \quad \text{if } \psi z = \sum z' \otimes z''$$

$$\text{and} \quad (\chi y) \wedge z = \chi(y \wedge z),$$

where $*$ and \wedge denote the loop and smash products respectively. Via a diagram chase precisely analogous to that in the proof of Theorem 1.4, [G, Proposition 8.2] implies the formulas

$$(Q^S y) \wedge z = \sum_i Q^{S+i}(y \wedge P_*^i z)$$

and, if $p > 2$,

$$(\beta Q^S y) \wedge z = \sum_i \beta Q^{S+i}(y \wedge P_*^i z) - \sum_i (-1)^{\deg y} Q^{S+i}(y \wedge P_*^i \beta z).$$

In particular, these results apply to $\wedge: QX \times QX' \rightarrow Q(X \wedge X')$ for any spaces X and X' . By [G, Lemma 8.7], the smash and composition products coincide and are commutative on $H_*QS^0 = H_*\tilde{F}$.

§2. Allowable structures over the Dyer-Lashof algebra

We here describe the Hopf algebra of homology operations on E_∞ spaces generated by the Q^s and βQ^s and develop analogs of the notions of unstable modules and algebras over the Steenrod algebra. The following definition determines the appropriate "admissible monomials".

Definition 2.1. (i) $p = 2$: Consider sequences $I = (s_1, \dots, s_k)$ such that $s_j \geq 0$. Define the degree, length, and excess of I by

$$d(I) = \sum_{j=1}^k s_j; \quad \ell(I) = k; \quad \text{and}$$

$$e(I) = s_k - \sum_{j=2}^k (2s_j - s_{j-1}) = s_1 - \sum_{j=2}^k s_j.$$

The sequence I determines the homology operation $Q^I = Q^{s_1} \dots Q^{s_k}$.

I is said to be admissible if $2s_j \geq s_{j-1}$ for $2 \leq j \leq k$.

(ii) $p > 2$: Consider sequences $I = (\epsilon_1, s_1, \dots, \epsilon_k, s_k)$ such that $\epsilon_j = 0$ or 1 and $s_j \geq \epsilon_j$. Define the degree, length, and excess of I by

$$d(I) = \sum_{j=1}^k [2s_j(p-1) - \epsilon_j]; \quad \ell(I) = k; \quad \text{and}$$

$$e(I) = 2s_k - \epsilon_1 - \sum_{j=2}^k [2ps_j - \epsilon_j - 2s_{j-1}] = 2s_1 - \epsilon_1 - \sum_{j=2}^k [2s_j(p-1) - \epsilon_j].$$

The sequence I determines the homology operation $Q^I = \beta^{\epsilon_1} Q^{s_1} \dots \beta^{\epsilon_k} Q^{s_k}$.

I is said to be admissible if $ps_j - \epsilon_j \geq s_{j-1}$ for $2 \leq j \leq k$.

(iii) **Conventions:** $b(I) = \epsilon_1$ if $p > 2$ and $b(I) = 0$ if $p = 2$.

The empty sequence I is admissible and satisfies $d(I) = 0$, $\ell(I) = 0$, $e(I) = \infty$, and $b(I) = 0$; it determines the identity homology operation $Q^I = 1$.

Definition 2.2. Let F denote the free associative algebra

generated by $\{Q^s | s \geq 0\}$ if $p = 2$ or by $\{Q^s | s \geq 0\} \cup \{\beta Q^s | s > 0\}$ if $p > 2$ (not β itself). For $q \geq 0$, define $J(q)$ to be the two-sided ideal of F generated by the Adem relations (and, if $p > 2$, the relations obtained by applying β to the Adem relations, with $\beta^2 = 0$) and by the relations $Q^I = 0$ if $e(I) < q$. Define $R(q)$ to be the quotient algebra $F/J(q)$, and observe that there are successive quotient maps $R(q) \rightarrow R(q+1)$. Let $R = R(0)$; R will be called the Dyer-Lashof algebra.

To avoid circularity, we have defined the $R(q)$ purely algebraically. The following theorem implies that this definition agrees with that naturally suggested by the geometry.

Theorem 2.3. (i) Let $i_q \in H_q S^q \subset H_q QS^q$ be the fundamental class if $q > 0$ and the class of the point other than the base-point if $q = 0$; then

$$\{Q^I i_q | I \text{ is admissible and } e(I) \geq q\}$$

is a linearly independent subset of $H_* QS^q$.

(ii) $J(q)$ coincides with the set $K(q)$ of all elements of F which annihilate every homology class of degree $\geq q$ of every E_∞ space (equivalently, of every infinite loop space).

(iii) $\{Q^I | I \text{ is admissible and } e(I) \geq q\}$ is a \mathbb{Z}_p -basis for $R(q)$.

(iv) $R(q)$ admits a unique structure of right A -module such that the Nishida relations are satisfied.

(v) $R = R(0)$ admits a structure of Hopf algebra and of unstable right coalgebra over A with coproduct defined on generators by

$$\psi Q^s = \sum_{i+j=s} Q^i \otimes Q^j \quad \text{and} \quad \psi \beta Q^{s+1} = \sum_{i+j=s} (\beta Q^{i+1} \otimes Q^j + Q^i \otimes \beta Q^{j+1})$$

and with augmentation defined on $R_0 = P\{Q^0\}$ by $\epsilon(Q^{0k}) = 1$, $k \geq 0$.

Proof: We shall prove (i) in §4. It is obvious from the Adem

relations that $R(q)$ is generated as a \mathbb{Z}_p -space by the set specified in (iii), and $J(q)$ is contained in $K(q)$ by (3) and (8) of Theorem 1.1. Therefore (i) implies (ii) and (iii). For (iv), the A operations on $1 \in R_0(q)$ are determined by $P_*^0 = 1$ and $R_i(q) = 0$ for $i < 0$ and the A operations on all elements $Q^I = Q^I \cdot 1$ with $\ell(I) > 0$ are determined from the Nishida relations by induction on $\ell(I)$. This action does give an A -module structure since if $f(q): R(q) \rightarrow H_*(QS^q)$ is defined by $f(q) Q^I = Q^I i_q$, then $f(q)$ is a monomorphism which commutes with the Steenrod operations. Let $f(0) = f$; since $\psi(1) = 1 \otimes 1$ and $\psi(i_0) = i_0 \otimes i_0$ and since $\varepsilon(i_0^k) = 1$, f commutes with the coproduct and augmentation. Here ψ is well-defined on R and R is a Hopf algebra since $J = J(0)$ is a Hopf ideal, $\psi(J) \subset F \otimes J + J \otimes F$, by commutativity of the following diagram (where π is the quotient map):

$$\begin{array}{ccccc}
 F & \xrightarrow{\pi} & R & \xrightarrow{f} & H_*QS^0 \\
 \psi \downarrow & & & & \downarrow \psi \\
 F \otimes F & \xrightarrow{\pi \otimes \pi} & R \otimes R & \xrightarrow{f \otimes f} & H_*QS^0 \otimes H_*QS^0
 \end{array}$$

Observe that this argument fails for $q > 0$ since $\psi i_q = i_q \otimes 1 + 1 \otimes i_q$. Since H_*QS^0 is an unstable right coalgebra over A , so is R .

Of course, we understand unstable right structures over A in the sense of homology: the dual object (if of finite type) is an unstable A -structure of the dual type, as defined by Steenrod [22,23]. We shall study the structure of R itself in the next section. The remainder of this section will be devoted to the study of structures over R . In order to deal with non-connected structures, we need some preliminaries.

Definition 2.4. A component coalgebra is a unital (and augmented) coalgebra C such that C is a direct sum of connected

coalgebras. Given such a C , define

$$\pi C = \{g \mid g \in C, \psi g = g \otimes g \text{ and } g \neq 0\}.$$

Clearly πC is a basis for C_0 . For $g \in \pi C$, define C_g to be the connected sub-coalgebra of C such that $g \in C_g$ and the set of positive degree elements of C_g is

$$\bar{C}_g = \{x \mid \psi x = x \otimes g + \sum x' \otimes x'' + g \otimes x, \deg x' > 0 \text{ and } \deg x'' > 0\}.$$

Then C is the direct sum of its components C_g for $g \in \pi C$. Note that $\varepsilon g = 1$ for $g \in \pi C$. πC contains the distinguished element $\phi = \eta(1)$, and $JC = \text{Coker } \eta$ may be identified with $\bar{C} \oplus (\bigoplus_{g \neq \phi} C_g) \subset C$.

If X is a based space, then H_*X is a component coalgebra; the base-point determines η and the components determine the direct sum decomposition. Indeed, there is an obvious identification of $\pi_0 X$ with πH_*X . As another example, we have the following observations on the structure of R .

Lemma 2.5. R is a component coalgebra. πR is the free monoid generated by Q^0 and the component $R[k]$ of $(Q^0)^k$, $k \geq 0$, is the sub unstable A -coalgebra of R spanned by

$$\{Q^I \mid I \text{ is admissible, } e(I) \geq 0, \text{ and } \ell(I) = k\}.$$

The product on R sends $R[k] \otimes R[l]$ to $R[k+l]$ for all k and l , and the elements Q^s and βQ^s are all indecomposable.

Definition 2.6. A component Hopf algebra B is said to be monoidal (resp., group-like) if πB is a monoid (resp., group) under the product of B . Equivalently, B is monoidal if all pairwise products of elements of πB are non-zero.

The proof of the following lemma requires only the defining formula

$$\eta \varepsilon = \phi(1 \otimes \chi) \psi \text{ for a conjugation.}$$

Lemma 2.7. A component Hopf algebra B admits a conjugation if and only if B is grouplike, and then $\chi g = g^{-1}$ if $g \in \pi B$ and

$$\chi^x = -g^{-1} x g^{-1} - \sum g^{-1} x' (\chi x'')$$

if $\deg x > 0$ and $\psi x = x \otimes g + \sum x' \otimes x'' + g \otimes x$ with $\deg x' > 0$ and $\deg x'' > 0$.

We can now define allowable structures over R , by which we simply mean those kinds of R -structures which satisfy the algebraic constraints dictated by the geometry.

Definition 2.8. A left R -module D is allowable if $J(q) D_q = 0$ for all $q \geq 0$. The category of allowable R -modules is the full subcategory of that of R -modules whose objects are allowable; it is an Abelian subcategory which is closed under the tensor product. An allowable R -algebra is an allowable R -module and a commutative algebra over R such that $Q^s x = x^p$ if $2s = \deg x$ [$Q^s x = x^2$ if $s = \deg x$]. An allowable R -coalgebra is an allowable R -module and a cocommutative component coalgebra over R . An allowable R -Hopf algebra (with conjugation) is a monoidal Hopf algebra (with conjugation) over R which is allowable both as an R -algebra and as an R -coalgebra. For any of these structures, an allowable AR -structure is an allowable R -structure and an unstable right A -structure of the same type such that the A and R operations satisfy the Nishida relations.

Theorem 1.1 implies that the homology of an E_∞ space is an allowable AR -Hopf algebra. Lemma 1.2 implies that the homology of an infinite loop space is an allowable AR -Hopf algebra with conjugation. Observe that a connected allowable AR -Hopf algebra is automatically an allowable AR -Hopf algebra with conjugation.

In order to take advantage of these definitions, we require five basic free functors, D, E, V, W , and G , of which E and W are essentially elaborations of D and V in the presence of coproducts. In addition, each of these functors has a

more elaborate counter-part, to be defined parenthetically, in the presence of Steenrod operations. The composite functors WE and GWE will describe $H_* C_\infty X$ and $H_* QX$, with all structure in sight, as functors of $H_* X$.

We shall describe our functors on objects and shall show that the given internal structures uniquely determine the required internal structures. The verifications (not all of which are trivial) that these structures are in fact well-defined and satisfy all of the requisite algebraic identities will be left to the reader, since these consistency statements obviously hold for those structures which can be realized geometrically. It is trivial to verify that our functors are indeed free, in the sense that they are adjoint to the forgetful functors going the other way. The functor V , which is a special case of the universal enveloping algebra functor on Abelian restricted Lie algebras, and the functor W occur in many other contexts in algebraic topology; they are discussed in detail in [11].

D: Z_p -modules (resp., unstable A -modules) to allowable R -modules (resp., AR -modules): Given M , define

$$DM = \bigoplus_{q \geq 0} R(q) \otimes M_q.$$

R acts on DM via the quotient maps $R \rightarrow R(q)$; thus DM is just the obvious quotient of the free R -module $R \otimes M$. The inclusion of M in DM is given by $m \rightarrow 1 \otimes m$. If A acts on M , then this action and the Nishida relations determine the action of A on DM by induction on the length of admissible monomials.

E: Cocommutative component coalgebras (resp., unstable A -coalgebras) to allowable R -coalgebras (resp., AR -coalgebras): Given C , define EC as an R -module, and as an A -module if A acts on C , by

$EC = DC/IR \cdot \text{Im} \eta = Z_p \oplus DJC$, $IR = \text{Ker } \epsilon$ and $JC = \text{Coker } \eta$.

The inclusion of C in EC is induced by that of C in DC . The coproduct on C and the diagonal Cartan formula determine the coproduct on EC . The unit of C and the augmentations of R and C determine the unit and augmentation of EC . Equivalently, EC is the obvious quotient component coalgebra of $R \otimes C$; thus

$$\pi EC = \{(Q^0)^k \otimes g \mid k \geq 0 \text{ and } g \in \pi C, k=0 \text{ if } g=\phi=\eta(1)\},$$

and the component of $(Q^0)^k \otimes g$ is the image of $R[k] \otimes C_g$ in EC if $g \neq \phi$ while the component of $1 \otimes \phi$ is the image of $R \otimes C_\phi$.

V: Allowable R-modules (resp., AR-modules) to allowable R-algebras (resp., AR-algebras): Given D , define

$$VD = AD/K$$

where AD is the free commutative algebra generated by D and K is the ideal of AD generated by

$$\{x^p - Q^s x \mid 2s = \deg x \text{ if } p > 2 \text{ or } s = \deg x \text{ if } p=2\}.$$

The R -action, and the A -action if A acts on D , are determined from the actions on $DCVD$ by the internal Cartan formulas (for R and A) and the properties required of the unit.

W: Allowable R-coalgebras (resp., AR-coalgebras) to allowable R-Hopf algebras (resp., AR-Hopf algebras): Given E , define WE as an R -algebra, and as an A -algebra if A acts on E , by

$$WE = VJE, JE = \text{Coker } \eta.$$

The inclusion of E in WE is given by $E = Z_p \oplus JE$ and $JE \subset VJE$. The coproduct and augmentation of WE are determined by those of E and the requirement that WE be a Hopf algebra (it is a well-defined Hopf algebra by [11, Proposition 12]). The components of WE are easily read off from the definition of $V_0 JE$.

G: Allowable R-Hopf algebras (resp., AR-Hopf algebras) to

allowable R-Hopf algebras (resp., AR-Hopf algebras) with conjugation:

Given W , define GW as follows. πW is a commutative monoid under the product in W and W_0 is its monoid ring. Let πGW be the commutative group generated by πW and let $G_0 W$ be its group ring. Let $\phi = \eta(1)$, let \bar{W} be the set of positive degree elements of W , and let \bar{W}^+ be the connected subalgebra $Z_p \phi \oplus \bar{W}$ of W . Define

$$GW = \bar{W}^+ \otimes_{G_0 W} W \cong W \otimes_{\pi W} G_0 W$$

as an augmented algebra. Embed W in GW as the subalgebra

$$(\bar{W} \otimes_{Z_p} \phi) \oplus (Z_p \phi \otimes W_0)$$

The coproduct on GW is determined by the requirements that W and $G_0 W$ be subcoalgebras and that GW be a Hopf algebra. The conjugation is given by Lemma 2.7. The R -action, and the A -action if A acts on W , are determined from the actions on $W \subset GW$ by commutation with χ and the Cartan formulas. If the product in WG is denoted by $*$, then the positive degree elements of the component of $f \in \pi GW$ are given by

$$(\bar{GW})_f = \bigoplus_{g \in \pi W} (\bar{W}_g \otimes g^{-1} * f) = \bigoplus_{g \in \pi W} \bar{W}_g * g^{-1} * f.$$

Observe that $GW = W$ if W is connected and that, as a ring, GW is just the localization of the ring W at the monoid πW .

§3. The dual of the Dyer-Lashof algebra

Since EH_*S^0 is the allowable AR-coalgebra $Z_p \oplus R$ (which should be thought of as $Z_p \cdot [0] \oplus R \cdot [1]$), a firm grasp on the structure of R is important to the understanding of $H_*C_{\infty}S^0$ and of H_*QS^0 . The coproduct and A-action on R are determined by the diagonal Cartan formula and the Nishida relations, but these merely give recursion formulas with respect to length, the explicit evaluation of which requires use of the Adem relations. To obtain precise information, we proceed by analogy with Milnor's computation of the dual of the Steenrod algebra [14]. In the case $p = 2$, the analogy is quite close; in the case $p > 2$, the Bocksteins introduce amusing complications. The structure of R^* , in the case $p = 2$, was first determined by Madsen [8]; his proofs are closer to the spirit of Milnor's work, but do not generalize readily to the case of odd primes.

By Lemma 2.5, $R = \bigoplus_{k \geq 0} R[k]$ as an A-coalgebra. Of course, $R[0] = Z_p$. We must first determine the primitive elements $PR[k]$ of the connected coalgebras $R[k]$, $k \geq 1$. To this end, define $P[k] = \{I \mid I \text{ is admissible, } e(I) \geq 0, \ell(I) = k, \text{ and } I \text{ ends with } 1\}$. We shall see that $\{Q^I \mid I \in P[k]\}$ is a basis for $PR[k]$. Define (inductively and explicitly) certain elements of $P[k]$ as follows:

(I) I_{jk} , $1 \leq j \leq k$, $p=2$: $I_{11} = (1)$, $I_{j,k+1} = (2^k - 2^{k-j}, I_{jk})$ if $j \leq k$, and $I_{k+1,k+1} = (2^k, I_{kk})$; then $d(I_{jk}) = 2^k - 2^{k-j}$, $e(I_{jk}) = 0$ if $j < k$, $e(I_{kk}) = 1$; $I_{jk} = (2^{k-1} - 2^{k-1-j}, 2^{k-2} - 2^{k-2-j}, \dots, 2^{j-1}, 2^{j-1}, \dots, 1)$.

(II) I_{jk} , $1 \leq j \leq k$, $p > 2$: $I_{11} = (0, 1)$, $I_{j,k+1} = (0, p^k - p^{k-j}, I_{jk})$ if $j < k$, and $I_{k+1,k+1} = (0, p^k, I_{kk})$; then $d(I_{jk}) = 2(p^k - p^{k-j})$, $e(I_{jk}) = 0$ if $j < k$, $e(I_{kk}) = 2$; $I_{jk} = (0, p^{k-1} - p^{k-1-j}, \dots, 0, p^{j-1}, 0, p^{j-1}, 0, p^{j-2}, \dots, 0, 1)$.

(III) J_{jk} , $1 \leq j \leq k$, $p > 2$: $J_{11} = (1, 1)$, $J_{j,k+1} = (0, p^k - p^{k-j}, J_{jk})$ if $j < k$, and $J_{k+1,k+1} = (1, p^k, I_{kk})$; then $d(J_{jk}) = 2(p^k - p^{k-j}) - 1$, $e(J_{jk}) = 1$; $J_{jk} = (0, p^{k-1} - p^{k-1-j}, \dots, 0, p^{j-1}, 1, p^{j-1}, 0, p^{j-2}, \dots, 0, 1)$.

(IV) K_{ijk} , $1 \leq i < j \leq k$, $p > 2$: $K_{i,j,k+1} = (0, p^k - p^{k-i} - p^{k-j}, K_{ijk})$ if $j \leq k$, and $K_{i,k+1,k+1} = (1, p^k - p^{k-i}, J_{ik})$; then $d(K_{ijk}) = 2(p^k - p^{k-i} - p^{k-j})$, $e(K_{ijk}) = 0$; $K_{ijk} = (0, p^{k-1} - p^{k-1-i} - p^{k-1-j}, \dots, 0, p^{j-1} - p^{j-1-i} - 1, 1, p^{j-1} - p^{j-1-i}, J_{i,j-1})$.

If we look back at the definition of the Q^S in terms of the Q_i in the proof of Theorem 1.1, we see that, when acting on a zero-dimensional class, our four classes of sequences correspond to sequences of operations of the respective forms

- (I) $Q_0 \cdots Q_0 Q_1 \cdots Q_1$
 (II) $Q_0 \cdots Q_0 Q_2^{(p-1)} \cdots Q_2^{(p-1)}$
 (III) $Q_{p-1} \cdots Q_{p-1} Q_2^{(p-1)} \cdots Q_2^{(p-1)}$
 (IV) $Q_0 \cdots Q_0 Q_{p-1} \cdots Q_{p-1} Q_2^{(p-1)} \cdots Q_2^{(p-1)}$

Many arguments in this section and the next can be illuminated by translation to lower indices.

Lemma 3.1. $P[k] = \{I_{jk} \mid 1 \leq j \leq k\}$ if $p=2$; $P[k] = \{I_{jk}, J_{jk}, K_{ijk} \mid 1 \leq j \leq k, 1 \leq i < j\}$ if $p > 2$. If $I \in P[k]$, then Q^I is primitive, $\psi Q^I = Q^I \otimes (Q^0)^k + (Q^0)^k \otimes Q^I$. **Proof:** Proceed by induction on k , the case $k = 1$ being trivial. Consider $I = (\epsilon, s, J) \in P[k]$, $k \geq 2$. Then, since I is admissible and $e(I) \geq 0$, $J \in P[k-1]$, $pr - \delta \geq s$ if $J = (\delta, r, K)$, and $2s - \epsilon \geq d(J)$. The first part follows inductively from these inequalities by a

trivial examination of cases. The second part is an easy calculation based on the facts that $\beta^\gamma Q^i Q^J = 0$ if $2i - \gamma < d(J)$ and that $\beta^\gamma Q^1 Q^0 = 0$ by the first Adem relation.

The computation of $R[k]^*$ as an algebra is based on a correspondence between addition of admissible sequences and multiplication of duals of admissible monomials. We first set up the required calculus of admissible sequences.

Definitions 3.2. The sum $I+J$ and difference $I-J$ of two sequences (as in Definition 2.1) of length k is defined termwise, under the conventions that $I+J$ is undefined if $p > 2$ and the i^{th} "Bockstein entry" ϵ_i is one in both I and J and that $I-J$ is undefined if any entry is less than zero. Observe that $e(I+J) = e(I) + e(J)$ and $d(I+J) = d(I) + d(J)$. If I and J are admissible, then $I+J$ is admissible but $I-J$ need not be admissible. In order to enumerate the admissible monomials when $p > 2$, consider all sequences $e = \{e_1, \dots, e_j\}$ with $1 \leq e_1 < \dots < e_j \leq k$ and define

$$I_e[k] = \{I \mid I \text{ is admissible, } e(I) \geq 0, \ell(I) = k, \text{ and } \epsilon_{k+1-a} = 1 \iff a \in e\}.$$

Write $I_e[k] = I[k]$ when e is empty. When $j \geq 1$, define $L_{e,k} \in I_e[k]$ by

$$L_{e,k} = \begin{cases} \epsilon_{e_1} \epsilon_{2k} + \dots + \epsilon_{e_{j-1}} \epsilon_{e_j k} & \text{if } j \text{ is even} \\ \epsilon_{e_1} \epsilon_{2k} + \dots + \epsilon_{e_{j-2}} \epsilon_{e_{j-1} k} + \epsilon_{e_j k} & \text{if } j \text{ is odd} \end{cases}$$

If $p=2$, write $I[k] = \{I \mid I \text{ is admissible, } e(I) \geq 0, \text{ and } \ell(I) = k\}$.

With these notations, we have the following two counting lemmas.

Lemma 3.3. Let N denote the set of non-negative integers. For $p \geq 2$ and $k \geq 1$, define $f: N^k \rightarrow I[k]$ by $f(n_1, \dots, n_k) = \sum_{j=1}^k n_j I_{j,k}$.

Then f is an isomorphism of sets.

Proof: For $p > 2$, omit the irrelevant zeroes corresponding to absence of Bocksteins. Then f is given explicitly by

$$f(n_1, \dots, n_k) = (s_1, \dots, s_k), \text{ where } s_{k+1-j} = \sum_{q=1}^{j-1} n_q (p^{j-1} - p^{j-1-q}) + \sum_{q=j}^k n_q p^{j-1}.$$

The required inverse to f is given by

$$f^{-1}(s_1, \dots, s_k) = (n_1, \dots, n_k) \text{ where } n_j = \begin{cases} p s_{k+1-j} - s_{k-j} & \text{if } 1 \leq j < k \\ s_k - \sum_{q=2}^k (p s_q - s_{q-1}) & \text{if } j = k \end{cases}$$

Lemma 3.4. For $p > 2$, $k \geq 1$, and each non-empty e , define $f_e: I[k] \rightarrow I_e[k]$ by $f_e(I) = I + L_{e,k}$. Then f_e is an isomorphism of sets.

Proof: Obviously f_e^{-1} must be given by $f_e^{-1}(J) = J - L_{e,k}$, $J \in I_e[k]$, and it suffices to show that $J - L_{e,k}$ is defined, admissible, and has non-negative excess. Write $L_{e,k} = (\delta_1, r_1, \dots, \delta_k, r_k)$ and $J = (\epsilon_1, s_1, \dots, \epsilon_k, s_k)$. Observe that

$$e(J) = 2s_k - \sum_{q=1}^k \epsilon_q - 2 \sum_{q=2}^k (p s_q - \epsilon_q - s_{q-1}) \geq 0 \text{ and } p s_q - \epsilon_q \geq s_{q-1}.$$

$L_{e,k}$ is the unique element of $I_e[k]$ such that, if e has $j = 2i - \epsilon$ elements, then $e(L_{e,k}) = \epsilon$ and $L_{e,k}$ ends with i . Explicitly, $\delta_q = \epsilon_q$ is determined by e , $r_k = i$, and $r_{q-1} = p r_q - \delta_q$ for $q \leq k$. The result follows from the inequalities satisfied by the entries of J .

As a final preliminary, we require an ordering of sequences.

Definition 3.5. For a sequence $I = (\epsilon_1, s_1, \dots, \epsilon_k, s_k)$, define $I_j = (\epsilon_j, s_j, \dots, \epsilon_k, s_k)$, $1 \leq j \leq k$, and similarly when $p=2$. Note that $e(I_j) = e(J_j)$ for all j implies $I=J$, and define a total ordering of the sequences of length k by $I < J$ if $e(I_j) < e(J_j)$ for the smallest j such that $e(I_j) \neq e(J_j)$. Observe that $I < I'$ and $J < J'$ implies $I+J < I'+J'$.

An easy inspection demonstrates the following lemma.

Lemma 3.6. If I is inadmissible and $Q^I = \sum \lambda_J Q^J$ where the J are admissible, then $\lambda_J \neq 0$ implies $J < I$. If $P_*^r Q^I = \sum \lambda_J Q^J$, where $r > 0$ and the J are admissible, then $\lambda_J \neq 0$ implies $J < I$.

$R^* = \prod_{k>0} R[k]^*$ as an A -algebra. In the dual basis to that of admissible monomials, define elements of $R[k]^*$ by

$$\begin{aligned}\xi_{0k} &= (Q^{0k})^* & \text{if } 0 \leq k \\ \xi_{jk} &= (Q^{I_{jk}})^* & \text{if } 1 \leq j \leq k \\ \tau_{jk} &= (Q^{J_{jk}})^* & \text{if } 1 \leq j \leq k \\ \sigma_{ijk} &= (Q^{K_{ijk}})^* & \text{if } 1 \leq i < j \leq k.\end{aligned}$$

To simplify statements of formulas, define $\xi_{jk} = 0$ if $j < 0$ or $j > k$, $\tau_{jk} = 0$ if $j < 1$ or $j > k$, and $\sigma_{ijk} = 0$ if $i < 1$, $j \leq i$, or $j > k$.

ξ_{0k} is the identity element of $R[k]^*$ and $\prod_{k>0} \xi_{0k}$ is the identity element of R^* . The augmentation of R^* is given by $\epsilon(\prod_{k>0} \lambda_k \xi_{0k}) = \lambda_0$. Of course, R^* is not a coalgebra since R_0 is not finite dimensional (although R_q is finite dimensional for $q > 0$). However, R^* does have a well-defined coproduct on positive degree elements and on finite linear combinations of the ξ_{0k} ; the latter is evidently given by

$$\psi \xi_{0k} = \sum \xi_{0, k-i} \otimes \xi_{0, i}$$

It is perhaps worth observing that although $\prod_{k=0}^n R[k]^*$ is a quotient augmented A -algebra of R^* and a coalgebra (dual to the quotient algebra $R/\sum_{m>n} R[m]$ of R) such that the product is a

morphism of coalgebras, $\prod_{k=0}^n R[k]^*$ is nevertheless not a Hopf algebra because its unit fails to be a morphism of coalgebras

(dually, $(Q^0)^{n+1} = 0$ but $\epsilon Q^0 = 1$).

We shall successively compute $R[k]^*$ as an algebra, compute the Steenrod operations on generators, and compute the coproduct on generators.

Theorem 3.7. If $p=2$, $R[k]^* = P\{\xi_{1k}, \dots, \xi_{kk}\}$ as an algebra. If $p>2$, let $M[k]$ be the subspace of $R[k]^*$ spanned by the set consisting of ξ_{0k} together with the monomials

$$\sigma_{e_1 e_2 k \dots e_{j-1} e_j k}, \quad 1 \leq e_1 < \dots < e_j < k \text{ and } j \text{ even,}$$

and

$$\sigma_{e_1 e_2 k \dots e_{j-2} e_{j-1} k \tau_{e_j k}}, \quad 1 \leq e_1 < \dots < e_j < k \text{ and } j \text{ odd.}$$

This set is linearly independent, and the product defines an isomorphism of Z_p -spaces

$$P\{\xi_{1k}, \dots, \xi_{kk}\} \otimes M[k] \rightarrow R[k]^*.$$

$R[k]^*$ is determined as an algebra by commutativity and the following relations:

- (i) $\tau_{ik} \tau_{jk} = \xi_{kk} \sigma_{ijk}$ if $i < j$ (and $\tau_{ik} \tau_{ik} = 0$);
- (ii) $\sigma_{ijk} \tau_{nk} = (\tau_{ik} \tau_{jk} \tau_{nk}) / \xi_{kk}$; and
- (iii) $\sigma_{ijk} \sigma_{mnk} = (\tau_{ik} \tau_{jk} \tau_{mk} \tau_{nk}) / \xi_{kk}^2$.

(In (ii) and (iii), the right sides are to be evaluated in terms of the basis monomials by use of (i); the numerators, if non-zero, are divisible by the non zero-divisor ξ_{kk} or ξ_{kk}^2 .)

Proof: By the counting lemmas, an admissible monomial I with $l(I) = k$ and $e(I) \geq 0$ can be uniquely expressed in the form

$$I = n_1 I_{1k} + \dots + n_k I_{kk} + L_{ek}, \quad n_q \geq 0 \text{ and } e = \{e_1, \dots, e_j\},$$

where L_{ek} is the sequence of all zeroes if $j=0$ (or if $p=2$).

Let $j=2i-\epsilon$, $\epsilon=0$ or 1 , and define $n(I)=i+\sum_{q=1}^n n_q$. Let λ_e denote the monomial in $M[k]$ corresponding to the sequence e . Let \langle, \rangle be the Kronecker product (that is, the evaluation pairing $R[k]^* \otimes R[k] \rightarrow Z_p$). We claim that

$$(1) \quad \langle \xi_{1k}^{n_1} \dots \xi_{kk}^{n_k} \lambda_e, Q^I \rangle = 1, \text{ and}$$

$$(2) \quad \langle \xi_{1k}^{n_1} \dots \xi_{kk}^{n_k} \lambda_e, Q^J \rangle \neq 0 \text{ and } J \neq I \text{ imply } J > I.$$

Let $\psi: R[k] \rightarrow R[k]^{n(I)}$ be the iterated coproduct. For any J ,

$$(3) \quad \psi Q^J = \sum_{\underline{1}} Q^{\underline{1}} \otimes \dots \otimes Q^{n(I)}, \quad \sum J_i = J.$$

Now (1) is immediate from the definition of the L_{ek} . Given J as in (2), we can obtain a summand

$$\lambda Q^{\underline{1}} \otimes \dots \otimes Q^{n(I)}, \quad \sum I_i = I \text{ and } \lambda \neq 0.$$

on the right side of (3) by applying Adem relations to put the $Q^{\underline{1}}$ in admissible form, and $J > I$ follows. If we express the

monomials $\xi_I = \xi_{1k}^{n_1} \dots \xi_{kk}^{n_k} \lambda_e$ in the ordered basis dual to that of admissible monomials,

$$\xi_I = \sum_J a_{IJ} (Q^J)^*,$$

then (1) and (2) state that (a_{IJ}) is an upper triangular matrix with ones along the main diagonal. Therefore $\{\xi_I\}$ is a basis for $R[k]^*$. It remains only to prove (i), (ii), and (iii). By inspection of the definitions, we have

$$I_{kk} + K_{ijk} = J_{ik} + J_{jk} \text{ if } i < j.$$

An easy dimensional argument shows that $\xi_{kk} \sigma_{ijk}$ is the only possible summand of $\tau_{ik} \tau_{jk}$, and this proves (i). Since

$$\xi_{kk} \sigma_{ijk} \tau_{nk} = \tau_{ik} \tau_{jk} \tau_{nk} \text{ and } \xi_{kk}^2 \sigma_{ijk} \sigma_{mnk} = \tau_{ik} \tau_{jk} \tau_{mk} \tau_{nk},$$

formulas (ii) and (iii) follow immediately from (i).

In order to determine the Steenrod operations on the generators of $R[k]^*$, we need to know all operations in $R[k]$ which hit any of the Q^I , $I \in P[k]$, from above; of course, we may restrict attention to the generators P_*^r and β of A . For dimensional arguments, it should be observed that R can be given a second grading by the number of Bocksteins which occur in monomials and that all structure (except, of course, action by β) preserves this grading.

Lemma 3.8. The following formulas are valid in $R[k]$, $k \geq 1$, and these formulas specify all operations βQ^J and $P_*^r Q^J$, $r \geq 0$, on basis elements Q^J , which have a summand of the form λQ^I with $0 \neq \lambda \in Z_p$ and $I \in P[k]$:

$$(i) \quad p > 2: \quad \beta Q^{I_{kk}} = Q^{J_{kk}} \text{ and } \beta Q^{J_{ik}} = Q^{K_{ikk}} \text{ if } 1 \leq i < k$$

$$(ii) \quad p \geq 2: \quad P_*^{p-k-1-j} Q^{I_{j+1,k}} = -Q^{I_{jk}} \text{ if } 1 \leq j < k$$

$$(iii) \quad p > 2: \quad P_*^{p-k-1-j} Q^{J_{j+1,k}} = -Q^{J_{jk}} \text{ if } 1 \leq j < k$$

$$(iv) \quad p > 2: \quad P_*^{p-k-1-i} Q^{K_{i+1,j,k}} = -Q^{K_{ijk}} \text{ if } 1 \leq i < j-1 < k$$

$$(v) \quad p > 2: \quad P_*^{p-k-1-j} Q^{K_{i,j+1,k}} = -Q^{K_{ijk}} \text{ if } 1 \leq i < j < k$$

$$(vi) \quad p \geq 2: \quad P_*^{p-k-1} Q^{I_{1k} + I_{jk}} = Q^{I_{jk}} \text{ if } 1 \leq j \leq k$$

$$(vii) \quad p > 2: \quad P_*^{p-k-1} Q^{I_{1k} + J_{jk}} = Q^{J_{jk}} \text{ if } 2 \leq j \leq k$$

$$(viii) \quad p > 2: \quad P_*^{p-k-1} Q^{I_{jk} + J_{1k}} = 2Q^{J_{jk}} \text{ if } 1 \leq j \leq k$$

$$(ix) \quad p > 2: \quad P_*^{p-k-1} Q^{I_{1k} + K_{ijk}} = Q^{K_{ijk}} \text{ if } 2 \leq i < j \leq k$$

$$(x) \quad p > 2: \quad P_*^{p^{k-1}} Q^{I_{ik} + K_{1jk}} = 2Q^{K_{ijk}} \quad \text{if } 1 < i < j < k$$

Proof: The statements about β are obvious. For the rest, we first reduce the problem to manageable proportions by a search of dimensions. Observe that

$$(a) \quad I \in P[k] \text{ implies } 2p^{k-2}(p^2 - p - 1) \leq d(I) \leq 2(p^k - 1) \quad [2^{k-1} \leq d(I) \leq 2^k - 1].$$

Since $R[k]^*$ is an unstable A -module, (a) implies that

$$P_*^{p^r}(Q^{I^*}) = 0 \quad \text{if } r \geq k \text{ and } I \in P[k]. \quad \text{For } r < k \text{ and } I \in P[k], \text{ we have}$$

$$(b) \quad d(I) + 2p^r(p-1) \leq 2(2p^k - p^{k-1} - 1) < 6p^{k-2}(p^2 - p - 1)[d(I) + 2^r < 3 \cdot 2^{k-1}].$$

Clearly (a) and (b) imply that if $P_*^{p^r} Q^J$ has a summand λQ^I with $\lambda \neq 0$ and $I \in P[k]$, then either $J \in P[k]$ or $J = J' + J''$ with both J' and J'' in $P[k]$.

Observe further that

$$d(I) + 2p^r(p-1) < 4p^{k-2}(p^2 - p - 1), \quad I \in P[k],$$

if either $p > 3$ and $r \leq k-2$ or $p = 3$ and $r \leq k-3$; thus the possibility $J = J' + J''$ is also ruled out in these cases. Simple dimensional arguments in the few remaining cases demonstrate that our list will be exhaustive provided that the following formulas also hold:

$$(xi) \quad p > 2: \quad P_*^{p^{k-1}} Q^{I_{jk} + K_{1ik}} = 0 \quad \text{if } 1 < i < j \leq k,$$

$$(xii) \quad p = 2: \quad P_*^{2^{k-j}} Q^{I_{1k} + K_{1k}} = 0 \quad \text{if } 2 \leq j \leq k.$$

To prove (ii) through (xii), observe that the Nishida relations can be described as follows on admissible monomials Q^J .

$$(4) \quad \text{Let } J = (\varepsilon, s, K); \text{ if } r \geq 0 \text{ and } s < p^r + \varepsilon, \text{ then } P_*^{p^r} Q^J = 0; \text{ if } r \geq 1 \text{ and } s \geq p^r + \varepsilon, \text{ then } P_*^{p^r} Q^J \equiv \beta^\varepsilon Q^{s-p^r+p^{r-1}} P_*^{p^{r-1}} Q^K \text{ modulo linear combinations of admissible monomials } Q^L \text{ such that } e(L) \leq e(J) - 2(p-\varepsilon) [e(L) \leq e(J) - 2].$$

Further, we have the particular Nishida relations

$$(5) \quad P_*^1 Q^s = (s-1)Q^{s-1} \quad \text{and} \quad P_*^1 \beta Q^s = s\beta Q^{s-1} - Q^{s-1}\beta \quad \text{if } s \geq 1.$$

Formulas (4) and (5) clearly imply

$$(6) \quad \text{If } e(J_j) < 2(p-\varepsilon), \quad 1 \leq j < k, \text{ then } P_*^{p^{k-1}} Q^J = (s_k - 1 + \varepsilon_k) Q^{J - I_{1k}}$$

$$[\text{If } e(J_j) < 2, \quad 1 \leq j < k, \text{ then } P_*^{p^{k-1}} Q^J = (s_k - 1) Q^{J - I_{1k}}],$$

$$\text{where } J = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k) \text{ and } J_j = (\varepsilon_j, s_j, \dots, \varepsilon_k, s_k).$$

In all cases (vi) through (xi), the hypothesis of (6) is satisfied, and this proves (vi), (vii), and (ix). In (viii), (x), and (xi), the sequence $J - I_{1k}$ obtained on the right side of (6) is not admissible. However, the only Adem relation required to reduce $J - I_{1k}$ to admissible form is

$$(7) \quad Q^{ps} \beta Q^s = \beta Q^{ps} Q^s \quad \text{if } s \geq 1.$$

The proofs of (ii) through (v) and (xii) are similar applications of (4) and (5); they are simplified by use of induction on k . The following Adem relation is needed in the proof of (xii).

$$(8) \quad Q^{ps+1} Q^s = 0 \quad \text{if } s \geq 0.$$

Because of the change of basis involved in our description of $R[k]^*$, our formulas simplify slightly upon dualization.

Theorem 3.9. The following list of relations specifies all non-trivial actions of the generators $P_*^{p^r}$, $r \geq 0$, and β of the Steenrod algebra on the generators $\{\xi_{jk}, \tau_{jk}, \sigma_{ijk}\}$ of $R[k]^*$.

$$(i) \quad p > 2: \quad \beta \tau_{kk} = \xi_{kk} \text{ and } \beta \sigma_{ikk} = -\tau_{ik} \quad \text{if } 1 < i < k$$

$$(ii) \quad p \geq 2: \quad P_*^{p^{k-1-j}} \xi_{jk} = -\xi_{j+1,k} \quad \text{if } 1 \leq j < k$$

$$(iii) \quad p > 2: \quad P_*^{p^{k-1-j}} \tau_{jk} = -\tau_{j+1,k} \quad \text{if } 1 \leq j < k$$

- (iv) $p > 2$: $P^p \sigma_{ijk}^{k-1-i} = -\sigma_{i+1,j,k}$ if $1 \leq i < j-1 < k$
- (v) $p > 2$: $P^p \sigma_{ijk}^{k-1-j} = -\sigma_{i,j+1,k}$ if $1 \leq i < j < k$
- (vi) $p \geq 2$: $P^p \xi_{jk}^{k-1} = \xi_{1k} \xi_{jk}$ if $1 \leq j \leq k$
- (vii) $p > 2$: $P^p \tau_{jk}^{k-1} = \xi_{1k} \tau_{jk} + \xi_{jk} \tau_{1k}$ if $1 \leq j \leq k$
- (viii) $p > 2$: $P^p \sigma_{ijk}^{k-1} = \xi_{1k} \sigma_{ijk} + \xi_{ik} \sigma_{1jk} - \xi_{jk} \sigma_{1ik}$ if $1 \leq i < j \leq k$, $\sigma_{11k} = 0$.

Proof. (i) is trivial since, as explained in [A, p. 207], the cohomology and homology Bocksteins are related by

$$\langle \beta \alpha, a \rangle = (-1)^{\deg \alpha + 1} \langle \alpha, \beta a \rangle.$$

Relations (ii) through (vi) are immediate from the corresponding numbered relations of the lemma, since $R[k]^*$ is one-dimensional in the degrees in which these relations occur. For (vii), we can certainly write

$$P^p \tau_{jk}^{k-1} = a \xi_{1k} \tau_{jk} + b \xi_{jk} \tau_{1k} \quad (\text{with } a = 0 \text{ if } j = 1).$$

For $j \geq 2$, $I_{1k} + J_{jk} < J_{1k} + J_{jk}$ and $\langle \xi_{1k} \tau_{jk}, Q^{J_{1k} + I_{jk}} \rangle = 1$, as can be seen by examination of $\psi Q^{J_{1k} + I_{jk}}$. By (vii) and (viii) of the lemma, and by formulas (1) and (2), we find

$$1 = \langle P^p \tau_{jk}^{k-1}, Q^{I_{1k} + J_{jk}} \rangle = a \quad \text{if } 2 \leq j \leq k$$

and

$$2 = \langle P^p \tau_{jk}^{k-1}, Q^{J_{1k} + I_{jk}} \rangle = a + b \quad \text{if } 1 \leq j \leq k.$$

This proves (vii). Similarly, for (viii), we can certainly write

$$P^p \sigma_{ijk}^{k-1} = a \xi_{1k} \sigma_{ijk} + b \xi_{ik} \sigma_{1jk} + c \xi_{jk} \sigma_{1ik} \quad (\text{with } a = 0 \text{ and } c = 0 \text{ if } i = 1).$$

$$I_{1k} + K_{ijk} < I_{ik} + K_{1jk} < I_{jk} + K_{1ik}, \quad \text{and } \langle \xi_{ik} \sigma_{1jk}, Q^{I_{jk} + K_{1ik}} \rangle = 0,$$

$$\langle \xi_{1k} \sigma_{ijk}, Q^{I_{ik} + K_{1jk}} \rangle = 1 \quad \text{and} \quad \langle \xi_{ik} \sigma_{1jk}, Q^{I_{jk} + K_{1ik}} \rangle = 1.$$

by examinations of coproducts. Now (ix), (x), and (xi) of the lemma, together with (1) and (2), imply (viii) by evaluation of $P^p \sigma_{ijk}^{k-1}$ on $Q^{I_{1k} + K_{ijk}}$, $Q^{I_{ik} + K_{1jk}}$, and $Q^{J_{jk} + K_{1ik}}$. Note that (viii) can be predicted from (vi) and (vii) by application of P^{2p} to the relation $\tau_{ik} \tau_{jk} = \xi_{kk} \sigma_{ijk}$.

We have the following immediate corollary.

Corollary 3.10. If $p = 2$, $R[k]^*$ is generated as an A-algebra by ξ_{1k} . If $p > 2$, $R[1]^*$ is generated as an A-algebra by τ_{11} and $R[k]^*$, $k \geq 2$, is generated as an A-algebra by ξ_{1k} and σ_{12k} .

In other words, $R[k]^*$ is a quotient A-algebra of $H^*K(Z_p, n)$ or of $H^*K(Z_p, m) \otimes H^*K(Z_p, n)$ for appropriate integers m and n .

Remarks 3.11. In order to obtain an upper bound on the spherical classes of H_*QS^0 by determination of its A-annihilated primitive elements, it would be desirable to have complete information on the A-module (rather than the A-algebra) generators of $R[k]^*$: we can add classes not in R to elements of R to obtain primitive classes of H_*QS^0 ; we cannot so obtain A-annihilated classes of H_*QS^0 unless the given class in R was A-annihilated. I have not carried out the necessary calculations. Madsen [8] has obtained considerable information in the case $p = 2$ and has used this information to retrieve Browder's results [4] on the Arf invariant.

It remains to compute the coproduct on generators of $R[k]^*$, and we need information about the products in R which hit any of the Q^I , $I \in P[k]$. Fortunately, we do not need complete information when $I = K_{ijk}$.

Lemma 3.12. Let I_{0k} denote the sequence of length k , $k \geq 0$, with all entries zero. Suppose that J and K are admissible sequences such that $Q^J Q^K$ has a summand λQ^I with $\lambda \neq 0$ and $I \in P[k]$. Then either $K \in P[i]$ or $K = I_{0i}$ for some $i < k$ and, in the latter case, $I = I_{jk}$ for some j . All possible choices for J and K when $I = I_{jk}$ or $I = J_{jk}$ are specified in the following relations; in (i) and (ii), if $h < i$ and $h < j$, then the asserted relations merely hold modulo the subspace of $R[k]$ spanned by the admissible monomials which do not end with Q^1 .

$$(i) \quad Q^{(p^i - p^{i-h})I_{k-i, k-i} + p^{i-h}I_{j-h, k-i}} Q^{I_{hi}} \equiv Q^{I_{jk}}, \quad 0 \leq h \leq i \leq k, \quad 0 \leq j-h \leq k-i.$$

$$(ii) \quad Q^{(p^i - p^{i-h})I_{k-i, k-i} + p^{i-h}I_{j-h, k-i}} Q^{J_{hi}} \equiv Q^{J_{jk}}, \quad 1 \leq h \leq i \leq k, \quad 0 \leq j-h \leq k-i.$$

$$(iii) \quad Q^{(p^i - 1)I_{k-i, k-i} + J_{j-i, k-i}} Q^{I_{ii}} = Q^{J_{jk}}, \quad 0 \leq i < j \leq k.$$

Proof. If (J, K) is admissible and in $P[k]$, then $K \in P[i]$ for some i ; such decompositions of I_{jk} and J_{jk} account for the relations with $h = i$ in (i) and with $h = j$ in (i) and (ii) and for all relations in (iii). If (J, K) is inadmissible, then $K \in P[i]$ or $K = I_{0i}$ for some i since the only Adem relations which have Q^1 appearing on the right side are

$$Q^p Q^0 = Q^{p-1} Q^1 \quad \text{and} \quad Q^p \beta Q^1 = \beta Q^p Q^1.$$

We claim that if (t, I_{hi}) is inadmissible, $0 \leq h \leq i$, then $Q^t Q^{I_{hi}}$ has no non-zero summand ending with Q^1 unless $t = p^i$ and $h < i$, when $Q^{I_{h+1, i+1}}$ is the only such summand. Indeed, $Q^t Q^{I_{ii}}$ can have no such summand because, in the Adem relation $Q^r Q^s = \sum \lambda_j Q^{r+s-j} Q^j$ for $r > ps$, $\lambda_j = 0$ unless $j > s$. The claim now follows by upwards induction on i and, for fixed i , downwards induction on h , via explicit calculation from the Adem relations and the inductive definition of the I_{hi} . The essential

fact is that $Q^p Q^{p^{i-1} - p^{i-1-h}}$ has the summand $Q^{p^i - p^{i-1-h}} Q^{p^{i-1}}$, $0 \leq h < i$. Note that, since $\beta Q^{I_{hi}} = 0$ for $h < i$, it follows that if (J, I_{hi}) is inadmissible, $0 \leq h \leq i$, and if any Bockstein entry ϵ_j in J is non-zero, then $Q^J Q^{I_{hi}}$ has no non-zero summand ending with Q^1 . We claim also that if (t, J_{hi}) is inadmissible, $1 \leq h \leq i$, then $Q^t Q^{J_{hi}}$ has no non-zero summand ending with Q^1 unless $t = p^i$, when $Q^{J_{h+1, i+1}}$ is the only such summand. The proof is again an easy double induction; the Adem relation (7) is used to prove the claim when $h = i$. A straightforward bookkeeping argument from our claims shows that the relations of (i) and (ii) with $h < i$ and $h < j$ give all possibilities for $Q^J Q^K$ to have a non-zero summand $\lambda Q^{I_{jk}}$ or $\lambda Q^{J_{jk}}$ when (J, K) is inadmissible.

In our formulas for the coproduct in R^* , the sums are to range over the integers; this makes sense in view of our convention that ξ_{jk} , τ_{jk} , and σ_{ijk} are zero except where explicitly specified otherwise. The formula for $\psi \sigma_{ijk}$ announced in [12] is incorrect; the correct formula given here is in fact somewhat simpler.

Theorem 3.13. The following formulas specify the coproduct on the generators of R^* .

$$(i) \quad \psi \xi_{jk} = \sum_{(h, i)} \xi_{k-i, k-i}^{p^i - p^{i-h}} \xi_{j-h, k-i}^p \otimes \xi_{hi}$$

$$(ii) \quad \psi \tau_{jk} = \sum_{(h, i)} \xi_{k-i, k-i}^{p^i - p^{i-h}} \xi_{j-h, k-i}^p \otimes \tau_{hi} + \sum_i \xi_{k-i, k-i}^{p^i - 1} \tau_{j-i, k-i} \otimes \xi_{ii}$$

$$(iii) \quad \psi \sigma_{ijk} = \sum_{(f, g, h)} \xi_{k-h, k-h}^{p^h - p^{h-f} - p^{h-g}} (\xi_{j-g, k-h}^p \xi_{i-f, k-h}^{p^{h-f}} - \xi_{j-f, k-h}^p \xi_{i-g, k-h}^{p^{h-g}}) \otimes \sigma_{fgh} \\ + \sum_{(g, h)} \xi_{k-h, k-h}^{p^h - p^{h-g} - 1} (\xi_{j-g, k-h}^p \tau_{i-h, k-h}^{h-g} - \xi_{i-g, k-h}^p \tau_{j-h, k-h}^{h-g}) \otimes \tau_{gh}$$

$$+ \sum_h \xi_{k-h, k-h}^{p^h-1} \sigma_{i-h, j-h, k-h} \otimes \xi_{hh}$$

Proof. Observe first that if $J = \sum_i n_i I_{ik} + L_{ek}$, then $e(J) = n_k + \epsilon$, where $\epsilon = e(L_{ek})$ is zero or one. In view of the lemma, (i) and (ii) will hold provided that the monomials to the left of the tensor signs are precisely dual to the corresponding admissible monomials Q^J . By (1) and (2) in the proof of Theorem 3.7, this will certainly hold if the J are maximal among all admissible sequences of the requisite degrees. A dimensional argument shows that, due to the multiple $p^i - p^{i-h}$ of $I_{k-i, k-i}$ which appears, the J actually have maximal excess among the admissible sequences of the requisite degrees. We prove (iii) by a trick. By the lemma, we can certainly write

$$\psi_{ijk} = \sum_{(f,g,h)} \alpha_{fgh} \otimes \sigma_{fgh} + \sum_{(g,h)} \beta_{gh} \otimes \tau_{gh} + \sum_h \gamma_h \otimes \xi_{hh}$$

(ξ_{gh} for $g < h$ cannot appear on the right because, as noted in the proof of the lemma, $Q^J Q^{I_{gh}}$ cannot have a non-zero summand $\lambda Q^{K_{ijk}}$ unless (J, I_{gh}) is admissible, when $g = h$.) We have $\tau_{ik} \tau_{jk} = \xi_{kk} \sigma_{ijk}$ and therefore $(\psi_{\tau_{ik}})(\psi_{\tau_{jk}}) = (\psi_{\xi_{kk}})(\psi_{\sigma_{ijk}})$. After expanding both sides by use of Theorem 3.7 and the fact that $R[k]^* \cdot R[l]^* = 0$ for $k \neq l$, we find that there is a unique solution for the unknowns α_{fgh} , β_{gh} , and γ_h , namely that specified in (iii).

§4. The homology of QX

In this section and the next, we shall compute $H_* QX$ and $H_* CX$ for any space X , where C is the monad associated to an E_∞ operad \mathcal{C} [see G, Construction 2.4]. We shall also compute the mod p Bockstein spectral sequences of QX and CX , hence our results will determine the integral homology groups of these spaces.

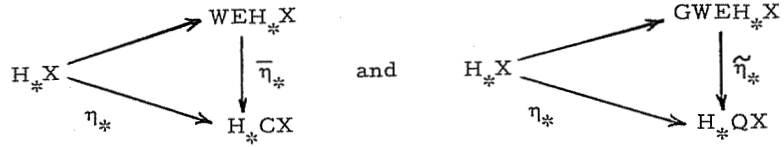
QX is the free infinite loop space generated by X in the sense that if $Y \in \mathcal{L}_\infty$ and $f: X \rightarrow Y_0$ is any map in \mathcal{J} , then there is a unique map $g: \{Q\Sigma^i X\} \rightarrow Y$ in \mathcal{L}_∞ such that $g_0 \circ \eta = f$, where $\eta: X \rightarrow QX$ is the natural inclusion [see G, p. 43]. Since, for all finite n , the composite

$$\Sigma^n X \xrightarrow{\Sigma^n \eta} \Sigma^n \Omega^n \Sigma^n X \xrightarrow{\lambda} \Sigma^n X$$

is the identity, where λ is the evaluation map, $\eta_*: H_* X \rightarrow H_* QX$ is a monomorphism. It is therefore reasonable to expect $H_* QX$ to be an appropriate free object generated by $H_* X$.

Similarly, for any operad \mathcal{C} , (CX, μ) is the free \mathcal{C} -space generated by X in the sense that if (Y, θ) is a \mathcal{C} -space and $f: X \rightarrow Y$ is a map in \mathcal{J} , then there is a unique map $g: CX \rightarrow Y$ of \mathcal{C} -spaces such that $g\eta = f$, $\eta: X \rightarrow CX$ [see G, p. 13, 16, 17]. Again, it is reasonable to expect $H_* CX$ to be an appropriate free object generated by $H_* X$, at least for nice operads \mathcal{C} .

We have constructed certain free functors WE and GWE in section 2 and, by freeness, there are unique morphisms $\bar{\eta}_*$ of allowable AR-Hopf algebras and $\tilde{\eta}_*$ of allowable AR-Hopf algebras with conjugation such that the following diagrams are commutative:



We have the following two theorems.

Theorem 4.1. For every space $X \in \mathcal{J}$ and every E_∞ operad \mathcal{C} , $\bar{\eta}_*: WEH_*X \rightarrow H_*CX$ is an isomorphism of AR-Hopf algebras.

Theorem 4.2. For every space $X \in \mathcal{J}$, $\tilde{\eta}_*: GWEH_*X \rightarrow H_*QX$ is an isomorphism of AR-Hopf algebras with conjugation.

The second theorem is a reformulation (and generalization) of the calculations of Dyer and Lashof [6].

By [G, Lemma 8.11], $CS^0 = \coprod_{j \geq 0} \mathcal{C}(j)/\Sigma_j$ for any operad \mathcal{C} (where \coprod denotes disjoint union). If \mathcal{C} is an E_∞ operad, the orbit space $\mathcal{C}(j)/\Sigma_j$ is just a $K(\Sigma_j, 1)$. Thus, as a very special case, Theorem 4.1 contains a concise reformulation of Nakaoka's results [16, 17, 18] on the homology of symmetric groups. An E_∞ operad \mathcal{C} should be thought of as a suitably coherent construction of universal bundles for symmetric groups; the simple statement that CS^0 is a \mathcal{C} -space contains a great deal of information that is usually obtained by more cumbersome algebraic techniques.

The elements of $H_*X \mathcal{C} H_*CX$ and of $H_*X \mathcal{C} H_*QX$ play a role in the homology of E_∞ spaces and of infinite loop spaces which is analogous to that played by the fundamental classes of $K(\pi, n)$'s in the cohomology of spaces. In particular, the following corollaries are analogs of the statement that the cohomology of any space can be represented, via the morphism induced by a map, as a quotient of a free unstable A-algebra.

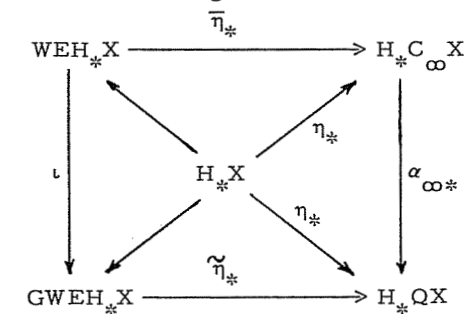
Corollary 4.3. If (X, θ) is a \mathcal{C} -space, where \mathcal{C} is an E_∞ operad, then $\theta_*: H_*CX \rightarrow H_*X$ represents H_*X as a quotient AR-Hopf algebra of the free allowable AR-Hopf algebra WEH_*X .

Proof. $\theta: CX \rightarrow X$ is the unique map of \mathcal{C} -spaces such that $\theta\eta = 1$.

Corollary 4.4. If Y is an infinite loop sequence, then $\xi_{\infty*}: H_*QY_0 \rightarrow H_*Y_0$ represents H_*Y_0 as a quotient AR-Hopf algebra with conjugation of the free allowable AR-Hopf algebra with conjugation $GWEH_*X$.

Proof. $\xi_{\infty}: QY_0 \rightarrow Y_0$ is the unique infinite loop map such that $\xi_{\infty}\eta = 1$; ξ_{∞} is defined explicitly in [G, p. 43].

Of course, Theorems 4.1 and 4.2 are not unrelated. By [G, Theorem 4.2], there is a morphism of monads $\alpha_{\infty}: C_{\infty} \rightarrow Q$. Thus $\alpha_{\infty}\eta = \eta$, $\alpha_{\infty*}: C_{\infty}X \rightarrow QX$ is a map of \mathcal{C}_{∞} -spaces for all X , and we have the following commutative diagram:



Here ι is the natural inclusion. Since ι is the identity if X is connected, Theorems 4.1 and 4.2, coupled with the Whitehead theorem for connected H-spaces, imply the following result.

Corollary 4.5. $\alpha_{\infty*}: C_{\infty}X \rightarrow QX$ is a weak homotopy equivalence for all connected spaces X .

The corollary was proven geometrically in [G, Theorem 6.1] by use of the much deeper fact that $\alpha_n : C_n X \rightarrow \Omega^n \Sigma^n X$ is a weak homotopy equivalence for all n and all connected X . We shall prove Theorem 4.1 and shall generalize the corollary by obtaining a homology approximation to QX , for arbitrary X , in the next section. We prove Theorem 4.2 and compute the Bockstein spectral sequence of CX and QX here.

For counting arguments, it will be useful to have explicit bases for $WEH_* X$ and $GWEH_* X$. Let tX be a basis for $JH_* X$ which contains the set of components of X , other than the component \emptyset of the base-point, regarded as homology classes of degree zero. Thus $tX \cup \{\emptyset\}$ is a basis for $H_* X$. Let $N\pi_0 X$ and $\tilde{N}\pi_0 X$ denote the free commutative monoid and the free commutative group generated by $\pi_0 X$, each subject to the single relation $\emptyset = 1$; let $Z_p N\pi_0 X$ and $Z_p \tilde{N}\pi_0 X$ denote their monoid and group rings. Let ATX be the free commutative algebra generated by the set

$$(1) \quad TX = \{Q^I x \mid x \in tX, I \text{ is admissible, } e(I) + b(I) > \deg x, \deg Q^I x > 0\}$$

(Recall the conventions, Definition 2.1(iii).) Then, as algebras,

$$(2) \quad WEH_* X = ATX \otimes Z_p N\pi_0 X \text{ and } GWEH_* X = ATX \otimes Z_p \tilde{N}\pi_0 X.$$

Note that the $Q^I x$ with $e(I) = \deg x$, $b(I) = 0$, and $\deg Q^I x > 0$ precisely account for all p -th powers of positive degree elements. Note also that Theorems 4.1 and 4.2 are correct in degree zero by comparison of (2) with [G, Proposition 8.14].

We need some preliminaries in order to prove Theorem 4.2 for non-connected spaces. The following well-known lemma clearly

implies that Theorem 4.2 will hold provided that it correctly describes the homology of the component $Q_\emptyset X$ of the base-point of QX .

Lemma 4.6. Let X be a homotopy associative H-space such that $\pi_0 X$ is a group under the induced product. Choose a point $a \in [a]$ for each component $[a]$ of X , write a^{-1} for the chosen point in $[a]^{-1}$, and let X_\emptyset denote the component of the identity element. Define $f: X \rightarrow X_\emptyset \times \pi_0 X$ by $f(x) = (x \cdot a^{-1}, [a])$ if $x \in [a]$. Then f is a homotopy equivalence with homotopy inverse g given by $g(y, [a]) = ya$. If left translation by any given element of X is homotopic to right translation by the same element, then f and g are H-maps.

To study $Q_\emptyset X$, which is the component $\Omega_\emptyset Q\Sigma X$ of the trivial loop in $Q\Sigma X$, observe that we may assume, without loss of generality, that all connected spaces Y in sight are sufficiently well-behaved locally to have universal covers $\pi: UY \rightarrow Y$. Of course, $\Omega\pi: \Omega UY \rightarrow \Omega_\emptyset Y$ is then a weak homotopy equivalence. We require two simple lemmas on universal covers.

Lemma 4.7. Let X be a homotopy associative H-space such that X is connected and $\pi_1 X$ is a free Abelian group. Then there exists a map $\rho: K(\pi_1 X, 1) \rightarrow X$ such that ρ induces an isomorphism on π_1 . The composite of the product on X and $\pi \times \rho$ is therefore a weak homotopy equivalence $UX \times K(\pi_1 X, 1) \rightarrow X$.

Proof. $K(\pi_1 X, 1)$ is the restricted Cartesian product of one copy of S^1 for each generator of $\pi_1 X$, restricted in the sense that all but finitely many coordinates of each point are at a chosen base-point in S^1 . Now ρ can be constructed by (transfinite) induction and use of the

product on X from any chosen representatives $S^1 \rightarrow X$ for the generators of $\pi_1 X$. Of course, if X is a monoid, we can use the product directly rather than inductively.

Lemma 4.8. Let (X, θ) be a connected \mathcal{C} -space, where \mathcal{C} is any operad. Then UX admits a structure of \mathcal{C} -space such that $\pi: UX \rightarrow X$ is a map of \mathcal{C} -spaces.

Proof. $UX = PX / (\sim)$, where two paths in X which start at $*$ are equivalent if they end at the same point and are homotopic with end-points fixed, and π is induced by the end-point projection. It is trivial to verify that the pointwise \mathcal{C} -space structure on PX of [G, Lemma 1.5] passes to the quotient space UX .

As a final preliminary, we have the following observation concerning the homology suspension.

Lemma 4.9. Let X be a space. Let $x \in H_0 \Omega X$ and $y \in H_* \Omega X$. Then, if $\varepsilon y = 0$, the loop product $x * y$ suspends to $(\varepsilon x)(\sigma_* y)$.

Proof. Let a and b be representative cycles in $C_* \Omega X$ for x and y . Let $i: \Omega X \rightarrow PX$ and $\pi: PX \rightarrow X$ be the inclusion and end-point projection. $\sigma_* y$ is the homology class of $\pi_* c$, where $c \in C_* PX$ is a chain such that $i_* b = dc$. ΩX acts on the left of PX by composition of paths, and $\pi(f * g) = \pi g$ for a loop f and path g . Now $d(i_* a * c) = i_*(a * b)$ and $\pi_*(i_* a * c) = (\varepsilon a)(\pi_* c)$. The result follows.

Proof of Theorem 4.2. If X is $(q-1)$ -connected, $q > 1$, then $\eta_*: H_* X \rightarrow H_* \Omega X$ is an isomorphism in degrees less than $2q$, as can easily be verified by inductive calculation of $H_* \Omega^i \Sigma^n X$ for $i \leq n$ in low degrees (by use of the Serre spectral sequence). Indeed, this is just the standard proof that $\eta_*: \pi_* X \rightarrow \pi_* \Omega X = \pi_*^S X$ is an isomorphism

in degrees less than $2q-1$ and an epimorphism in degree $2q-1$. Thus the theorem is trivially true in degrees less than $2q$ if X is $(q-1)$ -connected. We claim that if the theorem is true for ΣX in degrees less than n , then the theorem is true for X in degrees less than $n-1$. This will complete the proof since it will follow that the theorem for $\Sigma^q X$ in degrees less than $2q$ implies the theorem for X in degrees less than q , for all integers $q > 1$. We shall prove our claim by constructing a model spectral sequence $\{E^r\}$, mapping it into the Serre spectral sequence $\{E^r\}$ of the path space fibration over $UQ\Sigma X$, and invoking the comparison theorem [7, XI 11.1]. By [G, Proposition 8.14] and Lemma 4.7, we may write

$$H_* Q\Sigma X = H_* UQ\Sigma X \otimes H_* K(\tilde{N}\pi_0 X, 1).$$

Let $x' = x - (\varepsilon x)\beta$ for $x \in H_* X$. We take $t\Sigma X = \{\Sigma_* x' \mid x \in tX\}$ as our basis for $JH_* \Sigma X$, $\Sigma_*: \tilde{H}_* X \cong \tilde{H}_* \Sigma X$. We may then write

$$H_* K(\tilde{N}\pi_0 X, 1) = E\{\Sigma_* x' \mid x \in t_0 X\} \subset H_* Q\Sigma X.$$

Of course, if $p = 2$, this is not a sub-algebra and the squares $(\Sigma_* x')^2 = Q^1 \Sigma_* x'$ lie in $H_* UQ\Sigma X$. Define $\tilde{W}EH_* \Sigma X$ to be the sub-algebra of $WEH_* \Sigma X$ generated by the elements of $T\Sigma X$ of degree greater than one and, if $p = 2$, the squares $Q^1 \Sigma_* x'$, $x \in t_0 X$. Define

$$'E^2 = \tilde{W}EH_* \Sigma X \otimes (GWEH_* X)_\emptyset.$$

(If X is connected, $'E^2$ reduces to $WEH_* \Sigma X \otimes WEH_* X$.) The differentials of $\{E^r\}$ are specified by requiring $\{E^r\}$ to be a spectral sequence of differential algebras such that if $Q^I x \in TX$, then

$$\tau Q^I \Sigma_* x' = (-1)^{d(I)} Q^I x * [p^{l(I)}]_{ax}$$

and, if $p > 2$ and $\deg Q^s x = 2s - 1$,

$$\tau((Q^I \Sigma_* x')^{p-1} \otimes Q^I x_* [p^{\ell(I)} ax]) = (-1)^{d(I)+1} \beta Q^S Q^I x_* [p^{\ell(I)+1} ax].$$

Here $ax \in \pi_0 X$ denotes the component in which the homology class x lies ($\psi x = x \otimes ax + ax \otimes x$ plus other terms if $\deg x > 0$) and, for $a \in \pi_0 X$ and $n \in \mathbb{Z}$, $[na]$ denotes the n -th power of a in the group $\tilde{N}_{\pi_0} X \subset GWEH_* X$. An easy counting argument demonstrates that $\{E^T\}$ is isomorphic to a tensor product of elementary spectral sequences of the forms $E\{y\} \otimes P\{\tau y\}$ and, if $p > 2$, $P\{z\}/(z^p) \otimes [E\{\tau z\} \otimes P\{\tau(z^{p-1} \otimes \tau z)\}]$, where E and P denote exterior and polynomial algebras. Here y runs through

$$\{Q^I \Sigma_* x' \mid I \text{ admissible, } e(I) > \deg x, \deg Q^I \Sigma_* x' > 1 \text{ and odd if } p > 2\}$$

and, if $p > 2$, z runs through

$$\{Q^I \Sigma_* x' \mid I \text{ admissible, } e(I) > \deg x, \deg Q^I \Sigma_* x' \text{ even}\}$$

(Note that, if $p > 2$, $e(I) \equiv d(I) \pmod{2}$, hence $e(I) = \deg x + 1$ implies that $\deg Q^I \Sigma_* x'$ is even.) Of course, to the eyes of $\{E^T\}$, the base $\tilde{WEH}_* \Sigma X$ looks like a tensor product of exterior and truncated polynomial algebras rather than like a free commutative algebra. Clearly $E^\infty = Z_p$. By construction, there is a unique morphism of algebras $f: E^2 \rightarrow E^2$ such that the following diagram is commutative:

$$\begin{array}{ccc} E^2 = \tilde{WEH}_* \Sigma X \otimes (GWEH_* X)_\emptyset & \xrightarrow{f} & H_* UQ\Sigma X \otimes H_* \Omega UQ\Sigma X = E^2, \\ \downarrow \cap & & \downarrow \pi_* \otimes (\Omega\pi)_* \\ WEH_* \Sigma X \otimes GWEH_* X & \xrightarrow{\tilde{\eta}_* \otimes \tilde{\eta}_*} & H_* Q\Sigma X \otimes H_* QX \end{array}$$

Since $Q^I x = Q^I x'$ if $d(I) > 0$, by Theorem 1.1 (5), Lemma 4.9 implies that, for $\sigma_*: H_* QX \rightarrow H_* Q\Sigma X$,

$$\sigma_* (Q^I x_* [p^{\ell(I)} ax]) = (-1)^{d(I)} Q^I \Sigma_* x'$$

(the sign comes from $\sigma\beta = -\beta\sigma$). By the naturality of σ_* , the same formula holds for $\sigma_*: H_* \Omega UQ\Sigma X \rightarrow H_* UQ\Sigma X$, although here the elements $Q^S \Sigma_* x'$, $x \in t_0 X$, are of course not operations because the elements $\Sigma_* x'$ are not present in $H_* UQ\Sigma X$. By Theorem 1.1 (7) and the definition of $\{E^T\}$, f induces a morphism of spectral sequences. Since $f = f(\text{base}) \otimes f(\text{fibre})$, our claim and the theorem now follow directly from the comparison theorem.

The following observation on the structure of $H_* QX$ is sometimes useful. Note that $H_* Q_\emptyset X$ is the free commutative algebra generated by $\{y*(ay)^{-1} \mid y \in TX\}$, where ay is the component in which y lies. This description uses operations which occur in various components of QX . We can instead use just those operations which actually occur in the component $Q_\emptyset X$.

Lemma 4.10. $H_* Q_\emptyset X$ is the free commutative algebra generated by the union of the following three sets:

$$\{Q^I x \mid Q^I x \in TX \text{ and } ax = \emptyset\}$$

$$\{x*[-ax] \mid x \in tX, \deg x > 0, \text{ and } ax \neq \emptyset\}$$

$$\{Q^J (\beta^E Q^S x_* [-p \cdot ax]) \mid Q^J \beta^E Q^S x \in TX \text{ and } ax \neq \emptyset\}.$$

Proof. $[-p \cdot ax] = [-ax]* \dots * [-ax]$, and we therefore have

$$Q^J (\beta^E Q^S x_* [-p \cdot ax]) \equiv (Q^J \beta^E Q^S x) * [-p^{\ell(J)+1} \cdot ax],$$

modulo decomposable elements of $H_* Q_\emptyset X$, by the Cartan formula.

When $X = S^0$, the first two sets above are clearly empty.

We complete this section by computing the Bockstein spectral

sequences of H_*CX and of H_*QX . Let $\{E^rX\}$ denote the mod p Bockstein spectral sequence of a space X . A slight variant (when $p > 2$ and $r = 2$) of the proof of [A, Proposition 6.8] yields the following lemma.

Lemma 4.11. If (X, θ) is a ζ -space, where ζ is an E_∞ operad, then $\{E^rX\}$ is a spectral sequence of differential algebras such that if $y \in E_{2q}^{r-1}X$, then $\beta_r y^p = y^{p-1} \beta_{r-1} y$ if $p > 2$ or if $p = 2$ and $r > 2$, and $\beta_2 y^2 = y\beta y + Q^{2q}\beta y$ if $p = 2$ and $r = 2$.

Let $Y = CX$ or $Y = QX$, and let $\{E^rATX\}$ denote the restriction of $\{E^rY\}$ to ATX ; in both cases, we clearly have $E^rY = E^rATX \otimes_{H_0} Y$ for all $r \geq 1$. To describe E^rATX explicitly, we require some notations.

Definition 4.12. Let C_r , $r \geq 1$, be a basis for the positive degree elements of E^rX , and assume the C_r to be so chosen that

$$C_r = D_r \cup \beta_r D_r \cup C_{r+1},$$

where D_r , $\beta_r D_r$, and C_{r+1} are disjoint linearly independent subsets of E^rX such that $\beta_r D_r = \{\beta_r y \mid y \in D_r\}$ and C_{r+1} is a set of cycles under β_r which projects to the chosen basis for $E^{r+1}X$. Define A^rX , $r \geq 2$, to be the free strictly commutative algebra generated by the following set (strictness requires the squares of odd degree elements to be zero):

$$\bigcup_{1 \leq j < r} (S_{jr} \cup \beta_r S_{jr}) \cup C_r,$$

where $S_{jr} = \{y^{p^{r-j}} \mid y \in D_j, \text{ deg } y \text{ even}\}$ and

$$\beta_r S_{jr} = \begin{cases} \{y^{p^{r-j}-1} \beta_j y \mid y \in D_j, \text{ deg } y \text{ even}\} & \text{if } p > 2 \text{ or } j \geq 2 \\ \{y^{2^{r-j}-2} (y\beta y + Q^{2q}\beta y) \mid y \in D_1, \text{ deg } y = 2q\} & \text{if } p = 2 \text{ and } j = 1. \end{cases}$$

The proof of the following theorem is precisely analogous to the computation of the cohomology Bockstein spectral sequence of $K(Z_p^t, n)$ given in [A, Theorem 10.4] and will therefore be omitted. It depends only on Lemma 4.11, the fact that $\beta Q^{2s} = Q^{2s-1}$ if $p = 2$, and counting arguments.

Theorem 4.13. Define a subset SX of TX as follows:

- (a) $p = 2$: $SX = \{Q^I x \mid I = (2s, J), \text{ deg } Q^I x \text{ is even, } \ell(I) > 0\}$
- (b) $p > 2$: $SX = \{Q^I x \mid b(I) = 0, \text{ deg } Q^I x \text{ is even, } \ell(I) > 0\}$

Then $E^{r+1}ATX = P\{y^{p^r} \mid y \in SX\} \otimes E\{\beta_{r+1} y^{p^r} \mid y \in SX\} \otimes A^{r+1}X$ for all $r \geq 1$, where

$$\beta_{r+1} y^{p^r} = \begin{cases} y^{p^r-1} \beta y & \text{if } p > 2 \\ y^{2^r-2} (y\beta y + Q^{2q}\beta y) & \text{if } p = 2 \text{ and } \text{deg } y = 2q \end{cases}$$

Therefore $E^\infty ATX = A^\infty X$ is the free strictly commutative algebra generated by the positive degree elements of $E^\infty X$.

§5. The homology of CX and the spaces \overline{CX}

We first prove Theorem 4.1 and then construct a homology approximation $\overline{\alpha}_\infty: \overline{C}_\infty X \rightarrow Q_\emptyset X$ for arbitrary spaces X. The space $\overline{C}_\infty S^0$ will be a $K(\Sigma_\infty, 1)$, and this special case of our approximation theorem was first obtained by Priddy [20].

Observe that the maps $\overline{\eta}_*$ of Theorem 4.1 are natural in ζ as well as in X. In particular, the following result holds.

Lemma 5.1. If ζ and ζ' are E_∞ operads, then the following is a commutative diagram of morphisms of AR-Hopf algebras:

$$\begin{array}{ccccc}
 & & H_*(C \times C')X & & \\
 & \swarrow \pi_{1*} & \uparrow \overline{\eta}_* & \searrow \pi_{2*} & \\
 H_*CX & & & & H_*C'X \\
 & \nwarrow \overline{\eta}_* & & \nearrow \overline{\eta}_* & \\
 & & WEH_*X & &
 \end{array}$$

Moreover, π_{1*} and π_{2*} are isomorphisms for all spaces X.

Proof. $\zeta \times \zeta'$ is an E_∞ operad by [G, Definitions 3.5 and 3.8], and π_{1*} and π_{2*} are isomorphisms by [G, Proposition 3.10] and the proof of [G, Proposition 3.4].

By Theorem 4.2 and Figure 1, we already know that

$\overline{\eta}_*: WEH_*X \rightarrow H_*C_\infty X$ is a monomorphism (since $\alpha_{\infty*} \overline{\eta}_*$ is a monomorphism); by the lemma, we know this for every E_∞ operad ζ . In order to prove that $\overline{\eta}_*$ is an epimorphism, we need the following standard consequence of the properties of the transfer in the (mod p) homology of finite groups; a proof may be found in [5, p. 255].

Lemma 5.2. If π is a subgroup of the finite group Π and if the index of π in Π is prime to p, then the restriction

$$i_*: H_*(\pi; M) \rightarrow H_*(\Pi; M)$$

is an epimorphism for every $\mathbb{Z}_p \Pi$ -module M.

We shall also need the definition of wreath products.

Definition 5.3. Let π be a subgroup of Σ_n and let G be any monoid. Then the wreath product $\pi \int G$ is the semi-direct product of π and G^n determined by the permutation action of π on G^n ; explicitly if $\sigma \in \pi$ and $\tau_i \in G$, then, in $\pi \int G$,

$$(\tau_1, \dots, \tau_n)\sigma = \sigma(\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)}).$$

Embed G^n in G^{n+1} as $G^n \times \{e\}$ and embed Σ_n in Σ_{n+1} as the subgroup fixing the last letter; this fixes an embedding of $\Sigma_n \int G$ in $\Sigma_{n+1} \int G$, and $\Sigma_\infty \int G$ is defined to be the union of the $\Sigma_n \int G$ for finite n.

Proof of Theorem 4.1. Consider the monad (C, μ, η) associated to an E_∞ operad ζ . As in [G, p. 17], we write μ both for the ζ -action on CX and for the monad product $\mu: CCX \rightarrow CX$. Recall that, by [G, p. 13 and 14], CX is a filtered space such that the product $*$ and ζ -action μ restrict to give

$$*: F_j CX \times F_k CX \rightarrow F_{j+k} CX$$

and

$$\mu_k: \zeta(k) \times F_{j_1} CX \times \dots \times F_{j_k} CX \rightarrow F_j CX, \quad j = j_1 + \dots + j_k.$$

Indeed, $*$ is $\mu_2(c)$ for any fixed $c \in \zeta(2)$ and, if γ denotes the structural map of the operad [G, Definition 1.1], then

$$\mu_k(d; [e_1, y_1], \dots, [e_k, y_k]) = [\gamma(d; e_1, \dots, e_k); y_1, \dots, y_k]$$

for $d \in \zeta(k)$, $e_i \in \zeta(j_i)$, and $y_i \in X^{j_i}$. We define a corresponding algebraic filtration of WEH_*X by giving all elements of the image of

