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Frederick R. Cohen
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The Homology of Iterated Loop Spaces

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Preface

This volume is a collection of five papers (to be referred to as I-V). The first four together give a thorough treatment of homology operations and of their application to the calculation of, and analysis of internal structure in, the homologies of various spaces of interest. The last studies an up to homotopy notion of an algebra over a monad and the role of this notion in the theory of iterated loop spaces. I have established the algebraic preliminaries necessary to the first four papers and the geometric preliminaries necessary for all of the papers in the following references, which shall be referred to by the specified letters throughout the volume.

[A]. A general algebraic approach to Steenrod operations. Springer Lecture Notes in Mathematics Vol. 168, 1970, 153-231.

[G]. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics Vol. 271, 1972.

[G']. E_∞ spaces, group completions, and permutative categories. London Math. Soc. Lecture Note Series Vol. 11, 1974, 61-93.

In addition, the paper II here is a companion piece to my book (contributed to by F. Quinn, N. Ray, and J. Tornehave)

[R]. E_∞ Ring Spaces and E_∞ Ring Spectra.

With these papers, this volume completes the development of a comprehensive theory of the geometry and homology of iterated loop spaces. There are no known results in or applications of this area of topology which do not fit naturally into the framework thus established. However, there are several papers by other authors which seem to me to add significantly to the theory developed in [G]. The relevant references will be incorporated in the list of errata and addenda to [A], [G], and [G'] which concludes this volume.

The geometric theory of [G] was incomplete in two essential respects. First, it worked well only for connected spaces (see [G, p. 156-158]). It was the primary purpose of [G'] to generalize the theory to non-connected spaces. In particular, this allowed it to be applied to the classifying spaces of permutative categories and thus to algebraic K-theory. More profoundly, the ring theory of [R] and II was thereby made possible.

Second, the theory of [G] circumvented analysis of homotopy invariance (see [G, p. 158-160]). It is the purpose of Lada's paper V to generalize the theory of [G] to one based on homotopy invariant structures on topological spaces in the sense of Boardman and Vogt [Springer Lecture Notes in Mathematics, Vol. 347] [1]. In Boardman and

[1] Incidentally, the claim there (p. VII) that [G] failed to apply to non Σ-free operads is based on a misreading; see [G, p. 22].

Vogt's work, an action up to homotopy by an operad (or PROP) on a space was essentially an action by a larger, but equivalent, operad on the same space. In Lada's work, an action up to homotopy is essentially an action by the given operad on a larger, but equivalent, space. In both cases, the expansion makes room for higher homotopies. While these need not be made explicit in the first approach, it seems to me that the second approach is nevertheless technically and conceptually simpler (although still quite complicated in detail) since the expansion construction is much less intricate and since the problem of composing higher homotopies largely evaporates.

We have attempted to make the homological results of this volume accessible to the reader unfamiliar with the geometric theory in the papers cited above. In I, I set up the theory of homology operations on infinite loop spaces. This is based on actions by E_∞ operads $\mathcal{C}$ on spaces and is used to compute $H_*(CX; Z_p)$ and $H_*(QX; Z_p)$ as Hopf algebras over the Dyer-Lashof and Steenrod algebras, where CX and QX are the free $\mathcal{C}$-space and free infinite loop space generated by a space X. The structure of the Dyer-Lashof algebra is also analyzed. In II, I set up the theory of homology operations on E_∞ ring spaces, which are spaces with two suitably interrelated E_∞ space structures. In particular, the mixed Cartan formula and mixed Adem relations are proven and are shown to determine the multiplicative homology operations of the free E_∞ ring space $C(X^+)$ and the free E_∞ ring infinite loop space $Q(X^+)$ generated by an E_∞ space X. In the second half of II, homology operations on E_∞ ring spaces associated to matrix groups are analyzed and an exhaustive study is made of the homology of BSF and of such related classifying spaces as BTop (at $p > 2$) and BCoker J. Perhaps the most interesting feature of these calculations is the precise homological analysis of the infinite loop splitting $BSF = BCoker\, J \times BJ$ at odd primes and of the infinite loop fibration $BCoker\, J \to BSF \to BJ_\otimes$ at $p = 2$.

In III, Cohen sets up the theory of homology operations on n-fold loop spaces for $n < \infty$. This is based on actions by the little cubes operad $\mathcal{C}_n$ and is used to compute $H_*(C_nX; Z_p)$ and $H_*(\Omega^n\Sigma^nX; Z_p)$ as Hopf algebras over the Steenrod algebra with three types of homology operations. While the first four sections of III are precisely parallel to sections 1, 2, 4, and 5 of I, the construction of the unstable operations (for odd p) and the proofs of all requisite commutation formulas between them (which occupies the rest of III) is several orders of magnitude more difficult than the analogous work of I (most of which is already contained in [A]). The basic ingredient is a homological analysis of configuration spaces, which should be of independent interest. In IV, Cohen computes

$H_*(SF(n); Z_p)$ as an algebra for p odd and n even, the remaining cases being determined by the stable calculations of II. Again, the calculation is considerably more difficult than in the stable case, the key fact being that $H_*(SF(n); Z_p)$ is commutative even though $SF(n)$ is not homotopy commutative. Due to the lack of internal structure on $BSF(n)$, the calculation of $H_*(BSF(n); Z_p)$ is not yet complete.

In addition to their original material, I and III properly contain all work related to homology operations which antedates 1970, while II contains either complete information on or at least an introduction to most subsequent work in this area, the one major exception being that nothing will be said about BTop and BPL at the prime 2. Up to minor variants, all work since 1970 has been expressed in the language and notations established in I §1-§2 and II §1.

Our thanks to Maija May for preparing the index.

J. P. May
August 20, 1975

TABLE OF CONTENTS

The Homology of E_∞ Spaces

J.P. May

Homology operations on iterated loop spaces were first introduced, mod 2, by Araki and Kudo [1] in 1956; their work was clarified and extended by Browder [2] in 1960. Homology operations mod p, $p > 2$, were first introduced by Dyer and Lashof [6] in 1962. The work of Araki and Kudo proceeded in analogy with Steenrod's construction of the Sq^n in terms of $\cup_i$-products, whereas that of Browder and of Dyer and Lashof proceeded in analogy with Steenrod's later construction of the P^n in terms of the cohomology of the symmetric group Σ_p. The analogy was closest in the case of infinite loop spaces and, in [A], I reformulated the algebra behind Steenrod's work in a sufficiently general context that it could be applied equally well to the homology of infinite loop spaces and to the cohomology of spaces. Later, in [G], I introduced the notions of E_∞ operad and E_∞ space. Their use greatly simplifies the geometry required for the construction and analysis of the homology operations and, in the non-connected case, yields operations on a wider class of spaces than infinite loop spaces. These operations, and further operations on the homology of infinite loop spaces given by the elements of $H_*\tilde{F}$, will be analyzed in section 1.

Historically, the obvious next step after introduction of the homology operations should have been the introduction of the Hopf algebra of all homology operations and the analysis of geometrically allowable modules (and more complicated structrues) over this algebra, in analogy with the definitions in cohomology given by Steenrod [22] in 1961. However, this step seems not to have been taken until lectures of mine in 1968-69. The requisite definitions will be given in section 2. Since the idea that homology operations should satisfy Adem relations first appears in [6] (although these relations were not formulated or proven there), we call the resulting algebra of operations the Dyer-Lashof algebra; we denote it by R. The main point of section 2 is the explicit construction of free allowable structures over R.

During my 1968-69 lectures, Madsen raised and solved at the prime 2 the problem of carrying out for R the analog of Milnor's calculation of the dual of the Steenrod algebra A. His solution appears in [8]. Shortly after, I solved the problem at odd primes, where the structure of R^* turned out to be surprisingly complicated. The details of this computation ($p = 2$ included) will be given in section 3.

In section 4, we reformulate (and extend to general non-connected spaces X) the calculation of H_*QX, $QX = \varinjlim \Omega^n\Sigma^n X$, given by Dyer and Lashof [6]. Indeed, the definitions in section 2 allow us to describe H_*QX as the free allowable Hopf algebra with conjugation over R and A. With the passage of time, it has become possible to give considerably simpler details of proof than were available in 1962. We also compute the Bockstein spectral sequence of QX (for each prime) in terms of that of X.

Just as QX is the free infinite loop space generated by a space X, so CX, as constructed in [G, §2], is the free $\mathcal{C}$-space generated by X (where $\mathcal{C}$ is an E_∞ operad). In section 5, we prove that H_*CX is the

free allowable Hopf algebra (without conjugation) over R and A. The proof is quite simple, especially since the geometry of the situation makes half of the calculation an immediate consequence of the calculation of H_*QX. Although the result here seems to be new, in this generality, special cases have long been known. When X is connected, CX is weakly equivalent to QX by [G, 6.3]. When $X = S^0$, $CX = \coprod K(\Sigma_j, 1)$ and the result thus contains Nakaoka's calculations [16, 17, 18] of the homology of symmetric groups. We end section 5 with a generalization (from S^0 to arbitrary spaces X) of Priddy's homology equivalence $B\Sigma_\infty \to Q_0S^0$ [20].

In section 6, we describe how the iterated homology operations of an infinite loop space appear successively in the stages of its Postnikov decomposition.

In section 7, we construct and analyze homology operations analogous to the Pontryagin p^{th} powers in the cohomology of spaces. When $p = 2$, these operations were first introduced by Madsen [9].

Most of the material of sections 1-4 dates from my 1968-69 lectures at Chicago and was summarized in [12]. The material of section 5 dates from my 1971-72 lectures at Cambridge. The long delay in publication, for which I must apologize, was caused by problems with the sequel II (to be explained in its introduction).

CONTENTS

§1. Homology operations

We first define and develop the properties of homology operations on E_∞ spaces. We then specialize to obtain further properties of the resulting operations on infinite loop spaces. In fact, the requisite geometry has been developed in [G, §1,4,5, and 8] and the requisite algebra has been developed in [A, §1-4 and 9]. The proofs in this section merely describe the transition from the geometry to the algebra.

All spaces are to be compactly generated and weakly Hausdorff; $\mathcal{T}$ denotes the category of spaces with non-degenerate base-point [G, p.1]. All homology is to be taken with coefficients in Z_p for an arbitrary prime p; the modifications of statements required in the case p=2 are indicated inside square brackets.

We require some recollections from [G] in order to make sense of the following theorem. Recall that an E_∞ space (X,θ) is a $\mathcal{C}$-space over any E_∞ operad $\mathcal{C}$ [G, Definitions 1.1, 1.2, and 3.5]; θ determines an H-space structure on X with the base-point $* \in X$ as identity element and with $\theta_2(c)\colon X\times X \to X$ as product for any $c \in \mathcal{C}(2)$ [G, p.4]. Recall too that the category $\mathcal{C}[\mathcal{T}]$ of $\mathcal{C}$-spaces is closed under formation of loop and path spaces [G, Lemma 1.5] and has products and fibred products [G, Lemma 1.7].

Theorem 1.1. Let $\mathcal{C}$ be an E_∞ operad and let (X,θ) be a $\mathcal{C}$-space. Then there exist homomorphisms $Q^s\colon H_*X \to H_*X$, $s \geq 0$, which satisfy the following properties:

(1) The Q^s are natural with respect to maps of $\mathcal{C}$-spaces.

(2) Q^s raises degrees by $2s(p-1)$ [by s].

(3) $Q^s x = 0$ if $2s <$ degree (x) [if $s <$ degree (x)], $x \in H_*X$

(4) $Q^s x = x^p$ if $2s =$ degree (x) [if $s =$ degree (x)], $x \in H_*X$

(5) $Q^s \phi = 0$ if $s > 0$, where $\phi \in H_0(X)$ is the identity element.

(6) The external, internal, and diagonal Cartan formulas hold:

$$Q^s(x\otimes y) = \sum_{i+j=s} Q^i x\otimes Q^j y \text{ if } x\otimes y\in H_*(X\times Y),\ (Y,\theta')\in \mathcal{C}[\Upsilon];$$

$$Q^s(xy) = \sum_{i+j=s} (Q^i x)(Q^j y) \text{ if } x,\ y\in H_*X; \text{ and}$$

$$\psi(Q^s x) = \sum_{i+j=s} \Sigma Q^i x'\otimes Q^j x'' \text{ if } \psi x = \Sigma x'\otimes x'',\ x\in H_*X.$$

(7) The Q^s are stable and the Kudo transgression theorem holds: $Q^s\sigma_* = \sigma_* Q^s$, where σ_*: $\tilde{H}_*\ \Omega X\to H_*X$ is the homology suspension; if X is simply connected and if $x\in H_qX$ transgresses to $y\in H_{q-1}\Omega X$ in the Serre spectral sequence of the path space fibration π: $PX\to X$, then $Q^s x$ and $\beta Q^s x$ transgress to $Q^s y$ and $-\beta Q^s y$ and, if $p>2$ and $q=2s$, $x^{p-1}\otimes y$ "transgresses" to $-\beta Q^s y$, $d^{q(p-1)}(x^{p-1}\otimes y) = -\beta Q^s y$.

(8) The Adem relations hold: If $p\geq 2$ and $r>ps$, then

$$Q^r Q^s = \sum_i (-1)^{r+i}(pi-r,\ r-(p-1)s-i-1)Q^{r+s-i}Q^i;$$

if $p>2$, $r\geq ps$, and β denotes the mod p Bockstein, then

$$Q^r\beta Q^s = \sum_i (-1)^{r+i}(pi-r,\ r-(p-1)s-i)\beta Q^{r+s-i}Q^i$$

$$-\sum_i (-1)^{r+i}(pi-r-1,\ r-(p-1)s-i)Q^{r+s-i}\beta Q^i$$

(9) The Nishida relations hold: Let P^r_*: $H_*X\to H_*X$ be dual to P^r where $P^r = Sq^r$ if $p=2$ (thus $P^r = (P^r_*)^*$ on $H^*X = (H_*X)^*$). Then

$$P^r_* Q^s = \sum_i (-1)^{r+i}(r-pi,\ s(p-1)-pr+pi)Q^{s-r+i}P^i_*;$$

in particular, $\beta Q^s = (s-1)Q^{s-1}$ if p=2; if $p>2$,

$$P^r_*\beta Q^s = \sum_i (-1)^{r+i}(r-pi,\ s(p-1)-pr+pi-1)\beta Q^{s-r+i}P^i_*$$

$$+\sum_i (-1)^{r+i}(r-pi-1,\ s(p-1)-pr+pi)Q^{s-r+i}P^i_*\beta.$$

(In (8) and (9), $(i,j) = (i+j)!/i!j!$ if $i>0$ and $j>0$, $(i,0)=1=(0,i)$ if $i\geq 0$, and $(i,j)=0$ if $i<0$ or $j<0$; the sums run over the integers.)

Proof: The symmetric group Σ_p acts freely on the contractible space $\mathcal{C}(p)$, hence the normalized singular chain complex $C_*\mathcal{C}(p)$ is a Σ_p-free resolution of Z_p [7, IV 11]. Let W be the standard π-free resolution of Z_p [A, Definition 1.2], where π is cyclic of order p, and let j: $W\to C_*\mathcal{C}(p)$ be a morphism of π-complexes over Z_p. Let $(C_*X)^p$ denote the p-fold tensor product. We are given a Z_p-equivariant map θ_p: $\mathcal{C}(p)\times X^p\to X$, and we define θ_*: $W\otimes(C_*X)^p\to C_*X$ to be the following composite morphism of π-complexes:

$$W\otimes(C_*X)^p \xrightarrow{j\otimes\eta} C_*\mathcal{C}(p)\otimes C_*(X^p) \xrightarrow{\eta} C_*(\mathcal{C}(p)\times X^p) \xrightarrow{C_*\theta_p} C_*X.$$

Here η is the shuffle map; for diagram chases, it should be recalled that η: $C_*X\otimes C_*Y\to C_*(X\times Y)$ is a commutative and associative natural transformation which is chain homotopy inverse to the Alexander-Whitney map ξ. In view of [G, Lemmas 1.6 and 1.9 (i)], (C_*X,θ_*) is a unital and mod p reduced object of the category $\mathcal{C}(p,\infty)$ defined in [A, Definitions 2.1]. Moreover, $(X,\theta)\to(C_*X,\theta_*)$ is clearly the object map of a functor from $\mathcal{C}[\Upsilon]$ to the subcategory $\mathcal{P}(p,\infty)$ of $\mathcal{C}(p,\infty)$ defined in [A, Definitions 2.1]. Let $x\in H_qX$. As in [A, Definitions 2.2], we define

(i) $Q_i(x) = \theta_*(e_i\otimes x^p)$, θ_*: $H(W\otimes_\pi(H_*X)^p)\cong H(W\otimes_\pi(C_*X)^p)\to H_*X$,

and we define the desired operations Q^s by the formulas

(ii) $p=2$: $Q^s x = 0$ if $s<q$ and $Q^s x = Q_{s-q}(x)$ if $s\geq q$; and

(iii) $p>2$: $Q^s x = 0$ if $2s<q$ and $Q^s x = (-1)^s v(q)Q_{(2s-q)(p-1)}(x)$ if $2s>q$, where $v(q) = (-1)^{q(q-1)m/2}(m!)^q$, with $m=\frac{1}{2}(p-1)$.

The Q^s are homomorphisms which satisfy (1) through (5) by [A, Proposition 2.3 and Corollary 2.4]. Note that [A, Proposition 2.3] also implies that if $p > 2$, then $\beta Q^s x = (-1)^s v(q)\, Q_{(2s-q)(p-1)-1}(x)$ and the Q^s and βQ^s account for all non-trivial operations Q_i. For (6), recall that the product of $\mathcal{C}$-spaces (X,θ) and (Y,θ') is $(X\times Y,\tilde{\theta})$, where $\tilde{\theta}_p$ is the composite

$$\mathcal{C}(p)\times(X\times Y)^p \xrightarrow{\Delta\times u} \mathcal{C}(p)\times\mathcal{C}(p)\times X^p \xrightarrow{1\times t\times 1} \mathcal{C}(p)\times X^p\times \mathcal{C}(p)\times Y^p \xrightarrow{\theta_p\times\theta'_p} X\times Y$$

(Here Δ, u, and t are the diagonal and the evident shuffle and interchange maps.) Similarly, the tensor product of objects (K,θ_*) and (L,θ'_*) in $\mathcal{C}(p,\infty)$ is $(K\times L,\tilde{\theta}_*)$, where $\tilde{\theta}_*$ is the composite

$$W\otimes(K\otimes L)^p \xrightarrow{\psi\otimes U} W\otimes W\otimes K^p\otimes L^p \xrightarrow{1\otimes T\otimes 1} W\otimes K^p\otimes W\otimes L^p \xrightarrow{\theta_*\otimes\theta'_*} K\otimes L.$$

(Here ψ, U, and T are the coproduct on W and the evident shuffle and interchange homomorphisms.) Since $(j\otimes j)\psi$ is π-homotopic to $(\xi\circ C_*\Delta)j$, an easy diagram chase demonstrates that $\eta: C_*X\otimes C_*Y \to C_*(X\times Y)$ is a morphism in the category $\mathcal{C}(p,\infty)$. The external Cartan formula now follows from [A, Corollary 2.7]. By [G, Lemmas 1.7 and 1.9 (ii)], $\Delta: X\to X\times X$ is a map of $\mathcal{C}$-spaces and (C_*X,θ_*) is a Cartan object of $\mathcal{C}(p,\infty)$; the diagonal and internal Cartan formulas follow by naturality. Part (7) is an immediate consequence of [G, Lemma 1.5] and [A, Theorems 3.3 and 3.4]; the simple connectivity of X serves to ensure that $E^2 = H_*X\otimes H_*\Omega X$ in the Serre spectral sequence of $\pi: PX \to X$. For (8), note that the following diagram is commutative by [G, Lemma 1.4]:

$$\begin{array}{ccccc}
\mathcal{C}(p)\times\mathcal{C}(p)^p\times X^{p^2} & \xrightarrow{\gamma\times 1} & \mathcal{C}(p^2)\times X^{p^2} & \xrightarrow{\theta_{p^2}} & \\
\downarrow{\scriptstyle 1\times u} & & & & X \\
\mathcal{C}(p)\times[\mathcal{C}(p)\times X^p]^p & \xrightarrow{1\times(\theta_p)^p} & \mathcal{C}(p)\times X^p & \xrightarrow{\theta_p} &
\end{array}$$

An easy diagram chase demonstrates that (C_*X,θ_*) is an Adem object, in the sense of [A, Definition 4.1], and (8) follows by [A, Theorem 4.7]. Part (9) follows by the naturality of the Steenrod operations from [A, Theorem 9.4], which computes the Steenrod operations in $H_*(\mathcal{C}(p)\times_\pi X^p)$. As explained in [A, p.209], our formula differs by a sign from that obtained by Nishida [19].

Let $\mathcal{L}_\infty$ be the category of infinite loop sequences. Recall that an object $Y = \{Y_i\}$ in $\mathcal{L}_\infty$ is a sequence of spaces with $Y_i = \Omega Y_{i+1}$ and a morphism $g = \{g_i\}$ in $\mathcal{L}_\infty$ is a sequence of maps with $g_i = \Omega g_{i+1}$. Y_0 is said to be an infinite loop space, g_0 an infinite loop map. By the results of [10], these notions are equivalent for the purposes of homotopy theory to the more usual ones in which equalities are replaced by homotopies. By [G, Theorem 5.1], there is a functor $W_\infty: \mathcal{L}_\infty \to \mathcal{C}_\infty[\mathcal{T}]$, with $W_\infty Y = (Y_0,\theta_\infty)$ and $W_\infty g = g_0$, where $\mathcal{C}_\infty$ is the infinite little cubes operad of [G, Definition 4.1]. The previous theorem therefore applies to H_*Y_0; the resulting operations Q^s will be referred to as the loop operations. The relevant Pontryagin product is that induced by the loop product on $Y_0 = \Omega Y_1$. Note that there are two different actions of $\mathcal{C}_\infty$ on ΩY_0, one coming from [G, Lemma 1.5] and the other from the fact that ΩY_0 is again an infinite loop space; by [G, Lemma 5.6], these two actions are equivariantly homotopic, hence part (7) of the theorem does apply to the loop operations. Similarly, part (6) applies to the loop operations since, by [G, Lemma 5.7], the two evident actions of $\mathcal{C}_\infty$ on the product of two infinite loop spaces are in fact the same.

The recognition theorem [G, Theorem 14.4; G'] gives a weak homotopy equivalence between any given grouplike E_∞ space X and an infinite loop space B_0X; moreover, as explained in [G, p.153-155], the homology operations on H_*X coming via Theorem 1.1 from the given E_∞ space structure agree under the equivalence with the loop operations on H_*B_0X. Thus, in principle, it is only for non grouplike E_∞ spaces that the operations of Theorem 1.1 are more general than loop operations. In practice, the theorem gives considerable geometric freedom in the construction of the operations, and this freedom is often essential to the calculations.

The following additional property of the loop operations, which is implied by [G, Remarks 5.8], will be important in the study of non-connected infinite loop spaces. Recall that the conjugation χ on a Hopf algebra, if present, is related to the unit η, augmentation ε, product ϕ, and coproduct ψ by the formula $\eta\varepsilon = \phi(1\times\chi)\psi$.

Lemma 1.2. For $Y\in\mathcal{L}_\infty$, $Q^s\chi = \chi Q^s$ on H_*Y_0, where the conjugation is induced from the inverse map on $Y_0 = \Omega Y_1$.

In the next two sections, we will define and study the global algebraic structures which are naturally suggested by the results above. We make a preliminary definition here.

Definition 1.3. Let A be a Hopf algebra. Let A act on Z_p through its augmentation, $a\cdot 1 = \varepsilon(a)$, and let A act on the tensor product $M\otimes N$ of two left A-modules through its coproduct,

$$a(m\otimes n) = \Sigma(-1)^{\deg a''\deg m} a'm\otimes a''n \text{ if } \psi a = \Sigma a'\otimes a''.$$

A left or right structure (algebra, coalgebra, Hopf algebra, Hopf algebra with conjugation, etc.) over A is a left or right A-module and a structure of the specified type such that all of the structure maps are morphisms of A-modules.

We shall define a Hopf algebra R of homology operations in the next section and, if $Y\in\mathcal{L}_\infty$, H_*Y_0 will be a left Hopf algebra with conjugation over R. For any space X, H_*X is a left coalgebra over the opposite algebra A^0 of the Steenrod algebra; here the opposite algebra enters because dualization is contravariant. Henceforward, although we shall continue to write the Steenrod operations P^r_* on the left, we shall speak of right A-modules rather than of left A^0-modules. Thus H_*Y_0 is a right Hopf algebra with conjugation over A, and the Nishida relations give commutation formulas between the A and R operations on H_*Y_0.

There is yet another Hopf algebra which acts naturally and stably on H_*Y_0 namely $H_*\tilde{F}$ where $\tilde{F}$ is the monoid (under composition) of based maps of spheres. The precise definition of $\tilde{F}$ is given in [G, p.74], and it is shown there that composition of maps defines a natural action $c_\infty\colon Y_0\times\tilde{F}\to Y_0$ of $\tilde{F}$ on infinite loop spaces. The following theorem gives the basic properties of the induced action of $H_*\tilde{F}$ on H_*Y_0.

Theorem 1.4. For $Y\in\mathcal{L}_\infty$, $c_{\infty*}\colon H_*Y_0\otimes H_*\tilde{F}\to H_*Y_0$ gives H_*Y_0 a structure of right Hopf algebra with conjugation over $H_*\tilde{F}$. Moreover, $c_{\infty*}$ satisfies the following properties, where $c_{\infty*}(x\otimes f) = xf$:

(1) $c_{\infty*}$ is natural with respect to maps in $\mathcal{L}_\infty$.

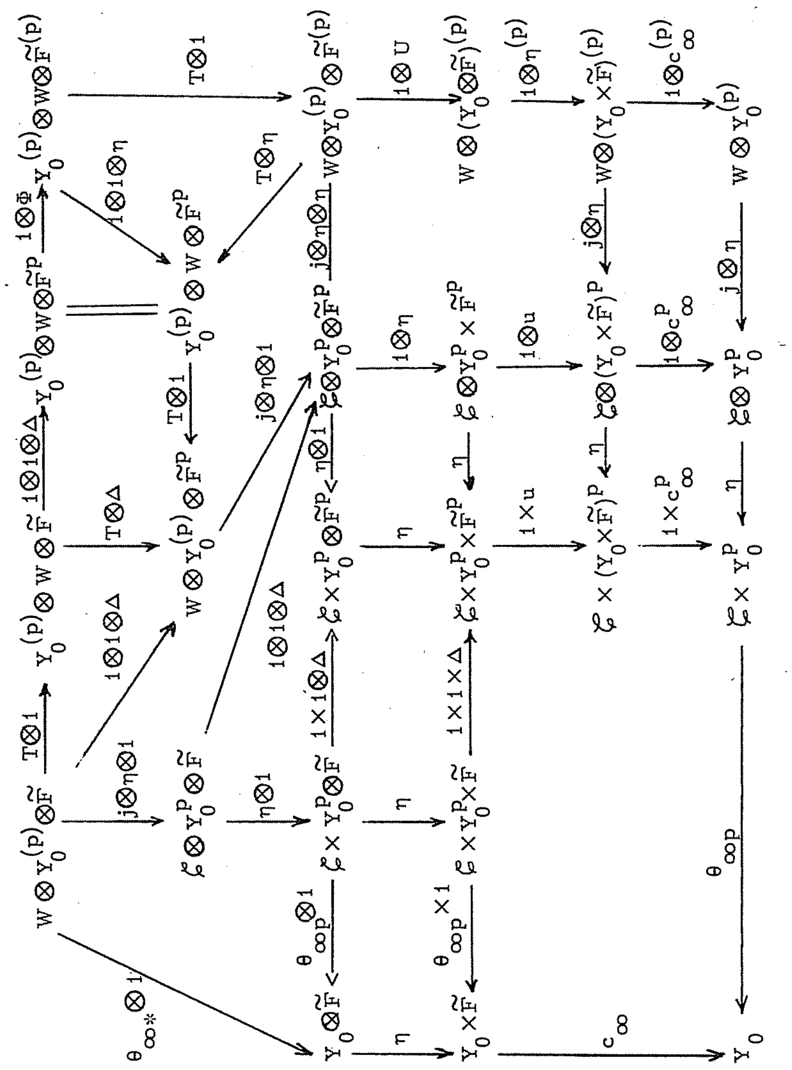
(2) $\sigma_*(xf) = (\sigma_*x)f$, where $\sigma_*\colon \tilde{H}_*\Omega Y_0\to H_*Y_0$ is the suspension.

(3) $P^r_*(xf) = \sum_{i+j=r}(P^i_*x)(P^j_*f)$ and $\beta(xf) = (\beta x)f+(-1)^{\deg x}x(\beta f)$.

(4) $(Q^s x)f = \sum_i Q^{s+i}(xP^i_*f)$ and, if $p>2$,

$$(\beta Q^s x)f = \sum_i \beta Q^{s+i}(xP^i_*f) - \sum_i(-1)^{\deg x}Q^{s+i}(x\,P^i_*\beta f).$$

<u>Proof</u>: Part (1) is trivial. The maps $* \to Y_0$ and $Y_0 \to *$ are infinite loop maps, hence the unit and augmentation of H_*Y_0 are morphisms of $H_*\tilde{F}$-modules. The loop product is a morphism of $H_*\tilde{F}$-modules by a simple diagram chase from [G, Lemma 8.8], and a similar lemma for the inverse map implies that the conjugation is a morphism of $H_*\tilde{F}$-modules. The coproduct on H_*Y_0 is a morphism of $H_*\tilde{F}$-modules and formula (3) holds because $c_{\infty *}$ is induced by a map. Formula (2) is an immediate consequence of [G, Lemmas 8.4 and 8.5]. For (4), consider the following diagram, in which we have abbreviated $\mathcal{C}$ for $\mathcal{C}_\infty(p)$ and X, $X^{(p)}$, and X^p for C_*X, $(C_*X)^p$, and $C_*(X^p)$:



Here Φ: $W \otimes C_*(X^p) \to W \otimes (C_*X)^p$ is given by [A, Lemma 7.1], and $(1 \otimes \eta)\ \Phi$ is π-homotopic to the identity by [A, Remarks 7.2]. The bottom left square is Σ_p-homotopy commutative by [G, Proposition 8.9], and the remaining parts of the diagram commute trivially. The shuffle and interchange homomorphisms U and T merely involve signs, the composite $c_\infty \eta$: $C_*Y_0 \otimes C_*\tilde{F} \to C_*Y_0$ induces $c_{\infty *}$, and the map induced on π-equivariant homology by $d = \Phi(1 \otimes \Delta)$ is explicitly computed in [A, Proposition 9.1]. Of course, it is the presence of d in the diagram which leads to the appearance of Steenrod operations in formula (4). The verification of this formula is now an easy direct calculation from the definition of the operations Q^s.

The essential part of the previous diagram is of course the geometric bottom left square. Henceforward, we shall omit the pedantic details in the passage from geometric diagrams to algebraic formulas.

We evaluate one obvious example of the operations on H_*Y_0 given by right multiplication by elements of $H_*\tilde{F}$.

Lemma 1.5. Let $[i] \in H_0\tilde{F}$ be the class represented by a map of degree i. Then $x\,[-1] = \chi x$ for all $x \in H_*Y_0$, $Y \in \mathcal{L}_\infty$.

Proof: Define f: $S^n \to S^n$ by $f(s_1, \ldots, s_n) = (1-s_1, s_2, \ldots, s_n)$, where $S^n = I^n/\partial I^n$. For any X, the inverse map on $\Omega^n X$ is given by $g \to g \circ f$, g: $S^n \to X$, and similarly on Y_0 by passage to limits via [G, p.74]. The result follows.

Recall that $QX = \varinjlim \Omega^n \Sigma^n X$ and $QX = \Omega Q \Sigma X$ [G, p.42]. As we shall see in II § 5, application of the results above to $Y_0 = QS^0$, where c_∞ reduces to the product on $\tilde{F}$, completely determines the composition product on $H_*\tilde{F}$.

Remarks 1.6. A functorial definition of a smash product between objects of $\mathcal{L}_\infty$ is given in [13], in which a new construction of the stable homotopy category is given. (In the language of [13], $\mathcal{L}_\infty$ is a category of coordinatized spectra; the smash product is constructed by passing to the category $\mathcal{S}$ of coordinate-free spectra, applying the smash product there, and then returning to $\mathcal{L}_\infty$.) For objects $Y, Z \in \mathcal{L}_\infty$ and elements

$x, y \in H_*Y_0$ and $z \in H_*Z_0$, [G, Lemma 8.1] and a similar lemma for the inverse map imply the formulas

$$(x * y) \wedge z = \sum (-1)^{\deg y \deg z'} (x \wedge z') * (y \wedge z'') \quad \text{if } \psi z = \sum z' \otimes z''$$

and

$$(\chi y) \wedge z = \chi(y \wedge z),$$

where $*$ and $\wedge$ denote the loop and smash products respectively. Via a diagram chase precisely analogous to that in the proof of Theorem 1.4, [G, Proposition 8.2] implies the formulas

$$(Q^s y) \wedge z = \sum_i Q^{s+i}(y \wedge P_*^i z)$$

and, if $p > 2$,

$$(\beta Q^s y) \wedge z = \sum_i \beta Q^{s+i}(y \wedge P_*^i z) - \sum_i (-1)^{\deg y} Q^{s+i}(y \wedge P_*^i \beta z).$$

In particular, these results apply to $\wedge : QX \times QX' \to Q(X \wedge X')$ for any spaces X and X'. By [G, Lemma 8.7], the smash and composition products coincide and are commutative on $H_*QS^0 = H_*\tilde{F}$.

§2. Allowable structures over the Dyer-Lashof algebra

We here describe the Hopf algebra of homology operations on E_∞ spaces generated by the Q^s and βQ^s and develop analogs of the notions of unstable modules and algebras over the Steenrod algebra. The following definition determines the appropriate "admissible monomials".

Definition 2.1. (i) $p = 2$: Consider sequences $I = (s_1, \ldots, s_k)$ such that $s_j \geq 0$. Define the degree, length, and excess of I by

$$d(I) = \sum_{j=1}^{k} s_j;\ \ell(I) = k;\ \text{and}$$

$$e(I) = s_k - \sum_{j=2}^{k} (2s_j - s_{j-1}) = s_1 - \sum_{j=2}^{k} s_j.$$

The sequence I determines the homology operation $Q^I = Q^{s_1} \ldots Q^{s_k}$. I is said to be admissible if $2s_j \geq s_{j-1}$ for $2 \leq j \leq k$.

(ii) $p > 2$: Consider sequences $I = (\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k)$ such that $\varepsilon_j = 0$ or 1 and $s_j \geq \varepsilon_j$. Define the degree, length, and excess of I by

$$d(I) = \sum_{j=1}^{k} [2s_j(p-1) - \varepsilon_j];\ \ell(I) = k;\ \text{and}$$

$$e(I) = 2s_k - \varepsilon_1 - \sum_{j=2}^{k} [2ps_j - \varepsilon_j - 2s_{j-1}] = 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} [2s_j(p-1) - \varepsilon_j].$$

The sequence I determines the homology operation $Q^I = \beta^{\varepsilon_1} Q^{s_1} \ldots \beta^{\varepsilon_k} Q^{s_k}$. I is said to be admissible if $ps_j - \varepsilon_j \geq s_{j-1}$ for $2 \leq j \leq k$.

(iii) Conventions: $b(I) = \varepsilon_1$ if $p > 2$ and $b(I) = 0$ if $p = 2$. The empty sequence I is admissible and satisfies $d(I) = 0$, $\ell(I) = 0$, $e(I) = \infty$, and $b(I) = 0$; it determines the identity homology operation $Q^I = 1$.

Definition 2.2. Let F denote the free associative algebra

generated by $\{Q^s | s \geq 0\}$ if $p = 2$ or by $\{Q^s | s \geq 0\} \cup \{\beta Q^s | s > 0\}$ if $p > 2$ (not β itself). For $q \geq 0$, define $J(q)$ to be the two-sided ideal of F generated by the Adem relations (and, if $p > 2$, the relations obtained by applying β to the Adem relations, with $\beta^2 = 0$) and by the relations $Q^I = 0$ if $e(I) < q$. Define $R(q)$ to be the quotient algebra $F/J(q)$, and observe that there are successive quotient maps $R(q) \to R(q+1)$. Let $R = R(0)$; R will be called the Dyer-Lashof algebra.

To avoid circularity, we have defined the $R(q)$ purely algebraically. The following theorem implies that this definition agrees with that naturally suggested by the geometry.

Theorem 2.3. (i) Let $i_q \in H_q S^q \subset H_q QS^q$ be the fundamental class if $q > 0$ and the class of the point other than the base-point if $q = 0$; then

$$\{Q^I i_q | I \text{ is admissible and } e(I) \geq q\}$$

is a linearly independent subset of $H_* QS^q$.

(ii) $J(q)$ coincides with the set $K(q)$ of all elements of F which annihilate every homology class of degree $\geq q$ of every E_∞ space (equivalently, of every infinite loop space).

(iii) $\{Q^I | I \text{ is admissible and } e(I) \geq q\}$ is a Z_p-basis for $R(q)$.

(iv) $R(q)$ admits a unique structure of right A-module such that the Nishida relations are satisfied.

(v) $R = R(0)$ admits a structure of Hopf algebra and of unstable right coalgebra over A with coproduct defined on generators by

$$\psi Q^s = \sum_{i+j=s} Q^i \otimes Q^j \text{ and } \psi \beta Q^{s+1} = \sum_{i+j=s} (\beta Q^{i+1} \otimes Q^j + Q^i \otimes \beta Q^{j+1})$$

and with augmentation defined on $R_0 = P\{Q^0\}$ by $\varepsilon(Q^{0k}) = 1$, $k \geq 0$.

Proof: We shall prove (i) in §4. It is obvious from the Adem