

# Characteristic classes

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# Contents

|   |    |
|---|----|
| Chapter 0. Introduction   | 5  |
| Chapter 1. Classical groups and bundle theory                       | 7  |
| 1. The classical groups   | 7  |
| 2. Fiber bundles  | 10 |
| 3. Principal bundles and homogeneous spaces                         | 13 |
| 4. Vector bundles, Stiefel and Grassmann manifolds                  | 16 |
| 5. The classification theorem and characteristic classes            | 19 |
| 6. Some homotopical properties of classifying spaces                | 22 |
| 7. Algebraic Tools  | 24 |
| 8. Spectral sequences   | 24 |
| Chapter 2. Cohomology of the Classical Groups and Stiefel Manifolds | 25 |
| 1. The complex and quaternionic Stiefel manifolds                   | 25 |
| 2. The real Stiefel manifolds                                       | 27 |
| Chapter 3. Chern classes  | 33 |
| Chapter 4. Symplectic classes                                       | 39 |
| Chapter 5. Stiefel-Whitney Classes                                  | 45 |
| Chapter 6. Steenrod Operations, the Wu formula, and BSpin           | 51 |
| Chapter 7. Euler and Pontrjagin classes in rings containing $1/2$   | 57 |
| Chapter 8. The Thom Isomorphism                                     | 65 |
| Chapter 9. Integral Euler, Pontrjagin and Stiefel-Whitney classes   | 75 |
| Chapter 10. Applications and Examples                               | 81 |
| Appendix A. Bott periodicity  | 89 |
| 1. Definition of the Maps   | 89 |
| 2. Commutative Diagrams in the $\phi_i$                             | 93 |
| 3. Proof of the Periodicity Theorem                                 | 96 |
| Appendix. Bibliography  | 97 |



## CHAPTER 0

# Introduction

These notes had their genesis in a class Peter May taught in the spring (?) quarter of 1974 at the University of Chicago. Robert Bruner was assigned the task of writing them up in a coherent fashion based on his class notes and Peter's notes. They were used in this handwritten form for many years at the University of Chicago. (What is the true version of this??) In the summer of 2012 Mike Catanzaro took on the task of T<sub>E</sub>Xing the notes. After that, May and Bruner undertook some reorganization and added a few items to make the notes more self contained.

The precipitating event in the decision to publish them was a question from a colleague about the cohomology of a particular homogeneous space. It became clear these basic results in algebraic topology should be available in textbook form. (???????)

Compare to Mimura and Toda???

### ADAPT THE FOLLOWING INTRODUCTORY SKETCH

We develop the classical theory of characteristic classes. Our procedure is simultaneously to compute the cohomology of the relevant classifying spaces and to display the standard axiomatically determined characteristic classes.

We first compute the homology and cohomology of Stiefel varieties and classical groups and then use the latter computations to pass to classifying spaces. Along the way, we compute the cohomologies of various homogeneous spaces, such as

$$Sp(n)/U(n), \quad U(2n)/Sp(n), \quad U(n)/O(n), \quad \text{and} \quad SO(2n)/U(n).$$

We also obtain the usual intrinsic characterizations, via the Thom isomorphism, of the Stiefel-Whitney and Euler classes.

Since we shall have a plethora of explicit calculations, some generic notational conventions will help to keep order.

We shall end up with the usual characteristic classes

$$w_i \in H^i(BO(n); \mathbb{F}_2), \text{ the } \textit{Stiefel-Whitney} \text{ classes}$$

$$c_i \in H^{2i}(BU(n); \mathbb{Z}), \text{ the } \textit{Chern} \text{ classes}$$

$$k_i \in H^{4i}(BSp(n); \mathbb{Z}), \text{ the } \textit{symplectic} \text{ classes}$$

$$P_i \in H^{4i}(BO(n); \mathbb{Z}), \text{ the } \textit{Pontryagin} \text{ classes}$$

$$\chi \in H^{2n}(BSO(2n); \mathbb{Z}), \text{ the } \textit{Euler} \text{ class.}$$

The  $P_i$  and  $\chi$  will be studied in coefficient rings containing  $1/2$  before being introduced integrally. We use the same notations for integral characteristic classes and for their images in cohomology with other coefficient rings.

Prerequisites: To do. (Just say "see the next chapter"?)



## Classical groups and bundle theory

We introduce the spaces we shall study and review the fundamentals of bundle theory in this chapter. Aside from a few arguments included for didactic purposes, proofs are generally sketched or omitted. However, Sections 3 and 6 contain some material either hard to find or missing from the literature, and full proofs of such statements have been supplied.

We assume once and for all that all spaces we consider are to be of the homotopy type of CW-complexes. This ensures that a weak homotopy equivalence, namely a map which induces isomorphisms of homotopy groups for all choices of basepoints, is a homotopy equivalence. By the basic results of Milnor [9] (see also Schon [11]), this is not a very restrictive assumption. We also assume that all spaces are paracompact. This ensures that all bundles are numerable (in the sense specified in Section 2). Since all metric spaces, all countable unions of compact spaces, and all CW-complexes (Miyazaki [10] or Fritsch and Piccinini [6, Thm 1.3.5]), are paracompact, this assumption is also not unduly restrictive.

### 1. The classical groups

All of our work will deal with the classical Lie groups and related spaces defined in this chapter. Good general references for this section are Adams [1] and Chevalley [4].

Let  $K$  denote any one of  $\mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ , the real numbers, complex numbers, or quaternions. For  $\alpha \in K$ , let  $\bar{\alpha}$  denote the conjugate of  $\alpha$ . A *right inner product space* over  $K$  is a right  $K$ -module  $W$ , together with a function  $(\ , \ ) : W \times W \rightarrow K$  which satisfies the following properties.

- (i)  $(x, y + y') = (x, y) + (x, y')$
- (ii)  $(x, y\alpha) = (x, y)\alpha$  for any  $\alpha \in K$
- (iii)  $(x, y) = \overline{(y, x)}$
- (iv)  $(x, x) \in \mathbb{R}$ ,  $(x, x) \geq 0$ , and  $(x, x) = 0$  if and only if  $x = 0$ .

The unmodified term *inner product space* will mean right inner product space. All inner product spaces will be finite or countably infinite dimensional; we write  $\dim W = \infty$  in the latter case.

We say that a  $K$ -linear transformation  $T : W \rightarrow W$  is of *finite type* if  $W$  contains a finite dimensional subspace  $V$  invariant under  $T$  such that  $T$  restricts to the identity on  $V^\perp$ .

The *classical groups* are

$$GL(W) = \{T : W \rightarrow W \mid T \text{ is invertible and of finite type}\},$$

$$U(W) = \{T \mid T \in GL(W) \text{ and } T \text{ is an isometry}\},$$

and, if  $K = \mathbb{R}$  or  $K = \mathbb{C}$ ,

$$\begin{aligned} SL(W) &= \{T \mid T \in GL(W) \text{ and } \det T = 1\} \\ SU(W) &= \{T \mid T \in U(W) \text{ and } \det T = 1\}. \end{aligned}$$

The finite type requirement assures that the determinant is well-defined. By choice of fixed orthonormal basis for  $W$ , we can identify  $GL(W)$  with the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

where  $A$  is an invertible  $n \times n$  matrix with  $n < \infty$ . Such a matrix is in  $U(W)$  if and only if  $A^{-1} = \overline{A^T}$ , where  $\overline{A}$  is obtained from  $A$  by conjugating each entry and  $(\ )^T$  denotes the transpose.

Topologize inner product spaces as the union (or colimit) of their finite dimensional subspaces. By choice of a fixed orthonormal basis and use of matrices, the classical groups of  $W$  may be topologized as subspaces of  $K^{n^2}$  when  $n = \dim W < \infty$ . The same topology may also be specified either in terms of norms of linear transformations or as the compact open topology obtained by regarding these groups as subsets of the space of maps  $W \rightarrow W$ . With this topology,  $G(W)$  is a Lie group ( $G = GL, U, SL$ , or  $SU$ ) and  $U(W)$  and  $SU(W)$  are compact. When  $\dim W = \infty$ ,  $G(W)$  is topologized as the union of its subgroups  $G(V)$ , where  $V$  runs through all finite dimensional subspaces of  $W$  or through those  $V$  in any expanding sequence with union  $W$ .

A standard theorem of linear algebra states that any element of  $GL(W)$  can be written uniquely as the product of a symmetric positive definite transformation and an element of  $U(W)$ , and similarly with  $GL$  and  $U$  replaced by  $SL$  and  $SU$ . It follows that the inclusions  $U(W) \hookrightarrow GL(W)$  and  $SU(W) \hookrightarrow SL(W)$  are homotopy equivalences. For our purpose, it suffices to restrict attention to  $U(W)$  and  $SU(W)$ .

A convenient framework in which to view the classical groups is as follows. Let  $\tilde{K}$  denote the category of finite or countably infinite dimensional inner product spaces of  $K$  with linear isometries as morphisms. Note that isometries need not be surjective. Then  $U$  and  $SU$  are functors from  $\tilde{K}$  to the category of topological groups. Obviously if  $V$  and  $W$  are objects in  $\tilde{K}$  of the same dimension, then there is an isomorphism  $V \cong W$  in  $\tilde{K}$  which induces isomorphisms  $U(V) \cong U(W)$  and  $SU(V) \cong SU(W)$ .

This formulation has the conceptual clarity common to basis free presentations and will be useful in our proof of Bott periodicity. However, for calculational purposes, it is more convenient to deal with particularly representatives of the classical groups. We define examples as follows, where  $K^n$  has its standard inner product.

- (i)  $O(n) := U(\mathbb{R}^n)$ ,  $O := U(\mathbb{R}^\infty)$  the *orthogonal* groups
- (ii)  $SO(n) := SU(\mathbb{R}^n)$ ,  $SO := SU(\mathbb{R}^\infty)$  the *special orthogonal* groups
- (iii)  $U(n) := U(\mathbb{C}^n)$ ,  $U := U(\mathbb{C}^\infty)$  the *unitary* groups
- (iv)  $SU(n) := SU(\mathbb{C}^n)$ ,  $SU := SU(\mathbb{C}^\infty)$  the *special unitary* groups
- (v)  $Sp(n) := U(\mathbb{H}^n)$ ,  $Sp := U(\mathbb{H}^\infty)$  the *symplectic* groups

There is another family of classical groups not included in this scheme, namely the *spinor* groups  $\text{Spin}(n)$  for  $n > 2$  and  $\text{Spin} = \text{Spin}(\infty)$ . We define  $\text{Spin}(n)$  to be the universal covering group of  $SO(n)$ . Each  $\text{Spin}(n)$  for  $n < \infty$  is a Lie group and  $\text{Spin} = \bigcup_n \text{Spin}(n)$ . Since  $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$ ,  $\text{Spin}(n)$  is a 2-fold cover of  $SO(n)$ .



An alternative description of the spinor groups in terms of Clifford algebras is given in Chevalley [4, p.65].

There are forgetful functors

$$(\ )^{\mathbb{R}} : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{R}} \quad \text{and} \quad (\ )^{\mathbb{C}} : \tilde{\mathbb{H}} \rightarrow \tilde{\mathbb{C}}.$$

If  $W$  is in  $\tilde{\mathbb{C}}$ , then  $W^{\mathbb{R}}$  is the underlying vector space with inner product the real part of the inner product of  $W$ . This induces an inclusion  $U(W) \subset SU(W^{\mathbb{R}})$ . Thus

$$U(n) \subset SO(2n) \quad \text{and} \quad U \subset SO.$$

Similarly, for  $W$  in  $\tilde{\mathbb{H}}$ , we have  $U(W) \subset SU(W^{\mathbb{C}})$  and thus

$$Sp(n) \subset SU(2n) \quad \text{and} \quad Sp \subset SU.$$

There are also extension of scalars functors

$$(\ )_{\mathbb{C}} : \tilde{\mathbb{R}} \rightarrow \tilde{\mathbb{C}} \quad \text{and} \quad (\ )_{\mathbb{H}} : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{H}}.$$

If  $W$  is in  $\tilde{\mathbb{R}}$ , then  $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$  with inner product

$$(v \otimes \alpha, w \otimes \beta) = \bar{\alpha}(v, w)\beta.$$

This induces inclusions  $U(W) \hookrightarrow U(W_{\mathbb{C}})$  and  $SU(W) \hookrightarrow SU(W_{\mathbb{C}})$  via  $T \mapsto T \otimes 1$ . Thus

$$O(n) \subset U(n), \quad O \subset U, \quad SO(n) \subset SU(n), \quad \text{and} \quad SO \subset SU.$$

Similarly, for  $W$  in  $\mathbb{C}$ ,  $W_{\mathbb{H}} = W \otimes_{\mathbb{C}} \mathbb{H}$  as a right  $H$ -space. In this case, the noncommutativity of  $\mathbb{H}$  requires careful attention; we are forced to the formula

$$(v \otimes \alpha, w \otimes \beta) = \bar{\alpha}(v, w)\beta.$$

for the inner product. This gives  $U(W) \subset U(W_{\mathbb{H}})$  and thus

$$U(n) \subset Sp(n) \quad \text{and} \quad U \subset Sp.$$

These inclusions are summarized in the following diagram, the vertical inclusions of which are given by extension of scalars.

$$\begin{array}{ccccccc} SO(n) & \longrightarrow & O(n) & & & & \\ \downarrow & & \downarrow & & & & \\ SU(n) & \longrightarrow & U(n) & \longrightarrow & SO(2n) & \longrightarrow & O(2n) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & Sp(n) & \longrightarrow & SU(2n) & \longrightarrow & U(2n) & \longrightarrow & SO(4n) & \longrightarrow & O(4n). \end{array}$$

In low dimensions, we have the following identifications:

- (i)  $SO(1) = SU(1) = e$  and  $O(1) = \mathbb{Z}/2\mathbb{Z}$
- (ii)  $SO(2) \cong U(1) = T^1$  (the circle group)
- (iii)  $Spin(3) \cong SU(2) \cong Sp(1) = S^3$  (the group of norm one quaternions)
- (iv)  $Spin(4) \cong Sp(1) \times Sp(1)$
- (v)  $Spin(5) \cong Sp(2)$
- (vi)  $Spin(6) \cong SU(4)$

Together with the 2-fold covers  $\text{Spin}(n) \rightarrow \text{SO}(n)$ , this list gives all local isomorphisms among the classical Lie groups.

The following theorem will be essential to our work. Recall that a torus is a Lie group isomorphic to  $T^n = (T^1)^n$ , for some  $n$ .

**THEOREM 1.1.** *A compact connected Lie group  $G$  contains maximal tori. Any two such are conjugate, and  $G$  is the union of its maximal tori.*

Actually, we shall only use particular maximal tori in our canonical examples of classical Lie groups. In  $U(n)$ , the subgroup of diagonal matrices is a maximal torus  $T^n$ . In  $SU(n)$ , the subgroup of diagonal matrices of determinant 1 is a maximal torus  $T^{n-1}$ . In  $Sp(n)$ , the subgroup of diagonal matrices with complex entries is a maximal torus  $T^n$ . In  $SO(2n)$  or  $SO(2n+1)$ , the subgroup of matrices of the form  $\text{diag}(A_1, A_2, \dots, A_n)$  or  $\text{diag}(A_1, A_2, \dots, A_n, 1)$  with each  $A_i \in \text{SO}(2) \cong T^1$  is a maximal torus  $T^n$ .

The quotient  $N/T$ , where  $T$  is a maximal torus is a compact Lie group  $G$  and  $N$  is the normalizer of  $T$  in  $G$ , is a finite group called the *Weyl group* of  $G$  and denoted  $W(G)$ . We shall say more about these groups where they are used.

## 2. Fiber bundles

Although our main interest will be in vector bundles, we prefer to view them in their proper general setting as examples of fiber bundles. This section and the next will give an exposition of the more general theory. We essentially follow Steenrod [12], but with a number of modifications and additions reflecting more recent changes in point of view.

Recall that a cover  $V_j$  of a space  $B$  is said to be *numerable* if it is locally finite and if each  $V_j$  is  $\lambda_j^{-1}([0, 1])$ , for some map  $\lambda_j : B \rightarrow I$ . Since every open cover of a paracompact space has a numerable refinement, we agree to restrict attention to numerable covers throughout. One motivation for doing so is the following standard result; see May [8, Sec 3.8].

**THEOREM 1.2.** *A map  $p : E \rightarrow B$  is a fibration if it restricts to a fibration  $p^{-1}(U) \rightarrow U$ , for all  $U$  in a numerable cover of  $B$ .*

Here, by a *fibration*, we understand a map  $p : E \rightarrow B$  which satisfies the homotopy covering property: for any map  $f : X \rightarrow E$  and homotopy  $h : X \times I \rightarrow B$  of  $pf$ , there is a homotopy  $H$  of  $f$  with  $pH = h$ . It follows that, for any basepoint in any fiber  $F = p^{-1}(b)$ , there is a long exact homotopy sequence

$$\cdots \longrightarrow \pi_n F \longrightarrow \pi_n E \longrightarrow \pi_n B \longrightarrow \pi_{n-1} F \longrightarrow \cdots$$

A fiber bundle is a locally trivial fibration with coordinate patches glued together continuously by means of some specified group. To be precise, recall that a (left) action by a topological group  $G$  on a space  $F$  is a map  $G \times F \rightarrow F$  such that  $g \cdot (g'f) = (gg') \cdot f$  and  $e \cdot f = f$ , where  $e$  is the identity element of  $G$ . The group  $G$  is said to act *effectively* on  $F$  if  $g \cdot f = g' \cdot f$  for all  $f \in F$  implies  $g = g'$ ; equivalently, the only element of  $G$  which acts trivially on  $F$  is  $e$ . The reader may want to think in terms of  $G = U(W)$  and  $F = W$  for some inner product space  $W$ .

**DEFINITION 1.3.** *A coordinate bundle  $\xi = (E, p, B, F, G, \{V_j, \phi_j\})$  is a map  $p : E \rightarrow B$ , an effective transformation group  $G$  of  $F$ , a numerable cover  $\{V_j\}$  of  $B$ , and homeomorphisms  $\phi_j : V_j \times F \rightarrow p^{-1}(V_j)$  such that the following properties hold.*

- (i)  $p \circ \phi_j : V_j \times F \longrightarrow V_j$  is the projection onto the first variable.
- (ii) If  $\phi_{j,x} : F \longrightarrow p^{-1}(x)$  is defined by  $\phi_{j,x}(f) = \phi_j(x, f)$ , then, for each  $x \in V_i \cap V_j$ ,  $\phi_{j,x}^{-1} \circ \phi_{i,x} : F \longrightarrow F$  coincides with operation by a (necessarily unique) element  $g_{ji}(x) \in G$ .
- (iii) The function  $g_{ji} : V_i \cap V_j \longrightarrow G$  is continuous.

Two coordinate bundles are strictly equivalent if they have the same base space  $B$ , total space  $E$ , projection  $p$ , fiber  $F$ , and group  $G$  and if the union of their atlases  $\{V_j, \phi_j\}$  and  $\{V'_k, \phi'_k\}$  is again the atlas of a coordinate bundle. A fiber bundle, or  $G$ -bundle with fiber  $F$ , is a strict equivalence class of coordinate bundles.

DEFINITION 1.4. A map  $(\tilde{f}, f)$  of coordinate bundles is a pair of maps  $f : B \longrightarrow B'$  and  $\tilde{f} : E \longrightarrow E'$ , such that the following properties hold.

- (i) For each  $x \in V_j \cap f^{-1}(V'_k)$ ,  $(\phi'_{k,x})^{-1} \tilde{f}_x \phi_{j,x} : F \longrightarrow F$  coincides with operation by a (necessarily unique) element  $\bar{g}_{kj}(x) \in G$ .
- (ii) The function  $\bar{g}_{kj} : V_j \cap f^{-1}(V'_k) \longrightarrow G$  is continuous.

Note that  $\tilde{f}$  is determined by  $f$  and the  $\bar{g}_{kj}$  via the formula

$$\tilde{f}y = \phi'_k(fx, \bar{g}_{kj}(x)\phi_{j,x}^{-1}(y)) \text{ for } x \in V_j \cap f^{-1}(V'_k) \text{ and } y \in p^{-1}(x).$$

If  $f$  is a homeomorphism, then so is  $\tilde{f}$  and  $(\tilde{f}^{-1}, f^{-1})$  is again a bundle map. Two coordinate bundles with the same base space, fiber, and group are said to be *equivalent* if there is a bundle map between them which is the identity on the base space. Two fiber bundles are said to be equivalent if they have equivalent representative coordinate bundles.

These notions can all be described directly in terms of *systems of transition functions*  $\{V_j, g_{ji}\}$ , namely a numerable cover  $\{V_j\}$  of  $B$  together with maps  $g_{ji} : V_i \cap V_j \longrightarrow G$  which satisfy the cocycle condition

$$g_{kj}(x)g_{ji}(x) = g_{ki}(x) \text{ for } x \in V_i \cap V_j \cap V_k$$

(from which  $g_{ii}(x) = e$  and  $g_{ij}(x) = g_{ji}(x)^{-1}$  follows). The maps  $g_{ji}$  of Definition 1.3 certainly satisfy this condition.

THEOREM 1.5. If  $G$  is an effective transformation group of  $F$ , then there exists one and, up to equivalence, only one  $G$ -bundle with fiber  $F$ , base space  $B$  and a given system  $\{V_j, g_{ji}\}$  of transition functions. If  $\xi$  and  $\xi'$  are  $G$ -bundles with fiber  $F$  over  $B$  and  $B'$  determined by  $\{V_j, g_{ji}\}$  and  $\{V'_j, g'_{ji}\}$  and if  $f : B \longrightarrow B'$  is any map, then a bundle map  $(\tilde{f}, f) : \xi \longrightarrow \xi'$  determines and is determined by maps  $\bar{g}_{kj} : V_j \cap f^{-1}(V'_k) \longrightarrow G$  such that

$$\bar{g}_{kj}(x)g_{ji}(x) = \bar{g}_{ki}(x) \text{ for } x \in V_i \cap V_j \cap f^{-1}(V'_k)$$

and

$$g'_{hk}(f(x)\bar{g}_{kj}(x)) = \bar{g}_{hj}(x) \text{ for } x \in V_j \cap f^{-1}(V'_k \cap V'_h).$$

When  $B = B'$  and  $f$  is the identity, these conditions  $\{\bar{g}_{kj}\}$  prescribe equivalence. When, further,  $\xi$  and  $\xi'$  have the same coordinate neighborhoods (as can always be arranged up to strict equivalence by use of intersections),  $\xi$  and  $\xi'$  are equivalent if and only if there exist maps  $\psi_j : V_j \longrightarrow G$  such that

$$g'_{ji}(x) = \psi_j^{-1}(x)g_{ji}(x)\psi_i(x) \text{ for } x \in V_i \cap V_j.$$

For the first statement,  $E$  can be constructed from  $\coprod V_j \times F$  by identifying  $(x, y) \in V_i \times F$  with  $(x, g_{ji}(x)y) \in V_j \times F$  whenever  $x \in V_i \cap V_j$ . For the second statement,  $\tilde{f}$  can and must be specified by the formula in Definition 1.4. For the last statement, set  $\psi_j = (\bar{g}_{jj})^{-1}$  and  $\bar{g}_{kj}(x) = \psi_j(x)^{-1}g_{kj}(x)$  to construct  $\{\psi_j\}$  from  $\{\bar{g}_{kj}\}$  and conversely. The requisite verifications are straightforward; see Steenrod [12, Sec. 2-3].

Fiber bundles are often just called  $G$ -bundles since the theorem makes clear that the fiber plays an auxiliary role. In particular, we have described equivalences independently of  $F$ , and the set of equivalence classes of  $G$ -bundles is thus the same for all choices of  $F$ . We shall return to this point in the next section, where we consider the canonical choice  $F = G$ . Note too that the effectiveness of the action of  $G$  on  $F$  is not essential to the construction. In other words, if in Definitions 1.3 and 1.4 we assume given maps  $g_{ji}$  and  $\bar{g}_{kj}$  with the prescribed properties, then we may drop the effectiveness since we no longer need the clauses (necessarily unique) in parts (ii).

The basic operations on fiber bundles can be described conveniently directly in terms of transition functions. The product  $\xi_1 \times \cdots \times \xi_n$  of  $G$ -bundles  $\xi_q$  with fibers  $F_q$  and systems of transition functions  $\{(V_j)_q, (g_{ij})_q\}$  is the  $G_1 \times \cdots \times G_n$ -bundle with fiber  $F_1 \times \cdots \times F_n$  and system of transition functions given by the evident  $n$ -fold products of neighborhoods and maps. Its total space, base space, and projection are also the obvious products.

For a  $G$ -bundle  $\xi$  with fiber  $F$ , base space  $B$ , and system of transition functions  $\{V_j, g_{ij}\}$  and for a map  $f : A \rightarrow B$ ,  $\{f^{-1}(V_j), g_{ij} \circ f\}$  is a system of transition functions for the *induced  $G$ -bundle*  $f^*\xi$  with fiber  $F$  over  $A$ . The total space of  $f^*\xi$  is the pullback of  $f$  and the projection  $p : E \rightarrow B$ . If  $(\tilde{f}, f) : \xi' \rightarrow \xi$  is any bundle map, then  $\xi'$  is equivalent to  $f^*\xi$ . A crucially important fact is that homotopic maps induce equivalent  $G$ -bundles; see Steenrod [12, p.53] or Dold [5]. This is the second place where numerability plays a role.

For our last construction, assume given a continuous homomorphism  $\gamma : G \rightarrow G'$ , a specified  $G$ -space  $F$  and a specified  $G'$ -space  $F'$ . If  $\xi$  is a  $G$ -bundle with fiber  $F$ , base space  $V$ , and a system of transition functions  $\{V_j, g_{ji}\}$ , then  $\{V_j, \gamma g_{ji}\}$  is a system of transition functions for the *coinduced  $G'$ -bundle*  $\gamma_*\xi$  with fiber  $F'$  over  $B$ . As one special case, suppose that  $F = F'$  and  $G$  acts on  $F$  through  $\gamma$ ,  $g \cdot f = (\gamma g) \cdot f$ . We then say that  $\gamma_*\xi$  is obtained from  $\xi$  by *extending its group* to  $G'$ . We say that the group of a  $G'$ -bundle  $\xi'$  with fiber  $F$  is *reducible to  $G$*  if  $\xi'$  is equivalent as a  $G'$ -bundle to some extended bundle  $\gamma_*\xi$ . Such an equivalence is called a *reduction* of the structural group. This language is generally only used when  $\gamma$  is the inclusion of a closed subgroup, in which case the last statement of Theorem 1.5 has the following immediate consequence.

**COROLLARY 1.6.** *Let  $H$  be a closed subgroup of  $G$ . A  $G$ -bundle  $\xi$  specified by a system of transition functions  $\{V_j, g_{ji}\}$  has a reduction to  $H$  if and only if there exist maps  $\psi_j : V_j \rightarrow G$  such that*

$$\psi_j(x)^{-1}g_{ji}(x)\psi_i(x) \in H \text{ for all } x \in V_i \cap V_j$$

A  $G$ -bundle is said to be trivial if it is equivalent to the  $G$ -bundle given by the projection  $B \times F \rightarrow F$  or, what amounts to the same thing, if its group can be reduced to the trivial group.

### 3. Principal bundles and homogeneous spaces

The key reason for viewing vector bundles in the context of fiber bundles is that the general theory allows the clearer understanding of the global structure of vector bundles that comes from the comparison of general fiber bundles to principal bundles.

**DEFINITION 1.7.** *A principal  $G$ -bundle is a  $G$ -bundle with fiber  $G$  and  $G$ -action  $g \cdot g' = gg'$ . The principal  $G$ -bundle specified by the same system of transition functions as a given  $G$ -bundle  $\xi$  is called its associated principal bundle and denoted  $\text{Prin } \xi$ . It is immediate from Theorem 1.5 that two  $G$ -bundles with same fiber are equivalent if and only if their associated principal bundles are equivalent. Two  $G$ -bundles with possibly different fibers are said to be associated if their associated principal bundles are equivalent.*

If  $\pi : Y \rightarrow B$  is a principal  $G$ -bundle, then  $G$  acts from the right on  $Y$  in such a way that the coordinate functions  $\phi_j : V_j \times G \rightarrow Y$  are  $G$ -maps, where  $G$  acts on  $V_j \times G$  by right translation of the second factor. Moreover,  $B$  may be identified with the orbit space of  $Y$  with respect to this action. The following description of the construction of general fiber bundles from principal bundles is immediate from the proof of Theorem 1.5.

**LEMMA 1.8.** *Let  $\pi : Y \rightarrow B$  be a principal  $G$ -bundle. The associated  $G$ -bundle  $p : E \rightarrow B$  with fiber  $F$  has total space*

$$E = Y \times_G F := \frac{Y \times F}{\sim}, \text{ where } (yg, f) \sim (y, gf).$$

*The map  $p$  is induced by passage to orbits from the projection  $Y \times F \rightarrow Y$ .*

The construction of  $\text{Prin } \xi$  from  $\xi$  is less transparent (and will not be used in our work). If  $\xi$  is given by  $p : E \rightarrow B$  and has fiber  $F$ , call a map  $\psi : F \rightarrow p^{-1}(x)$  *admissible* if  $\phi_{j,x}^{-1} \circ \psi : F \rightarrow F$  coincides with action by an element of  $G$ , where  $x \in V_j$ , and note that admissibility is independent of the choice of coordinate neighborhood  $V_j$ . The total space  $Y$  of  $\text{Prin } \xi$  is the set of admissible maps  $F \rightarrow E$ . Its projection to  $B$  is induced by  $p$  and its right  $G$ -action is given by composition of maps. Provided that the topology on  $G$  coincides with that obtained by regarding it as a subspace of the spaces of maps  $F \rightarrow F$  (with the compact open topology),  $Y$  is topologized as a subspace of the space of maps  $F \rightarrow E$ . This proviso is satisfied in all of our examples.

The following consequence of Corollary 1.6 is often useful.

**PROPOSITION 1.9.** *A principal  $G$ -bundle  $\pi : Y \rightarrow B$  is trivial if and only if it admits a cross section  $\sigma : B \rightarrow Y$ .*

**PROOF.** Necessity is obvious. Given  $\sigma$  and an atlas  $\{V_j, \phi_j\}$ , the maps  $\psi_j : V_j \rightarrow G$  given by  $\psi_j(x) = \phi_{j,x}^{-1} \circ \sigma(x)$  satisfy

$$\psi_j(x)^{-1} g_{ji}(x) \psi_i(x) = e \text{ for } x \in V_i \cap V_j. \quad \square$$

Another useful fact is that the local continuity conditions (iii) in Definitions 1.3 and 1.4 can be replaced by a single global continuity condition in the case of principal bundles.

DEFINITION 1.10. Let  $Y$  be a right  $G$ -space and let  $\text{Orb } Y$  denote the subspace of  $Y \times Y$  consisting of all pairs of points in the same orbit under the action of  $G$ .  $Y$  is said to be a principal  $G$ -space if  $yg = y$  for any one  $y \in Y$  implies  $g = e$  and if  $\tau : \text{Orb } Y \rightarrow G$  specified by  $\tau(y, yg) = g$  is continuous. Let  $B = Y/G$  with projection  $\pi : Y \rightarrow B$ .  $Y$  is said to be locally trivial if  $B$  has a numerable cover  $\{V_j\}$  together with homeomorphisms  $\phi_j : V_j \times G \rightarrow \pi^{-1}(V_j)$  such that  $\pi\phi_j$  is the projection on  $V_j$  and  $\phi_j$  is a right  $G$ -map.

PROPOSITION 1.11. A map  $\pi : Y \rightarrow B$  is a principal  $G$ -bundle if and only if  $Y$  is a locally trivial principal  $G$ -space,  $B = Y/G$ , and  $\pi$  is the projection onto orbits. If  $\pi : Y \rightarrow B$  and  $\pi' : Y' \rightarrow B'$  are principal  $G$ -bundles, then maps  $\tilde{f} : Y \rightarrow Y'$  and  $f : B \rightarrow B'$  specify a bundle map  $\pi \rightarrow \pi'$  if and only if  $\tilde{f}$  is a right  $G$ -map and  $f$  is obtained from  $\tilde{f}$  by passage to orbits.

PROOF. Since any right  $G$ -map  $G \rightarrow G$  is left multiplication by an element of  $G$ , conditions (i) and (ii) of Definition 1.3 and 1.4 certainly hold for  $\{V_j, \phi_j\}$  as in the previous definition. It is only necessary to relate the continuity conditions (iii) to the continuity of  $\tau$ . Since

$$\phi_i(x, e) = \phi_{i,x}(e) = \phi_{j,x}(g_{ji}(x)) = \phi_j(x, e)g_{ji}(x) \text{ for } x \in V_i \cap V_j,$$

the following diagram commutes, where  $\omega(h, g) = g^{-1}hg$ :

$$\begin{array}{ccc} (V_i \cap V_j) \times G & \xrightarrow{(\phi_i, \phi_j)} & \text{orb } \pi^{-1}(V_i \cap V_j) \\ g_{ij} \times 1 \downarrow & & \downarrow \tau \\ G \times G & \xrightarrow{\omega} & G \end{array}$$

Moreover,  $(\phi_i, \phi_j)$  is a homeomorphism. It follows that  $\tau$  is continuous if and only if all  $g_{ji}$  are so. Similarly, with the notations of Definition 1.4 (iii), the following diagram commutes:

$$\begin{array}{ccc} (V_j \cap f^{-1}(V'_k)) \times G & \xrightarrow{(\tilde{f}\phi_j, \phi'_k f)} & \text{orb}(\pi')^{-1}(f(V_j) \cap V'_k) \\ \bar{g}_{kj} \times 1 \downarrow & & \downarrow \tau \\ G \times G & \xrightarrow{\omega} & G \end{array}$$

Therefore, the  $\bar{g}_{kj}$  are continuous if  $\tau$  is so.  $\square$

If  $H$  is a closed subgroup of a topological group  $G$ , we denote by  $G/H$  the space of left cosets  $gH$  in  $G$  with the quotient topology. Such a coset space is called a *homogeneous space*. The basic method in our study of the cohomology of classical groups will be the inductive analysis of various bundles relating such spaces. We need some preliminary observations in order to state the results which provide the requisite bundles. Subgroups are understood to be closed throughout.

We let  $G$  act on  $G/H$  by left translation. Let  $H_0 \subset G$  be the subset of those elements  $g$  which act trivially on  $G/H$ . Explicitly,  $H_0$  is a closed normal subgroup of  $G$  contained in  $H$  and is the largest subgroup of  $H$  which is normal in  $G$ . The factor group  $G/H_0$  acts effectively on  $G/H$ .

Note that  $G$ , and thus also  $G/H_0$ , acts *transitively* on  $G/H$ . That is, for every pair of cosets  $x, x'$ , there exists  $g$  such that  $g \cdot x = x'$ . Conversely if  $G$  acts transitively on a space  $X$  and if  $H$  is the *isotropy group* of a chosen basepoint

$x \in X$ , namely the subgroup of elements which fix  $x$ , then  $H$  is a closed subgroup of  $G$  and the map  $p : G \rightarrow X$  specified by  $p(g) = gx$  induces a continuous bijection  $q : G/H \rightarrow X$ . By the definition of the quotient topology,  $q^{-1}$  is continuous if and only if  $p$  is an open map. This is certainly the case when  $G$  is compact Hausdorff. We shall make frequent use of such homeomorphisms  $q$  and shall generally regard them as identifications.

In many cases of interest to us, the group  $H_0$  is trivial by virtue of the following observation.

LEMMA 1.12. *For  $K = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ , the largest subgroup of  $U(K^{n-1})$  which is normal in  $U(K^n)$  is the trivial group.*

PROOF.  $U(K^n)/U(K^{n-1})$  is homeomorphic to the unit sphere  $S^{dn-1}$ ,  $d = \dim_{\mathbb{R}} K$ , on which  $U(K^n)$  itself acts effectively.  $\square$

We need one other concept. Let  $p : G \rightarrow G/H$  be the quotient map. A *local cross section* for  $H$  in  $G$  is a neighborhood  $U$  of the basepoint  $eH$  in  $G/H$  together with a map  $f : U \rightarrow G$  such that  $pf = 1$  on  $U$ . When  $G$  is a Lie group, a local cross section always exists by Chevalley [4, p.110]. By Cartan, Moore, et al [3, p.5.10], the infinite classical groups are enough like Lie groups that essentially the same argument works for such  $G$  and reasonable  $H$ . The idea is that if  $G$  has a Lie algebra  $\mathfrak{G}$  and  $H$  has a Lie algebra  $\mathfrak{H} \subset \mathfrak{G}$  with  $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{H}^\perp$ , then the exponential local homeomorphism  $\exp : (\mathfrak{G}, \mathfrak{H}) \rightarrow (G, H)$  can be used to show that there is a homeomorphism  $\phi : V \times H \rightarrow W$  specified by  $\phi(v, h) = \exp(v)h$ , where  $V$  is a suitably small open neighborhood of 0 in  $\mathfrak{H}^\perp$  and  $W$  is an open neighborhood of  $e$  in  $G$ . Then  $U = p(W)$  and  $f(u) = \exp(v)$  if  $p\phi(v, h) = u$  specify the required local cross section.

PROPOSITION 1.13. *If  $H$  has a local cross section in  $G$ , then  $p : G \rightarrow G/H$  is a principal  $H$ -bundle.*

PROOF. If  $f : U \rightarrow G$  is a local cross section, then  $\{g(U) \mid g \in G\}$  is an open cover of  $G/H$  and the right  $H$ -maps  $\phi_h : g(U) \times H \rightarrow p^{-1}(g(U))$  specified by  $\phi_g(gu, h) = gf(u)h$  are homeomorphisms. The continuity of  $\tau : \text{Orb } G \rightarrow H$  is obvious, because the conclusion is immediate from Proposition 1.11.  $\square$

The proposition admits the following useful generalization.

PROPOSITION 1.14. *If  $H$  has a local cross section in  $G$  and  $\pi : Y \rightarrow B$  is a principal  $G$ -bundle, then the projection  $q : Y \rightarrow Y/H$  is a principal  $H$ -bundle.*

PROOF. Let  $\{V_j, \phi_j\}$  be an atlas for  $\pi$ . With the notations of the previous proof, let  $W_{j,g} = q\phi_j(V_j \times p^{-1}(gU)) \subset Y/H$ . Then the right  $H$ -maps  $\omega_{j,g} : W_{j,g} \times H \rightarrow q^{-1}(W_{j,g})$  specified by

$$\omega_{j,g}(\phi_j(x, \phi_g(gu, e))H, h) = \phi_j(x, gf(u))h$$

are homeomorphisms, where  $q(y) = yH$ . The conclusion is again immediate from Proposition 1.11.  $\square$

These principal bundles appear in conjunction with associated bundles with homogeneous spaces as fibers.

LEMMA 1.15. *If  $\pi : Y \rightarrow B$  is a principal  $G$ -bundle, then passage to orbits yields a  $G/H_0$  bundle  $Y/H \rightarrow B$  with fiber  $G/H$ .*

PROOF. This is immediate by passage to orbits on the level of coordinate functions.  $\square$

This lends to the following generalization of Proposition 1.9.

PROPOSITION 1.16. *If  $H$  has a local cross section in  $G$  and  $\pi : Y \rightarrow B$  is a principal  $G$ -bundle, then  $\pi$  admits a reduction of its structure group to  $H$  if and only if the orbit bundle  $Y/H \rightarrow B$  admits a cross section.*

PROOF. We use Corollary 1.6 and the notations of the previous two proofs. Given  $\psi_j : V_j \rightarrow G$  such that  $\psi_j(x)^{-1}g_{ji}(x)\psi_i(x) \in H$  for  $x \in V_i \cap V_j$ , the formula  $\sigma(x) = \phi_j(x, \psi_j(x))H$  for  $x \in V_j$  specifies a well-defined global cross section  $\sigma : B \rightarrow Y/H$ . Conversely, given  $\sigma$ , the maps  $\psi_j : V_j \rightarrow G$  specified by  $\psi_j(x) = gf(u)$  if  $\sigma(x) = \phi_j(x, gf(u))H$  satisfy the cited condition.  $\square$

finally, we note that Lemma 1.15 and Proposition 1.13 together imply most of the following generalization of the latter.

PROPOSITION 1.17. *Let  $J \subset H \subset G$  and let  $H$  admit a local cross section in  $G$ . Then the inclusion of cosets  $G/H \hookrightarrow G/H$  is an  $H/J_0$ -bundle with fiber  $H/J$ , where  $J_0$  is the largest subgroup of  $J$  which is normal in  $H$ . Moreover, left translation by elements of  $G$  specifies self maps of this bundle, and its associated principal bundle is  $G/J_0 \rightarrow G/H$ .*

#### 4. Vector bundles, Stiefel and Grassmann manifolds

By a *vector bundle*, we understand a  $U(W)$ -bundle  $\xi$  with fiber  $W$ , where  $W$  is any finite dimensional inner product space over  $K = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ . When  $W = K^n$ , we refer to  $\xi$  as a (real, complex, or quaternionic)  $n$ -plane bundle. Taking the group to be  $U(W)$  rather than  $GL(W)$  implies that we can give  $\xi$  an inner product metric. That is, we can transport the inner product of  $W$  onto fibers  $p^{-1}(x)$  by means of the coordinate functions, and the positive definite quadratic forms given by the real numbers  $(y, y)$  for  $y \in p^{-1}(x)$  then specify a continuous function  $\mu : E \rightarrow \mathbb{R}$ . The map  $\mu$  is generally called a *Euclidean metric* in the real case and a *Hermitian metric* in the complex case.

Vector bundles are our basic objects of study, and we need various operations on them. We give a generic construction. Recall that  $\tilde{K}$  denotes the category of finite or countably infinite dimensional inner product spaces over  $K$  with linear isometries as morphisms (section 1). Let  $\tilde{K}^* \subset \tilde{K}$  denote the subcategory containing all objects and the linear isometric isomorphisms. This is a topological category, i.e. a category enriched over topological spaces. That is, its hom sets are spaces and composition is continuous. A functor between topological categories is said to be continuous if it induces continuous maps on hom sets. Suppose we're given such a functor

$$T : \tilde{K}_1^* \times \cdots \times \tilde{K}_n^* \rightarrow \tilde{K}^*$$

(where  $K$  and each  $K_q$  is one of  $\mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ ). We allow contravariance; that is, some of the  $\tilde{K}^*$  may be replaced by their opposites, and then  $U(W_q)$  is to be replaced by its opposite below. Then  $T$  gives continuous homomorphisms

$$T : U(W_1) \times \cdots \times U(W_n) \rightarrow UT(W_1, \dots, W_n), \text{ for } W_q \in \tilde{K}_q^*.$$

Suppose given vector bundles  $\xi_q$  with fiber  $W_q$  over base spaces  $B_q$  and suppose that  $\xi_q$  is specified by a system of transition functions  $\{V_{j,q}, (g_{ij})_q\}$ . We obtain



a vector bundle with fiber  $T(W_1, \dots, W_n)$  over  $B_1 \times \dots \times B_n$  by virtue of the system of transition functions obtained by composing products of maps  $(g_{ij})_q : V_{i,q} \cap V_{j,q} \rightarrow U(W_q)$ , or  $(g_{ji})_q^{-1}$  if  $T$  is contravariant in the  $q^{\text{th}}$  variable, with the above maps  $T$ . This defines the external operation on bundles determined by  $T$ . With  $B_1 = \dots = B_n = B$ , the pullback of this external operation along the diagonal  $\Delta : B \rightarrow B^n$  gives the corresponding internal operation. The notation  $T(\xi_1, \dots, \xi_n)$  will be reserved for the internal operation.

The most important example is the *Whitney sum*  $\xi \oplus \xi'$  obtained via  $\oplus : \tilde{K}^* \times \tilde{K}^* \rightarrow \tilde{K}^*$ . Note that if  $\eta$  is a sub-bundle of a vector bundle  $\xi$ , so that each fiber of  $\eta$  is a sub-inner product space of the corresponding fiber of  $\xi$ , then  $\eta$  has a complement  $\eta^\perp$  such that  $\xi = \eta \oplus \eta^\perp$ . An  $n$ -plane bundle  $\xi$  is trivial if and only if it admits  $n$  orthonormal cross sections and, by fiberwise Gram-Schmidt orthonormalization, this holds if and only if  $\xi$  admits  $n$  linearly independent cross sections. Indeed, a nowhere zero cross section prescribes a  $K$ -line sub-bundle, and the conclusion follows by induction.

Other examples are given by such functors as Hom, the tensor product, and exterior powers. Note in particular that our general context includes such operations as

$$\otimes_R : \tilde{\mathbb{H}}^* \times \tilde{\mathbb{H}}^* \rightarrow \tilde{\mathbb{R}}^* \quad \text{and} \quad \otimes_R : \tilde{\mathbb{H}}^* \times \tilde{\mathbb{R}}^* \rightarrow \tilde{\mathbb{H}}^*.$$

These will be useful in the study of Bott periodicity.

We next recall some of the classical examples of homogeneous spaces.

**DEFINITION 1.18.** *Let  $W$  be an inner product space over  $K$ . A  $q$ -frame in  $W$  is an ordered  $q$ -tuple of orthonormal vectors. A  $q$ -plane in  $W$  is a sub inner product space of dimension  $q$  over  $K$ . Let  $V_q(W)$  be the set of  $q$ -frames in  $W$  topologized as a subspace of  $W^q$ . Let  $G_q(W)$  be the set of  $q$ -planes in  $W$ , let  $\pi : V_q(W) \rightarrow G_q(W)$  be the map which sends a  $q$ -frame to the  $q$ -plane it spans, and give  $G_q(W)$  the resulting quotient topology. The spaces  $V_q(W)$  and  $G_q(W)$  are called the Stiefel manifolds and Grassmann manifolds of  $W$ . Clearly  $U(W)$  acts transitively on these spaces. If  $x \in V_q(W)$  spans  $X \in G_q(W)$ , then  $U(X^\perp)$  fixes  $X$ . There result homeomorphisms*

$$V_q(W) \cong U(W)/U(X^\perp) \quad \text{and} \quad G_q(W) \cong U(W)/U(X) \times U(X^\perp),$$

and  $\pi : V_q(W) \rightarrow G_q(W)$  is a principal  $U(X)$ -bundle. The associated bundle with fiber  $X$  has total space

$$\{(Y, w) \mid Y \text{ is a } q\text{-plane in } W, w \text{ is a vector in } Y\}$$

topologized as a subspace of  $G_q(W) \times W$  and given the evident projection to  $G_q(W)$ . These are the classical universal vector bundles.

We also need the oriented variants of these spaces.

**DEFINITION 1.19.** *An orientation of an inner product space  $Y$  of dimension  $q$  over  $\mathbb{R}$  or  $\mathbb{C}$  is an equivalence class of  $q$ -frames, where  $q$ -frames  $y$  and  $y'$  are equivalent if the element  $g \in U(Y)$  such that  $gy = y'$  has determinant one. Let  $\tilde{G}_q(W)$  be the set of oriented  $q$ -planes in  $W$ , let  $\pi : V_q(W) \rightarrow \tilde{G}_q(W)$  send a  $q$ -frame to the oriented  $q$ -plane it determines, and give  $\tilde{G}_q(W)$  the resulting quotient topology. Let  $p : \tilde{G}_q(W) \rightarrow G_q(W)$  be given by neglect of orientation. The space  $\tilde{G}_q(W)$  is called the oriented Grassmann manifold of  $W$ . For any closed  $X \in \tilde{G}_q(W)$ , there is a homeomorphism*

$$\tilde{G}_q(W) \cong U(W) / (SU(W) \times U(X^\perp));$$

$\pi : V_q(W) \longrightarrow \tilde{G}_q(W)$  is a principal  $SU(X)$ -bundle and  $p : \tilde{G}_q(W) \longrightarrow G_q(W)$  is a principal  $S^{d-1}$ -bundle, where  $S^{d-1}$  is  $\mathbb{Z}/2\mathbb{Z}$  in the real case and the circle group in the complex case. The associated bundle of  $\pi$  with fiber  $X$  has total space

$$\{(Y, w) \mid Y \text{ is an oriented } q\text{-plane in } W, w \text{ is a vector in } Y.\}$$

The following lemma will imply that, when  $\dim W = \infty$ , the bundles  $V_q(W) \rightarrow G_q(W)$  and  $V_q(W) \rightarrow \tilde{G}_q(W)$  are in fact “universal” in the sense to be discussed in the following section.

LEMMA 1.20. *If  $\dim W = \infty$ , then  $V_q(W)$  is contractible for all  $q$ .*

PROOF. By May [?, I.1.3][JPM: which 1977 I.1.3 is this?], the spaces of linear isometries  $Y \rightarrow W$  is contractible for all inner product spaces  $Y$ . Let  $X \subset W$  have dimension  $q$ . Since  $\dim X^\perp = \infty$ , the inclusion  $X^\perp \hookrightarrow W$  is homotopic through isometries to an isomorphism, hence the inclusion  $U(X^\perp) \hookrightarrow U(W)$  is homotopic to a homeomorphism and is thus a homotopy equivalence. Therefore,  $\pi_* V_q(W) = 0$  by the long exact sequence of homotopy groups of the fibration sequence  $U(X^\perp) \rightarrow U(W) \rightarrow V_q(W)$ . The conclusion follows.  $\square$

REMARK 1.21. *For  $n$  finite or  $n = \infty$ ,  $G_1(K^{n+1})$  is the projective space  $KP^n$  of lines through the origin in  $K^{n+1}$  and  $V_1(K^{n+1})$  is the unit sphere  $S^{d(n+1)-1}$  in  $K^{n+1}$ ,  $d = \dim_{\mathbb{R}} K$ . The principal  $S^{d-1}$ -bundles  $S^{d(n+1)-1} \rightarrow KP^n$  obtained by sending a point to the line it determines are called Hopf bundles. The associated line bundles are called the canonical line bundles over projective spaces. However, the reader should be warned that, in the complex case, some authors take the conjugates of these line bundle to be “canonical”.*

For calculational purposes, we record the canonical examples of Stiefel manifolds and the various bundles relating them.

DEFINITION 1.22. *For  $0 < q \leq n$ , the Stiefel manifold  $V_q(K^n)$  is the homogeneous space  $U(K^n)/U(K^{n-q})$ . That is,*

$$\begin{aligned} V_q(\mathbb{R}^n) &= O(n)/O(n-q) \\ V_q(\mathbb{C}^n) &= U(n)/U(n-q) \\ V_q(\mathbb{H}^n) &= Sp(n)/Sp(n-q). \end{aligned}$$

LEMMA 1.23. *For  $q < n$ , the natural maps*

$$SO(n)/SO(n-q) \rightarrow V_q(\mathbb{R}^n)$$

and

$$SU(n)/SU(n-q) \rightarrow V_q(\mathbb{C}^n)$$

are homeomorphisms. There are canonical homeomorphisms

$$\begin{array}{lll} V_n(\mathbb{R}^n) = O(n) & V_n(\mathbb{C}^n) = U(n) & V_n(\mathbb{H}^n) = Sp(n) \\ V_{n-1}(\mathbb{R}^n) = SO(n) & V_{n-1}(\mathbb{C}^n) = SU(n) & \\ V_1(\mathbb{R}^n) = S^{n-1} & V_1(\mathbb{C}^n) = S^{2n-1} & V_1(\mathbb{H}^n) = S^{4n-1}. \end{array}$$

The results of the previous section have the following immediate consequence.

PROPOSITION 1.24. *For  $0 < p < q \leq n$ , there is a commutative diagram*

$$\begin{array}{ccccc}
 U(K^{n-q}) & \xlongequal{\quad} & U(K^{n-q}) & & \\
 \downarrow & & \downarrow & & \\
 U(K^{n-p}) & \longrightarrow & U(K^n) & \longrightarrow & V_p(K^n) \\
 \downarrow & & \downarrow & & \parallel \\
 V_{q-p}(K^{n-p}) & \longrightarrow & V_q(K^n) & \longrightarrow & V_p(K^n)
 \end{array}$$

*in which the left two columns are principal  $U(K^{n-q})$ -bundles, the map between these columns is a bundle map, the middle row is a principal  $U(K^{n-p})$ -bundle, and the bottom row is its associated bundle with fiber  $V_{q-p}(K^{n-p})$ .*

The case  $p = q - 1$  is of particular interest since it allows inductive study of these spaces for fixed  $n$ .

### 5. The classification theorem and characteristic classes

While vector bundles are our ultimate objects of study (although they themselves are of greatest interest as a tool for the study of manifolds), our calculational focus will be on classifying spaces. It is our belief that this focus yields the most efficient proofs and the greatest insight, [[OMIT? : particularly as there is yet no satisfactory intrinsic calculational theory of characteristic classes for less structured classes of bundles and fibrations than vector bundles (such as  $PL$  and topological sphere bundles and spherical fibrations)]]].

DEFINITION 1.25. *A universal bundle for a topological group  $G$  is a principal  $G$ -bundle  $\pi : EG \rightarrow BG$  such that  $EG$  is contractible. Any such base space  $BG$  is called a classifying space for  $G$ .*

Proposition 1.17 and Lemma 1.20 imply that if  $G$  is a closed subgroup of  $U(K^q)$ , then the principal  $G$ -bundle

$$V_q(K^q \oplus K^\infty) = U(K^q \oplus K^\infty)/U(K^\infty) \longrightarrow U(K^q \oplus K^\infty)/G \times U(K^\infty)$$

is universal. This is especially pertinent, since any compact Lie group embeds in some  $U(\mathbb{C}^q)$  by the Peter-Weyl Theorem ([7, Thm. 1.15]).

THEOREM 1.26 (The Classification Theorem). *If  $G$  acts effectively on a space  $F$ , then equivalence classes of  $G$ -bundles over  $X$  with fiber  $F$  are in natural one-to-one correspondence with homotopy classes of maps  $X \rightarrow BG$ .*

The correspondence assigns to a map  $f : X \rightarrow BG$  the  $G$ -bundle with fiber  $F$  associated to the induced principal  $G$ -bundle  $f^*\pi$ . Every topological group has a universal bundle  $\pi$ , and any two universal bundles are canonically equivalent. In particular, any two classifying spaces for  $G$  are canonically homotopy equivalent.

The classification theorem admits various proofs, such as those of Steenrod [12], Dold [5][OB is this the right Dold?] [[MC: Don't know]], and tom Dieck [13], the last being particularly elegant. The proof given in May [8] has the advantage that it applies equally well to the classification of fibrations and to both bundles and fibrations with various kinds of additional structure.

Let  $[X, Y]$  denote the set of homotopy classes of maps  $X \rightarrow Y$ . When  $X$  and  $Y$  have basepoints (denoted  $*$ ), let  $[X, Y]_0$  denote the set of based homotopy

classes of based maps  $X \rightarrow Y$ . If  $* \rightarrow X$  is a cofibration (as always holds if  $X$  is a CW-complex), then  $\pi_q Y$  acts on  $[X, Y]_0$  since evaluation at  $*$  is a fibration  $Y^X \rightarrow Y$ . The action is trivial if  $Y$  is an H-space by Whitehead [14, p. 119]. Neglect of basepoints defines a bijection from the orbit set  $[X, Y]_0/\pi_1(Y)$  to  $[X, Y]$ . We shall see in the next section that  $BG$  is simply connected if  $G$  is connected. Thus  $[X, BG] = [X, BG]_0$  if  $G$  is connected or if  $BG$  is an H-space.

Two familiar examples are

$$B\mathbb{Z}/2\mathbb{Z} = BO(1) = \mathbb{R}P^\infty = K(\mathbb{Z}/2\mathbb{Z}, 1)$$

and

$$BT^1 = BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2),$$

where  $K(\pi, n)$  denotes a space with  $n^{\text{th}}$  homotopy group  $\pi$  and remaining homotopy groups zero. Such Eilenberg-MacLane spaces represent cohomology,

$$H^n(X; \pi) = [X, K(\pi, n)]_0$$

and we conclude from the classification theorem that  $O(1)$ -bundles and  $U(1)$ -bundles over  $X$  are in natural one-to-one correspondence with elements of the cohomology groups  $H^1(X; \mathbb{Z}/2\mathbb{Z})$  and  $H^2(X; \mathbb{Z})$  respectively.

Many important properties of classifying spaces can be deduced directly from the classification theorem. Some details of proofs may help clarify the translation back and forth between bundle theory and homotopy theory.

**PROPOSITION 1.27.** *Up to homotopy, passage to classifying spaces specifies a product-preserving functor from topological spaces to topological spaces.*

**PROOF.** If  $\pi_i : EG_i \rightarrow BG_i$  is a universal bundle for  $G_i, i = 1, 2$ , then  $\pi_1 \times \pi_2 : EG_1 \times EG_2 \rightarrow BG_1 \times BG_2$  is clearly a universal bundle for  $G_1 \times G_2$ . Therefore  $BG_1 \times BG_2$  is a classifying space for  $G_1 \times G_2$ . If  $\gamma : G \rightarrow G'$  is a continuous homomorphism, coinduction assigns a principal  $G'$ -bundle  $\gamma_*\xi$  to a principal  $G$ -bundle  $\xi$  (see section 2). Any map  $B\gamma : BG \rightarrow BG'$  also converts principal  $G$ -bundles to principal  $G'$ -bundles, via composition with classifying maps. Since these constructions are both natural with respect to the operation of pulling a bundle back along a map, it suffices to specify  $B\gamma$  to be the classifying map of the principal  $G'$ -bundle coinduced from the universal  $G$ -bundle to ensure that  $B\gamma$  induces  $\gamma_*$ .  $\square$

While the proposition suffices for our purposes and will lead to the classification of particular maps  $B\gamma$  in the next section, the precise construction of classifying spaces and universal bundles by use of the “geometric bar construction” yields much sharper results. Indeed, with this construction,  $E$  and  $B$  are functors before passage to homotopy,  $EG_1 \times EG_2$  and  $BG_1 \times BG_2$  are naturally homeomorphic to  $E(G_1 \times G_2)$  and  $B(G_1 \times G_2)$ , and,  $B$  is homotopy preserving in the sense that if  $\gamma$  and  $\gamma'$  are homotopic through homomorphisms, then  $B\gamma$  is homotopic to  $B\gamma'$ . Much more is true, and the reader is referred to May [8, Sec. 7-8] for an exposition.

It is immediate from the proposition and the generic construction of operations on vector bundles in the previous section that if  $T$  is a continuous functor of the sort considered there, then

$$\begin{aligned} BT : BU(W_1) \times \cdots \times BU(W_n) &\simeq B(U(W_1) \times \cdots \times U(W_n)) \\ &\rightarrow BU(T(W_1, \dots, W_n)) \end{aligned}$$

induces the corresponding external operation on bundles via composition with the products of classifying maps. If  $f_q : X \rightarrow BU(W_q)$  classifies  $\xi_q$ , then the following composite classifies the internal operation  $T(\xi_1, \dots, \xi_n)$ :

$$X \xrightarrow{\Delta} X^n \xrightarrow{f_1 \times \dots \times f_n} BU(W_1) \times \dots \times BU(W_n) \xrightarrow{BT} BU(T(W_1, \dots, W_n)).$$

If  $T$  is contravariant in the  $q^{\text{th}}$  variable, then, to arrange that  $BU(W_q)^{\text{op}}$  rather than  $BU(W_q)$  appears as the  $q^{\text{th}}$  space in the domain of  $BT$ , we must precompose with  $B\chi : BU(W_q) \rightarrow BU(W_q)^{\text{op}}$ , where  $\chi : G \rightarrow G^{\text{op}}$  is the anti-isomorphism  $\chi(g) = g^{-1}$ .

We shall study such operations via canonical examples. In particular, for any classical group  $G$  ( $G = O, U$ , etc.), Whitney sums are induced by the maps

$$\rho_{mn} : BG(m) \times BG(n) \rightarrow BG(m+n)$$

obtained from the block sum of matrix homomorphisms

$$G(m) \times G(n) \rightarrow G(m+n).$$

The inclusion  $G(n) \hookrightarrow G(n+1)$  is block sum with  $I \in G(1)$ , hence the corresponding map

$$i_n : BG(n) \rightarrow BG(n+1)$$

induces addition of trivial line bundles.

Say that two vector bundles  $\xi$  and  $\xi'$  are *stably equivalent* if  $\xi \oplus \epsilon$  is equivalent to  $\xi' \oplus \epsilon'$  for some trivial bundles  $\epsilon$  and  $\epsilon'$ . With our explicit Grassmann manifold construction of classifying spaces, we have that  $i_n$  is an inclusion and the union of the  $BG(n)$  is a classifying space for the infinite classical group  $G$ .

The natural map  $BG(n) \rightarrow BG$  induces the transformation which sends an  $n$ -plane bundle to its stable equivalence class. For a finite dimensional CW-complex  $X$ , a map  $X \rightarrow BG$  necessarily factors through some  $BG(n)$ . For such spaces,  $[X, BG]$  is in natural one-to-one correspondence with the set of stable equivalence classes of  $G(n)$ -bundles over  $X$ .

The map  $BO(n) \rightarrow BU(n)$  induced by the inclusion  $O(n) \rightarrow U(n)$  represents complexification of real vector bundles, and similarly for our other forgetful maps and extension of scalars maps between classical groups. In sum, we may think of the classification theorem as providing an equivalence between the theory of vector bundles and the study of classifying spaces of classical groups.

We shall exploit this equivalence for the study of characteristic classes.

**DEFINITION 1.28.** *Let  $G$  be a topological group. A characteristic class  $c$  for  $G$ -bundles associates to each  $G$ -bundle  $\xi$  over  $X$  a cohomology class  $c(\xi) \in H^*(X)$  (for a given cohomology theory  $H^*$ ) naturally with respect to  $G$ -bundle maps; that is, if  $(f, f) : \xi \rightarrow \xi'$  is a map of  $G$ -bundles, then  $f^*c(\xi') = c(\xi)$ .*

We have not mentioned a fiber. While the definition implicitly assumes a fixed choice, any choice will do. We adopt the convention that associated bundles have the same characteristic classes since they have the same classifying maps.

**LEMMA 1.29.** *Characteristic classes for  $G$ -bundles are in one-to-one correspondence with elements of  $H^*BG$ .*

**PROOF.** We may restrict attention to principal  $G$ -bundles. If  $\pi : EG \rightarrow BG$  is universal,  $c$  is a characteristic class, and  $f : X \rightarrow BG$  classifies  $\xi$ , then  $c(f) =$

$f^*c(\pi)$ . Thus  $c$  is completely determined by  $c(\pi) \in H^*BG$ . Conversely,  $\gamma \in H^*BG$  determines a characteristic class  $c$  by  $c(\pi) = \gamma$  and naturality.  $\square$

Categorically, this is a special case of the Yoneda lemma: natural transformations  $[?, Y] \rightarrow F(?)$  are in one-to-one correspondence with elements of  $F(Y)$  for any set-valued contravariant homotopy functor  $F$ .

The lemma is the philosophical basis for our calculations. Observe that we may study the effect on characteristic classes of operations on vector bundles by calculating the induced map on the cohomology of the relevant classifying spaces. For example, the effect of Whitney sum on characteristic classes can be deduced from the map  $p_{mn}^*$ .

## 6. Some homotopical properties of classifying spaces

We collect a few miscellaneous facts about classifying spaces for later use.

Let  $\pi : EG \rightarrow BG$  be a universal  $G$ -bundle. If  $h : EG \times I \rightarrow EG$  is a contracting homotopy,  $h(y, 0) = *$  and  $h(y, 1) = y$ , let  $\tilde{h} : EG \rightarrow PBG$  be the map specified by  $\tilde{h}(y)(t) = \pi h(y, t)$ , where  $PBG$  is the space of paths in  $BG$  which start at  $\pi(*)$ . Let  $p : PBG \rightarrow BG$  be the end-point projection, so that  $p^{-1}\pi(*)$  is the loop space  $\Omega BG$ . Then  $p\tilde{h} = \pi$  and  $\tilde{h}$  restricts to a map  $\zeta : G \rightarrow \Omega BG$ , where  $G$  is identified with the fiber  $\pi^{-1}(\pi(*))$ . Thus the following diagram commutes:

$$\begin{array}{ccccc} G & \longrightarrow & EG & \xrightarrow{\pi} & BG \\ \zeta \downarrow & & \downarrow \tilde{h} & & \parallel \\ \Omega BG & \longrightarrow & PBG & \xrightarrow{p} & BG. \end{array}$$

By composition of long exact homotopy sequences, this yields the following result.

**PROPOSITION 1.30.**  *$BG$  is a connected space and  $\pi_{n+1}BG$  is naturally isomorphic to  $\pi_n G$  for  $n \geq 0$ . The map  $\zeta : G \rightarrow \Omega BG$  is a homotopy equivalence.*

In particular,  $BG$  is simply connected if  $G$  is connected.

We have observed that  $BG$  is a functor of  $G$ . We need several results about the behavior of this functor on particular kinds of maps. These will all be consequences of the following criterion for recognizing when a map of classifying spaces is  $B\gamma$  for some homomorphism  $\gamma$ .

**LEMMA 1.31.** *Let  $\gamma : G \rightarrow G'$  be a continuous homomorphism and let  $\pi : EG \rightarrow BG$  and  $\pi' : EG' \rightarrow BG'$  be universal bundles. If  $f : EG \rightarrow EG'$  is any map such that  $f(yg) = f(y)\gamma(g)$  for all  $y \in EG$  and  $g \in G$ , then the map  $BG \rightarrow BG'$  obtained from  $f$  by passage to orbits is in the homotopy class  $B\gamma$ .*

**PROOF.** Regarding  $G'$  as a left  $G$ -space via  $\gamma$ , we see by inspection of definitions that  $\gamma_*\pi$  is the principal  $G'$ -bundle  $EG \times_G G' \rightarrow BG$ . By Proposition 1.11, the  $G'$ -map

$$f \times 1 : EG \times_G G' \rightarrow EG' \times_{G'} G' = EG'$$

gives a bundle map  $\gamma_*(\pi) \rightarrow \pi'$ . The conclusion follows.  $\square$

As a first example, we have the following observation.

**LEMMA 1.32.** *If  $g \in G$  and  $\gamma_g : G \rightarrow G$  is given by conjugation,  $\gamma_g(h) = g^{-1}hg$ , then  $B\gamma_g$  is the identity map of  $BG$ .*

PROOF. The map  $EG \rightarrow EG$  given by right multiplication by  $g$  satisfies the prescribed equivariance property and induces the identity map on  $BG$ .  $\square$

Henceforward in this section, let  $H$  be a closed subgroup with a local cross section in  $G$  and let  $i : H \rightarrow G$  denote the inclusion. By Proposition 1.14,  $EG \rightarrow EG/H$  is a universal  $H$ -bundle. Applying Lemma 1.31 to the identity map of  $EG$ , we obtain the following observation.

LEMMA 1.33. *The map  $Bi : BH \rightarrow BG$  is the bundle*

$$BH = EG/H \rightarrow EG/G = BG$$

with fiber  $G/H$ .

Now assume further that  $H$  is normal in  $G$  with quotient  $K = G/H$ ; that is, assume given an extension

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1.$$

Let  $EG \rightarrow BG$  and  $EK \rightarrow BK$  be universal bundles for  $G$  and  $K$  and take  $EG \rightarrow EG/H = BH$  to be the universal bundle for  $H$ . We may assume given a map  $Ej : EG \rightarrow EK$  such that  $(Ej)(yg) = (Ej)(y)j(g)$ . Since  $j(h) = e$  for  $h \in H$ ,  $Ej$  factors through  $BH$  and we obtain a bundle map

$$\begin{array}{ccc} BH & \longrightarrow & EK \\ Bi \downarrow & & \downarrow \\ BG & \xrightarrow{Ej} & BK. \end{array}$$

In particular, this square is a pullback.

Recall that the homotopy fiber  $F(f)$  of a (based) map  $f : X \rightarrow Y$  is defined by the pullback diagram

$$\begin{array}{ccc} F(f) & \longrightarrow & PY \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Equivalently,  $F(f)$  is the actual fiber over the basepoint of the map  $NX \rightarrow Y$  obtained by turning  $f$  into a fibration via the standard mapping path fibration construction. There is thus a long exact homotopy sequence

$$\cdots \rightarrow \pi_{n+1}Y \rightarrow \pi_n F(f) \rightarrow \pi_n X \rightarrow \pi_n Y \rightarrow \cdots.$$

Using the diagram above Proposition 1.30, we see by the universal property of pullbacks that there is a map  $\theta : BH \rightarrow F(Bj)$  such that the following diagram commutes.

$$\begin{array}{ccccc} K & \longrightarrow & BH & \xrightarrow{Bi} & BG \\ \zeta \downarrow & & \downarrow \theta & & \parallel \\ \Omega BK & \longrightarrow & F(Bj) & \longrightarrow & BG \end{array}$$

By the five lemma,  $\theta$  induces an isomorphism on homotopy groups and is thus a homotopy equivalence. This conclusion may be restated as follows.

PROPOSITION 1.34. *Up to homotopy, the sequence*

$$BH \xrightarrow{Bi} BG \xrightarrow{Bj} BK$$

*is a fiber sequence.*

Finally, retaining the hypotheses above, assume further that  $K$  is discrete. For  $g \in G$ , we have the conjugation homomorphism  $\gamma_g : H \rightarrow H$ . On the other hand,  $Bi : BH \rightarrow BG$  is a principal  $K$ -bundle, hence we have a right action of  $K$  on  $BH$ . In fact,  $Bi$  is a regular cover and  $K$  is its group of covering transformations. We have the following generalization of Lemma 1.32.

LEMMA 1.35. *If  $k = gH \in K = G/H$ , then the covering transformation  $k : BH \rightarrow BH$  is homotopic to  $B\gamma_g : BH \rightarrow BH$ .*

PROOF. The action of  $K$  on  $BH = EG/H$  is induced from the action of  $G$  on  $EG$ , and right translation by  $g$  gives a map  $EG \rightarrow EG$  with the equivariance property prescribed for  $\gamma_g$  in Lemma 1.31.  $\square$

We shall later apply this to the extension

$$1 \longrightarrow SO(n) \longrightarrow O(n) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

It also applies to the extensions

$$1 \longrightarrow T \longrightarrow N \longrightarrow W(G) \longrightarrow 1,$$

where  $T$  is a maximal torus in a compact Lie group  $G$ ,  $N$  is the normalizer of  $T$  in  $G$ , and  $W(G)$  is the Weyl group. The covering transformations of  $BT$  given by the elements of  $W(G)$  are of fundamental importance of the following result.

LEMMA 1.36. *For  $\sigma \in W(G)$ , the following diagram is homotopy commutative:*

$$\begin{array}{ccc} BT & \xrightarrow{Bi} & BG \\ \sigma \downarrow & & \parallel \\ BT & \xrightarrow{Bi} & BG \end{array}$$

PROOF. If  $g \in N \subset G$  has image  $\sigma$ , then  $\sigma \simeq B\gamma_g$ . Since  $\gamma_g i \simeq i\gamma_g$  and  $B\gamma_g \simeq 1$  on  $BG$  by Lemma 1.32, the conclusion follows.  $\square$

## 7. Algebraic Tools

LEMMA 1.37. *Suppose that  $A$  is a graded commutative ring with a multiplicative exhaustive filtration  $A = F_0 \supset F_1 \supset \dots$ . If the associated graded  $E^0 A$  is a free commutative algebra on elements  $\{x_\alpha + F_{s_\alpha}\}$  then  $A$  is a free commutative algebra on  $\{x_\alpha\}$ .*

Define simple system of generators and state the analogous theorem for them.

Notation for polynomial, exterior, divided polynomial rings and tings with simple systems of gernators.

Duality of Hopf algebras for these special cases.

## 8. Spectral sequences

Clear statements of the Serre spectral sequence and E.M. ss.

Include facts about transgression and the edge homs. and interaction with Steenrod operations.



## Cohomology of the Classical Groups and Stiefel Manifolds

With the exception of characteristic classes, which have traditional indexings we shall respect, *we shall index homology and cohomology classes by their degrees.* That is, given a sequence of classes in degrees  $2i - 1$ , say, we shall label them  $x_{2i-1}$  rather than  $x_i$ . Moreover, when we have a canonical map  $X \rightarrow Y$  and given homology classes of  $X$  or cohomology classes of  $Y$ , we shall generally use the same notations for the images of these classes in the homology of  $Y$  or the cohomology of  $X$ .

We fix the canonical fundamental classes

$$i_n \in H_n(S^n; \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad \iota_n \in H^n(S^n; \mathbb{Z}) \cong \mathbb{Z}$$

and use the same notations for their images in homology or cohomology with other coefficients. Explicitly,  $i_n$  is the image of the identity map  $S^n \rightarrow S^n$  under the Hurewicz homomorphism  $\pi_n(S^n) \rightarrow H_n(S^n; \mathbb{Z})$ , and  $\iota_n$  is the dual generator characterized by  $\iota_n(i_n) = 1$  under the evaluation pairing  $H^n(S^n; \mathbb{Z}) \otimes H_n(S^n; \mathbb{Z}) \rightarrow \mathbb{Z}$ . We often write  $i$  or  $\iota$  when the dimension  $n$  is clear from the context. *We do not assume prior knowledge of any further explicit calculations of homology or cohomology groups.* We do assume that the reader is familiar with the Künneth and universal coefficient theorems.

### 1. The complex and quaternionic Stiefel manifolds

In this section, we compute the homology and cohomology of the complex and quaternionic Stiefel manifolds inductively, ending with the homology and cohomology of the complex and symplectic classical groups. All homology and cohomology groups are to be taken with integer coefficients. The only tool we shall need is the Serre spectral sequence in the special cases (with fiber and base space a sphere, respectively) which give the Gysin and Wang exact sequences.

**THEOREM 2.1.** *Let  $1 \leq q \leq n$ . As algebras, [BOB deg symbol?]*

$$H^*(V_q(\mathbb{C}^n)) = E\{y_{2i-1} \mid n - q < i \leq n\}, \quad \deg y_{2i-1} = 2i - 1,$$

and

$$H^*(V_q(\mathbb{H}^n)) = E\{z_{4i-1} \mid n - q < i \leq n\}, \quad |z_{4i-1}| = 4i - 1.$$

**PROOF.** We treat the complex case. The symplectic case is entirely similar. Abbreviate  $V_q = V_q(\mathbb{C}^n)$ . We proceed by induction on  $q$ . We begin with  $V_1 = S^{2n-1}$ , taking  $y_{2n-1} = \iota_{2n-1}$ . This completes the case  $n = 1$ , so we assume that  $n \geq 2$ . Assume the result for  $V_{q-1}$  and consider the Serre spectral sequence  $\{E_r\}$  of the bundle  $V_q \rightarrow V_{q-1}$  with fiber  $V_1(\mathbb{C}^{n-q+1}) = S^{2(n-q+1)-1}$ . It is clear from the

long exact homotopy sequences of such bundles that the  $V_q$  are simply connected. As algebras, we have

$$E_2 = H^*(V_{q-1}) \otimes H^*(S^{2(n-q+1)-1}).$$

All differentials on the  $x$  axis (or base axis)  $E_2^{*0}$  are zero since they land in the fourth quadrant, which consists of zero groups. The rows  $E_2^{*q}$  are zero for  $0 < q < 2(n - q + 1) - 1$ . Therefore the only generator that might possibly support a non-trivial differential is the fundamental class  $\iota$ , and it can only “transgress”, that is, map from the  $y$  axis (or fiber axis) to the  $x$ -axis via  $d_{2(n-q+1)}$ . By the induction hypothesis,  $H^p(V_{q-1}) = 0$  for  $0 < p < 2(n - q + 2) - 1$ . In particular,  $H^{2(n-q+1)}(V_{q-1}) = 0$ , so that  $d_{2(n-q+1)}(\iota)$  lands in a zero group. Therefore  $E_2 = E_\infty$  and  $H^*(V_q)$  is as claimed.  $\square$

Hereafter, we shall summarize arguments like the one above by asserting that  $E_2 = E_\infty$  (or the spectral sequence collapses at  $E_2$ ) for dimensional reasons.

REMARK 2.1. *As suggested by the notation, we may choose the generators  $y_{2i-1}$  and  $z_{4i-1}$  consistently as  $q$  and  $n$  vary. To be precise about this, consider the commutative diagram*

$$\begin{array}{ccccc} S^{2(n-q)-1} & \xlongequal{\quad} & S^{2(n-q)-1} & & \\ \downarrow \iota & & \downarrow \iota & & \\ V_q(\mathbb{C}^{n-1}) & \xrightarrow{\quad \iota \quad} & V_{q+1}(\mathbb{C}^n) & \xrightarrow{\quad \pi \quad} & S^{2n-1} \\ \downarrow \pi & & \downarrow \pi & & \downarrow \\ V_{q-1}(\mathbb{C}^{n-1}) & \xrightarrow{\quad \iota \quad} & V_q(\mathbb{C}^n) & \xrightarrow{\quad \pi \quad} & S^{2n-1}. \end{array}$$

Here the  $\pi$  are projections of bundles and the  $\iota$  are inclusions of fibers. We assume the  $y_{2i-1}$  have been chosen compatibly in  $H^*(V_q(\mathbb{C}^m))$  for  $m - q < i \leq m < n$ . We have seen that the spectral sequences of the two columns satisfy  $E_2 = E_\infty$  for dimensional reasons, and a similar argument shows that the Serre spectral sequences of the two rows also satisfy  $E_2 = E_\infty$ . By consideration of the edge homomorphism,  $\iota^* : H^j(V_{q-1}(\mathbb{C}^{n-1})) \rightarrow H^j(V_q(\mathbb{C}^n))$  is an isomorphism for  $j < 2n - 1$ . We specify generators  $y_{2i-1}$  for  $H^*(V_q(\mathbb{C}^n))$  by requiring  $\iota^*(y_{2i-1}) = y_{2i-1}$  for  $i < n$  and taking  $y_{2n-1} = \pi^*(\iota_{2n-1})$ . It follows inductively from the diagram that  $\pi^*(y_{2i-1}) = y_{2i-1}$  in  $H^*(V_{q+1}(\mathbb{C}^n))$  for  $n - q < i \leq n$ . It also follows that  $\iota^*(y_{2n-1}) = 0$  in  $H^*(V_q(\mathbb{C}^{n-1}))$ .

THEOREM 2.2. *As Hopf algebras,*

$$\begin{aligned} H^*(U(n)) &= E\{y_{2i-1} \mid 1 \leq i \leq n\} \quad \text{and} \quad H^*(U) = E\{y_{2i-1} \mid i \geq 1\}, \\ H^*(SU(n)) &= E\{y_{2i-1} \mid 2 \leq i \leq n\} \quad \text{and} \quad H^*(SU) = E\{y_{2i-1} \mid i \geq 2\}, \\ H^*(Sp(n)) &= E\{z_{4i-1} \mid 1 \leq i \leq n\} \quad \text{and} \quad H^*(Sp) = E\{z_{4i-1} \mid i \geq 1\}. \end{aligned}$$

*In each case, the corresponding homology Hopf algebra is the exterior Hopf algebra on the dual generators.*

PROOF. First consider the case of finite  $n$ . As algebras, the stated cohomologies are immediate from Theorem 2.1. We must show that the generators are primitive. We proceed by induction on  $n$ , the case  $n = 1$  being trivial. Thus assume the result for  $n - 1$  and let  $\iota : U(n - 1) \rightarrow U(n)$  be the standard inclusion. By Remark 2.1,

$\iota^*$  is an isomorphism in degrees less than  $2n - 1$ . By the induction hypothesis, this implies that  $y_{2i-1}$  is primitive for  $i < n$ . Write

$$\psi(y_{2n-1}) = y_{2n-1} \otimes 1 + \sum y' \otimes y'' + 1 \otimes y_{2n-1}, \deg y' > 0 \text{ and } \deg y'' > 0.$$

Since  $\iota^*(y_{2n-1}) = 0$  and  $(\iota^* \otimes \iota^*)\psi = \psi\iota^*$ , we must have that  $(\iota^* \otimes \iota^*)(\sum y' \otimes y'') = 0$ . Since  $\iota^* \otimes \iota^*$  is an isomorphism in the pairs of degrees in which the elements  $y' \otimes y''$  occur, it follows that  $\sum y' \otimes y'' = 0$ . The  $y_{2i-1}$  in  $H^*(SU(n))$  are the images of the  $y_{2n-1}$  in  $H^*(U(n))$  and are thus also primitive. The proof that the  $z_{4i-1}$  are primitive is similar.

By use of the Poincaré duality theorem, the universal coefficients theorem, or the homology version of the calculation of Theorem 2.1, we see that  $H_*(U(n))$  is a free graded  $\mathbb{Z}$ -module of finite type and that  $H^*(U(n))$  and  $H_*(U(n))$  are dual to one another as  $\mathbb{Z}$ -modules. Since locally finite exterior Hopf algebras are self-dual, this implies the homology statement.

Finally, for the infinite classical groups, the homology statement is immediate since homology commutes with colimits, and the cohomology statements follow by duality.  $\square$

## 2. The real Stiefel manifolds

The cohomology of the real Stiefel manifolds is complicated by the failure of the dimensional argument used to prove Theorem 2.1 and by the presence of 2-torsion. Accordingly, we will calculate separately the cohomology at the prime 2 (that is with  $\mathbb{F}_2$  coefficients) and away from the prime 2 (that is, with coefficients in a ring in which 2 is invertible). This is a standard technique which will occur over and over in our work.

We fix  $n$ , and we often write  $V_q$  for  $V_q(\mathbb{R}^n)$  in this section. We need a little homotopical input, namely a calculation of the homotopy groups of the  $V_q$  in the ‘‘Hurewicz dimension’’, the largest dimension  $j$  below which all homotopy groups vanish, so that  $\pi_j$  maps isomorphically to  $H_j$  (if  $j > 1$  or  $\pi_1$  is Abelian).

LEMMA 2.2. *For  $1 \leq q \leq n$ ,  $V_q(\mathbb{R}^n)$  is  $(n - q - 1)$ -connected and*

$$\pi_{n-q}(V_q(\mathbb{R}^n)) = \begin{cases} \mathbb{Z} & \text{if } q < n \text{ and } q = 1 \text{ or } n - q \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q = n \text{ or } n - q \text{ is odd.} \end{cases}$$

PROOF. Since  $V_1 = S^{n-1}$  and  $V_n = O(n)$ , we may assume that  $1 < q < n$  and thus  $n \geq 3$ . The main work is in the case  $q = 2$ . Here the fibration

$$S^{n-2} \xrightarrow{\iota} V_2 \xrightarrow{\pi} V_1 = S^{n-1}$$

yields the exact sequence

$$\pi_{n-1}(S^{n-1}) \xrightarrow{\partial} \pi_{n-2}(S^{n-2}) \xrightarrow{\iota_*} \pi_{n-2}(V_2) \longrightarrow 0,$$

and we must compute  $\partial$ . For any  $r$ , let  $E^r \subset S^r$  be the upper hemisphere,

$$E^r = \left\{ x = (x_1, \dots, x_{r+1}) \mid \sum x_i^2 = 1 \text{ and } x_1 \geq 0 \right\},$$

and let  $S^{r-1}$  be the equator,  $S^{r-1} = \{x \mid x \in E^r \text{ and } x_1 = 0\}$ . Give  $S^r$  the base-point  $*$  =  $(1, 0, \dots, 0)$ . Represent elements of  $V_2$  by  $2 \times n$  matrices  $A$  such that  $AA^t$  is the  $2 \times 2$  identity matrix. This makes sense since such matrices specify pairs of orthonormal vectors in  $\mathbb{R}^n$ . (We note parenthetically that this identifies  $V_2(\mathbb{R}^n)$  with the unit sphere bundle of the tangent bundle of  $S^{n-1}$ .)

Define  $\mu : E^{n-1} \longrightarrow V_2$  by  $\mu(x) = (\mu_{ij}(x))$ , where

$$\mu_{ij}(x) = \delta_{ij} - 2x_i x_j \quad \text{for } 1 \leq i \leq 2 \quad \text{and } 1 \leq j \leq n.$$

Here  $\delta_{ij} = 1$  if  $i = j$  and 0 if  $i \neq j$ . Represent  $\pi : V_2 \longrightarrow S^{n-1}$  by  $\pi(a_{ij}) = (a_{1j})$ . The composite  $\pi\mu$  sends  $S^{n-2}$  to the basepoint  $* \in S^{n-1}$  and restricts to a homeomorphism  $E^{n-1} - S^{n-2} \longrightarrow S^{n-1} - *$ . Therefore

$$\pi\mu : (E^{n-1}, S^{n-2}) \longrightarrow (S^{n-1}, *)$$

represents a generator of  $\pi_{n-1}(S^{n-1})$  and the restriction

$$\tau : S^{n-2} \longrightarrow \pi^{-1}(*) = S^{n-2}$$

of  $\mu$  represents a generator of  $\text{Im}(\partial) \subset \pi_{n-2}(S^{n-2})$ . Write  $y_j = x_{j+1}$ , so that

$$\tau(y_1, \dots, y_{n-1}) = (1 - 2y_1^2, -2y_1 y_2, \dots, -2y_1 y_{n-1}).$$

Then  $\tau$  sends  $S^{n-3}$  to the basepoint  $* \in S^{n-2}$ , restricts to a homeomorphism  $E^{n-2} - S^{n-3} \longrightarrow S^{n-2} - *$ , and satisfies  $\tau(-y) = \tau(y)$ . Therefore  $\tau$  represents the sum of the identity map and the antipodal map (or the negative of this sum depending on orientations). The antipodal map on  $S^{n-2}$  has degree  $(-1)^{n-1}$ , hence  $\tau$  has degree  $\pm(1 + (-1)^{n-1})$ . Thus  $\partial = 0$  if  $n$  is even and  $\partial i_{n-1} = \pm 2i_{n-2}$  if  $n$  is odd. This proves the lemma for  $q = 2$ .

For  $q = 3$ , the commutative diagram

$$\begin{array}{ccccc} S^{n-3} = V_1(\mathbb{R}^{n-2}) & \longrightarrow & V_2(\mathbb{R}^{n-1}) & \longrightarrow & V_1(\mathbb{R}^{n-1}) = S^{n-2} \\ \parallel & & \downarrow & & \downarrow \iota \\ S^{n-3} = V_1(\mathbb{R}^{n-2}) & \longrightarrow & V_3(\mathbb{R}^n) & \longrightarrow & V_2(\mathbb{R}^n) = V_2 \end{array}$$

gives rise to the commutative diagram

$$\begin{array}{ccccccc} \pi_{n-2}(S^{n-2}) & & & & & & \\ \downarrow \iota_* & \searrow \partial & & & & & \\ \pi_{n-2}(V_2) & \xrightarrow{\partial'} & \pi_{n-3}(S^{n-3}) & \longrightarrow & \pi_{n-3}(V_3) & \longrightarrow & 0. \end{array}$$

Here  $\text{Im}(\partial) = \text{Im}(\partial')$  since  $\iota_*$  is an epimorphism. We know  $\partial$  from the case  $q = 2$ , and the result for  $q = 3$  follows.

Finally, for  $q \geq 3$ , the long exact homotopy sequence of the fibration

$$V_{q-2}(\mathbb{R}^{n-2}) \longrightarrow V_q \longrightarrow V_2$$

and the fact that  $\pi_i(V_2) = 0$  for  $i < n - 2$  show that  $\pi_i(V_{q-2}(\mathbb{R}^{n-2})) \cong \pi_i(V_q)$  for  $i < n - 3$ . The result follows from known cases by induction on  $n$ .  $\square$

We begin our homological calculations by considering  $V_2$ . Since  $V_2(\mathbb{R}^2) = O(2)$ , which is homeomorphic to  $S^0 \times S^1$ , we already know  $H^*(V_2(\mathbb{R}^2); \mathbb{Z})$ .

**LEMMA 2.3.** *Assume that  $n \geq 3$ . If  $n$  is even, then*

$$H^*(V_2(\mathbb{R}^n); \mathbb{Z}) = E\{x_{n-2}, x_{n-1}\}.$$

*If  $n$  is odd, then*

$$H^*(V_2(\mathbb{R}^n); \mathbb{Z}) = E\{x_{n-1}, x_{2n-3}\}/(2x_{n-1}, x_{n-1}x_{2n-3}).$$

PROOF. In the Serre spectral sequence of the fibration  $S^{n-2} \rightarrow V_2 \rightarrow S^{n-1}$ , we have  $E_2 = H^*(S^{n-1}) \otimes H^*(S^{n-2})$ . For dimensional reasons, the only possible non-zero differential is the transgression  $d_{n-1}(\iota_{n-2})$ , which must be a multiple of  $\iota_{n-1}$ . If  $n$  is even, we have

$$\mathbb{Z} = \pi_{n-2}(V_2) \cong H_{n-2}(V_2; \mathbb{Z}) \cong H^{n-2}(V_2; \mathbb{Z})$$

by the Hurewicz and universal coefficient theorems. Therefore  $d_{n-1}(\iota_{n-2}) = 0$  and  $E_2 = E_\infty$ . The algebra structure of  $H^*(V_2; \mathbb{Z})$  is as stated since the squares of the generators lie in zero groups. If  $n$  is odd,

$$\mathbb{Z}/2\mathbb{Z} = \pi_{n-2}(V_2) \cong H_{n-2}(V_2; \mathbb{Z})$$

and

$$H^{n-1}(V_2; \mathbb{Z}) \cong \text{Hom}(H_{n-1}(V_2; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_{n-2}(V_2; \mathbb{Z}), \mathbb{Z}).$$

Since  $\text{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ , we must have  $d_{n-1}(\iota_{n-2}) = \pm 2\iota_{n-1}$ . Then  $E_n = E_\infty$  has the stated form, the class  $\iota_{n-1}$  giving rise to the class  $x_{n-1}$  and the class  $\iota_{n-1} \otimes \iota_{n-2}$  giving rise to the class  $x_{2n-3}$ , while  $x_{n-1}x_{2n-3} = 0$  since this product lies in a zero group.  $\square$

Now we consider mod 2 cohomology. Here separate consideration of cases as in the previous lemma is no longer necessary.

**THEOREM 2.3.** *The cohomology  $H^*(V_q(\mathbb{R}^n); \mathbb{F}_2)$  has a simple system of generators  $\{x_i \mid n - q \leq i < n\}$ .*

PROOF. We proceed by induction on  $q$ , beginning with  $V_1 = S^{n-1}$ . Assume the result for  $V_{q-1}$  and consider the Serre spectral sequence in mod 2 cohomology of the bundle  $V_q \rightarrow V_{q-1}$  with fiber  $S^{n-q}$ . We have  $E_2 = H^*(V_{q-1}; \mathbb{F}_2) \otimes H^*(S^{n-q}; \mathbb{F}_2)$  since  $\pi_1(V_{q-1}) = 0$  if  $q < n$  and since  $V_n \rightarrow V_{n-1}$  is the trivial double cover  $O(n) \rightarrow SO(n)$ . All differentials except  $d_{n-q+1}$  are zero for dimensional reasons. If  $d_{n-q+1}(\iota_{n-q})$  were non-zero, we would have  $H^{n-q}(V_q; \mathbb{F}_2) = 0$ . However,  $H^{n-q}(V_q; \mathbb{F}_2) = \mathbb{F}_2$  by Lemma 2.2 and the Hurewicz and universal coefficient theorems. Therefore  $E_2 = E_\infty$  and the conclusion follows.  $\square$

Here we cannot conclude that the squares of the generators  $x_i$  are zero, since they land in non-zero groups in general. In fact, we shall see later that these squares are non-zero whenever they land in non-zero groups.

**REMARK 2.4.** *Exactly as in Remark 2.1, the  $x_i$  may be chosen consistently as  $q$  and  $n$  vary. To be precise, consider the canonical maps*

$$\iota : V_{q-1}(\mathbb{R}^{n-1}) \rightarrow V_q(\mathbb{R}^n) \quad \text{and} \quad \pi : V_{q+1}(\mathbb{R}^n) \rightarrow V_q(\mathbb{R}^n).$$

*Then  $\iota^*(x_i) = x_i$  for  $i < n-1$ ,  $\pi^*(x_i) = x_i$  for  $i > n-q$ , and  $\pi^*(x_{n-q}) = 0$ . The key point is that  $E_2 = E_\infty$  in the Serre spectral sequence of the fibration  $V_{q-1}(\mathbb{R}^{n-1}) \xrightarrow{\iota} V_q(\mathbb{R}^n) \xrightarrow{\pi} S^{n-1}$ , by dimensional reasons, so that  $\iota^*$  is an isomorphism in degrees less than  $n$ .*

**THEOREM 2.4.** *The cohomology Hopf algebra  $H^*(SO(n); \mathbb{F}_2)$  has a simple system of primitive generators  $\{x_i \mid 1 \leq i < n\}$ , and  $H_*(SO(n); \mathbb{F}_2)$  is an exterior algebra on the dual generators.*

PROOF. We need only check that the  $x_i$  are primitive, and the argument is exactly the same as in the proof of Theorem 2.2.  $\square$

Since  $O(n)$  is the disjoint union of  $SO(n)$  and  $t \cdot SO(n)$  for any element  $t \in O(n)$  of determinant  $-1$  which we can choose to satisfy  $t^2 = 1$ , there is no problem in reading off the homology and cohomology of  $O(n)$  from that of  $SO(n)$ .

LEMMA 2.5. *With coefficients in any commutative ring  $R$ ,*

$$H_*(O(n)) = H_*(SO(n)) \oplus \chi_0 H_*(SO(n))$$

and

$$H^*(O(n)) = H^*(SO(n)) \oplus x_0 H^*(SO(n))$$

where  $\chi_0 \in H_0(SO(n))$  satisfies  $\chi_0^2 = 1$  and  $x_0 \in H^0(O(n))$  satisfies  $x_0(\chi_0) = 1$  and  $x_0^2 = x_0$ .

PROOF. For any space  $X$ ,  $H_0(X)$  is the free  $R$ -module generated by  $\pi_0(X)$ , and we take  $\chi_0$  to be the class of  $t$ . In the cohomology of any space,  $x^2 = x$  if  $x \in H^0(X)$  is dual to an element of  $\pi_0(X) \subset H_0(X)$ . [[Better, if  $x$  is the characteristic function of any set of components.]]  $\square$

We now turn to cohomology away from the prime 2. In the following proofs, we often abbreviate  $H^*(X) = H^*(X; R)$  when the ring  $R$  of coefficients is clear from the context.

THEOREM 2.5. *The integral homology and cohomology of  $V_q(\mathbb{R}^n)$  has no odd torsion. If  $R$  is a commutative ring in which 2 is invertible, then*

$$H^*(V_q(\mathbb{R}^n); R) = E\{x_{4i-1} \mid n - q < 2i < n\} \otimes E\{y_{n-1} \mid n \text{ even}\} \otimes E\{x_{n-q} \mid n - q \text{ even}\}.$$

PROOF. The first statement follows from the second and the universal coefficient theorem: if  $H_*(V_q; \mathbb{Z})$  had any odd torsion, then  $H^*(V_q; \mathbb{Z}[1/2])$  would have odd torsion. The second statement is clear for  $V_1 = S^{n-1}$  and follows from Lemma 2.3 and the universal coefficient theorem for  $V_2$ . For the general case, assume that  $q > 2$  and proceed by induction on  $q$ .

If  $n - q$  is even, then the  $E_2$ -term  $H^*(V_{q-1}) \otimes H^*(S^{n-q})$  of the Serre spectral sequence of the fibration  $S^{n-q} \rightarrow V_q \rightarrow V_{q-1}$  has the form claimed. By a check of dimensions, the only differential on generators that might conceivably be non-zero is  $d_{n-q+1}(t_{n-q})$ , but this lands in the group  $H^{n-q+1}(V_{q-1})$ , which is zero by the induction hypothesis. Thus  $E_2 = E_\infty$ . There is no multiplicative extension problem since the square of  $x_{n-q}$  lies in a zero group.

If  $n - q$  is odd, then the  $E_2$ -term  $H^*(V_{q-2}) \otimes H^*(V_2(\mathbb{R}^{n-q+2}))$  of the Serre spectral sequence of the fibration  $V_2(\mathbb{R}^{n-q+2}) \rightarrow V_q \rightarrow V_{q-2}$  has the form claimed. Since  $H^*(V_2(\mathbb{R}^{n-q+2})) = E\{x_{2n-2q+1}\}$ , the only differential that might conceivably be non-zero is  $d_{2n-2q+2}(x_{2n-2q+1})$ , but this lands in the group  $H^{2n-2q+2}(V_{q-2})$ , which is zero by the induction hypothesis. Thus  $E_2 = E_\infty$ , and the conclusion follows.  $\square$

REMARK 2.6. *Again, the generators  $x_{4i-1}$ ,  $y_{n-1}$  ( $n$  even), and  $x_{n-q}$  ( $n - q$  even) can be chosen consistently as  $q$  and  $n$  vary. However, the proof in this case is more subtle than in the previous cases. To make this precise, consider the canonical maps*

$$\iota : V_{q-1}(\mathbb{R}^{n-1}) \rightarrow V_q(\mathbb{R}^n) \quad \text{and} \quad \pi : V_{q+1}(\mathbb{R}^n) \rightarrow V_q(\mathbb{R}^n).$$

*We claim that generators can be so chosen that*

- (a)  $\iota^*(x_{4n-1}) = x_{4i-1}$  for  $n - q < 2i < n - 1$ ,  $\iota^*(x_{2n-3}) = 0$  if  $n$  is odd,  $\iota^*(y_{n-1}) = 0$  if  $n$  is even, and  $\iota^*(x_{n-q}) = 0$  if  $n - q$  is even.
- (b)  $\pi^*(x_{4i-1}) = x_{4i-1}$  for  $n - q < 2i < n$ ,  $\pi^*(y_{n-1}) = y_{n-1}$  if  $n$  is even, and  $\pi^*(x_{n-q}) = 0$  if  $n - q$  is even.

We assume inductively that such generators have been chosen for  $V_q(\mathbb{R}^m)$  for  $m < n$  and all  $q$ , and we consider the diagram

$$\begin{array}{ccccc}
S^{n-q-1} & \xlongequal{\quad} & S^{n-q-1} & & \\
\downarrow \iota & & \downarrow \iota & & \\
V_q(\mathbb{R}^{n-1}) & \xrightarrow{\quad \iota \quad} & V_{q+1}(\mathbb{R}^n) & \xrightarrow{\quad \pi \quad} & S^{n-1} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \\
V_{q-1}(\mathbb{R}^{n-1}) & \xrightarrow{\quad \iota \quad} & V_q(\mathbb{R}^n) & \xrightarrow{\quad \pi \quad} & S^{n-1}.
\end{array}$$

We prove (a) by studying the Serre spectral sequence of the bottom row in two cases. In each case, (b) can be derived from (a) by inspection of the induced diagram on cohomology and use of the induction hypothesis.

For the first case, assume that  $n$  is even. Here

$$E_2 = E\{\iota_{n-1}\} \otimes (E\{x_{4i-1} \mid n - q < 2i < n\} \otimes E\{x_{n-q} \mid q \text{ even}\})$$

By the calculation of  $H^*(V_q)$  in Theorem 2.5, we must have  $E_2 = E_\infty$ . By the description of  $\iota^*$  and  $\pi^*$  in terms of edge homomorphisms, we can choose  $x_{4i-1}$  and  $x_{n-q}$  in  $H^*(V_q(\mathbb{R}^n))$  that map under  $\iota^*$  to the elements with the same name in  $H^*(V_{q-1}(\mathbb{R}^{n-1}))$ , and we can take  $y_{n-1} = \pi^*(\iota_{n-1})$ . This ensures that  $\iota^*(y_{n-1}) = 0$ .

For the second case, assume that  $n$  is odd. Here

$$E_2 = E\{\iota_{n-1}\} \otimes (E\{x_{4i-1} \mid n - q < 2i < n\} \otimes E\{y_{n-1}\} \otimes E\{x_{n-q} \mid q \text{ odd}\}).$$

The only non-zero differential is  $d_{n-1}$ . Consider the map of fibrations

$$\begin{array}{ccccc}
V_{q-1}(\mathbb{R}^{n-1}) & \xrightarrow{\quad \iota \quad} & V_q(\mathbb{R}^n) & \xrightarrow{\quad \pi \quad} & S^{n-1} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \\
S^{n-2} & \xrightarrow{\quad \iota \quad} & V_2(\mathbb{R}^n) & \xrightarrow{\quad \pi \quad} & S^{n-1}.
\end{array}$$

By the proof of Lemma 2.3,  $d_{n-1}(\iota_{n-2}) = \pm 2\iota_{n-1}$  in the spectral sequence of the bottom row. Since  $\pi^*(\iota_{n-2}) = y_{n-2}$  by the first case, it follows by naturality that  $d_{n-1}(y_{n-2}) = \pm 2\iota_{n-1}$  in the spectral sequence of the top row. If  $n \equiv 1 \pmod{4}$ , we have a second generator  $x_{n-2}$  in  $H^{n-2}(V_q(\mathbb{R}^{n-1}))$ . By dimensional reasons,  $d_{n-1}(x_{n-2}) = r\iota_{n-1}$  for some  $r \in R$ . We agree to replace our previously chosen generator  $x_{n-2}$  by  $x_{n-2} + sy_{n-2}$ , where  $s = \mp 2^{-1}r \in R$ . By the consistency of the  $x_{n-2}$  as  $q$  varies in (b), we have the same constant  $s$  for each  $q$  and so retain the consistency in (b) after the change. We also retain the consistency in (a) since  $\iota^*(y_{n-1}) = 0$ . Now  $d_{n-1}(x_{n-2}) = 0$  and, for any  $q$ ,

$$E_\infty = E\{x_{4i-1} \mid n - q < 2i < n - 1\} \otimes E\{\iota_{n-1} \otimes y_{n-2}\} \otimes E\{x_{n-q} \mid q \text{ odd}\}.$$

We can choose  $x_{4i-1}$  and  $x_{n-q}$  in  $H^*(V_q(\mathbb{R}^n))$  that map under  $\iota^*$  to the elements with the same name in  $H^*(V_{q-1}(\mathbb{R}^{n-1}))$  and can choose  $x_{2n-3}$  in filtration  $n - 1$

that projects to  $\iota_{n-1} \otimes y_{n-2}$  in  $E_\infty$ . By the edge homomorphism description of  $\iota^*$ ,  $\iota^*(x_{2n-3}) = 0$ .

**THEOREM 2.6.** *If  $R$  is a commutative ring in which 2 is invertible, then, as Hopf algebras,*

$$H^*(SO(n); R) = E\{x_{4i-1} \mid 0 < 2i < n\} \otimes E\{y_{n-1} \mid n \text{ even}\}$$

and  $H^*(SO; R) = E\{x_{4i-1} \mid i \geq 1\}$ . In both cases, the corresponding homology Hopf algebra is the exterior algebra on the dual generators.

**PROOF.** We must show that the generators  $x_{4i-1}$  and  $y_{n-1}$  are primitive. We proceed by induction on  $n$ . The result is clear if  $n = 2$  or  $n = 3$ , when there is only one generator. When  $n$  is even,  $y_{n-1}$  is primitive since  $\iota^*(y_{n-1}) = 0$  and  $\iota^*$  is a monomorphism in degrees less than  $n - 1$ . In general,  $x_{4i-1} \in H^*(SO(n))$  comes from  $x_{4i-1} \in H^*(SO(n+1))$  and maps to a primitive element (or zero) in  $H^*(SO(n-1))$ . If we write

$$\psi(x_{4i-1}) = x_{4i-1} \otimes 1 + \sum x' \otimes x'' + 1 \otimes x_{4i-1},$$

then  $\sum x' \otimes x''$  comes from an element of  $H^*(SO(n+1)) \otimes H^*(SO(n+1))$  and maps to zero in  $H^*(SO(n-1)) \otimes H^*(SO(n-1))$ . Since  $\iota : SO(n-1) \rightarrow SO(n+1)$  induces a monomorphism in cohomology in degrees less than  $4i - 1$ ,  $\sum x' \otimes x'' = 0$ .

The homology statement follows essentially as in Theorem 2.2. Since  $R$  is a module over the PID  $\mathbb{Z}[1/2]$ , it suffices by universal coefficients to assume that  $R = \mathbb{Z}[1/2]$ . Since each  $H_q(SO(n); R)$  is a finitely generated  $R$ -module and

$$H^q(SO(n); R) \cong \text{Hom}_R(H_q(SO(n); R), R) \oplus \text{Ext}_R^1(H_{q-1}(SO(n); R), R)$$

is a free  $R$ -module, the Ext term must be zero and each  $H_q(SO(n); R)$  must be a free  $R$ -module. Therefore  $H_*(SO(n); R)$  is dual to  $H^*(SO(n); R)$ . The statements about  $SO$  follow in homology by passage to colimits and in cohomology by dualization.  $\square$



## CHAPTER 3

### Chern classes

Take homology and cohomology with integer coefficients in this chapter. From the Serre spectral sequence of the universal bundle of  $S^1 = U(1)$ , we see that the transgression  $\tau : H^1(S^1) \rightarrow H^2(BU(1))$  is an isomorphism, hence so is the suspension  $\sigma : H^2(BU(1)) \rightarrow H^1(S^1)$ . The *canonical generator* of  $H^2(BU(1))$  is the class that suspends to the canonical generator  $\iota_1 \in H^1(S^1)$ .

Let

$$\begin{aligned} i_n &: U(n-1) \rightarrow U(n), \\ j_n &: SU(n) \rightarrow U(n), \text{ and} \\ p_{i,j} &: U(i) \times U(j) \rightarrow U(i+j) \end{aligned}$$

be the canonical maps. Let  $i_n$ ,  $j_n$ , and  $p_{i,j}$  also denote the induced maps on classifying spaces. We shall prove the following basic result in this chapter.

**THEOREM 3.1.** *There are unique classes  $c_i \in H^{2i}(BU(n))$ , called the Chern classes, which satisfy the following four axioms.*

- (i)  $c_0 = 1$  and  $c_i = 0$  if  $i > n$ .
- (ii)  $c_1 \in H^2(BU(1))$  is the canonical class.
- (iii)  $i_n^*(c_i) = c_i$  (hence  $i_n^*(c_n) = 0$ ).
- (iv)  $p_{i,j}^*(c_k) = \sum_{a+b=k} c_a \otimes c_b$ .

Moreover,  $H^*(BU(n)) = P\{c_1, \dots, c_n\}$ . In  $H^*(BSU(n))$ , define  $c_i = j_n^*(c_i)$ . Then, with  $c_1 = 0$ , (i), (iii), and (iv) again hold and  $H^*(BSU(n)) = P\{c_2, \dots, c_n\}$ .

The uniqueness implies that, proceeding inductively on  $n$ , if there are classes  $d_i$  satisfying (i)-(iv), where the right sides of the formulas in (iii) and (iv) involve Chern classes already known to be uniquely characterized, then  $d_i = c_i$ .

The following is an immediate reinterpretation of the theorem in the language of characteristic classes of complex vector bundles.

**THEOREM 3.2.** *There are unique characteristic classes  $c_i(\xi) \in H^{2i}(B)$  for complex  $n$ -plane bundles  $\xi : E \rightarrow B$  such that*

- (i)  $c_0(\xi) = 1$  and  $c_i(\xi) = 0$  if  $i > n$ .
- (ii)  $c_1(\xi) \in H^2(BU(1))$  is the canonical class when  $\xi$  is the universal line bundle.
- (iii)  $c_i(\eta \oplus \varepsilon) = c_i(\eta)$  if  $\eta$  is an  $(n-1)$ -plane bundle and  $\varepsilon$  is the trivial line bundle.
- (iv)  $c_k(\eta \oplus \xi) = \sum_{a+b=k} c_a(\eta) \otimes c_b(\xi)$  if  $\eta$  is an  $i$ -plane bundle and  $\xi$  is a  $j$ -plane bundle, where  $i+j = n$ .

All characteristic classes of  $n$ -plane bundles  $\xi$  can be expressed uniquely as polynomials in the  $c_i(\xi)$ . If the structure group of  $\xi$  can be reduced to  $SU(n)$ ,  $c_1(\xi) = 0$ .

With the Grassmannian construction of classifying spaces, we have  $BU(1) = \mathbb{C}P^\infty$ . The universal bundle is the evident orbit projection of  $S^\infty$  onto  $\mathbb{C}P^\infty$ . Its restriction over  $\mathbb{C}P^n$  is the Hopf bundle  $\xi_n : S^{2n+1} \rightarrow \mathbb{C}P^n$ . We have defined the Chern classes so that  $c_1(\xi_n)$  is the canonical generator of  $H^2(\mathbb{C}P^n)$ .

Theorem 3.2.(iv) directly implies its analogue for the external direct sum  $\eta \times \xi$  of bundles  $\eta$  over  $X$  and  $\xi$  over  $Y$ , namely

$$c_k(\eta \times \xi) = \sum_{a+b=k} c_a(\eta) \otimes c_b(\xi),$$

where we implicitly use the canonical map  $H^*(X) \otimes H^*(Y) \rightarrow H^*(X \times Y)$ .

The original statement of Theorem 3.2.(iv) can be put in a convenient alternative form by writing  $H^{**}(X)$  for the product over  $q \geq 0$  of the groups  $H^q(X)$ . The elements with 0th term 1 form a group under the product induced by the cup product. We define the *total Chern class*  $c(\xi) \in H^{**}(X)$  of a bundle  $\xi$  over  $X$  to be the class whose even components are the  $c_i(\xi)$  and odd components 0. The equations in 3.2.(iv) then take the pleasing form

$$c(\eta \oplus \xi) = c(\eta)c(\xi).$$

This implies the *Whitney duality theorem* relating the Chern classes of the tangent bundle  $\tau$  and normal bundle  $\nu$  over a complex manifold  $M$  of (complex) dimension  $n$  embedded in  $\mathbb{C}^q$  for some  $q$ . The sum  $\tau \oplus \nu$  is the trivial  $q$ -plane bundle  $\varepsilon$ , and  $c_i(\varepsilon) = 0$  for  $i > 0$ . This follows both from the axioms and from the fact that the classifying map of  $\varepsilon$  is null homotopic. Thus the following result is immediate.

**THEOREM 3.3 (Whitney duality).** *Let  $M$  be a complex manifold with tangent bundle  $\tau$  and normal bundle  $\nu$ . Then*

$$c(\tau)c(\nu) = 1.$$

Writing  $c_i = c_i(\tau)$  and  $\bar{c}_i = c_i(\nu)$ , we can solve for  $\bar{c}_i$  in terms of  $c_i$ . Thus

$$\begin{aligned} \bar{c}_1 &= -c_1, \\ \bar{c}_2 &= -c_2 - c_1\bar{c}_1 = c_1^2 - c_2, \\ \bar{c}_3 &= -c_3 - c_2\bar{c}_1 - c_1\bar{c}_2 = -c_1^3 + 2c_1c_2 - c_3, \end{aligned}$$

and so on. This can be used, for example, to prove non-embedding theorems. If we find that  $\bar{c}_{q+1} \neq 0$ , then  $M$  cannot embed in  $\mathbb{C}^q$ .

Since the classifying space functor converts extensions of groups to fibrations of spaces, we have the fibration sequence

$$BSU(n) \xrightarrow{j^n} BU(n) \xrightarrow{\det} BU(1),$$

where  $\det$  denotes the classifying map of the determinant homomorphism  $\det : U(n) \rightarrow U(1)$ . Since  $BU(1) = K(\mathbb{Z}, 2)$ , we may think of the classifying map  $\det$  as an element of the group  $H^2(BU(n))$ . The map is the identity map when  $n = 1$ , and this means that the element is the canonical class  $c_1$ . For  $n > 1$ ,  $\det^* : H^2(BU(1)) \rightarrow H^2(BU(n))$  is an isomorphism, by Theorem 3.1, and therefore the classifying map  $\det$  may be identified with the first Chern class in general.

Let  $\xi$  be a complex  $n$ -plane bundle over a space  $X$  and also write  $\xi$  for its classifying map  $X \rightarrow BU(n)$ . When  $n = 1$ , this homotopy class of maps may be regarded as an element of  $H^2(X)$ , namely the Chern class  $c_1(\xi)$ . Thus equivalence classes of line bundles are determined by their first Chern classes. When  $n > 1$ , the homotopy class  $\det \circ \xi : X \rightarrow BU(1)$  regarded as an element of  $H^2(X)$  is again

the Chern class  $c_1(\xi)$ , and thus  $c_1(\xi) = 0$  if and only if the composite  $\det \circ \xi$  is null homotopic. This holds if and only if there is a map  $\tilde{\xi} : X \rightarrow BSU(n)$  such that  $j_n \circ \tilde{\xi}$  is homotopic to  $\xi$ . In turn, this holds if and only if the structure group of the bundle  $\xi$  can be reduced to  $SU(n)$ . This gives a bundle theoretic interpretation of  $c_1(\xi)$ .

**COROLLARY 3.1.** *The first Chern class  $c_1(\xi)$  is the obstruction to reducing the structure group of a complex  $n$ -plane bundle  $\xi$  to  $SU(n)$ .*

Let  $\phi_n : T^n \rightarrow U(n)$  be the inclusion of the canonical maximal torus of diagonal matrices in  $U(n)$  and also write  $\phi_n$  for the induced map of classifying spaces  $BT^n \rightarrow BU(n)$ . The Weyl group of  $U(n)$  is the symmetric group  $\Sigma_n$  which we may represent by permutation matrices, These act on  $T^n$  by permuting the entries of diagonal matrices. Since the classifying space functor commutes with products, the Künneth theorem gives an isomorphism

$$H^*(BT^n) \cong H^*(BU(1)) \otimes \cdots \otimes H^*(BU(1)) = P\{x_1, \dots, x_n\},$$

where there are  $n$  copies of  $BU(1)$  and we write  $x_i$  for the element  $c_1$  in the  $i$ th copy. The action of  $\Sigma_n$  on  $T^n$  induces an action on  $BT^n$  and hence on  $H^*(BU(n))$ . The action of  $\sigma \in \Sigma_n$  is the map of polynomial rings given on generators by  $c(x_i) = x_{\sigma(i)}$ . [[PROBABLY IT IS  $c(x_{\sigma(i)}) = x_i$ .]]

By Lemma 1.36, for  $\sigma \in \Sigma_n$  regarded as a map  $BT^n \rightarrow BT^n$ , we have  $\phi_n \circ \sigma \simeq \phi_n$ . Passing to cohomology, this implies that the elements of the image of  $\phi_n^* : H^*(BU(n)) \rightarrow H^*(BT^n)$  are fixed under the action of  $\Sigma_n$ .

We will prove Theorem 3.1 by proving the following theorem.

**THEOREM 3.4.** *The ring homomorphism*

$$\phi_n^* : H^*BU(n) \rightarrow (H^*BT^n)^{\Sigma_n}$$

*is an isomorphism.*

In fact, quite generally, if  $G$  is a compact, connected, Lie group with maximal torus  $T$  and Weyl group  $W$  and if  $H_*(G)$  has no torsion for any prime  $p$  that divides the order of  $W$ , then

$$H^*(BG) \cong H^*(BT)^W.$$

However, we will not pursue such a general approach here. See, e.g. Borel [2] [BOB which Borel?].

Before proving Theorem 3.4, we show how it implies Theorem 3.1. Let us write  $S = H^*(BT^n)^{\Sigma_n}$  for the subring of elements of  $H^*(BT^n)$  fixed under the action of  $\Sigma_n$ . Then  $S$  is the polynomial ring on the  $n$  elementary symmetric functions  $\sigma_i = \sigma_i(x_1, \dots, x_n)$ . Thus

$$\begin{aligned} \sigma_1 &= x_1 + \cdots + x_n, \\ \sigma_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \\ &\vdots \\ \sigma_n &= x_1 \cdots x_n. \end{aligned}$$

It is convenient to set  $\sigma_0 = 1$  and  $\sigma_i = 0$  if  $i > n$ . We define the Chern classes  $c_i \in H^{2i}(BU(n))$  by

$$c_i = (\phi_n^*)^{-1}(\sigma_i).$$

Since  $\phi_n^* : H^*(BU(n)) \rightarrow S$  is an isomorphism, it is obvious that

$$H^*(BU(n)) = P\{c_1, \dots, c_n\}.$$

It is also obvious that  $c_0 = 1$ ,  $c_1$  is the canonical class when  $n = 1$ , and  $c_i = 0$  when  $i > n$ . We have the following commutative diagrams:

$$\begin{array}{ccc} BT^{n-1} & \xrightarrow{\phi_{n-1}} & BU^{n-1} \\ \downarrow h_n & & \downarrow j_n \\ BT^n & \xrightarrow{\phi_n} & BU(n) \end{array} \quad \begin{array}{ccc} BT^i \times BT^j & \xrightarrow{\phi_i \times \phi_j} & BU(i) \times BU(j) \\ \parallel & & \downarrow p_{ij} \\ BT^{i+j} & \xrightarrow{\phi_{i+j}} & BU(i+j). \end{array}$$

Here  $h_n : BT^{n-1} \rightarrow BT^n$  is induced by the inclusion  $T^{n-1} \rightarrow T^n$  of the first  $n-1$  factors. Clearly  $h_n^*(x_i) = x_i$  for  $i < n$  and  $h_n^*(x_n) = 0$ . Therefore  $h_n^*(\sigma_i) = \sigma_i$ . This implies that  $i_n^*(c_i) = c_i$  by the diagram on the left above. Since the  $\phi_k^*$  are all monomorphisms, the diagram on the right shows that  $p_{ij}^*$  is a monomorphism and that

$$(\phi_i^* \otimes \phi_j^*)(p_{ij}^*(c_k)) = \phi_{i+j}^*(c_k) = \sigma_k(x_1, \dots, x_{i+j}).$$

As a matter of algebra, it is easy to verify that

$$\sigma_k(x_1, \dots, x_{i+k}) = \sum_{a+b=k} \sigma_a(x_1, \dots, x_i) \sigma_b(x_{i+1}, \dots, x_{i+k})$$

Therefore

$$(\phi_i^* \otimes \phi_j^*)(p_{ij}^*(c_k)) = \sum_{a+b=k} \phi_i^*(c_a) \phi_j^*(c_b) = (\phi_i^* \otimes \phi_j^*) \left( \sum_{a+b=k} c_a \otimes c_b \right).$$

Since  $\phi_i^* \otimes \phi_j^*$  is a monomorphism, this proves that  $p_{ij}^*(c_k) = \sum_{a+b=k} c_a \otimes c_b$ .

The uniqueness of the  $c_i$  is proven by induction on  $n$ . There is nothing to show if  $n = 1$ . Assume the uniqueness of the  $c_i$  in  $H^*(BU(m))$  for  $m < n$ . Then, for  $i < n$ , the  $c_i$  in  $H^*(BU(n))$  are uniquely determined since  $i_n^*(c_i)$  is prescribed and  $i_n^*$  is an isomorphism in degrees less than  $2n$ . For  $i = n$ ,  $p_{1,n-1}^*(c_n)$  is prescribed and  $p_{1,n-1}^*$  is a monomorphism.

For the statements about  $BSU(n)$  in Theorem 3.1, we will see in Theorem 1.37 below that  $H^*(BSU(n))$  is a polynomial ring on generators of degree  $2i$ ,  $2 \leq i \leq n$ . Certainly  $j_n^*(c_1) = 0$ . Either by naturality from the proof of Theorem 1.37 or by the Serre spectral sequence of the fibration  $BSU(n) \rightarrow BU(n) \rightarrow BU(1)$ , we see that  $j_n^* : H^*(BU(n)) \rightarrow H^*(BSU(n))$  is an epimorphism, and it follows that its restriction to  $P\{c_2, \dots, c_n\}$  is an isomorphism. The commutative diagrams

$$\begin{array}{ccc} BSU(n-1) & \xrightarrow{j_{n-1}} & BU(n-1) \\ \downarrow i_{n-1} & & \downarrow i_n \\ BSU(n) & \xrightarrow{j_n} & BU(n) \end{array} \quad \begin{array}{ccc} BSU(i) \times BSU(l) & \xrightarrow{j_i \times j_l} & BU(i) \times BU(l) \\ \downarrow p_{il} & & \downarrow p_{il} \\ BSU(i+l) & \xrightarrow{j_{i+l}} & BU(i+l) \end{array}$$

show that the  $c_i$  in  $H^*(BSU(n))$  behave the same way under  $i_n^*$  and the  $p_{ij}^*$  as do the  $c_i$  in  $H^*(BU(n))$ . This completes the proof of Theorem 3.1.

We insert a further important property of Chern classes that follows from our way of defining them.

**COROLLARY 3.2.** *If  $\bar{\xi}$  is the conjugate bundle of a complex  $n$ -plane bundle  $\xi$ , then  $c_i(\bar{\xi}) = (-1)^i c_i(\xi)$ .*

PROOF. Let  $c : U(n) \rightarrow U(n)$  be the homomorphism given by complex conjugation of matrix entries. Then  $\bar{\xi}$  is classified by the composite of the classifying map of  $\xi$  and the induced map  $c : BU(n) \rightarrow BU(n)$ . On diagonal matrices,  $c$  restricts to the product of  $n$  copies of the conjugation map  $c : S^1 \rightarrow S^1$ , which has degree  $-1$ . Therefore, in  $H^*(BT^n)$ ,  $c^*(x_i) = -x_i$ . This implies that  $c^*(\sigma_i) = (-1)^i \sigma_i$ , and the conclusion follows from the definition of the Chern classes.  $\square$



## Symplectic classes

In this chapter we will show that  $H^*BSp(n)$  has polynomial generators analogous to the Chern classes in  $H^*BU(n)$ . Let  $i_n : Sp(n-1) \rightarrow Sp(n)$ ,  $\mu_n : Sp(n) \rightarrow SU(2n)$ ,  $\nu_n : U(n) \rightarrow Sp(n)$ , and  $p_{ij} : Sp(i) \times Sp(j) \rightarrow Sp(i+j)$  be the canonical maps and let  $\phi_n : T^n \rightarrow Sp(n)$  be the maximal torus consisting of diagonal matrices with complex entries. Let  $i_n, \mu_n, \nu_n, p_{ij}$ , and  $\phi_n$  also denoted the induced map on classifying spaces. Recall that  $\mu_1 : Sp(1) \rightarrow SU(2)$  is an isomorphism of topological groups.

**THEOREM 4.1.** *There exist unique classes  $k_i \in H^{4i}BSp(n)$ , called the symplectic classes, which satisfy the following axioms.*

- (i)  $k_1 \in H^4BSp(1)$  is  $-c_2$  under the identification  $\mu_1 : Sp(1) \xrightarrow{\cong} SU(2)$ .
- (ii)  $i_n^*(k_i) = k_i$ .
- (iii)  $p_{ij}^*(k_l) = \sum_{a+b=l} k_a \otimes k_b$
- (iv)  $k_0 = 1$  and  $k_i = 0$  for  $i > n$

Furthermore,  $H^*BSp(n) = P\{k_1, \dots, k_n\}$  and

- (v)  $\mu_n^*(c_i) = 0$  if  $i$  is odd, and  $\mu_n^*(c_{2i}) = (-1)^i k_i$
- (vi)  $\nu_n^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$

**PROOF.** We proceed as in the case of  $BU(n)$ . Consider the maximal torus  $\phi_n : T^n \rightarrow Sp(n)$ . By Theorem ??,  $H^*BSp(n) = P\{z_1, \dots, z_n\}$  with  $|z_i| = 4i$ , and  $H^*BT^n = P\{y_1, \dots, y_n\}$  with  $|y_i| = 2$ . Since  $H^*(Sp(n)/T^n)$  is finitely generated, Theorem ?? implies  $\phi_n^* : H^*BSp(n) \rightarrow H^*BT^n$  is a monomorphism. As in the complex case,  $T^n$  is invariant under permutations of coordinates. Hence the image of  $\phi_n^*$  is contained in the symmetric polynomials in  $H^*BT^n$ . Define  $c : Sp(n) \rightarrow Sp(n)$  to be conjugation by the matrix

$$\begin{pmatrix} j & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

Clearly  $c(T^n) = T^n$ . In fact,  $c|_{T^n}$  is complex conjugation of the first factor since  $-j\alpha j = \bar{\alpha}$ , for  $\alpha \in \mathbb{C}$ . Hence  $Bc^* : H^*BT^n \rightarrow H^*BT^n$  is given by  $Bc^*(y_1) = -y_1$ , and  $Bc^*(y_i) = y_i$  for  $i > 1$ . We have shown that  $Bc^* : H^*BSp(n) \rightarrow H^*BSp(n)$  is the identity. Since we can do this for each coordinate it follows that the image of  $\phi_n^*$  in  $H^*BT^n$  is contained in the subgroup of polynomials in  $y_1, \dots, y_n$  invariant under permutations and sign reversals,  $y_i \mapsto -y_i$ . This subring is generated by the

symmetric polynomials in the  $y_i^2$ ,  $\sigma_i(y_1^2, \dots, y_n^2)$ . Thus

$$\phi_n^* : H^* BSp(n) \longrightarrow P\{\sigma_i(y_1^2, \dots, y_n^2)\} \subset H^* BT^n.$$

The generators of  $H^* BSp(n)$  and of  $P\{\sigma_i(y_1^2, \dots, y_n^2)\}$  lie in the same dimensions. Hence we conclude as in Theorem 3.1 that  $\phi_n^*$  is an isomorphism between  $H^* BSp(n)$  and  $P\{\sigma_i(y_1^2, \dots, y_n^2)\}$ . We therefore define

$$k_i = (\phi_n^*)^{-1}(\sigma_i(y_1^2, \dots, y_n^2)).$$

Clearly  $H^* BSp(n) = P\{k_1, \dots, k_n\}$ . Properties (ii)-(iv) and uniqueness follow exactly as in Theorem 3.1. To prove (i), consider the following commutative diagrams ( $\mathbb{H}$  has  $\mathbb{C}$ -basis  $\{i, j\}$  as a right  $\mathbb{C}$ -module)

$$\begin{array}{ccc} T^1 & \xrightarrow{\alpha} & T^2 \\ \phi_1 \downarrow & & \downarrow \phi_2 \\ Sp(1) & \xrightarrow{\mu_1} SU(2) \xrightarrow{j_2} & U(2) \end{array}$$

and the induced diagram on classifying spaces in cohomology

$$\begin{array}{ccccc} H^* BT^1 & \xleftarrow{\alpha^*} & & & H^* BT^2 \\ \phi_1^* \uparrow & & & & \uparrow \phi_2^* \\ H^* BSp(1) & \xleftarrow{\mu_1^*} & H^* BSU(2) & \xleftarrow{j_2^*} & H^* BU(2) \end{array}$$

where  $\alpha(\lambda) = (\lambda, \bar{\lambda})$  so that  $\alpha^*(y_1) = y_1$ ,  $\alpha^*(y_2) = -y_1$ . Clearly  $\alpha^* \phi_2^*(c_2) = \alpha^*(\sigma_2(y_1, y_2)) = \alpha^*(y_1 y_2) = -y_1^2$ . By commutativity,  $\phi_1^* \mu_1^*(c_2) = \phi_1^* \mu_1^* j_2^*(c_2) = -y_1^2$ . Since  $\phi_1^*$  is a monomorphism,  $\mu_1^*(c_2) = -k_1$ , proving (i). Since (i) is the  $n = 1$  case of (v), we prove (v) by induction on  $n$ . First, it is clear that  $\mu_n^*(c_i) = 0$  for odd  $i$  since  $H^* BSp(n)$  is zero in degrees not divisible by 4. Now consider the commutative diagram

$$\begin{array}{ccc} Sp(1) \times Sp(n-1) & \xrightarrow{\mu_1 \times \mu_{n-1}} & SU(2) \times SU(2n-2) \\ p_{1,n-1} \downarrow & & \downarrow p_{2,2n-2} \\ Sp(n) & \xrightarrow{\mu_n} & SU(2n). \end{array}$$

Since  $p_{1,n-1}^*$  is a monomorphism it is sufficient to show that  $p_{1,n-1}^* \mu_n^*(c_{2i}) = p_{1,n-1}^*((-1)^i k_i)$ . This is immediate:

$$\begin{aligned} (\mu_1^* \otimes \mu_{n-1}^*)(p_{2,2n-2}^*(c_{2i})) &= (\mu_1^* \otimes \mu_{n-1}^*)(1 \otimes c_{2i} + c_2 \otimes c_{2i-2}) \\ &= (-1)^i (1 \otimes k_i + k_1 \otimes k_{i-1}), \text{ by inductive hypothesis} \\ &= p_{1,n-1}^*((-1)^i k_i), \end{aligned}$$

proving (v).



We also prove (vi) by induction on  $n$ . To start, we note that  $v_1 = \phi_1 : U(1) = T^1 \rightarrow Sp(1)$  and hence  $v_1^*(k_1) = c_1^2$  as required. Consider the commutative diagram

$$\begin{array}{ccc} U(1) \times U(n-1) & \xrightarrow{\nu_1 \times \nu_{n-1}} & Sp(1) \times Sp(n-1) \\ \downarrow p_{1,n-1} & & \downarrow p_{1,n-1} \\ U(n) & \xrightarrow{\nu_n} & Sp(n). \end{array}$$

As above, it is sufficient to show that  $p_{1,n-1}^* \nu_n^*(k_i) = p_{1,n-1}^* (\sum_{a+b=2i} (-1)^{a+i} c_a c_b)$ . We simply calculate:

$$\begin{aligned} p_{1,n-1}^* \nu_n^*(k_i) &= (\nu_1^* \otimes \nu_{n-1}^*) p_{1,n-1}^*(k_i) = (\nu_1^* \otimes \nu_{n-1}^*)(1 \otimes k_i + k_1 \otimes k_{i-1}) \\ &= 1 \otimes \sum_{a+b=2i} (-1)^{a+i} c_a c_b + c_1^2 \otimes \sum_{a+b=2i-2} (-1)^{a+i-1} c_a c_b \end{aligned}$$

by inductive hypothesis, while

$$\begin{aligned} p_{1,n-1}^* \left( \sum_{a+b=2i} (-1)^{a+i} c_a c_b \right) &= \sum_{a+b=2i} (-1)^{a+i} (1 \otimes c_a c_b + c_1 \otimes c_{a-1} c_b + c_1 \otimes c_a c_{b-1} + c_1^2 \otimes c_{a-1} c_{b-1}) \\ &= \sum_{a+b=2i} (-1)^{a+i} (1 \otimes c_a c_b + c_1^2 \otimes c_{a-1} c_{b-1}) \end{aligned}$$

since the middle terms cancel. This proves (vi), completing the proof.  $\square$

If  $\xi$  is a  $U(n)$  bundle, write  $\xi_{\mathbb{H}}$  for the  $Sp(n)$  bundle given by the inclusion  $U(n) \rightarrow Sp(n)$ . If  $\xi$  is an  $Sp(n)$  bundle, write  $\xi^{\mathbb{C}}$  for the  $SU(2n)$  bundle given by  $Sp(n) \rightarrow SU(2n)$ . If  $\xi$  is an  $Sp(n)$  bundle over  $A$ , we define the *total symplectic class*  $k(\xi)$  of  $\xi$  to be  $k_0(\xi) + k_1(\xi) + k_2(\xi) + \dots \in H^{**}A$ . The following corollary is an immediate consequence of Theorem 4.1.

**COROLLARY 4.2.** *All characteristic classes of  $Sp(n)$  bundles can be expressed as polynomials in the symplectic classes. If  $\xi : E \rightarrow A$  is an  $Sp(n)$  bundle, the symplectic classes  $k_i(\xi) \in H^{4i}(A)$  satisfy:*

- (i) if  $\xi$  is an  $Sp(1) \cong SU(2)$  bundle then  $k_1(\xi) = -c_2(\xi^{\mathbb{C}})$ .
- (ii)  $k_i(\xi) = k_i(\xi \oplus 1)$ , where  $1$  denotes the trivial  $Sp(1)$  bundle over  $A$ .
- (iii)  $k(\xi \oplus \eta) = k(\xi)k(\eta)$ , for any  $Sp$  bundle  $\eta$ .
- (iv)  $k_0(\xi) = 1$  and  $k_i(\xi) = 0$  for  $i > n$ .
- (v)  $c_i(\xi^{\mathbb{C}}) = 0$  if  $i$  is odd and  $c_{2i}(\xi^{\mathbb{C}}) = (-1)^i k_i(\xi)$ .
- (vi) if  $\xi$  is a  $U(n)$  bundle then

$$k_i(\xi_{\mathbb{H}}) = \sum_{a+b=2i} (-1)^{a+i} c_a(\xi) c_b(\xi)$$

We are now able to compute the cohomology of the homogeneous spaces  $U(2n)/Sp(n)$ ,  $SU(2n)/Sp(n)$ , and  $Sp(n)/U(n)$ .

**COROLLARY 4.3.** *As algebras,  $H^*U(2n)/Sp(n) = E\{x_{4i-3} \mid 1 \leq i \leq n\}$  and  $H^*SU(2n)/Sp(n) = E\{x_{4i-3} \mid 2 \leq i \leq n\}$  with  $|x_{4i-3}| = 4i - 3$ . The induced homomorphisms  $H^*U(2n)/Sp(n) \rightarrow H^*U(2n)$  and  $H^*SU(2n)/Sp(n) \rightarrow H^*SU(2n)$  take  $x_{4i-3}$  to  $x_{4i-3}$ .*

PROOF. Consider the Eilenberg-Moore spectral sequence of  $U(2n)/Sp(n) \rightarrow BSp(n) \rightarrow BU(2n)$ . We must calculate  $E_2 = \text{Tor}_{H^*BU(2n)}(\mathbb{Z}, H^*BSp(n))$ . Let  $A$  be  $H^*BU(2n) = P\{c_1, \dots, c_{2n}\}$  and let  $B$  be  $H^*BSp(n) = P\{k_1, \dots, k_n\}$ . Of course  $B$  is an  $A$ -module via  $\mu_n^* : A \rightarrow B$ . Let  $X$  be the Koszul resolution of  $\mathbb{Z}$  over  $A$ ;  $X = E_A\{y_1, y_2, \dots, y_{2n}\}$ , with  $|y_i| = (-1, 2i)$  and differential  $d : X \rightarrow X$  given by  $d(y_i) = c_i$ . Now  $X \otimes_A B = E_B\{y_1, \dots, y_{2n}\}$  with differential  $d(y_i) = \mu_n^*(c_i)$ . Hence we can split  $X \otimes_A B$  as  $X_1 \otimes_A X_2$ , where  $X_1$  is  $E_B\{y_1, y_3, \dots, y_{2n-1}\}$  with zero differential and  $X_2$  is  $E_B\{y_2, y_4, \dots, y_{2n}\}$  with differential  $d(y_{2i}) = (-1)^i k_i$ . Clearly  $X_2$  is the Koszul resolution of  $\mathbb{Z}$  over  $B$ . It follows that  $H(X_1) = X_1$  and  $H(X_2) = \mathbb{Z}$ . Since  $X_1$  is a free  $B$ -module, the Kunneth theorem implies that  $E_2 = H(X_1 \otimes X_2) = H(X_1) \otimes H(X_2) = E_B\{y_{2i-1} \mid 1 \leq i \leq n\} \otimes \mathbb{Z} = E\{y_{2i-1} \mid 1 \leq i \leq n\}$ . Since  $E_2$  lies in the second quadrant, every differential must send each generator to zero. Hence  $E_2 = E_\infty$  and  $H^*U(2n)/Sp(n) = E\{y_{2i-1}\} = E\{x_{4i-3} \mid 1 \leq i \leq n\}$ , where we have renamed the generators so that the subscript will reflect the degree. The second statement is immediate from the map of fibrations

$$\begin{array}{ccccc} U(2n) & \longrightarrow & EU(2n) & \longrightarrow & BU(2n) \\ \downarrow & & \downarrow & & \parallel \\ U(2n)/Sp(n) & \longrightarrow & BSp(n) & \longrightarrow & BU(2n) \end{array}$$

since we have chosen as generators for  $H^*U(2n)$  the suspensions of the Chern classes (Corollary ??). The computation of  $H^*SU(n)/Sp(n)$  is completely analogous.  $\square$

COROLLARY 4.4. *As an algebra,  $H^*(Sp(n)/U(n); R) = P_R\{c_1, \dots, c_n\}/I$  for any coefficient ring  $R$ , where  $I = \text{im}(\nu_n^*)$  is generated by*

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid 1 \leq i \leq n \right\}.$$

*The induced homomorphism  $H^*(BU(n); R) \rightarrow H^*(Sp(n)/U(n); R)$  is the obvious epimorphism.*

PROOF. First take  $R$  to be a field and let all cohomology have coefficients in  $R$ . In the Eilenberg-Moore spectral sequence of  $Sp(n)/U(n) \rightarrow BU(n) \rightarrow BSp(n)$ ,  $E_2 = \text{Tor}_{H^*BSp(n)}(R, H^*BU(n))$ , where  $H^*BU(n)$  is an  $H^*BSp(n)$  module via  $\nu_n^* : H^*BSp(n) \rightarrow H^*BU(n)$ . By Theorem ??,  $H^*BU(n)$  is free over  $H^*BSp(n)$ . Therefore,  $E_2 = R \otimes_{H^*BSp(n)} H^*BU(n)$  and hence  $H^*Sp(n)/U(n) = E_\infty = E_2$ , proving the first statement when  $R$  is a field. Since  $H^*(Sp(n)/U(n); \mathbb{Z}/p\mathbb{Z})$  is concentrated in even degrees, the Bockstein spectral sequence collapses ( $E_1 = E_\infty$ ) for each prime  $p$ . It follows that  $H^*(Sp(n)/U(n); \mathbb{Z})$  is torsion-free and hence  $H^*(Sp(n)/U(n); R) = H^*(Sp(n)/U(n); \mathbb{Z}) \otimes R$  for any ring  $R$  by the universal coefficient theorem. Now take  $R = \mathbb{Z}$ . The natural homomorphism

$$\mathbb{Z} \otimes_{H^*BSp(n)} H^*BU(n) = P\{c_i\} \longrightarrow H^*Sp(n)/U(n)$$

is an isomorphism when tensored with any field, hence must be an isomorphism. This proves the first statement when  $R = \mathbb{Z}$ . The general result follows by universal coefficients since  $H^*Sp(n)/U(n)$  is torsion-free. The second statement is immediate since the induced homomorphism  $H^*BU(n) \rightarrow H^*Sp(n)/U(n)$  is the composite

$$H^*BU(n) \rightarrow \mathbb{Z} \otimes_{H^*BSp(n)} H^*BU(n) \rightarrow H^*Sp(n)/U(n). \quad \square$$

The next two corollaries are analogous to Corollaries ?? and ?? and are proved in exactly the same manner.

COROLLARY 4.5. *As an algebra,*

$$H^*BSp = P\{k_1, k_2, \dots\}.$$

*As coalgebras,*

$$\begin{aligned} H_*BSp(n) &= \Gamma\{\kappa_1, \kappa_2, \dots, \kappa_n\} \\ \text{and } H_*BSp &= \Gamma\{\kappa_1, \kappa_2, \dots\}, \end{aligned}$$

where  $\kappa_i$  is dual to  $k_i$ .

We will show in the Appendix that  $BSp$  is an H-space with product corresponding to the Whitney sum of bundles. As for  $BU$  the induced coproduct on  $H^*BSp$  will be given by  $\psi(k_n) = \sum_{i+j=n} k_i \otimes k_j$ .

COROLLARY 4.6. *As Hopf algebras,*

$$H^*Sp(n) = E\{x_3, x_7, \dots, x_{4n-1}\} \text{ and } H^*Sp = E\{x_3, x_7, \dots\}$$

where  $x_{4i-1} = \sigma^*(k_i)$  in each case. Dually, as Hopf algebras,

$$H_*Sp(n) = E\{a_3, a_7, \dots, a_{4n-1}\} \text{ and } H_*Sp = E\{a_3, a_7, \dots\}$$

where  $\sigma_*(x_{4i-1}) = \kappa_i$  and  $a_i$  is dual to  $x_i$  in each case.

Moreover, the homomorphisms induced in homology and cohomology by the inclusions  $Sp(n-1) \rightarrow Sp(n) \rightarrow Sp$  send each generator to the generators of the same name if is present and to zero otherwise.

COROLLARY 4.7. *As algebras,*

$$\begin{aligned} H^*U/Sp &= E\{x_1, x_5, x_9, \dots\}, \\ H^*SU/Sp &= E\{x_5, x_9, \dots\}, \\ \text{and } H^*Sp/U &= P\{c_1, c_2, \dots\}/I \end{aligned}$$

where  $I = \text{im } \nu^*$  is generated by

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid i \geq 1 \right\}.$$

PROOF. This is immediate by passage to limits since the maps induced in cohomology by  $U(n)/Sp(n) \rightarrow U(n+1)/Sp(n+1)$ ,  $SU(n)/Sp(n) \rightarrow SU(n+1)/Sp(n+1)$ , and  $Sp(n)/U(n) \rightarrow Sp(n+1)/U(n+1)$  are the obvious epimorphisms.  $\square$

We can now compute the homomorphisms induced in cohomology by each of the maps in the bundles

$$\begin{aligned} U &\xrightarrow{\nu} Sp \xrightarrow{q} U \\ Sp/U &\xrightarrow{i} BU \xrightarrow{\nu} BSp \\ Sp &\xrightarrow{\mu} U \xrightarrow{p} U/Sp \\ U/Sp &\xrightarrow{j} BSp \xrightarrow{\mu} BU. \end{aligned}$$

We consider only the infinite case since the results for the finite case then follow trivially.

COROLLARY 4.8. *With the above notation,*

- (i)  $\nu^*(x_{4i-1}) = (-1)^i 2x_{4i-1}$ , and  $q^* = 0$
- (ii)  $i^*(c_i) \equiv c_i \pmod{I}$ , where  $I$  is the ideal defined in Corollary 4.4, and  $\nu^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$
- (iii)  $\mu^*(x_{4i-3}) = 0$ ,  $\mu^*(x_{4i-1}) = (-1)^i x_{4i-1}$ , and  $p^*(x_{4i-3}) = x_{4i-3}$
- (iv)  $j^* = 0$ ,  $\mu^*(c_{2i-1}) = 0$ , and  $\mu^*(c_{2i}) = (-1)^i k_i$ .

PROOF. First, note that the composites  $q\nu$ ,  $\nu i$ , and  $\mu j$  are null homotopic. We have already determined  $i^*(c_i)$ ,  $\nu^*(k_i)$ ,  $p^*(x_{4i-3})$ , and  $\mu^*(c_i)$  in Theorem 4.1 and Corollaries 4.3 and 4.4. By naturality of suspension,

$$\nu^*(x_{4i-3}) = \nu^* \sigma^*(k_i) = \sigma^* \nu^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} \sigma^*(c_a c_b).$$

Since  $\sigma^*$  annihilates products,  $\nu^*(x_{4i-3}) = (-1)^i 2\sigma^*(c_{2i}) = (-1)^i 2x_{4i-1}$ . Thus,  $\nu^* : H^*Sp \rightarrow H^*U$  is a monomorphism. For  $\nu^*q^*$  to be zero, we must have  $q^* = 0$ . Now,  $\mu^*(x_{4i-3}) = \mu^*p^*(x_{4i-3}) = 0$ , while  $\mu^*(x_{4i-1}) = \mu^*\sigma^*(c_{2i}) = \sigma^*((-1)^i k_i) = (-1)^i x_{4i-1}$ . Finally,  $\mu^* : H^*BU \rightarrow B^*BSp$  is an epimorphism and thus  $j^* = 0$ .  $\square$

## Stiefel-Whitney Classes

In this chapter, all cohomology will have  $\mathbb{Z}/2$  coefficients. Let  $i_n : BO(n-1) \rightarrow BO(n)$ ,  $i_n : BSO(n-1) \rightarrow BSO(n)$ ,  $j_n : BSO(n) \rightarrow BO(n)$ ,  $\mu_n : BU(n) \rightarrow BSO(2n)$ ,  $\nu_n : BO(n) \rightarrow BU(n)$ , and  $p_{ij} : BO(i) \times BO(j) \rightarrow BO(i+j)$  be the canonical maps. Recall that  $O(1) = \mathbb{Z}/2\mathbb{Z}$  and hence,  $BO(1) = K(\mathbb{Z}/2\mathbb{Z}, 1) \cong \mathbb{R}P^\infty$ . There is an obvious inclusion of  $(\mathbb{Z}/2\mathbb{Z})^n$  into  $O(1)^n$  as diagonal matrices with entries  $\pm 1$  on the diagonal. This subgroup will play the role of the maximal torus played in the unitary and symplectic groups. Let  $\phi_n : BO(1)^n \rightarrow BO(n)$  be the map induced by the inclusion  $O(1)^n \subset O(n)$ . Denote by  $c_i \in H^{2i}(BU(n); \mathbb{Z}/2\mathbb{Z})$ , the mod 2 reduction of  $c_i \in H^{2i}(BU(n); \mathbb{Z})$ .

**THEOREM 5.1.** *There exist unique classes  $w_i \in H^i BO(n)$ , called the Stiefel-Whitney classes, which satisfy the following axioms.*

- (i)  $w_1 \in H^1 BO(1) = H^1 \mathbb{R}P^\infty$  is the unique non-zero element
- (ii)  $i_n^*(w_i) = w_i$
- (iii)  $p_{ij}^*(w_k) = \sum_{a+b=k} w_a \otimes w_b$
- (iv)  $w_0 = 1$  and  $w_i = 0$  for  $i > n$

Moreover,  $H^* BO(n) = P\{w_1, \dots, w_n\}$  and, if we let  $w_i = j_n^*(w_i) \in H^* BSO(n)$ , then  $H^* BSO(n) = P\{w_2, \dots, w_n\}$  and (with  $w_1 = 0$ ) properties (ii) - (iv) remain valid. Furthermore,

- (v)  $\mu_n^*(w_{2i-1}) = 0$  and  $\mu_n^*(w_{2i}) = c_i$  where  $\mu_n : U(n) \rightarrow O(2n)$  or  $\mu_n : U(n) \rightarrow SO(2n)$
- (vi)  $\nu_n^*(c_i) = w_i^2$  where  $\nu_n : O(n) \rightarrow U(n)$  or  $\nu_n : SO(n) \rightarrow SU(n)$ .

**PROOF.** We will show that  $H^* BO(n)$  is a polynomial algebra  $P\{y_1, \dots, y_n\}$  in several steps. Then we will be able to use the techniques used in the unitary and symplectic case to explicitly define the Stiefel-Whitney classes.

We know that  $H^* SO(n) = E\{x_1, \dots, x_{n-1}\}$  as a coalgebra from Chapter 2. If the  $x_i$  are universally transgressive then Theorem ?? implies that  $H^* BSO(n) = P\{y_2, \dots, y_n\}$  with  $\sigma^* y_i = x_{i-1}$ . To show that the  $x_i$  are universally transgressive, consider the map of fibrations

$$\begin{array}{ccccc} SO(n) & \longrightarrow & ESO(n) & \longrightarrow & BSO(n) \\ \downarrow f_1 & & \downarrow F & & \parallel \\ SO(n)/SO(q) & \longrightarrow & BSO(q) & \longrightarrow & BSO(n). \end{array}$$

Recall that  $H^* SO(n)/SO(q) = E\{x_q, \dots, x_{n-1}\}$  and  $f_1^*(x_i) = x_i$ . In the Serre sequence of the bottom row,  $d_r x_q = 0$  for dimensional reasons where  $r < q + 1$ . Hence, in the Serre sequence of the top row,  $d_r x_q = d_r E_r(f)(x_q) = E_r(f)(d_r x_q) = 0$

for  $r < q + 1$ . Now  $x_q$  must be transgressive,  $d_{q+1}(x_q) \neq 0$ , since  $H^*ESO(n)$  is trivial. Hence  $H^*BSO(n) = P\{y_2, \dots, y_n\}$ .

The Gysin sequence will enable us to finish showing that  $H^*BO(n) = P\{y_1, \dots, y_n\}$ .

LEMMA 5.2. *Let  $S^n \rightarrow E \xrightarrow{p} B$  be a fibration with trivial local coefficients and let  $H^* = H^*(-; R)$  for any commutative ring  $R$ . Then there is an exact sequence (the Gysin sequence)*

$$\dots \longrightarrow H^i E \longrightarrow H^{i-n} B \xrightarrow{\cdot \gamma} H^{i+1} B \xrightarrow{p^*} H^{i+1} E \longrightarrow \dots$$

where  $\cdot \gamma$  denotes multiplication by an element  $\gamma \in H^{n+1} B$ .

PROOF. Consider the Serre spectral sequence of  $p$ . We have

$$E_2^{r,s} = \begin{cases} H^r B & s = 0 \\ H^r B \otimes \langle \iota \rangle & s = n \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $d_{n+1}$  is the only nontrivial differential. Let  $\gamma = d_{n+1}(\iota)$  in  $E_2^{n+1,0} = H^{n+1} B$ . We have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(d_{n+1}) & \longrightarrow & E_2^{i,n} & \xrightarrow{d_{n+1}} & E_2^{i+n+1,0} & \longrightarrow & \text{Cok}(d_{n+1}) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & E_\infty^{i,0} & \longrightarrow & H^i B & \xrightarrow{\cdot \gamma} & H^{n+i+1} B & \longrightarrow & E_\infty^{i+n+1,0} & \longrightarrow & 0 \end{array}$$

and  $0 \rightarrow E_\infty^{i,n} \rightarrow H^i E \rightarrow E_\infty^{i-n,n} \rightarrow 0$ . Since  $p^* : H^i B \rightarrow H^i E$  factors as  $H^i B = E_2^{i,0} \rightarrow E_\infty^{i,0} \subset H^i E$ , splicing these exact sequences together yields the result.  $\square$

Consider the Gysin sequence of  $S^n \rightarrow BSO(n) \rightarrow BSO(n+1)$ . We have

$$H^0 BSO(n+1) \xrightarrow{d_{n+1}(\iota)} H^{n+1} BSO(n+1) \xrightarrow{i_{n+1}^*} H^{n+1} BSO(n).$$

Since  $H^*BSO(n+1) = P\{y_2, \dots, y_{n+1}\}$ ,  $i_{n+1}^*$  has nonzero kernel in degree  $n+1$ . Hence  $d_{n+1}(\iota) \neq 0$  in the Serre sequence of  $S^n \rightarrow BSO(n) \rightarrow BSO(n+1)$ . Now consider the Serre sequence of  $S^n \rightarrow BO(n) \rightarrow BO(n+1)$  (note that the local coefficients are trivial because the only automorphism of  $H^n S^n = \mathbb{Z}/2\mathbb{Z}$  is the identity). From the diagram

$$\begin{array}{ccccc} S^n & \longrightarrow & BSO(n) & \xrightarrow{i_n} & BSO(n+1) \\ \parallel & & \downarrow j_n & & \downarrow j_{n+1} \\ S^n & \longrightarrow & BO(n) & \xrightarrow{i_n} & BO(n+1) \end{array}$$

we observe that  $0 \neq d_{n+1}(\iota) \in H^{n+1} BO(n+1)$  since  $j_{n+1}^* d_{n+1}(\iota) = d_{n+1}(\iota) \neq 0$  in  $H^{n+1} BSO(n+1)$ .

We now have all the preliminary results needed to show  $H^*BO(n) = P\{y_1, \dots, y_n\}$ . Recall that  $BO(1) \cong \mathbb{R}P^\infty$  and hence  $H^*BO(1) = P\{y_1\}$ . Assume for induction that  $H^*BO(n) = P\{y_1, \dots, y_n\}$ . Let  $y_{n+1} = d_{n+1}(\iota) \in H^{n+1} BO(n+1)$ . Since  $y_{n+1} \neq 0$ , exactness of the Gysin sequence implies that  $i_{n+1}^* H^i BO(n+1) \rightarrow H^i BO(n)$  is an isomorphism for  $i \leq n$  and an epimorphism for all  $i$ . This implies that multiplication by  $y_{n+1}$  is a monomorphism  $H^i BO(n+1) \rightarrow H^{i+n+1} BO(n+1)$  for all  $i$ . It follows that  $H^*BO(n+1) = P\{y_1, \dots, y_{n+1}\}$ , completing the induction.

Now all of Theorem 5.1 except (v) and (vi) goes through exactly as in the unitary case with two minor changes. First, replace the maximal torus  $T^n \subset U(n)$  by  $O(1)^n \subset O(n)$ . Second, replace the proof that  $j_n^* : H^*BU(n) \rightarrow H^*SU(n)$  is an epimorphism by the following. Clearly,  $j_1^* : H^*BO(1) \rightarrow H^*BSO(1)$  is an epimorphism. Assume for induction that  $j_n^*$  is an epimorphism and consider the diagram

$$\begin{array}{ccccc} S^n & \longrightarrow & BSO(n) & \xrightarrow{i_n} & BSO(n+1) \\ \parallel & & \downarrow j_n & & \downarrow j_{n+1} \\ S^n & \longrightarrow & BO(n) & \xrightarrow{i_n} & BO(n+1). \end{array}$$

Since  $i_n^*$  is an isomorphism in degrees less than  $n+1$ ,  $y_i \in \text{im}(j_{n+1}^*)$  for  $i \leq n$ . Now in the Serre spectral sequences of the two bundles  $d_{n+1}(\iota_n) = y_{n+1}$ . Hence  $j_{n+1}(y_{n+1}) = j_{n+1}(d_{n+1}(\iota_n)) = d_{n+1}(\iota_n) = y_{n+1}$ . Hence  $j_{n+1}^*$  is an epimorphism.

We prove (v) by induction on  $n$ . When  $n=1$ ,  $\mu_1 : U(1) \rightarrow SO(2)$  is an isomorphism and hence  $\mu_1^*(w_2) = c_1$ . Since  $H^*BU(n)$  lies in even degrees,  $\mu_n^*(w_{2i-1}) = 0$ . Consider the diagram

$$\begin{array}{ccc} S^{2n-1} & \xlongequal{\quad} & S^{2n-1} \\ \downarrow & & \downarrow \\ BU(n-1) & \xrightarrow{i_{2n-1}\mu_{n-1}} & BSO(2n-1) \\ \downarrow i_n & & \downarrow i_{2n} \\ BU(n) & \xrightarrow{\mu_n} & BSO(2n). \end{array}$$

If  $i < n$  then  $\mu_n^*(w_{2i}) = c_i$  by inductive hypothesis, since  $i_n^*$  is an isomorphism in degree  $2i$ . It is clear that  $\iota_{2n-1}$  must transgress to  $c_n$  and  $w_{2n}$  in  $H^*BU(n)$  and  $H^*BSO(2n)$ . By naturality of transgression,  $\mu_n^*(w_{2n}) = c_n$ .

To prove (vi), consider the diagram

$$\begin{array}{ccc} BO(1)^n & \xrightarrow{\nu_1^n} & BU(1)^n \\ \phi_n \downarrow & & \downarrow \phi_n \\ BO(n) & \xrightarrow{\nu_n} & BU(n) \end{array}$$

From the Serre spectral sequence of  $U(1) \cong U(1)/O(1) \rightarrow BO(1) \xrightarrow{\nu_1} BU(1)$  it is clear that  $\nu_1^*$  is a monomorphism. Hence  $\nu_1^*(c_1) = w_1^2$ . If we write  $H^*BO(1)^n$  as  $P\{x_1, \dots, x_n\}$  then  $\phi_n^*\nu_n^*(c_i) = (\nu_1^n)^*\phi_n^*(c_i) = \sigma_i(x_1^2, \dots, x_n^2)$ . With  $\mathbb{Z}/2\mathbb{Z}$  coefficients,  $\sigma_i(x_1^2, \dots, x_n^2) = (\sigma_i(x_1, \dots, x_n))^2$  and hence  $\nu_n^*(c_i) = w_i^2$ . From the diagram

$$\begin{array}{ccc} BSU(n) & \xrightarrow{j_n} & BU(n) \\ \nu_n \uparrow & & \uparrow \nu_n \\ BSO(n) & \xrightarrow{j_n} & BO(n) \end{array}$$

it is clear that  $\nu_n^*(c_i) = w_i^2$  for the special groups also.  $\square$

If  $\xi$  is an  $O(n)$  (resp.  $SO(n)$ ) bundle, let  $\xi_{\mathbb{C}}$  denote its complexification, given by  $\nu_n : O(n) \rightarrow U(n)$ . If  $\xi$  is a  $U(n)$  bundle, let  $\xi^{\mathbb{R}}$  denote the underlying real bundle, given by  $\mu_n : U(n) \rightarrow SO(2n)$ . If  $\xi$  is an  $O(n)$  bundle over  $A$ , define the *total Stiefel-Whitney class*  $w(\xi)$  to be

$$w_0(\xi) + w_1(\xi) + \cdots \in H^{**}(A).$$

Of course we have the following corollary.

**COROLLARY 5.3.** *All  $\mathbb{Z}/2\mathbb{Z}$  characteristic classes of  $O(n)$  and  $SO(n)$  bundles can be expressed as polynomials in the Stiefel-Whitney classes. Let  $\xi$  be an  $O(n)$  bundle over  $A$ . The Stiefel-Whitney classes  $w_i(\xi) \in H^i(A; \mathbb{Z}/2\mathbb{Z})$  satisfy:*

- (i)  $w_1(\gamma) \in H^1 BO(1)$  is the unique nonzero element if  $\gamma$  is the nontrivial  $O(1)$  bundle over  $BO(1)$ .
- (ii)  $w_i(\xi) = w_i(\xi \oplus 1)$ , where  $1$  denotes the trivial line bundle over  $A$ .
- (iii)  $w(\xi \oplus \eta) = w(\xi)w(\eta)$  for any  $O(m)$  bundle  $\eta$  over  $A$ .
- (iv)  $w_0(\xi) = 1$ ,  $w_i(\xi) = 0$  if  $i > n$ , and  $w_1(\xi) = 0$  if and only if  $\xi$  is an  $SO(n)$  bundle.
- (v) if  $\xi$  is a  $U(n)$  bundle, then  $w_{2i-1}(\xi^{\mathbb{R}}) = 0$  and  $w_{2i}(\xi^{\mathbb{R}}) = c_i(\xi)$ .
- (vi)  $c_i(\xi_{\mathbb{C}}) = w_i(\xi)^2$ .

Note that in (v) and (vi) the Chern classes referred to are the mod 2 reductions of the ordinary Chern classes. The last statement in (iv) follows from the fact that

$$BSO(n) \xrightarrow{j_n} BO(n) \xrightarrow{B \det} BO(1)$$

is a fibration.

**COROLLARY 5.4.** *As algebras,  $H^*BO = P\{w_1, w_2, \dots\}$  and  $H^*BSO = P\{w_2, w_3, \dots\}$ . As coalgebras,  $H_*BO(n) = \Gamma\{\omega_1, \omega_2, \dots, \omega_n\}$ ,  $H_*BSO(n) = \Gamma\{\omega_2, \omega_3, \dots, \omega_n\}$ ,  $H_*BO = \Gamma\{\omega_1, \omega_2, \dots\}$ , and  $H_*BSO = \Gamma\{\omega_2, \omega_3, \dots\}$  where  $\omega_i$  is dual to  $w_i$ . The homomorphisms induced in homology and cohomology by the natural inclusions send each generator to the generator of the same name if present and to zero otherwise.*

**PROOF.**  $H_*BO(n)$  and  $H_*BSO(n)$  are as stated by duality. The consistency of the generators follows from the consistency of the Stiefel-Whitney classes in  $H^*BO(n)$  and  $H^*BSO(n)$  under the inclusion maps. The statements about  $H^*BO$ ,  $H^*BSO$ ,  $H_*BO$ , and  $H_*BSO$  now follow by passage to limits.  $\square$

We will show in the Appendix that  $BO$  and  $BSO$  are  $H$ -spaces with multiplication corresponding to Whitney sum. It will then follow from Theorem ??(ii)]thm:4.1 that the coproduct on  $H^*BO$  and  $H^*BSO$  is given by  $\psi(w_n) = \sum w_i \otimes w_{n-i}$ .

In Appendix ??, we will give an alternative description of  $H_*BO$  which is conceptually simpler.

**COROLLARY 5.5.** *As Hopf algebras  $H^*SO(n) = \Delta\{x_1, \dots, x_{n-1}\}$  and  $H^*SO = \Delta\{x_1, x_2, \dots\}$  where  $x_{i-1} = \sigma^*w_i$ . Dually,  $H_*SO(n) = E\{a_1, \dots, a_{n-1}\}$  and  $H_*SO = E\{a_1, a_2, \dots\}$  as algebras, where  $\sigma_*a_{i-1} = \omega_i$  and  $a_i$  is dual to  $x_i$ .*

**PROOF.** Recall that the notation  $\Delta\{x_1, \dots, x_{n-1}\}$  means that  $\{x_1, \dots, x_{n-1}\}$  is a simple system of primitive generators. In the Eilenberg-Moore spectral sequence of the universal bundle  $SO(n) \rightarrow ESO(n) \rightarrow BSO(n)$  we have

$$\mathrm{Tor}_{P\{w_2, \dots, w_n\}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = E\{x_1, \dots, x_{n-1}\}.$$



Since the Stiefel-Whitney classes suspend,  $E_2 = E_\infty$ . However, since we have  $\mathbb{Z}/2\mathbb{Z}$  coefficients, we can only conclude that  $\{x_1, \dots, x_{n-1}\}$  is a simple system of generators. The  $x_i$  are shown to be primitive as in Corollary ???. The computation of  $H^*SO$  is identical. The statements about  $H_*SO(n)$  and  $H_*SO$  follow by duality.  $\square$

We shall complete the determination of  $H^*SO(n)$  as an algebra in the next section.

**PROPOSITION 5.6.** *As an algebra,  $H^*SO(2n)/U(n) = \Delta\{x_2, x_4, \dots, x_{2n-2}\} \subset \Delta\{x_1, \dots, x_{2n-1}\} = H^*SO(2n)$ . The indicated inclusion is the homomorphism induced in cohomology by the natural map  $SO(2n) \rightarrow SO(2n)/U(n)$ .*

**PROOF.** Consider the Eilenberg-Moore spectral sequence of

$$SO(2n)/U(n) \longrightarrow BU(n) \xrightarrow{\mu_n} BSO(2n).$$

The  $E_2$  term is the homology of  $X = E_{H^*BU(n)}\{a_2, a_3, \dots, a_{2n}\}$  with respect to the differential  $d(a_{2i-1}) = 0$  and  $d(a_{2i}) = c_i$  where  $a_i$  has degree  $(-1, i)$ . Now  $X$  splits as the tensor product of  $E_{H^*BU(n)}\{a_3, a_5, \dots, a_{2n-1}\}$  with zero differential and  $E_{H^*BU(n)}\{a_2, a_4, \dots, a_{2n}\}$  with differential  $d(a_{2i}) = c_i$ . We recognize the second factor as the Koszul resolution of  $H^*BU(n)$ . Hence, the homology of  $X$  is  $E_{H^*BU(n)}\{a_3, a_5, \dots, a_{2n-1}\} \otimes_{H^*BU(n)} \mathbb{Z}/2\mathbb{Z} = E\{a_3, a_5, \dots, a_{2n-1}\}$ . Since everything lies in even degrees,  $E_2 = E_\infty$  and thus  $H^*SO(2n)/U(n) = \Delta\{a_3, a_5, \dots, a_{2n-1}\} = \Delta\{x_2, x_4, \dots, x_{2n-2}\}$ , where we have renamed the elements so their subscripts reflect their degrees. The second statement follows by naturality of the Eilenberg-Moore spectral sequence from the diagram

$$\begin{array}{ccccc} SO(2n) & \longrightarrow & ESO(2n) & \longrightarrow & BSO(2n) \\ \downarrow & & \downarrow & & \parallel \\ SO(2n)/U(n) & \longrightarrow & BU(n) & \longrightarrow & BSO(2n). \end{array}$$

**PROPOSITION 5.7.** *As algebras,  $H^*U(n)/O(n) = E\{w_1, \dots, w_n\} = H^*BO(n)/(w_i^2)$  and  $H^*SU(n)/SO(n) = E\{w_2, \dots, w_n\} = H^*BSO(n)/(w_i^2)$ . The maps induced in cohomology by the natural maps  $U(n)/O(n) \rightarrow BO(n)$  and  $SU(n)/SO(n) \rightarrow BSO(n)$  are the obvious epimorphisms.*

**PROOF.** Consider the Eilenberg-Moore spectral sequence of

$$U(n)/O(n) \longrightarrow BO(n) \xrightarrow{\nu_n} BU(n).$$

By Theorem ??,  $H^*BO(n)$  is a free  $H^*BU(n)$  module and hence  $H^*U(n)/O(n) = E_\infty = E_2 = \mathbb{Z}/2\mathbb{Z} \otimes_{H^*BU(n)} H^*BO(n) = H^*BO(n)/(w_i^2) = E\{w_1, \dots, w_n\}$ . Since the homomorphism  $H^*BO(n) \rightarrow H^*O(n)/U(n)$  is the composite

$$H^*BO(n) \rightarrow E_2^{0,*} \rightarrow E_\infty^{0,*} \rightarrow H^*U(n)/O(n),$$

it is clearly the evident epimorphism. The homogeneous space  $SU(n)/SO(n)$  is handled similarly.  $\square$

Note that since  $E_\infty^{p,q} = 0$  for  $p \neq 0$ , we can conclude that  $H^*U(n)/O(n)$  is actually an exterior algebra rather than just an algebra with a simple system of generators. There is then no extension problem in determining  $H^*U(n)/O(n)$  from

$E_\infty$ . Alternatively, we can note that the algebra homomorphism  $H^*BO(n) \rightarrow H^*U(n)/O(n)$  determines the algebra structure of  $H^*U(n)/O(n)$ .

By passage to limits, we obtain the following corollary.

**COROLLARY 5.8.** *As an algebra,  $H^*SO/U = \Delta\{x_2, x_4, \dots\} \subset \Delta\{x_1, x_2, x_3, \dots\} = H^*SO$ , where the homomorphism induced by the natural map  $SO \rightarrow SO/U$  is the indicated inclusion. As algebras,  $H^*U/O = E\{w_1, w_2, \dots\} = H^*BO/(w_i^2)$  and  $H^*SU/SO = E\{w_2, w_3, \dots\} = H^*BSO/(w_i^2)$  where the maps induced in cohomology by the natural maps  $U/O \rightarrow BO$  and  $SU/SO \rightarrow BSO$  are the evident epimorphisms.*

Consider the fibrations

$$SO \xrightarrow{\nu} SU \xrightarrow{q} SU/SO \qquad SU/SO \xrightarrow{i} BSO \xrightarrow{\nu} BSU$$

$$U \xrightarrow{\mu} SO \xrightarrow{p} SO/U \qquad SO/U \xrightarrow{j} BU \xrightarrow{\mu} BSO.$$

We know that  $i^*(w_i) = w_i$  and  $\nu^*(c_i) = w_i^2$ . By naturality of suspension,  $\nu^*(x_{2i-1}) = \nu^*\sigma^*(c_i) = \sigma^*(w_i^2) = 0$ . It can be shown that  $q^* = 0$  also, by showing that  $SU \rightarrow SU/SO \rightarrow BSO$  is equivalent to a fibration and noting that  $i^*$  is an epimorphism. We have shown that  $\mu^*(w_{2i-1}) = 0, \mu(w_{2i}) = c_i$ . Since  $\mu^*$  is an epimorphism,  $j^* = 0$ . By naturality of suspension,  $\mu^*(x_{2i}) = \sigma^*(0) = 0$  and  $\mu^*(x_{2i-1}) = \sigma^*(c_i) = x_{2i-1}$ . Finally, we know that  $p^*(x_{2i}) = x_{2i}$ .

## Steenrod Operations, the Wu formula, and BSpin

We have not yet determined the algebra structure of  $H^*(SO(n); \mathbb{F}_2)$  and  $H^*(SO(2n)/U(n); \mathbb{F}_2)$ . The Steenrod operations will enable us to do this. We will also use them in calculating  $H^*(BSpin; \mathbb{F}_2)$ . They provide important relations between Stiefel-Whitney classes by means of the Wu formula.

Let  $p$  be a prime. The Steenrod operations are natural transformations

$$\begin{aligned}\mathcal{P}^i &: H^*(X; \mathbb{F}_p) \rightarrow H^*(X; \mathbb{F}_p) \quad (i \geq 0) \\ \beta &: H^*(X; \mathbb{F}_p) \rightarrow H^*(X; \mathbb{F}_p)\end{aligned}$$

such that

- (i)  $\mathcal{P}^i$  has degree  $2i(p-1)$  if  $p > 2$ , degree  $i$  if  $p = 2$
- (ii)  $\mathcal{P}^0 = 1$
- (iii)  $\mathcal{P}^i(x) = 0$  if  $p > 2$  and  $|x| < 2i$  or if  $p = 2$  and  $|x| < i$
- (iv)  $\mathcal{P}^i(x) = x^p$  if  $p > 2$  and  $|x| = 2i$  or if  $p = 2$  and  $|x| = i$
- (v)  $\mathcal{P}^i(xy) = \sum \mathcal{P}^j(x)\mathcal{P}^{i-j}(y)$ .

When  $p = 2$ ,  $\mathcal{P}^i$  is denoted  $Sq^i$ . The following lemma will be extremely useful.

**LEMMA 6.1.** *Let  $|x| = 1$ . If  $p > 2$ , then  $\mathcal{P}^i(x) = 0$  for  $i > 0$ . If  $p = 2$ , then  $\mathcal{P}^i(x^k) = (i, k-i)x^{i+k}$ . Let  $|y| = 2$ . If  $p > 2$ , then  $\mathcal{P}^i(y^k) = (i, k-i)y^{k+i(p-1)}$ . If  $p = 2$  and  $\beta(y) = 0$ , then  $\mathcal{P}^{2i}(y^k) = (i, k-i)y^{k+i}$  and  $\mathcal{P}^{2i+1}(y^k) = 0$ .*

Note that for  $i, j \geq 0$ ,  $(i, j)$  denotes the binomial coefficient  $\frac{(i+j)!}{i!j!}$  where  $0! = 1$  as usual. As a convenience, we define  $(i, j)$  to be 0 if  $i < 0$  or  $j < 0$ .

This lemma determines all Steenrod operations in  $BT^n$  and  $BO(1)^n$ , and hence in all classifying spaces of classifying Lie groups since either  $H^*BG \rightarrow H^*BT^n$  or, when  $p = 2$ ,  $H^*BG \rightarrow H^*BO(1)^n$  is a monomorphism for some  $n$ . By suspension, the Steenrod operations in the classical Lie groups are also determined.

The following lemma simplifies the calculation of binomial coefficients mod  $p$ .

**LEMMA 6.2.** *Let  $i = \sum a_k p^k$  and  $j = \sum b_k p^k$ , be the  $p$ -adic expansion of  $i$  and  $j$ . Then  $(i, j) \equiv \prod_k (a_k, b_k) \pmod{p}$  and  $(i, j) \equiv 0 \pmod{p}$  if and only if  $a_k + b_k \geq p$  for some  $k$ .*

**PROOF.** Look at the coefficient of  $x^i y^j$  in  $(x+y)^{i+j} \pmod{p}$ . □

We can now derive the *Wu formula*. In the rest of this chapter, all cohomology will have  $\mathbb{Z}/2\mathbb{Z}$  coefficients.

**THEOREM 6.3.** *The Steenrod operations in  $H^*BO(n)$  are given by*

$$\begin{aligned}Sq^i w_j &= \sum_{t=0}^i (t, j-i-1) w_{i-t} w_{j+t} \\ &= w_i w_j + (1, j-i-1) w_{j-1} w_{j+1} + \cdots + (i, j-i-1) w_{i+j}\end{aligned}$$

for  $0 \leq i < j$ .

REMARK 6.4. *The proof we are about to give applies with minor modifications to show that*

$$\mathrm{Sq}^{2i}c_j = \sum_{t=0}^i (t, j-i-1)c_{i-t}c_{j+t} \text{ in } H^*(BU(n); \mathbb{F}_2) \text{ if } i < j$$

and

$$\mathrm{Sq}^{4i}k_j = \sum_{t=0}^i (t, j-i-1)k_{i-t}k_{j+t} \text{ in } H^*(BSP(n); \mathbb{F}_2) \text{ if } i < j.$$

Clearly  $\mathrm{Sq}^i c_j = 0$  for odd  $i$  and  $\mathrm{Sq}^i k_j = 0$  for  $i \equiv 0 \pmod{4}$ .

PROOF. Since  $\mathrm{Sq}^0 w_1 = w_1$  is the only relevant operation in  $H^*BO(1)$ , the theorem is true when  $n = 1$ . Consider  $p = p_{1,n} : BO(1) \times BO(n) \rightarrow BO(n+1)$  and assume the theorem holds for  $BO(n)$ . Recall that  $p^*$  is a monomorphism and  $p^*(w_j) = 1 \otimes w_j + w_1 \otimes w_{j-1}$ . Now,

$$p^* \mathrm{Sq}^i w_j = \mathrm{Sq}^i p^* w_j = 1 \otimes \mathrm{Sq}^i w_j + w_1 \otimes \mathrm{Sq}^i w_{j-1} + w_1^2 \otimes \mathrm{Sq}^{i-1} w_{j-1}$$

since  $\mathrm{Sq}^1 w_1 = w_1^2$ . On the other hand,

$$\begin{aligned} & p^* \sum_{t=0}^i (t, j-i-1) w_{i-t} w_{j+t} \\ &= \sum_{t=0}^i (t, j-i-1) (1 \otimes w_{i-t} + w_1 \otimes w_{i-t-1}) (1 \otimes w_{j+t} + w_1 \otimes w_{j+t-1}) \\ &= \sum_{t=0}^i (t, j-i-1) (1 \otimes w_{i-t} w_{j+t} + w_1 \otimes (w_{i-t-1} w_{j+t} + w_{i-t} w_{j+t-1}) + w_1^2 \otimes w_{i-t-1} w_{j+t-1}) \\ &= 1 \otimes \mathrm{Sq}^i w_j + w_1 \otimes \sum_{t=0}^i (t, j-i-1) (w_{i-t-1} w_{j+t} + w_{i-t} w_{j+t-1}) + w_1^2 \otimes \mathrm{Sq}^{i-1} w_{j-1}. \end{aligned}$$

Hence the formula will be proved if we show that

$$\mathrm{Sq}^i w_{j-1} = \sum_{t=0}^i (t, j-i-1) (w_{i-t-1} w_{j+t} + w_{i-t} w_{j+t-1}).$$

This sum is

$$\begin{aligned} & \sum_{t=0}^{i-1} (t, j-i-1) w_{i-t-1} w_{j+t} + \sum_{t=0}^i (t, j-i-1) w_{i-t} w_{j+t-1} \\ &= \sum_{t=1}^i (t-1, j-i-1) w_{i-t} w_{j+t-1} + \sum_{t=0}^i (t, j-i-1) w_{i-t} w_{j+t-1}. \end{aligned}$$

Now

$$(t, j-i-2) = (t, j-i-1) - (t-1, j-i-1) \equiv (t, j-i-1) + (t-1, j-i-1) \pmod{2}.$$

Our sum is therefore

$$\sum_{t=0}^i (t, j-i-2) w_{i-t} w_{j+t-1} = \mathrm{Sq}^i w_{j-1}$$

as desired.  $\square$

Clearly the same formula (with  $w_1 = 0$ ) holds in  $H^*BSO(n)$ . We can use it to compute the cohomology algebras of  $SO(n)$  and  $SO(2n)/U(n)$ .

**COROLLARY 6.5.** *In  $H^*SO(n)$ ,  $\text{Sq}^i x_j = (i, j - i)x_{j+i}$  for  $i \leq j$  (where  $x_k = 0$  if  $k > n - 1$ ).*

**PROOF.** Since  $x_j = \sigma^8(w_{j+1})$ ,  $\text{Sq}^i x_j = \sigma^* \text{Sq}^i w_{j+1} = \sigma^* \sum_{t=0}^i (t, j-i)w_{i-t}w_{j+1+t} = (i, j-i)x_{i+j}$ , the last equality holding because  $\sigma^*$  annihilates decomposable elements (see the Appendix).  $\square$

**COROLLARY 6.6.** *In  $H^*SO(n)$ ,  $x_j^2 = x_{2j}$  if  $2j < n$  and  $x_j^2 = 0$  if  $2j \geq n$ . Hence, if  $s_i$  is minimal so that  $2^{s_i}(2i-1) \geq n$ , then  $H^*SO(n) = P\{x_{2i-1} \mid 1 \leq i \leq [n/2]\}/(x_{2i-1}^2)$  as a Hopf algebra and  $H^*SO(2n)/U(n)$  is the subalgebra generated by the elements  $x_{2i-1}^2 = x_{4i-2}$ . Thus  $H^*SO = P\{x_{2i-1} \mid i \geq 1\}$  and  $H^*SO/U = P\{x_{2i-1}^2 = x_{4i-2} \mid i \geq 1\} \subset H^*SO$ .*

**PROOF.** Corollary 6.5 implies  $x_j^2 = \text{Sq}^j(x_j) = x_{2j}$ , where  $x_{2j} = 0$  if  $2j \geq n$ . The rest follows immediately.  $\square$

In order to determine the 2-torsion in  $H^*(SO(n); \mathbb{Z})$  and related spaces we will need to know the Bockstein. Recall that  $\beta = \text{Sq}^1$ .

**COROLLARY 6.7.** *In  $H^*SO(n)$ ,  $\beta(x_{2i-1}) = 0$  and  $\beta(x_{2j-1}) = x_{2j}$ . In  $H^*SO(2n)/U(n)$ ,  $\beta = 0$ . In  $H^*BO(n)$ ,  $\beta(w_j) = w_1 w_j + (j-1)w_{j+1}$  and in  $H^*BSO(n)$ ,  $\beta(w_j) = (j+1)w_{j+1}$ .*

The  $E_2$  term of the mod 2 Bockstein spectral sequence of  $X$  is the homology of  $H^*(X; \mathbb{Z}/2\mathbb{Z})$  with respect to the differential  $\beta$ . We will denote it by  $H^\beta X$ . In all the cases we are interested in, the torsion has order 2. That is,  $E_2 = E_\infty$ .

[BOB check formatting and wording of this prop please]

**PROPOSITION 6.8.** *We compute the following algebras.*

(i) *As algebras*

$$H^\beta SO(2n) = E\{x_{2n-1}, x_{2i-1}x_{2i} + x_{4i-1} \mid 1 \leq i \leq n-1\}$$

$$\text{and } H^\beta SO(2n+1) = E\{x_{2i-1}x_{2i} + x_{4i-1} \mid 1 \leq i \leq n-1\}.$$

(ii) *As an algebra*

$$H^\beta BO(n) = P\{w_{2i}^2 \mid 1 \leq [n/2]\}.$$

(iii) *As algebras*

$$H^\beta BSO(2n) = P\{w_{2n}, w_{2i}^2 \mid 1 \leq i \leq n\}$$

$$\text{and } H^\beta BSO(2n+1) = P\{w_{2i}^2 \mid 1 \leq i \leq n-1\}.$$

(iv) *As algebras*

$$H^\beta U(2n)/O(2n) = E\{w_1, w_{2i}w_{2i+1} \mid 1 \leq i \leq n-1\}$$

$$\text{and } H^\beta U(2n+1)/O(2n+1) = E\{w_1, w_{2i}w_{2i+1} \mid 1 \leq i \leq n\}.$$

(v) As algebras

$$H^\beta SU(2n)/SO(2n) = E\{w_{2n}, w_{2i}w_{2i+1} \mid 1 \leq i \leq n-1\}$$

$$\text{and } H^\beta SU(2n+1)/SO(2n+1) = E\{w_{2i}w_{2i+1} \mid 1 \leq i \leq n\}.$$

(vi) As an algebra

$$H^\beta SO(2n)/U(n) = H^*SO(2n)/U(n),$$

thus  $H^*(SO(2n)/U(n); \mathbb{Z})$  has no 2-torsion.

PROOF. We prove each case separately.

- (i)  $\{x_{2i-1}, x_{2i}, x_{2i-1}x_{2i} + x_{4i-1}\}$  is a basis for a subcomplex of  $H^*SO(n)$  and  $H^*SO(n)$  is the tensor product of these subcomplexes. The result follows immediately.
- (ii)  $H^*BO(2n)$  is the tensor product of subcomplexes  $P\{w_{2i}, w_{2i+1} + w_1w_{2i}\}$ ,  $1 \leq i < n$ , and  $P\{w_1, w_{2n}\}$ .  $H^*BO(2n+1)$  is the tensor product of  $P\{w_{2i}, w_{2i+1} + w_1w_{2i}\}$ ,  $i \leq n$  and  $P\{w_1\}$ . Now  $H^\beta P\{w_{2i}, w_{2i+1} + w_1w_{2i}\} = P\{w_{2i}^2\}$ ,  $H^\beta P\{w_1, w_{2n}\} = P\{w_{2n}^2\}$  and  $H^\beta P\{w_1\} = 0$ . The result follows.
- (iii) The relevant subcomplexes are  $P\{w_{2i}w_{2i+1}\}$  and, for  $H^*BSO(2n)$ ,  $P\{w_{2n}\}$ . The result follows as in (ii).
- (iv) As complexes,

$$H^*U(2n)/O(2n) = E\{w_1, w_{2n}\} \otimes \left( \bigotimes_{i=1}^{n-1} E\{w_{2i}, w_{2i+1} + w_1w_{2i}\} \right)$$

and

$$H^*U(2n+1)/O(2n+1) = E\{w_1\} \otimes \left( \bigotimes_{i=1}^n E\{w_{2i}, w_{2i+1} + w_1w_{2i}\} \right).$$

(v) As complexes,

$$H^*SU(2n)/SO(2n) = \left( \bigotimes_{i=1}^{n-1} E\{w_{2i}, w_{2i+1}\} \right) \otimes E\{w_{2n}\}$$

and

$$H^*SU(2n+1)/SO(2n+1) = \bigotimes_{i=1}^n E\{w_{2i}, w_{2i+1}\}.$$

(vi) We have shown this in Corollary 6.7. □

COROLLARY 6.9. *In all of the above cases  $E_2 = E_\infty$  and hence all torsion is of order 2.*

PROOF. In case (i), this follows from our calculation of  $H^*(SO(n); \mathbb{F}_2)$ . In cases (ii) and (iii),  $E_2$  is concentrated in even degrees and hence  $E_2 = E_\infty$ . Cases (iv) and (v) follow from our calculation of the relevant cohomology algebras away from 2 in Chapter 6. That the only torsion is 2-torsion in cases (ii)-(vi) will also follow from Chapter 6. □

Recall that  $\text{Spin}(n)$  is defined to be the universal covering group of  $SO(n)$  ( $n \geq 3$ ). We therefore have a short exact sequence  $\mathbb{Z}/2\mathbb{Z} \rightarrow \text{Spin}(n) \rightarrow SO(n)$  (and in the limit,  $\mathbb{Z}/2\mathbb{Z} \rightarrow \text{Spin} \rightarrow SO$ ) with  $\mathbb{Z}/2\mathbb{Z}$  normal in  $\text{Spin}(n)$  (or  $\text{Spin}$ ). In the next chapter we will see that  $H^*(\text{Spin}(n); \Lambda) = H^*(B\text{Spin}(n); \Lambda) = H^*(BSO(n); \Lambda)$  if 2 is invertible in  $\Lambda$ . In this chapter we will calculate  $H^*(\text{Spin}; \mathbb{Z}/2\mathbb{Z})$  and  $H^*(B\text{Spin}; \mathbb{Z}/2\mathbb{Z})$ . The cohomology of  $\text{Spin}(n)$  and  $B\text{Spin}(n)$  with  $\mathbb{Z}/2\mathbb{Z}$  coefficients is known but is more difficult to compute.

Before we can calculate  $H^*B\text{Spin}$  we need to define an alternative set of polynomial generators for  $H^*BSO$ . Let  $u_i = w_i$  if  $i \neq 2^i + 1$  and let  $u_2 = w_2$ ,  $u_{2^j i + 1} = \text{Sq}^{2^j} u_{2^j + 1} = \text{Sq}^{2^j} \text{Sq}^{2^{j-1}} \cdots \text{Sq}^2 \text{Sq}^1 u_2$ . By the Wu formula, we have

$$\begin{aligned} u_2 &= w_2 \\ u_3 &= w_3 \\ u_5 &= w_5 + w_2 w_3 \\ &\vdots \end{aligned}$$

It is easy to see by induction that in general

$$u_i = w_i + \text{decomposables.}$$

From this it follows that  $H^*BSO = P\{u_i \mid i \geq 2\}$ . The importance of these classes lies in the fact that the classes  $u_{2^i+1}$  are tied to  $w_2$  by Steenrod operations. Hence if  $f : X \rightarrow BSO$  satisfies  $f^*(w_2) = f^*(u_2)$ , then it must also satisfy  $f^*(u_{2^i+1}) = 0$  for all  $i$ .

[BOB these should stay as  $\mathbb{Z}/2\mathbb{Z}$ 's?] Since we have a short exact sequence  $\mathbb{Z}/2\mathbb{Z} \rightarrow \text{Spin} \rightarrow SO$ , the induced sequence  $B\mathbb{Z}/2\mathbb{Z} = K(\mathbb{Z}/2\mathbb{Z}, 1) \rightarrow B\text{Spin} \rightarrow BSO$  is a fibration (see [BOB ref/cite for this?]). The  $E_2$  term of the Serre spectral sequence is  $E_2 = H^*BSO \otimes H^*B\mathbb{Z}/2\mathbb{Z} = P\{u_i \mid i \geq 2\} \otimes P\{\iota\}$ , where  $\iota \in H^1 B\mathbb{Z}/2\mathbb{Z}$  is the fundamental class. (Recall that  $B\mathbb{Z}/2\mathbb{Z} = K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^\infty$ , so  $H^*B\mathbb{Z}/2\mathbb{Z} = P\{\iota\} = P\{w_1\}$ .) Since  $\text{Spin}$  is simply connected,  $B\text{Spin}$  is 2-connected and hence  $H^1 B\text{Spin} = H^2 B\text{Spin} = 0$ . It follows that  $\iota$  (and  $w_2 = u_2$ ) do not survive to  $E_\infty$ . The only differential which can kill them is  $d_2$ . Hence,  $d_2(\iota) = w_2 = u_2$ . In terms of suspension and transgression,  $\tau(\iota) = u_2$  and  $\sigma^*(u_2) = \iota$ . Now  $\sigma^*$  and its inverse  $\tau$ , commute with Steenrod squares (see the Appendix). Hence

$$\begin{aligned} \tau(\iota^{2^i}) &= \tau(\text{Sq}^{2^{i-1}} \text{Sq}^{2^{i-2}} \cdots \text{Sq}^2 \text{Sq}^1 \iota) \\ &= \text{Sq}^{2^{i-1}} \cdots \text{Sq}^1 u_2 \\ &= u_{2^i+1}. \end{aligned}$$

Since  $\iota^{2^i} \in E_2^{0, 2^i}$ ,  $\tau(\iota^{2^i}) = d_{2^i+1}(\iota^{2^i})$ . This is all we need to compute  $E_\infty$ .

**THEOREM 6.10.**  $H^*B\text{Spin} = P\{u_j \mid j \neq 2^i + 1\}$  and  $H^*\text{Spin} = P\{x_{2^i-1} \mid i \geq 2\}$ . The suspension homomorphism is  $\sigma^*(u_j) = x_{j-1}$  where  $x_{2^i}$  is defined as  $x_{2^i}^2$ . The maps induced in cohomology by the natural maps  $\text{Spin} \rightarrow SO$  and  $B\text{Spin} \rightarrow BSO$  are the evident epimorphisms.

**PROOF.** In the Serre spectral sequence of  $B\mathbb{Z}/2\mathbb{Z} \rightarrow B\text{Spin} \rightarrow BSO$  we know that  $d_{2^i+1}(\iota^{2^i}) = u_{2^i+1}$ . Loosely speaking, this means that in  $E_{2^i+1}$  term,  $u_{2^i+1}$  and all odd powers of  $\iota^{2^i}$  are killed, and in the other terms of the spectral sequences

nothing happens. Thus, in the  $E_\infty$  term we have only  $P\{u_j \mid j \neq 2^i + 1\}$ . We prove this by showing inductively that all terms following  $E_{2^{i-1}+1}$  up to and including  $E_{2^i+1}$  are  $P\{\iota^{2^i}\} \otimes P\{u_j \mid j \neq 2, 3, 5, \dots, 2^{i-1} + 1\}$  and that the differentials are zero except for  $d_{2^i+1}$  which we computed above. This is certainly true for  $i = 0$ . Assuming the hypothesis holds up to  $E_{2^i+1}$ , we have

$$\begin{aligned} d_{2^i+1}(\iota^{n2^i}) &= n\iota^{(n-1)2^i} u_{2^i+1}x \\ &= \begin{cases} 0 & n \text{ even} \\ \iota^{(n-1)2^i} u_{2^i+1}x & n \text{ odd} \end{cases} \end{aligned}$$

Hence,  $\ker(d_{2^i+1}) = P\{\iota^{2^{i+1}}\} \otimes P\{u_j \mid j \neq 2, \dots, 2^{i-1} + 1\}$  while  $\text{im}(d_{2^i+1})$  is the ideal generated by  $u_{2^i+1}$ . Thus,  $E_{2^{i+2}}$  is as claimed. For dimensional reasons, all differentials must be zero until the term  $E_{2^{i+1}+1}$ . This completes the induction.

From the Eilenberg-Moore spectral sequence of  $\text{Spin} \rightarrow \text{ESpin} \rightarrow \text{BSpin}$ , we have  $E_\infty = E\{x_i \mid i \neq 2^j\}$  and hence  $H^*\text{Spin} = \Delta\{x_i \mid i \neq 2^j\}$  with  $\sigma^*u_i = x_{i-1}$ . By naturality of suspension it follows that the homomorphism  $H^*SO \rightarrow H^*\text{Spin}$  sends  $x_i \rightarrow x_i$ . Therefore  $x_{2^i} = x_i^2$  in  $H^*\text{Spin}$  and hence  $H^*\text{Spin} = P\{x_{2^i-1} \mid i \geq 2\}$ .  $\square$

REMARK 6.11. *Note that it would be equivalent to say that  $H^*\text{BSpin} = P\{w_j \mid j \neq 2^i + 1\} = P\{w_4, w_6, w_7, w_8, w_{10}, \dots\}$ . Theorem 6.10 says that a  $\text{Spin}$  bundle (that is, an  $SO$ -bundle whose classifying map can be lifted to  $\text{Spin}$ ) must have  $u_2, u_3, u_5, u_9$ , etc all zero. Of course, if  $u_2(\xi) = 0$  then  $u_{2^i+1}(\xi) = \text{Sq}^{2^{i-1}} \cdots \text{Sq}^1 u_2(\xi) = 0$  also. It is possible to show by a deeper analysis of classifying spaces that  $\text{BSpin}$  is the homotopy theoretic fiber of  $u_2 = w_2 : \text{BSO} \rightarrow K(\mathbb{Z}/2\mathbb{Z}, 2)$ . This gives us the reverse implication: if  $w_2(\xi) = 0$ , then  $\xi$  is a  $\text{Spin}$ -bundle. These remarks apply equally well to  $\text{Spin}(n)$  and  $SO(n)$ .*

We now prove the following lemma which was used above.

[BOB should we put this in front of Lem 5.2?]

LEMMA 6.12. *If  $i = \sum a_k p^k$  and  $j = \sum b_k p^k$ , then  $(i, k) = \prod (a_k, b_k)$ .*

PROOF. Consider the polynomial algebra  $P_{\mathbb{Z}/p\mathbb{Z}}\{x, y\}$  over  $\mathbb{Z}/p\mathbb{Z}$  in  $x$  and  $y$ . The coefficient of  $x^i y^j$  in  $(x + y)^{i+j}$  can be expressed in two ways by the following sequence of identities:

$$\begin{aligned} \sum_{s+t=i+j} (s, t) x^s y^t &= (x + y)^{i+j} \\ &= \prod_k (x + y)^{(a_k + b_k) p^k} \\ &= \prod_k (x^{p^k} + y^{p^k})^{a_k + b_k} \\ &= \prod_k \sum_{s_k + t_k = a_k + b_k} (s_k, t_k) x^{s_k p^k} y^{t_k p^k} \\ &= \sum_{s_k + t_k = a_k + b_k} \left( \prod_k (s_k, t_k) \right) x^{\sum s_k p^k} y^{\sum t_k p^k}, \text{ for each } k. \end{aligned}$$

Comparing coefficients we that  $(i, j) = \prod (a_k, b_k) \pmod p$ .  $\square$

(cHECK OUT LAST PAGE OF NOTES AND INSERT)????????.



## Euler and Pontrjagin classes in rings containing $1/2$

In this chapter, *all cohomology will have coefficients in a ring  $\Lambda$  containing  $1/2$* . The important examples are  $\mathbb{Z}/p\mathbb{Z}$ ,  $p > 2$ ,  $\mathbb{Q}$ , and the universal example  $\mathbb{Z}[1/2]$ . We will define the canonical polynomial generators, the Euler class, and the Pontrjagin classes, for  $H^*(BO(n); \Lambda)$  and  $H^*(BSO(n); \Lambda)$ . In the following chapters, we will see that they are the  $\Lambda$ -reductions of integral Euler and Pontrjagin classes. However, since the presence of 2-torsion in the integral cohomology obscures their formal properties, they are most naturally studied away from the prime 2. Let  $i_n, j_n, p_{ij}, \mu_n$ , and  $\nu_n$  be as in Chapter 4.

**THEOREM 7.1.** *There exist unique classes  $P_i \in H^{4i}BO(n)$  and  $P_i \in H^{4i}BSO(n)$ , called the Pontrjagin classes, and a unique class  $\chi \in H^nBSO(n)$ , called the Euler class, such that*

- (i)  $P_1 = j_2^*(P_1) = c_1^2 \in H^4BU(1) = H^4BSO(2)$  and  $\chi = c_1 \in H^2BU(1) = H^2BSO(2)$
- (ii)  $i_n^*(P_i) = P_i, i_n^*(\chi) = 0$ , and  $j_n^*(P_i) = P_i$
- (iii)  $p_{ij}^*(P_k) = \sum_{a+b=k} P_a \otimes P_b$  and  $p_{ij}^*(\chi) = \chi \otimes \chi$
- (iv)  $P_0 = 1, P_i = 0$  if  $i > [n/2]$ , while  $\chi = 0$  if  $n$  is odd and  $\chi^2 = P_k$  if  $n = 2k$ .

Moreover,  $H^*BO(2n) = H^*BO(2n+1) = H^*BSO(2n+1) = P\{P_1, \dots, P_n\}$  and  $H^*BSO(2n) = P\{P_1, \dots, P_{n-1}, \chi\}$ . In addition,

- (v)  $\nu_n^*(c_{2i+1}) = 0$  and  $\nu_n^*(c_{2i}) = (-1)^i P_i$  where  $\nu_n : BO(n) \rightarrow BU(n)$ ,  $\nu_n : BSO(n) \rightarrow BU(n)$ , or  $\nu_n : BSO(n) \rightarrow BSU(n)$ .
- (vi)  $\mu_n^*(P_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$  where  $\mu_n : BU(n) \rightarrow BO(2n)$  or  $\mu_n : BU(n) \rightarrow BSO(2n)$  and  $\mu_n^*(\chi) = c_n$  for  $\mu_n : BU(n) \rightarrow BSO(2n)$ .

**PROOF.** We will prove the result for  $BSO(n)$  first and use it to obtain the result for  $BO(n)$ . Let  $\phi_{2n+\epsilon} : T^n \rightarrow SO(2n+\epsilon)$ ,  $\epsilon = 0$  or  $1$ , be the maximal torus consisting of matrices of the form

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \\ & & & & 1 \end{pmatrix} \text{ or } \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \\ 0 & & & & 1 \end{pmatrix}$$

in  $SO(2n)$  or  $SO(2n+1)$ , where each  $A_i \in SO(2) = T^1$ . In Corollary ??, we saw that  $H^*SO(2n) = E\{x_3, x_7, \dots, x_{4n-5}, y_{2n-1}\}$  and  $H^*SO(2n+1) = E\{x_3, x_7, \dots, x_{4n-1}\}$ . Hence  $H^*BSO(2n) = P\{z_4, z_8, z_{4n-4}, z'_{2n}\}$  and  $H^*BSO(2n+1) = P\{z_4, z_8, \dots, z_{4n}\}$ .

Since  $H^*BSO(2n + \epsilon)$  is a polynomial algebra on  $n$  generators, it follows as before that  $\phi_{2n+\epsilon}^*$  is a monomorphism. We can permute the factors of  $T^n$  by conjugation by an element of  $SO(2n + \epsilon)$ . Hence  $\text{im } \phi_{2n+\epsilon}^*$  is contained in the symmetric polynomials. If  $\epsilon = 1$ , the matrix

$$\bar{A} = \begin{pmatrix} I & & & & & & & 0 \\ & \ddots & & & & & & \\ & & I & & & & & \\ & & & A & & & & \\ & & & & I & & & \\ & & & & & \ddots & & \\ & & & & & & I & \\ 0 & & & & & & & -1 \end{pmatrix} \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is in  $SO(2n + 1)$ . Conjugation by  $\bar{A}$  has the effect of complex conjugation of a factor of  $T^n$ , which induces negation of the corresponding polynomial generator in  $H^*BT^n$ . Hence,  $\text{im } \phi_{2n+1}^*$  is contained in the subalgebra generated by the  $\sigma_i(y_1^2, \dots, y_n^2) \in H^*BT^n$ . Since  $H^*BSO(2n + 1) = P\{z_4, z_8, \dots, z_{4n}\}$  it follows that  $\phi_{2n+1}^* : H^*BSO(2n + 1) \rightarrow P\{\sigma_i(y_1^2, \dots, y_n^2) \mid i = 1, \dots, n\}$  is an isomorphism. Hence, we define  $P_i$  to be  $(\phi_{2n+1}^*)^{-1}(\sigma_i(y_1^2, \dots, y_n^2))$  and  $\chi$  to be 0. When  $\epsilon = 0$ , the matrix corresponding to complex conjugation by one factor of  $T^n$  is not in  $SO(2n)$ . However, the matrices corresponding to complex conjugation of an even number of factors of  $T^n$  are in  $SO(2n)$ . Hence,  $\text{im } (\phi_{2n}^*)$  is contained in the subalgebra invariant under permutations and an even number of sign changes. This has as generators  $\sigma_i(y_1^2, \dots, y_n^2)$ ,  $1 \leq i \leq n - 1$ , and  $\sigma_n(y_1, \dots, y_n)$ . Since the generators of  $H^*BSO(2n)$  lie in the same degrees,  $\phi_{2n}^*$  is an isomorphism onto this subalgebra. Accordingly, we define  $P_i = (\phi_{2n}^*)^{-1}(\sigma_i(y_1^2, \dots, y_n^2))$  and  $\chi = (\phi_{2n}^*)^{-1}(\sigma_n(y_1, \dots, y_n))$ .

Since  $O(n)/SO(n) = \mathbb{Z}/2\mathbb{Z}$ , we have a 2-fold covering (principal  $\mathbb{Z}/2\mathbb{Z}$ -bundle)  $\mathbb{Z}/2\mathbb{Z} \rightarrow BSO(n) \rightarrow BO(n)$ . We now proceed to calculate  $H^*BO(n)$  from  $H^*BSO(n)$  using this covering. We may as well take any covering  $p : E \rightarrow B$  whose group of covering transformations,  $\pi$ , is finite. As usual we have  $p_* : C_*E \rightarrow C_*B$ , where  $C_*$  denotes the singular chain complex functor. However, since  $p$  is a covering we also have a map  $p_! : C_*B \rightarrow C_*E$  where  $p_!(c)$  is the sum of all the lifts of  $c$  (of course, one has to do this for simplices and extend by linearity). It is easy to see that  $p_!$  commutes with the boundary operator. The composite  $p_*p_! : C_*B \rightarrow C_*B$  is multiplication by  $|\pi|$ , the order of  $\pi$  and the composite  $p_!p_*$  takes  $x$  to  $\sum_{g \in \pi} gx$ . Letting  $p_!$  also denote the map induced in cohomology we have the relations  $p_!p_* = |\pi| : H^*B \rightarrow H^*B$  and  $p_*p_! = \sum_{g \in \pi} g : H^*E \rightarrow H^*E$ . The homomorphism  $p_! : H^*E \rightarrow H^*B$  is called the *transfer* homomorphism.

**LEMMA 7.2.** *If  $|\pi|$  is invertible in  $\Lambda$  then  $p^*H^*(B; \Lambda) \rightarrow H^*(E; \Lambda)^\pi$  is an isomorphism (where  $H^*(E; \Lambda)^\pi$  denotes the elements fixed by  $\pi$ ).*

**PROOF.** Multiplication by  $|\pi|$  is monic so  $p^*$  is monic. If  $x \in (H^*E)^\pi$  then  $p_*p_!(\frac{1}{|\pi|}x) = x$  so  $p^*$  maps onto  $(H^*E)^\pi$ . Now  $|\pi|p^*(x) = p_*p_!p^*(x) = \sum gp^*(x)$ . Therefore  $p^*(x) = \frac{1}{|\pi|} \sum_{g \in \pi} gp^*(x)$ , which is obviously invariant under  $\pi$ . That is,  $p^*(H^*B) \subset (H^*E)^\pi$ .  $\square$



We wish to compare the maximal tori of  $SO(2n)$  and  $U(2n)$  under the map  $\nu_{2n}$ . However, this is not possible using the standard maximal tori. Hence, we use a different maximal torus for  $U(2n)$ . This presents no problems because all maximal tori induce the same homomorphism in cohomology. To see this we need only recall that any two maximal tori are conjugate and that conjugation

$$\begin{array}{ccc} & G & H^*BG \\ \phi_1 \nearrow & \downarrow g(\cdot)g^{-1} & \downarrow 1 \\ T & \xrightarrow{\phi_2} G & H^*BG \xrightarrow{B\phi_1^*} H^*BT \\ & & \searrow B\phi_2^* \end{array}$$

by  $g \in G$  induces the identity map  $H^*BG \rightarrow H^*BG$ . The maximal torus we need is obtained from the standard one by conjugating with

$$\tilde{A} = \begin{pmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix}$$

where  $A = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \in U(2)$ . Since  $\nu_2\phi_2 : T^1 \rightarrow U(2)$  and  $\overline{\phi_2} = A\phi_2A^{-1} : T^2 \rightarrow U(2)$  are given by

$$\begin{aligned} \nu_2\phi_2(\alpha + i\beta) &= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \\ \text{and } \overline{\phi_2}(\lambda_1, \lambda_2) &= 1/2 \begin{pmatrix} \lambda_1 + \lambda_2 & i(\lambda_2 - \lambda_1) \\ -i(\lambda_2 - \lambda_1) & \lambda_1 + \lambda_2 \end{pmatrix} \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccc} T^n & \xrightarrow{\gamma^n} & (T^2)^n \\ \phi_2^n \downarrow & & \downarrow \overline{\phi_2}^n \\ SO(2)^n & \xrightarrow{\nu_2^n} & U(2)^n \\ \rho \downarrow & & \downarrow q \\ SO(2n) & \xrightarrow{\nu_{2n}} & U(2n) \end{array}$$

where  $\gamma(c) = (c, \bar{c})$ . Let  $H^*BT^n = P\{x_1, \dots, x_n\}$  and  $H^*BT^{2n} = P\{x_1, y_1, \dots, x_n, y_n\}$  so that  $(\gamma^n)^*(x_i) = x_i$  and  $(\gamma^n)^*(y_i) = -x_i$ . Then

$$\begin{aligned} \phi_{2n}^*\nu_{2n}^*(c_i) &= (\gamma^n)^*\overline{\phi_{2n}^*}(c_i) \\ &= (\gamma^n)^*(\sigma_i(x_1, y_1, \dots, x_n, y_n)) \\ &= \sigma_i(x_1, -x_1, \dots, x_n, -x_n). \end{aligned}$$

Using the relation

$$\sigma_i(x_1, -x_1, \dots, x_n, -x_n) = \sigma_i(x_2, -x_2, \dots, x_n, -x_n) - x_1^2\sigma_{i-2}(x_2, -x_2, \dots, x_n, -x_n)$$

it is easy to see that this is 0 if  $i$  is odd and is  $(-1)^k\sigma_k(x_1^2, \dots, x_n^2)$  if  $i = 2k$ . This proves (v).

Again for (vi) we need only consider  $\mu_n : U(n) \rightarrow SO(2n)$ . We have a commutative diagram

$$\begin{array}{ccc}
 & Sp(n) & \\
 \phi_n \nearrow & & \nwarrow \nu_n \\
 T^n & \xrightarrow{\phi_n} & U(n) \\
 \phi_{2n} \searrow & & \swarrow \mu_n \\
 & SO(2n) & 
 \end{array}$$

Now  $\mu_n^* P_i$  is the unique element which is mapped to  $\phi_{2n}^*(P_i) = \sigma_i(y_1^2, \dots, y_n^2)$ . Hence  $\mu_n^*(P_i) = \nu_n^*(k_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$ . Finally  $\phi_{2n}^*(\chi) = \sigma_n(y_1, \dots, y_n) = \phi_n^*(c_n)$  and therefore  $\mu_n^*(\chi) = c_n$ . This completes the proof of Theorem 7.1.  $\square$

If  $\xi$  is an  $O(n)$  bundle, let  $\xi_{\mathbb{C}}$  denote the  $U(n)$  bundle induced by  $\nu_n : O(n) \rightarrow U(n)$ . If  $\xi$  is a  $U(n)$  bundle, let  $\xi^{\mathbb{R}}$  denote the  $SO(2n)$  bundle induced by  $\mu_n : U(n) \rightarrow SO(2n)$ . Define the *total Pontrjagin class*  $P(\xi)$  of an  $O(n)$  bundle  $\xi$  to be  $1 + P_1(\xi) + P_2(\xi) + \dots$ . Recall that  $\Lambda$  is a ring in which 2 is invertible.

**COROLLARY 7.3.** *All  $\Lambda$ -characteristic classes of  $O(n)$  bundles can be expressed as polynomials in the Pontrjagin classes. All  $\Lambda$ -characteristic classes of  $SO(n)$  bundles can be expressed as polynomials in the Pontrjagin classes and the Euler class. These classes satisfy:*

- (i) *if  $\xi$  is an  $SO(2) = U(1)$  bundle, then  $P_1(\xi) = (c_1(\xi_{\mathbb{C}}))^2$  and  $\chi(\xi) = c_1(\xi_{\mathbb{C}})$ . If  $\xi'$  denotes  $\xi$  regarded as an  $O(2)$  bundle then  $P_i(\xi') = P_i(\xi)$ .*
- (ii)  *$P_i(\xi \oplus 1) = P_i(\xi)$ ,  $\chi(\xi \oplus 1) = 0$ , and  $P_i(\xi') = P_i(\xi)$  if  $\xi$  is an  $SO(n)$  bundle,  $\xi'$  is  $\xi$  regarded as an  $O(n)$  bundle and 1 denotes the trivial  $SO(1)$  or  $O(1)$  bundle.*
- (iii)  *$P(\xi \oplus \eta) = P(\xi)P(\eta)$  if  $\xi$  and  $\eta$  are  $O(n)$  and  $O(m)$  bundles (or  $SO(n)$  and  $SO(m)$  bundles).  $\chi(\xi \oplus \eta) = \chi(\xi)\chi(\eta)$  if  $\xi$  and  $\eta$  are  $SO(n)$  and  $SO(m)$  bundles.*
- (iv)  *$P_0(\xi) = 1$  and  $P_i(\xi) = 0$  if  $i > [n/2]$  and  $\xi$  is an  $SO(n)$  or  $O(n)$  bundle.  $\chi(\xi) = 0$  if  $\xi$  is an  $SO(2n+1)$  bundle and  $\chi(\xi)^2 = P_n(\xi)$  if  $\xi$  is an  $SO(2n)$  bundle.*
- (v)  *$c_{2i+1}(\xi_{\mathbb{C}}) = 0$  and  $c_{2i}(\xi_{\mathbb{C}}) = (-1)^i P_i(\xi)$  if  $\xi$  is an  $O(n)$  bundle or  $SO(n)$  bundle.*
- (vi)  *$P_i(\xi^{\mathbb{R}}) = \sum_{a+b=2i} (-1)^{a+i} c_a(\xi) c_b(\xi)$  and  $\chi(\xi^{\mathbb{R}}) = c_n(\xi)$  if  $\xi$  is a  $U(n)$  bundle.*

**REMARK 7.4.** *By (v) we could have defined  $P_i(\xi)$  as  $(-1)^i c_{2i}(\xi_{\mathbb{C}})$ . This is often done. Many properties of the  $P_i$  follow directly from the analogous properties of the  $c_i$ .*

*The formula for  $P_i(\xi^{\mathbb{R}})$  in (vi) can be written*

$$\begin{aligned}
 1 - P_1(\xi^{\mathbb{R}}) + P_2(\xi^{\mathbb{R}}) - \dots &= (1 + c_1(\xi) + c_2(\xi) + \dots)(1 - c_1(\xi) + c_2(\xi) - \dots) \\
 &= c(\xi)c(\bar{\xi}),
 \end{aligned}$$

*since this product has non zero components only in degrees  $4i$ .*

From (iii) and (iv) it follows that if  $\chi(\xi) \neq 0$  (in  $\Lambda$ -cohomology) then  $\xi$  cannot split as the Whitney sum of two odd dimensional bundles.

Since  $c_{2i+1}(\xi_{\mathbb{C}}) = 0$  in  $\Lambda$ -cohomology by (v), it follows that  $c_{2i+1}(\xi_{\mathbb{C}})$  has order a power of two in integral cohomology (take  $\Lambda = \mathbb{Z}[1/2]$  to see this). We will improve this result in Chapter 8.

COROLLARY 7.5. *With coefficients in which  $2^{-1}$  exists, we have the following.*

(i) *As Hopf algebras,*

$$H^*SO(2n) = E\{x_3, x_7, \dots, x_{4n-5}\} \otimes E\{y_{2n-1}\},$$

$$H^*SO(2n+1) = E\{x_3, x_7, \dots, x_{4n-1}\},$$

and  $H^*SO = E\{x_{4i-1} \mid i \geq 1\}$  where  $x_{4i-1} = \sigma^*(P_i)$  and  $y_{2n-1} = \sigma^*(\chi)$ .

(ii) *As algebras*

$$H^*BSO = H^*BO = P\{P_i \mid i \geq 1\}.$$

(iii) *As algebras*

$$H^*U(n)/O(n) = E\{x_{4i-3} \mid 1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor\},$$

$$H^*SU(n)/SO(n) = E\{x_{4i-3} \mid 2 \leq i \leq n/2\} \otimes P\{\chi\}/(\chi^2) \text{ (n even),}$$

and  $H^*SU(n)/SO(n) = E\{x_{4i-3} \mid 2 \leq i \leq n/2\}$  (n odd).

The natural maps  $U(n) \rightarrow U(n)/O(n)$  and  $SU(n) \rightarrow SU(n)/SO(n)$  send  $x_{4i-3}$  to itself and the natural map  $SU(n)/SO(n) \rightarrow BSO(n)$  sends  $\chi$  to itself.

(iv) *As algebras,*

$$H^*U/O = E\{x_{4i-3} \mid i \geq 1\},$$

$$\text{and } H^*SU/SO = E\{x_{4i-3} \mid i \geq 2\}.$$

Again the natural maps  $U \rightarrow U/O$  and  $SU \rightarrow SU/Sp$  send  $x_{4i-3}$  to itself.

(v) *As an algebra,*

$$H^*SO(2n)/U(n) = P\{c_1, \dots, c_n\}/I$$

where  $I$  is the ideal generated by  $c_n$  and

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid 1 \leq i \leq n-1 \right\}.$$

The natural map  $SO(2n)/U(n) \rightarrow BU(n)$  sends  $c_i$  to  $c_i \pmod I$ .

(vi) *As an algebra*

$$H^*SO/U = P\{c_i \mid i \geq 1\}/I$$

where  $I$  is the ideal generated by

$$\left\{ \sum_{a+b=2i} (-1)^{a+i} c_a c_b \mid 1 \leq i \leq n-1 \right\}$$

and the natural map  $SO/U \rightarrow BU$  sends  $c_i$  to  $c_i \pmod I$ .

Further, none of the above spaces has  $p$ -torsion for odd primes  $p$ .

PROOF. (i) and (ii) follow in exactly the same manner as the analogous results in previous chapters. Once we have shown (iii) and (v), (ii) and (vi) will also follow in standard fashion. Having shown (i) through (vi) it will follow that none of these spaces have  $p$ -torsion for odd primes  $p$  since we may take  $\Lambda = \mathbb{Z}[1/2]$  and  $p$ -torsion in integral cohomology would have  $p$ -torsion in  $\Lambda$ -cohomology. Recall that we have shown in Corollary 4.4 that  $P\{c_i\}/I$  is torsion-free.

To prove (iii), consider the Eilenberg-Moore spectral sequence of  $U(n)/O(n) \rightarrow BO(n) \rightarrow BU(n)$ . The  $E_2$  term is the homology of  $E_{P\{c_i\}}\{a_i\} \otimes H^*BO(n)$  where  $1 \leq i \leq n$ ,  $d(a_i) = \nu_n^*(c_i)$ , and  $|a_i| = (-1, 2i)$ . Hence,  $E_2 = E_\infty$  and the result follows. Now in the Eilenberg-Moore spectral sequence of  $SU(n)/SO(n) \rightarrow BSO(n) \rightarrow BSU(n)$  for odd  $n$ , the  $E_2$  term is the same except that  $x_1$  is not present. Hence the result follows as above. If  $n$  is even, however, we have  $E_2 = E\{x_{4i-3} \mid 2 \leq i \leq n/2\} \otimes P\{\chi\}/(\chi^2)$  since  $\chi^2 = P_{n/2}$  is in the image of  $\nu_n^*$  while  $\chi$  is not. Clearly  $E_2 = E_\infty$  since the generators lie in  $E_2^{0,*}$  and  $E_2^{-1,*}$ . Now  $E_\infty$  is not free so our previous results do not allow us to conclude that  $H^* = H^*SU(n)/SO(n) \cong E_\infty$ . However,  $E_\infty$  is close enough to being free. Let  $F_0 \subset F_{-1} \subset F_{-2} \subset \dots$  be the filtration of  $H^*$  which gives  $E_\infty : E_\infty^{p,q} = F_p H^{p+q}/F_{p+1} H^{p+q}$ . Since  $E\{x_{4i-3} \mid 2 \leq i \leq n/2\}$  is free there is a homomorphism  $E\{x_{4i-3}\} \rightarrow H^*$  which projects to the obvious homomorphism  $E\{x_{4i-3}\} \rightarrow E_\infty$ . This and the inclusion  $P\{\chi\}/(\chi^2) = F_0 \subset H^*$  define a homomorphism  $f : E\{x_{4i-3}\} \otimes P\{\chi\}/(\chi^2) \rightarrow H^*$  which is filtration preserving if the domain is given the obvious filtration. Clearly,  $f$  induces an isomorphism of associated graded algebras, hence must be an isomorphism. Therefore,  $H^*SU(n)/SO(n)$  is as claimed. In general this shows that ‘deviation from freeness’ when confined to the ‘bottom’ filtration does not prevent the conclusion  $A \cong E^0 A$ .

To prove (v) consider the Eilenberg-Moore spectral sequence of  $SO(2n)/U(n) \rightarrow BU(n) \rightarrow BSO(2n)$ , where the last map is  $\mu_n$ . As in Corollary 4.4 we conclude that  $H^*BU(n)/\text{im}(\mu_n^*) \rightarrow H^*SO(2n)/U(n)$  is an isomorphism. It is clear that  $\text{im}(\mu_n^*) = I$  is as stated.  $\square$

Note that once we have shown that  $BO$  and  $BSO$  are H-spaces whose product is given by the Whitney sum map it will follow from Theorem 7.1 that the coproduct on  $H^*BO = H^*BSO$  is given by  $\psi(P_n) = \sum_{i+j=n} P_i \otimes P_j$ .

We now calculate  $H^*B\text{Spin}(n)$  and  $H^*\text{Spin}(n)$  (with  $\Lambda$  coefficients,  $1/2 \in \Lambda$ ). We have the fibration

$$B\mathbb{Z}/2\mathbb{Z} \rightarrow B\text{Spin}(n) \rightarrow BSO(n).$$

Since  $O(1) = \mathbb{Z}/2\mathbb{Z}$  we know from Theorem 7.1 that  $H^*B\mathbb{Z}/2\mathbb{Z} = H^*BO(1) = H^0BO(1) = \Lambda$ . Hence in the Serre sequence of the above fibration  $E_2 = E_2^{*,0} = H^*BSO(n)$ . It follows immediately that  $E_2 = E_\infty = H^*B\text{Spin}(n)$ . This establishes the first part of the next result.

PROPOSITION 7.6. *The natural maps  $B\text{Spin}(n) \rightarrow BSO(n)$  and  $\text{Spin}(n) \rightarrow SO(n)$ , and their limits  $B\text{Spin} \rightarrow BSO$  and  $\text{Spin} \rightarrow SO$  induce isomorphisms in  $\Lambda$ -cohomology.*

PROOF. We have just shown that  $B\text{Spin}(n) \rightarrow BSO(n)$  is a  $\Lambda$ -cohomology isomorphism. By the map of fibrations

$$\begin{array}{ccccc} \text{Spin}(n) & \longrightarrow & E\text{Spin}(n) & \longrightarrow & B\text{Spin}(n) \\ \downarrow & & \downarrow & & \downarrow \\ SO(n) & \longrightarrow & ESO(n) & \longrightarrow & BSO(n) \end{array}$$

and the naturality of the Eilenberg-Moore spectral sequence, it follows that  $\text{Spin}(n) \rightarrow SO(n)$  is also an isomorphism in  $\Lambda$ -cohomology. The statement about the limit spaces is an obvious consequence.  $\square$



## The Thom Isomorphism

We define Thom spaces and orientations, then prove the Thom Isomorphism Theorem. After a discussion of orientations and orientability, we show that the Stiefel-Whitney and Euler classes of a bundle can be defined in terms intrinsic to the bundle. We then define the homology tangent bundle of a topological manifold, define orientations in this context, and give a version of the Thom Isomorphism Theorem for topological manifolds. We then prove the topological invariances of the Stiefel-Whitney and Euler classes of a manifold by showing that the tangent bundle of a smooth manifold is equivalent to its homology tangent bundle.

Let  $H^*$  denote cohomology with coefficients in a ring  $\Lambda$  (commutative with unit). Given a principal  $O(n)$ -bundle  $\xi : E \rightarrow B$  we have the associated vector bundle

$$\mathbb{R}^n \rightarrow E' = E \times_{O(n)} \mathbb{R}^n \rightarrow B$$

and the associated disk and sphere bundles

$$D^n \rightarrow D_\xi = E \times_{O(n)} D^n \rightarrow B$$

and

$$S^{n-1} \rightarrow S_\xi = E \times_{O(n)} S^{n-1} \rightarrow B.$$

If we let  $E'_0$  denote the complement of the zero cross-section in  $E'$  then we have an obvious map of relative fibrations

$$\begin{array}{ccccc} (\mathbb{R}^n, \mathbb{R}^n - 0) & \longrightarrow & (E', E'_0) & \longrightarrow & B \\ \uparrow & & \uparrow & & \parallel \\ (D^n, S^{n-1}) & \longrightarrow & (D_\xi, S_\xi) & \longrightarrow & B \end{array}$$

which is easily seen to be a homotopy equivalence.

**DEFINITION 8.1.** *The quotient space  $D_\xi/S_\xi$  (collapse  $S_\xi$  to a point) is called the Thom space of  $\xi$  and is customarily denoted  $M(\xi)$  or  $T(\xi)$ . We shall use  $T(\xi)$  or  $T_\xi$ .*

The Thom space of the universal  $G$ -bundle is often denoted  $TG$  or  $MG$ , e.g.,  $TU(n)$ ,  $MU(n)$ ,  $TU$ ,  $MU$ . These universal Thom spaces are the representing spaces for bordism and cobordism just as  $BU$ ,  $BO$ , etc. are representing spaces for complex or real  $K$ -theory.

Observe that  $T(\xi)$  is homotopy equivalent to the mapping cone of  $S_\xi \rightarrow B$ . Clearly the Thom space is a functorial construction: a map of fibrations  $f : (D_\xi, S_\xi) \rightarrow (D_{\xi'}, S_{\xi'})$  induces a map  $\bar{f} : T(\xi) \rightarrow T(\xi')$ . Fiber homotopic maps  $F_1$  and  $F_2$  induce homotopic maps  $\bar{F}_1$  and  $\bar{F}_2$  (this follows directly from the fact that  $T(\xi)$  may be obtained as the mapping cone of  $S_\xi \rightarrow B$ ). It follows that, up

to homotopy,  $T(\xi)$  is an invariant of the fiber homotopy type of  $(D_\xi, S_\xi) \rightarrow B$  or  $S_\xi \rightarrow B$ .

Frequently, we are given an  $n$ -plane bundle with group  $GL(n, \mathbb{R})$  rather than  $O(n)$ . However, since  $GL(n, \mathbb{R})/O(n)$  is contractible [Steenrod], the natural map  $BO(n) \rightarrow BGL(n, \mathbb{R})$  is a homotopy equivalence and hence there is no significant distinction between  $O(n)$  bundles  $GL(n, \mathbb{R})$  bundles.

Note also that the above discussion applies to  $U(n)$  bundles and  $Sp(n)$  bundles. Recall  $\widetilde{H}^*T(\xi) \cong H^*(D_\xi, S_\xi)$ .

**DEFINITION 8.2.** A  $\Lambda$ -orientation (orientation if  $\Lambda = \mathbb{Z}$ ) of  $\xi$  is a class  $u_\xi \in H^n T(\xi) = H^n(D_\xi, S_\xi)$  which restricts to a generator of  $H^n(F_b D_\xi, F_b S_\xi) = H^n(D^n, S^{n-1}) = \Lambda$  under the inclusion  $(F_b D_\xi, F_b S_\xi) \subset (D_\xi, S_\xi)$  for every fiber  $(F_b D_\xi, F_b S_\xi) = \xi^{-1}(b)$ ,  $\xi : (D_\xi, S_\xi) \rightarrow B, b \in B$ .

If we have a bundle map

$$\begin{array}{ccc} (D_{\xi'}, S_{\xi'}) & \xrightarrow{\tilde{f}} & (D_\xi, S_\xi) \\ \xi' \downarrow & & \downarrow \xi \\ B' & \xrightarrow{f} & B \end{array}$$

and a  $\Lambda$ -orientation  $u \in H^n(D_\xi, S_\xi)$  of  $\xi$ , then  $\tilde{f}^*(u) \in H^n(D_{\xi'}, S_{\xi'})$  is a  $\Lambda$ -orientation of  $\xi'$ . In fact, this holds whenever we have a map of fibrations  $\tilde{f} : S_{\xi'} \rightarrow S_\xi$  which has degree  $\pm 1$  on each fiber.

If  $\alpha : \Lambda_1 \rightarrow \Lambda_2$  is a ring homomorphism (preserving unit) and if  $u \in H^n(D_\xi, S_\xi; \Lambda_1)$  is a  $\Lambda_1$  orientation of  $\xi$  then  $\alpha_*(u) \in H^n(D_\xi, S_\xi; \Lambda_2)$  is a  $\Lambda_2$ -orientation of  $\xi$ . Hence, an orientation (that is a  $\mathbb{Z}$ -orientation) provides us with a  $\Lambda$ -orientation for any coefficient ring  $\Lambda$ .

The following theorem is fundamental.

**THEOREM 8.3.** *The following are equivalent:*

- (1)  $\xi$  has a  $\Lambda$ -orientation  $u_\xi \in H^n(d_\xi, S_\xi)$ .
- (2) there is a class  $\bar{u} \in H^n(D_\xi, S_\xi)$  which restricts to a generator of  $H^n(F_b D_\xi, F_b S_\xi)$  for some  $b$  in each path component of  $B$ .
- (3) the local coefficient system  $\mathcal{H}^*(F_b D_\xi, F_b S_\xi)$  of  $\xi$  is trivial.

If (1)-(3) hold then the homomorphism  $\Phi : H^k B \rightarrow H^{n+k}(D_\xi, S_\xi)$  defined by  $\Phi(\alpha) = \xi^*(\alpha) \cup u_\xi$  is an isomorphism where  $\xi : D_\xi \rightarrow B$  and  $\cup : H^* D_\xi \otimes H^*(D_\xi, S_\xi)$ . The map  $\Phi$  is natural with respect to orientation preserving bundle maps.

**PROOF.** Obviously (1) implies (2).

Consider the Serre sequence of  $(D_\xi, S_\xi) \rightarrow B$ . We have  $E_2 = H^*(B; \mathcal{H}^*(D^n, S^{n-1}))$  and since  $\mathcal{H}^i(D^n, S^{n-1}) = 0$  if  $i \neq n$ ,  $E_2 = E_\infty$ . Clearly we may assume  $B$  is connected. Then  $E_2^{0,*} = H^0(B; \mathcal{H}^*(D^n, S^{n-1})F(H^*(D^n, S^{n-1})))$ , the elements of  $H^*(D^n, S^{n-1})$  fixed by  $\pi_1(B)$ .

Assume (2) holds and let  $i_b : (D^n, S^{n-1}) \rightarrow (D_\xi, S_\xi)$  be the inclusion of the fiber over  $b \in B$ . Since  $i_b^*$  factors as

$$H^n(D_\xi, S_\xi) = E_\infty^{0,n} = E_2^{0,n} = F(H^n(D^n, S^{n-1})) \subset H^n(D^n, S^{n-1})$$

and is onto by assumption, all elements of  $H^n(D^n, S^{n-1})$  are fixed by  $\pi_1 B$  and the local coefficient system is trivial. Thus (2) implies (3).

Assume (3) holds. Then  $E - 2 = H^*B \otimes \iota_n$  where  $\iota_n \in E_2^{0,n} = H^n(D^n, S^{n-1})$  is a generator. Obviously,  $H^*(D_\xi, S_\xi) = E_\infty = E - 2$ . Let  $u_\xi$  correspond to  $1 \otimes \iota_n$  under these identifications. Then  $u_\xi$  restricts to a generator of each fiber by the naturality of the Serre spectral sequence applied to the diagram

$$\begin{array}{ccc} (D^n, S^{n-1}) & \xrightarrow{i_b} & (D_\xi, S_\xi) \\ \downarrow & & \downarrow \\ \{b\} & \longrightarrow & B. \end{array}$$

Hence (3) implies (1).

Now assume (1)-(3) hold. There is a pairing of spectral sequences  $E_r(D_\xi) \otimes E_r(D_\xi, S_\xi) \rightarrow E_r(D_\xi, S_\xi)$  which yields the cap product on  $E_\infty$ . Now  $E_r^{p,q}(D_\xi) = 0$  if  $q \neq 0$  and  $E_r^{*,0}(D_\xi) = H^*B$ . On  $E_2$  the pairing is just left multiplication by  $H^*B$  and  $\Phi$  corresponds to the isomorphism sending  $x$  to  $x \otimes \iota_n$ . Since this is an isomorphism,  $\Phi$  is also.

The statement that  $\Phi$  is natural is expressed by the equality  $\tilde{f}^*\xi^*(x) \cup \bar{f}^*u_\xi = \xi'^*f^*(x) \cup u'_\xi$  when  $\bar{f}^*u_\xi = u'_\xi$  in the situation

$$\begin{array}{ccc} T_{\xi'} & \xrightarrow{\bar{f}} & T_\xi \\ \uparrow & & \uparrow \\ D_{\xi'} & \xrightarrow{\tilde{f}} & D_\xi \\ \downarrow \xi' & & \downarrow \xi \\ B' & \xrightarrow{f} & B \end{array}$$

□

Clearly we may also think of  $\Phi$  as an isomorphism  $H^k B \rightarrow H^{n+k}(T(\xi))$ .

By condition (3) we see that any bundle has a  $\mathbb{Z}/2\mathbb{Z}$  orientation since the only automorphism of  $\mathbb{Z}/2\mathbb{Z}$ . (Insert rest of sentence from notes)

A word about the connection between this notion of orientation and orientations of vector spaces is in order. An orientation of  $\mathbb{R}^n$  is an equivalence class of ordered bases, where two bases are equivalent if the linear transformation sending one to the other has positive determinant. Taking  $A = \mathbb{Z}$ , this is the same as a choice of generator for  $H^n(\mathbb{R}^n, \mathbb{R}^n - 0) = \mathbb{Z}$ . The correspondence is given by taking a simplex linearly embedded in  $\mathbb{R}^n$  representing the chosen generator of  $H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$  and letting the  $i^{\text{th}}$  basis vector of the corresponding basis be the vector connecting vertex  $i - 1$  and vertex  $i$ . Since  $(D^n, S^{n-1}) \subset (\mathbb{R}^n, \mathbb{R}^n - 0)$  is a homotopy equivalence it is equivalent to supply a generator of  $H^n(D^n, S^{n-1})$ . Letting  $i_b : (D^n, S^{n-1}) \rightarrow (D_\xi, S_\xi)$  be the inclusion of the fiber over  $b \in B$  we see that an orientation  $u \in H^n(D_\xi, S_\xi)$  provides us with orientations  $i_b^*(u)$  of each fiber. These orientations are locally compatible in the sense that for each  $b \in B$  there is a neighborhood  $V$  of  $b$  and a class  $\pi \in H^n(\xi^{-1}(V), \xi^{-1}(V) \otimes S_\xi)$  such that  $\pi$  restricts to the chosen orientation of each fiber lying over a point of  $V$ . Conversely, given orientations of the fibers which are locally compatible in this sense we can recover an orientation  $u \in H^n(D_\xi, S_\xi)$  inducing them by Mayer-Vietoris sequences and a limit argument [Spanier, p.212]. If one considers orientations with respect

to a generalized cohomology theory, orientations as in Definition 8.2 are no longer equivalent to locally compatible families of orientations (see [Adams, Stable Homotopy and Generalised Homology, pp. 253-255]). Definition 8.2 is the one which generalizes.

Observe that any two  $\Lambda$ -orientations of a bundle differ by a unit of  $\Lambda$ . Hence  $\mathbb{Z}/2\mathbb{Z}$ -orientations are unique, while  $\mathbb{Z}$ -orientations occur in pairs  $\{u, -u\}$ .

**DEFINITION 8.4.** *A bundle is  $\Lambda$ -orientable if it has a  $\Lambda$ -orientation. A  $\Lambda$ -oriented bundle is a pair  $(\xi, u_\xi)$  consisting of a bundle  $\xi$  and a particular  $\Lambda$ -orientation  $u_\xi \in H^n T(\xi)$ .*

These ideas can be expressed most clearly in terms of classifying maps and characteristic classes. We need the following fact.

**THEOREM 8.5.** *Let  $\Lambda = \mathbb{Z}$ . Then conditions (1)-(3) of Theorem 8.3 are equivalent to either of the following:*

- (1') *the classifying map  $\xi$  can be lifted to  $BSO(n)$*
- (2')  *$w_1(\xi) = 0$ .*

**PROOF.** The equivalence of (1') and (2') was demonstrated in Corollary 5.3.(iv). As noted before (1') implies (3) since if  $w_1(\xi) \neq 0$  then  $\xi$  is not orientable. Let  $\xi : E \rightarrow B$  have classifying map  $f : B \rightarrow BO(n)$ . Since  $w_1(\xi) \neq 0$ ,  $f^* : H^1(BO(n); \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(B; \mathbb{Z}/2\mathbb{Z})$  is nonzero. By universal coefficients and the Hurewicz theorem,  $f_* : \pi_1 B \rightarrow \pi_1 BO(n) = \mathbb{Z}/2\mathbb{Z}$  is nonzero. Pick  $\alpha : S^1 \rightarrow B$  such that  $f_*(\alpha) \neq 0$ . Then  $\alpha^*\xi$  is the nontrivial  $O(n)$  bundle over  $S^1$ . Now if  $\xi$  were orientable,  $\alpha^*\xi$  would be also. Since there is only one nontrivial  $O(n)$  bundle over  $S^1$ , it remains only to exhibit a nonorientable  $O(n)$  bundle over  $S^1$ . Since the transformation

$$\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

in  $O(n)$  induces the nontrivial automorphism of  $H^n(D^n, S^{n-1})$ , this is easy.  $\square$

**REMARK 8.6.**

- (1) *Condition (1') is equivalent to saying that the group of the bundle can be reduced to  $SO(n)$ .*
- (2) *If we are given an  $n$ -plane bundle  $\xi : B \rightarrow B$  with group  $GL(n, \mathbb{R})$ , let  $E_0$  be the complement of the 0-cross section  $B \rightarrow E$ . Then  $(E, E_0) \rightarrow B$  is a bundle pair with fiber  $(\mathbb{R}^n, \mathbb{R}^n - 0)$ . Although we cannot define a Thom space from the pair  $(E, E_0)$  as we did with  $(D_\xi, S_\xi)$ , Theorem 8.3 holds with the pair  $(D_\xi, S_\xi)$  replaced by  $(E, E_0)$  and Theorem 8.5 holds with  $SO(n)$  replaced by  $SL(n, \mathbb{R})$ .*

Recall the fibration sequence

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow BSO(n) \longrightarrow BO(n) \longrightarrow B\mathbb{Z}/2\mathbb{Z}.$$

For any space  $X$ , we have an exact sequence

$$[X, \mathbb{Z}/2\mathbb{Z}] \xrightarrow{i} [X, BSO(n)] \xrightarrow{j} [X, BO(n)] \xrightarrow{k} [X, B\mathbb{Z}/2\mathbb{Z}].$$

The elements of  $[X, BSO(n)]$  correspond to *oriented* bundles since an orientation of the universal  $SO(n)$  bundle induces an orientation on each  $SO(n)$ -bundle which is preserved by  $SO(n)$  bundle maps. The image of  $[x, BSO(n)]$  in  $[X, BO(n)]$  consists of *orientable* bundles. The various lifts of a map  $f : X \rightarrow BO(n)$  to  $BSO(n)$  correspond to the various choices of orientation.  $\mathbb{Z}/2\mathbb{Z}$  acts on  $[X, BSO(n)]$  by reversing orientations (sending  $(\xi, u_\xi)$  to  $(\xi, -u_\xi)$ ). To prove the last statement, recall the situation of Lemma ???. Let  $\xi : E \rightarrow BSO(n)$  be the universal  $SO(n)$  bundle, let  $\psi : BSO(n) \rightarrow BSO(n)$  give the action of  $\mathbb{Z}/2\mathbb{Z} = O(n)/SO(n)$  on  $BSO(n)$  and let  $\xi' : E' \rightarrow BSO$  be classified by  $\psi$ . Then we have an  $SO(n)$  bundle map

$$\begin{array}{ccc} E' & \xrightarrow{\xi'} & E \\ \downarrow & & \downarrow \xi \\ BSO(n) & \xrightarrow{\psi} & BSO(n). \end{array}$$

We want to show that  $\xi'$  is the same  $O(n)$  bundle as  $\xi$  but with the opposite orientation. This is accomplished by exhibiting an  $O(n)$  bundle map

$$\begin{array}{ccc} E' \times_{SO(n)} S^{n-1} & \longrightarrow & E \times_{SO(n)} S^{n-1} \\ \downarrow & & \downarrow \\ BSO(n) & \xrightarrow{1} & BSO(n) \end{array}$$

covering the identity and noting that it has degree -1 on the fibers. Details are similar to Lemma ???.

We can identify  $[X, \mathbb{Z}/2\mathbb{Z}]$  with the set of orientations of the trivial  $O(n)$  bundle over  $X$  in an obvious way. Then the sequence (\*) can be interpreted as

$$\left\{ \begin{array}{c} \text{orientations} \\ \text{of the} \\ \text{trivial bundle} \end{array} \right\} \xrightarrow{i} \left\{ \begin{array}{c} \text{oriented} \\ \text{n-bundles} \end{array} \right\} \xrightarrow{j} \{ \text{n-bundles} \} \xrightarrow{k} \left\{ \begin{array}{c} \text{two-fold} \\ \text{covers} \end{array} \right\}$$

which may be split into

$$\left\{ \begin{array}{c} \text{orientations} \\ \text{of the} \\ \text{trivial bundle} \end{array} \right\} \xrightarrow{i} \left\{ \begin{array}{c} \text{oriented} \\ \text{n-bundles} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{orientable} \\ \text{n-bundles} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{orientable} \\ \text{n-bundles} \end{array} \right\} \longrightarrow \{ \text{n-bundles} \} \xrightarrow{k} \left\{ \begin{array}{c} \text{two-fold} \\ \text{covers} \end{array} \right\}$$

where  $\text{im}(j) \subset [X, BO(n)]$  is identified with the set of orientable bundles. The map  $k$  takes an  $O(n)$  bundle  $E \rightarrow B$  to the double cover

$$E \times_{O(n)} O(n)/SO(n) = E/SO(n) \longrightarrow B.$$

By exactness of (\*), we observe the amusing fact that, for connected base space  $B$ , the (principal)  $O(n)$ -bundle  $E \rightarrow B$  is orientable if and only if  $E$  is disconnected (since  $SO(n)$  is connected,  $E$  is disconnected if and only if  $E/SO(n)$  is disconnected).

A remarkable feature of the Thom isomorphism is that it enables us to define Stiefel-Whitney classes and an Euler class in terms intrinsic to the bundle.

THEOREM 8.7.

$$w_i = \Phi^{-1} \text{Sq}^i \Phi(1) \in H^i(BO(n); \mathbb{Z}/2\mathbb{Z})$$

Note: Clearly this implies that  $w_i(\xi) = \Phi^{-1} \text{Sq}^i u_\xi$  for any  $O(n)$ -bundle  $\xi$ .

PROOF. We shall prove this by explicitly analyzing the Thom space  $TO(n)$ . Let  $\gamma : E \rightarrow BO(n)$  be the universal  $O(n)$  bundle. Then  $S_\gamma = E \times_{O(n)} S^{n-1} = E \times_{O(n)} O(n)/O(n-1) = E/O(n-1) = BO(n-1)$  and there is an equivalence

$$\begin{array}{ccc} S_\gamma & \xlongequal{\quad} & BO(n-1) \\ \downarrow & & \downarrow i \\ BO(n) & \xlongequal{\quad} & BO(n) \end{array}$$

of fibrations. From the diagram

$$\begin{array}{ccc} S_\gamma & \longrightarrow & B \\ \downarrow & & \parallel \\ D_\gamma & \xrightarrow{\gamma} & B \end{array}$$

in which  $\gamma : D_\gamma \rightarrow B$  is an equivalence, we see that the inclusion  $S_\gamma \rightarrow D_\gamma$  is equivalent to  $BO(n-1) \xrightarrow{i} BO(n)$ . Since  $i^*$  is an epimorphism in  $\mathbb{Z}/2\mathbb{Z}$ -cohomology, the long exact sequence becomes

$$0 \longrightarrow H^*TO(n) \xrightarrow{\gamma^{*-1}p^*} H^*BO(n) \xrightarrow{i^*} H^*BO(n-1) \longrightarrow 0,$$

where  $p : D_\gamma \rightarrow TO(n)$  is the natural map and  $H^*$  denotes  $H^*(-; \mathbb{Z}/2\mathbb{Z})$ . Hence,  $p^*$  is a monomorphism which identifies  $H^*TO(n)$  with the ideal in  $H^*BO(n)$  generated by  $w_n$ . It follows that  $p^*u_\gamma = \gamma^*(w_n)$ . We want to show  $\Phi(w_i) = \text{Sq}^i u_\gamma$ . Since  $p^*$  is a monomorphism it is enough to show  $p^*\Phi(w_i) = p^*\text{Sq}^i u_\gamma$ . But

$$\begin{aligned} p^*\Phi(w_i) &= p^*(\gamma^*(w_i)u_\gamma) \\ &= \gamma^*(w_i)\gamma^*(u_\gamma) \end{aligned}$$

and

$$\begin{aligned} p^*\text{Sq}^i u_\gamma &= \gamma^*\text{Sq}^i w_n \\ &= \gamma^*(w_i w_n) \end{aligned}$$

so we are done.  $\square$

DEFINITION 8.8. The Euler class  $\chi$  of an oriented bundle  $(\xi, u_\xi)$  is defined to be  $\Phi^{-1}(u_\xi^2) \in H^*B$ .

Note that we could have defined  $\chi$  as  $(\xi^*)^{-1}p^*u_\xi$

$$\begin{array}{ccc} H^*D_\xi & \xleftarrow{p^*} & H^*T_\xi \\ \cong \downarrow & & \\ & (\xi^*)^{-1} & \\ & \downarrow & \\ & H^*B & \end{array}$$

since  $\Phi((\xi^*)^{-1}p^*u_\xi) = p^*(u_\xi)u_\xi = u_\xi^2$ .

If  $\xi$  is an  $n$ -bundle with  $n$  odd then

$$u_\xi^2 = (-1)^{n^2} u_\xi^2 = -u_\xi^2.$$

Thus  $2u_\xi^2 = 0$  and hence  $2\chi(\xi) = 0$ . With  $\mathbb{Z}/2\mathbb{Z}$  coefficients,  $u_\xi^2 = \text{Sq}^n u_\xi$ . Hence the mod 2 reduction of  $\chi$  is  $w_n$ .

The next thing we must do is check that this definition agrees with the definition given in Chapter 6 when  $1/2 \in \Lambda$ . In the following theorem, let  $\chi$  denote the Euler class defined in Chapter 6. Assume  $\Lambda$  contains  $1/2$ .

**THEOREM 8.9.** *The  $\Lambda$ -reduction of  $\Phi^{-1}(u_\xi^2)$  is  $\chi \in H^n(BSO(n); \Lambda)$  for an appropriately chosen orientation  $u_\gamma$  of the universal bundle  $\gamma : ESO(n) \rightarrow BSO(n)$ .*

**PROOF.** Let  $H^*$  denote  $H^*(-; \Lambda)$ . As in Theorem 8.7  $S_\gamma \rightarrow D_\gamma$  is equivalent to  $i : BSO(n-1) \rightarrow BSO(n)$  so we have the long exact sequence

$$\dots \longrightarrow H^{i-1}BSO(n-1) \longrightarrow H^iTSO(n) \xrightarrow{p^*} H^iBSO(n) \xrightarrow{i^*} H^iBSO(n-1) \longrightarrow \dots$$

The even and odd cases differ:  $H^*BSO(2k) = P\{P_1, \dots, P_{k-1}, \chi\}$  and  $H^*BSO(2k+1) = P\{P_1, \dots, P_k\}$ .

When  $n = 2k+1$ ,  $2u_\xi^2 = 0$ . Since 2 is invertible in  $\Lambda$ ,  $u_\gamma^2 = 0$ . Thus  $\Phi^{-1}(u_\gamma^2) = 0$  which is correct, since  $\chi = 0$  in  $H^*BSO(2k+1)$ .

When  $n = 2k$  we have the exact sequence

$$0 = H^{2k-1}BSO(n-1) \longrightarrow B^{2k}TSO(n) \xrightarrow{p^*} H^{2k}BSO(n) \xrightarrow{i^*} H^{2k}BSO(n-1).$$

The kernel of  $i^*$  is generated by  $\chi$  so  $p^*u_\chi = \pm\chi$ . Choose  $u_\gamma$  so that  $p^*u_\gamma = \chi$ . By the remark following Definition 8.8, this implies  $\Phi(x) = u_\gamma^2$ .  $\square$

Note that this determines a choice of canonical orientation for  $SO(2k)$ -bundles. Further properties of the integral Euler class  $\chi$  will be developed in the next chapter.

The method just used to define Stiefel-Whitney classes and the Euler class can be used in other situations. For generality, let  $F^*$  denote a cohomology theory which has a product and let  $\psi : F^* \rightarrow F^*$  be a cohomology operation of degree  $k$  ( $\psi : F^i \rightarrow F^{i+k}$ ). Now the Thom isomorphism (Theorem 8.3) says that  $H^*T_\xi$  is a free  $H^*B$  module with generator  $u_\xi \in H^nT_\xi$ . Hence let us assume given a bundle  $\xi : E \rightarrow B$  with an  $F$ -orientation  $u_\xi \in F^nT_\xi$  which defines a Thom isomorphism  $\Phi : F^iB \rightarrow F^{i+n}T_\xi$  by  $\Phi(x) = \xi^*(x)u_\xi$ . Clearly  $\Phi$  will be natural with respect to bundle maps which preserve the  $F$ -orientation.

For the purposes of the discussion it is sufficient to say that  $F^*(\mathbb{R}^n, \mathbb{R}^n - 0) \cong F^*(D^n, S^{n-1})$  is a free module on a generator of  $F^n(D^n, S^{n-1})$  over the ring of coefficients  $\pi_*F$  and that an  $F$ -orientation is a class which restricts to a generator under the inclusion of any fiber. See [Adams, Stable Homotopy and Generalized Homology] and [May,  $E_\infty$ -Ring spaces and  $E_\infty$ -Ring Spectra]. If  $F^* = H^*(-; \Lambda)$  this is clearly the ordinary notion of orientation.

The class  $\Phi^{-1}\psi(u_\xi) \in F^k(B)$  is a characteristic class  $c_\psi(\xi)$  of the bundle  $\xi$ . The naturality of  $\psi$  and  $\Phi$  implies the naturality of  $c_\psi$ . Here are our two examples:

- (i)  $F^* = H^*(-; \mathbb{Z}/2\mathbb{Z})$ ,  $\psi = \text{Sq}^i$  and our orientation is the unique  $\mathbb{Z}/2\mathbb{Z}$  orientation of an  $O(n)$ -bundle (so all bundle maps are orientation preserving). Then  $c_\psi = w_i$ , the  $i^{\text{th}}$  Stiefel-Whitney class.
- (ii)  $F^* = H^*(-; \mathbb{Z}/2\mathbb{Z})$ ,  $\psi : H^n \rightarrow H^{2n}$  by  $\psi(x) = x^2$  and our orientation is the ordinary orientation of a vector bundle. Then  $c_\psi = \chi$ , the Euler class. Of course this example works equally well in the general situation:

if  $u_\xi \in F^n T_\xi$  is an  $F$ -orientation of  $\xi$  define  $\chi_F(\xi)$  to be  $\Phi^{-1}(u_\xi^2)$ . The remark following Definition 8.8 applies here also to show that  $\chi_F(\xi) = (\xi^*)^{-1}p^*(u_\xi)$ ,  $p : D_\xi \rightarrow T_\xi$  the natural map.

We have noted that the Thom space is an invariant of the fiber homotopy type of a bundle. Theorems 8.7 and 8.9 therefore show that the Stiefel-Whitney (and Euler) classes of a bundle depend only on the (oriented) fiber homotopy type of the bundle. This is of particular importance in the study of manifolds. Let  $M$  be a smooth manifold. We define characteristic classes of  $M$  by use of the tangent bundle  $\tau$  of  $M$ . Thus we can speak of the Stiefel-Whitney classes of  $M$  ( $w_i(M) = w_i(\tau)$ ), the Euler class of  $M$  if  $M$  is oriented ( $\xi(M) = \xi(\tau)$ ), etc. Now it turns out that any *topological* manifold has a “homology tangent bundle” for which we can define a Thom isomorphism in the presence of an orientation. Thus we can define Stiefel-Whitney (and Euler) class for (oriented) manifolds using the formulas of Theorems 8.7 and 8.9. Further, if  $M$  has a smooth structure then the tangent bundle and the homology tangent bundle are equivalent in an appropriate sense. It follows that the new definition extends the previous one and that *the Stiefel-Whitney and Euler classes of a manifold  $M$  depend only upon the topological type of  $M$* . We proceed to the detail.

Let  $M$  be a topological manifold without boundary (add an open collar if  $M$  has a boundary). Let  $\Delta \subset M \times M$  be the diagonal  $\{(m, m)\}$ . Let

$$\begin{array}{c} (M \times M, M \times M - \Delta) \\ \downarrow \pi \\ M \end{array}$$

be projection on the first factor. Then  $\pi$  has fibers  $\pi^{-1}(m) = (m \times M, m \times M - m \times m) \approx (M, M - m)$ . If  $V$  is a coordinate neighborhood of  $m$  then  $\{V, M - m\}$  is an excisive couple. Hence  $H^*(M, M - m) \cong H^*(V, V - m) \cong H^*(\mathbb{R}^n, \mathbb{R}^n - 0)$ . Further,  $\pi$  is locally trivial [Spanier, p.293]. If  $m, m' \in M$  are in the same component of  $M$  then  $M - m \approx M - m'$ . Thus, if  $M$  is connected,  $\pi$  is a fiber bundle pair with fiber  $(M, M - m)$  for any  $m \in M$ .

DEFINITION 8.10.  $\pi$  is called the homology tangent bundle of  $M$ . A  $\Lambda$ -orientation of  $M$  is a class  $u \in H^n(M \times M, M \times M - \Delta; \Lambda)$  such that  $i_m^*(u)$  is a generator of  $H^n(m \times M, m \times M - m \times m) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$  for the inclusion  $i_m : (m \times M, m \times M - m \times m) \subset (M \times M, M \times M - \Delta)$  of each fiber.

The proof of Theorem 8.3 goes through to give the following version of the Thom Isomorphism Theorem.

THEOREM 8.11. Let  $M$  be an  $n$ -manifold without boundary. The following conditions are equivalent:

- (1)  $M$  has a  $\Lambda$ -orientation  $u \in H^n(M \times M, M \times M - \Delta; \Lambda)$
- (2) there is a class  $\bar{u} \in H^n(M \times M, M \times M - \Delta; \Lambda)$  such that  $i_m^*(\bar{u})$  is a generator of  $H^n(m \times M, m \times M - m \times m; \Lambda)$  for some  $m$  in each component of  $M$ .
- (3) the local coefficient system  $\mathcal{H}^n(M, M - m; \Lambda)$  is trivial.

If (1)-(3) hold then the homomorphism

$$\Phi : H^i(M; \Lambda) \longrightarrow H^{i+n}(M \times M, M \times M - \Delta; \Lambda)$$



defined by  $\Phi(x) = \pi^*(x)u$  is an isomorphism, where  $\pi^*(x) \in H^*(M \times M; \Lambda)$ .  $\Phi$  is natural with respect to orientation preserving maps.

Taking Theorem 8.7 and 8.9 as our cue we now define characteristic class of  $M$ .

**DEFINITION 8.12.** *The  $i^{\text{th}}$  Stiefel-Whitney class of  $M$ ,  $w_i(M) \in H^i(M; \mathbb{Z}/2\mathbb{Z})$  is  $\Phi^{-1}\text{Sq}^i u$  where  $u$  is the unique  $\mathbb{Z}/2\mathbb{Z}$ -orientation of  $M$ . If  $(M, u)$  is an oriented manifold,  $u \in H^n(M \times M, M \times M - \Delta)$ , the Euler class  $\chi$  of  $(M, u)$  is  $\Phi^{-1}(u^2) \in H^n(M; \mathbb{Z})$ .*

It is easy to check that if we choose the opposite orientation  $-u$  of  $M$  then the Euler class changes sign. Also, if  $n$  is odd then  $2u^2 = 0$  so  $2\chi = 0$ . Since  $H^n(M; \mathbb{Z})$  is torsion free this implies that  $\chi = 0$ .

It remains to prove that these definitions coincide with our earlier definitions when  $M$  has a smooth structure. Let  $M$  be a smooth  $n$ -manifold with tangent bundle  $\tau : E \rightarrow M$ . Choose a Euclidean metric  $\mu : E \rightarrow \mathbb{R}^+$  for  $\tau$  (a Riemannian metric on  $M$ ). The metric provides us with an exponential map defined on a neighborhood of the zero cross section of  $E$  as follows. For  $v \in E$ , let  $g : \mathbb{R} \rightarrow M$  be the unique geodesic with  $g(0) = \tau(v)$  and  $g'(0) = v$ . Then if  $\mu(v)$  is small enough that  $g(1)$  is defined, we let  $\exp(v) = g(1)$ . Recall that for each fiber  $\tau^{-1}(x)$  there is a small disk  $D_{\epsilon, x} = \{v \in \tau^{-1}(x) \mid \mu(v) \leq \epsilon\}$  such that  $\exp|_{D_{\epsilon, x}}$  is a diffeomorphism. Let  $\epsilon : M \rightarrow \mathbb{R}$  be a positive continuous function such that for each  $\alpha \in M$ ,  $\exp_{D_{\epsilon(x), x}}$  is a diffeomorphism. Let  $D_\tau = \{v \in E \mid \mu(v) \leq \epsilon(\tau(v))\}$  and let  $S_\tau = \{v \in E \mid \mu(v) = \epsilon(\tau(v))\}$ . Then  $D_\tau$  is the  $n$ -disk bundle and  $S_\tau$  be the  $(n-1)$ -sphere bundle associated to  $\tau$ . We define a map

$$\psi : (D_\tau, S_\tau) \longrightarrow (M \times M, M \times M - \Delta)$$

by

$$\psi(v) = (\tau(v), \exp(v)).$$

Since  $\psi$  maps fibers to fibers it is a map of fibrations

$$\begin{array}{ccc} (D^n, S^{n-1}) & \longrightarrow & (M, M - x) \\ \downarrow & & \downarrow \\ (D_\tau, S_\tau) & \xrightarrow{\psi} & (M \times M, M \times M - \Delta) \\ \downarrow \tau & & \downarrow \pi \\ M & \xlongequal{\quad} & M. \end{array}$$

Now  $\psi$  restricted to a fiber is a homotopy equivalence so by the five lemma  $\psi$  is a homotopy equivalence. Hence, orientations of the tangent bundle  $\tau$  and the homology tangent bundle  $\pi$  correspond via  $\psi^*$  and we have a commutative diagram

$$\begin{array}{ccc} & H^i(M) & \\ \swarrow \Phi & & \searrow \Phi \\ H^{n+i}(M \times M, M \times M - \Delta) & \xrightarrow{\psi^*} & H^{n+i}(D_\tau, S_\tau) \end{array}$$

if we choose orientations  $u$  and  $u_\tau$  so that  $u_\tau = \psi^*(u)$ . Clearly this implies that the two definitions of Stiefel-Whitney and Euler classes coincide. Since the homology

tangent bundle depends only upon the topological structure of  $M$ , it follows that the Stiefel-Whitney classes and the Euler class are topological invariants; they are independent of any possible smooth structures on  $M$ .

## Integral Euler, Pontrjagin and Stiefel-Whitney classes

Let  $H^*$  denote integral cohomology and let  $\Lambda$  be a ring in which 2 is invertible. As before, we shall use the following names for in the indicated natural maps:

$$\begin{aligned} j_n : BSO(n) &\rightarrow BO(n) & i_n : BSO(n-1) &\rightarrow BSO(n) & i_n : BO(n-1) &\rightarrow BO(n) \\ p_{ij} : BSO(i) \times BSO(j) &\rightarrow BSO(i+j) & p_{ij} : BO(i) \times BO(j) &\rightarrow BO(i+j) \\ \mu_n : BU(n) &\rightarrow BSO(2n) & \nu_n : BSO(n) &\rightarrow BSU(n) & \nu_n : BO(n) &\rightarrow BU(n). \end{aligned}$$

By our calculation of the mod 2 Bockstein spectral sequence of  $BO(n)$  and  $BSO(n)$  (Proposition 6.8) we know that all torsion on  $H^*BO(n)$  and  $H^*BSO(n)$  has order 2 and that

$$\begin{aligned} \frac{H^*BO(2n)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} &\cong \frac{H^*BO(2n+1)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} \cong \frac{H^*BSO(2n+1)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} \\ &\cong P\{w_2^2, w_4^2, \dots, w_{2n}^2\} \end{aligned}$$

and

$$\frac{H^*BSO(2n)}{\text{torsion}} \otimes \mathbb{Z}/2\mathbb{Z} \cong P\{w_2^2, \dots, w_{2(n-1)}^2, w_{2n}\}.$$

This tells us that there are classes of infinite order which reduce mod 2 to the classes of  $w_{2i}^2$  (and for  $BSO(2n)$ ,  $w_{2n}$ ) and which generate the torsion free part modulo torsion and elements divisible by 2. We will give a more precise description shortly.

The following trivial fact will be used repeatedly.

**LEMMA 9.1.** *Let  $B$  be a space such that all torsion elements of  $H^*B$  have order 2. Let  $K_1$  denote the kernel of  $H^*B \rightarrow H^*(B; \mathbb{Z}/2\mathbb{Z})$  and let  $K_2$  be the kernel of  $H^*B \rightarrow H^*(B; \mathbb{Z}[1/2])$ . Then  $K_1 \cap K_2 = 0$ . Hence, an element of  $H^*B$  is completely determined by its images in  $H^*(B; \mathbb{Z}/2\mathbb{Z})$  and  $H^*(B; \mathbb{Z}/2\mathbb{Z})$*

**PROOF.**  $K_1$  is the set of elements divisible by 2 and  $K_2$  is the set of torsion elements. Since all torsion has order 2, no torsion element is divisible by 2 □

**DEFINITION 9.2.** *The  $i^{\text{th}}$  Pontrjagin class  $P_i$  in  $H^*BO(n)$  or  $H^*BSO(n)$  is defined to be  $(-1)^i \nu_n^*(c_{2i})$ .*

Note that  $\nu_n^*(c_i)$  reduces mod 2 to  $w_i^2$  by Theorem ??(vi)]thm:4.1. Since  $w_{2i}^2$  survives to  $E_\infty$  of the Bockstein spectral sequence,  $\nu_n^*(c_{2i})$  has infinite order. Since  $w_{2i+1}^2 = \beta(w_{2i}, w_{2i+1})$ ,  $\nu_n^*(c_{2i+1})$  has order 2.

**THEOREM 9.3.** *The integral Pontrjagin classes have the following properties:*

- (i) *The mod 2 reduction of  $P_i$  is  $w_{2i}^2$ . The  $\Lambda$ -reduction of  $P_i$  is  $P_i$ . The integral classes  $P_i$  are characterized by this condition.*

- (ii)  $i_n^*(P_i) = P_i$  and  $j_n^*(P_i) = P_i$ .
- (iii)  $p_{ij}^*(P_k) \equiv \sum_{a+b=k} P_a \otimes P_b$  modulo 2-torsion.
- (iv)  $\nu_n^*(c_{2i}) = (-1)^i P_i$
- (v)  $\mu_n^*(P_i) = \sum_{a+b=2i} (-1)^{a+i} c_a c_b$
- (vi)  $H^*BO(2n + \epsilon)$  and  $H^*BSO(2n + \epsilon)$ ,  $\epsilon = 0$  or  $1$ , contain  $P\{P_1, \dots, P_n\}$  as a subalgebra.

PROOF. (i)  $\nu_n^*(c_{2i}) = w_{2i}^2 \pmod{2}$  by Theorem ??(vi)]thm:4.1 and  $\nu_n^*(c_{2i}) = (-1)^i P_i$  in  $\Lambda$ -coefficients by Theorem ??(v)]thm:6.1. Hence the first two statements follow. The third statement is immediate from Lemma 9.1.

(ii) This follows immediately from the corresponding statement about the Chern class  $c_{2i}$ .

(iii) This is easy:

$$\begin{aligned}
p_{ij}^*(P_k) &= p_{ij}^*((-1)^k \nu_n^*(c_{2k})) \\
&= (-1)^k (\nu_i^* \otimes \nu_j^*) p_{ij}^*(c_{2k}) \\
&= (-1)^k (\nu_i^* \otimes \nu_j^*) \sum_{a+b=2k} c_a \otimes c_b \\
&= \sum_{a+b=k} P_a \otimes P_b + \sum_{a+b=k} \nu_i^*(c_{2a+1}) \otimes \nu_j^*(c_{2b-1}).
\end{aligned}$$

We can drop the sign  $(-1)^k$  in the second sum because the elements involved have order 2. Part (iii) is now immediate since the second sum has order 2.

- (iv) This is the definition.
- (v) This holds in  $\mathbb{Z}[1/2]$  coefficients. Since the homomorphism  $H^*BU(n) \rightarrow H^*(BU(n); \mathbb{Z}[1/2])$  is monic, it must also hold in integral coefficients.
- (vi) Since there are no relations among the  $P_i$  in  $\Lambda$ -cohomology, part (i) implies that there are no relations among the  $P_i$  in integral cohomology.  $\square$

Let the total Pontrjagin class of an  $O(n)$  bundle  $\xi$  be  $P(\xi) = 1 + P_1(\xi) + P_2(\xi) + \dots$ . If  $\xi$  is an  $O(n)$ -bundle, let  $\xi_{\mathbb{C}}$  be its complexification. If  $\xi$  is a  $U(n)$ -bundle, let  $\xi^{\mathbb{R}}$  be the underlying  $SO(2n)$  bundle. We have the following interpretation of Theorem 9.3 in terms of characteristic classes.

COROLLARY 9.4. *The Pontrjagin classes satisfy:*

- (i) *the mod 2 reduction of  $P_i(\xi)$  is  $(w_{2i}(\xi))^2$ . The  $\Lambda$ -reduction of  $P_i(\xi)$  is  $P_i(\xi)$ .*
- (ii)  *$P_i(\xi \oplus 1) = P_i(\xi)$  if 1 is the trivial line bundle. If  $\xi$  is an  $SO(n)$  bundle and  $\xi'$  is  $\xi$  regarded as an  $O(n)$  bundle, then  $P_i(\xi) = P_i(\xi')$ .*
- (iii)  *$P(\xi \oplus \eta) \equiv P(\xi)P(\eta)$  modulo 2-torsion*
- (iv)  *$c_{2i}(\xi_{\mathbb{C}}) = (-1)^i P_i(\xi)$*
- (v)  *$c_{2i}(\xi_{\mathbb{C}}) = (-1)^i P_i(\xi)$*
- (vi)  *$P_i(\xi^{\mathbb{R}}) = \sum_{a+b=2i} c_a(\xi) c_b(\xi)$ , or, equivalently,  $1 - P_1(\xi^{\mathbb{R}}) + P_2(\xi^{\mathbb{R}}) - \dots = c(\xi)c(\bar{\xi})$ .*

Note the following useful fact. If we work mod 2, then  $\nu_n^*(c_{2i+1}) = w_{2i+1}^2 = \beta(w_{2i} w_{2i+1})$ . Therefore,  $\nu_n^*(c_{2i+1})$  has order 2 and so does  $c_{2i+1}(\xi_{\mathbb{C}})$ .

The Euler class  $\chi \in H^*BSO(n)$  was defined in Definition ???. Using Theorem 8.9 we choose the orientation of the universal  $SO(2n)$  bundle which makes  $\chi$  reduce to the Euler class in  $H^{2n}(BSO(2n); \Lambda)$  defined in Chapter 6. In odd dimensions,  $\chi = -\chi$  so it does not matter how we orient the universal  $SO(2n+1)$  bundle.

**THEOREM 9.5.** *The Euler class  $\chi \in H^*BSO(n)$  satisfies:*

- (i) *The mod 2 reduction of  $\chi$  is  $w_n$ . The  $\Lambda$  reduction of  $\chi$  is  $\chi$  (which is zero in  $H^*BSO(2n+1)$ ). The class  $\chi$  is characterized by these two conditions. In  $H^*BSO(2n)$ ,  $\chi$  has infinite order, while  $2\chi = 0$  in  $H^*BSO(2n+1)$ .*
- (ii)  $i_n^*(\chi) = 0$
- (iii)  $p_{ij}^*(\chi) = \chi \otimes \chi$
- (iv)  $\nu_n^*(c_n) = (-1)^{[n/2]}\chi^2$
- (v)  $\mu_n^*(\chi) = c_n$  ( $\mu_n : BU(n) \rightarrow BSO(2n)$ )
- (vi)  $\chi^2 = P_n$  in  $H^*BSO(2n)$ .  $H^*BSO(2n)$  contains  $P\{P_1, \dots, P_{n-1}, \chi\}$  as a subalgebra.  $H^*BSO(2n+1)$  contains  $P\{P_1, \dots, P_n\} \otimes_{\mathbb{Z}/2\mathbb{Z}}\{\chi\}$  as a subalgebra.
- (vii) *The canonical automorphism  $BSO(n) \rightarrow BSO(n)$  obtained from the bundle  $O(n)/SO(n) = \mathbb{Z}/2\mathbb{Z} \rightarrow BSO(n) \rightarrow BO(n)$  sends  $\chi$  to  $-\chi$ .*

**PROOF.** (i) For the first two statements see Theorem 8.9 and the remarks following Definition 8.8. The third statement is implied by Lemma 9.1. Since  $\chi \in H^*(BSO(2n); \mathbb{Z}[1/2])$  is nonzero,  $\chi \in H^*BSO(2n)$  has infinite order. Since  $\chi \in H^*(BSO(2n+1); \mathbb{Z}[1/2])$  is zero,  $\chi \in H^*BSO(2n+1)$  is torsion and hence of order 2.

- (ii) Mod 2,  $i_n^*(w_n) = 0$  and with  $\Lambda$  coefficients  $i_n^*(\chi) = 0$ . The result follows by Lemma 9.1.
- (iii) Mod 2,  $p_{ij}^*(w_{i+j}) = w_i \otimes w_j$  by Theorem ??(iii)]thm:4.1. With  $\Lambda$  coefficients  $p_{ij}^*(\chi) = \chi \otimes \chi$  by Theorem ??(iii)]thm:6.1. By Lemma ??(i)]lem:8.1, this proves the result.
- (iv) Mod 2,  $\nu_n^*(c_n) = w_n^2 = -w_n^2$ . With  $\Lambda$  coefficients  $\nu_{2n}^*(c_{2n}) = (-1)^n P_n = (-1)^n \chi^2$  while  $\nu_{2n+1}^*(c_{2n+1}) = 0 = \chi$ . By Lemma 9.1 and part (i) the result follows.
- (v) This is true with  $\mathbb{Z}[1/2]$  coefficients. Since  $H^*BU(n) \rightarrow H^*(BU(n); \mathbb{Z}[1/2])$  is a monomorphism, it also holds integrally.
- (vi) With  $\Lambda$  coefficients  $\chi^2 = P_n$  in  $H^*BSO(2n)$ . Mod 2,  $\chi$  is  $w_{2n}$  and  $P_n$  is  $w_{2n}^2$  in  $H^*BSO(2n)$ . Lemma 9.1 completes the proof that  $\chi^2 = P_n$  in  $H^*BSO(2n)$ . Since there are no relations among  $P_1, \dots, P_{n-1}$  and  $\chi$  in  $H^*(BSO(2n); \Lambda)$  there can be none in  $H^*BSO(2n)$ . In  $H^*BSO(2n+1)$  we can write any relation between  $P_1, \dots, P_n$  and  $\chi$  in the form  $\chi A + B = 0$  where  $B$  does not involve  $\chi$ . Since  $\chi$  reduces to 0 in  $\Lambda$  coefficients and there are no relations between the  $P_i$  in  $H^*(BSO(2n+1); \Lambda)$ ,  $B$  must be trivial and hence our relation has the form  $\chi A = 0$ . Mod 2, the  $P_i$  and  $\chi$  reduce to  $w_2^2, \dots, w_{2n}^2$  and  $w_{2n+1}$  among which there are no relations. Hence  $A$  must be divisible by 2:  $\chi A = 2\chi A'$ . But  $2\chi = 0$  and thus the relation is trivial.

- (vii) In Chapter 7, we showed that  $\mathbb{Z}/2\mathbb{Z}$  acts by reversing orientations. Since  $\chi = (\gamma^*)^{-1}(p^*u_\gamma)$  where  $(\gamma, u_\gamma)$  is the universal  $SO(n)$  bundle

$$BSO(n) \xleftarrow{\gamma} D_\gamma \xrightarrow{p} TSO(n)$$

and the result follows immediately.  $\square$

COROLLARY 9.6. *The Euler class of an  $SO(n)$ -bundle satisfies:*

- (i) *The mod 2 reduction of  $\chi(\xi)$  is  $w_n(\xi)$ . The  $\Lambda$ -reduction of  $\chi(\xi)$  is  $\chi(\xi)$ . If  $\xi$  is odd dimensional then  $2\chi(\xi) = 0$ .*
- (ii)  *$\xi(\xi \oplus 1) = 0$  where 1 denotes the trivial line bundle.*
- (iii)  *$\xi(\xi \oplus \eta) = \chi(\xi)\chi(\eta)$ .*
- (iv) *If  $\xi$  is an  $SO(n)$  bundle then  $c_n(\xi_{\mathbb{C}}) = (-1)^{[n/2]}\chi(\xi)^2$ .*
- (v) *If  $\xi$  is a  $U(n)$  bundle then  $\chi(\xi^{\mathbb{R}}) = c_n(\xi)$ .*
- (vi) *If  $\xi$  is an  $SO(2n)$  bundle then  $\chi(\xi)^2 = P_n(\xi)$ .*
- (vii) *If  $\tilde{\xi}$  denotes  $\xi$  with the opposite orientation then  $\chi(\tilde{\xi}) = -\chi(\xi)$ .*

The Euler class and the Pontrjagin classes are the most important integral characteristic classes. They account for all of the torsion-free classes. We now consider a family of torsion classes. In  $H^*(BSO(n); \mathbb{Z}/2\mathbb{Z})$  we have  $\beta(w_{2i}) = w_{2i+1}$ . Hence, there exists a (necessarily unique) class of order 2 in  $H^*BSO(n)$  which reduces mod 2 to  $w_{2i+1}$ .

DEFINITION 9.7. *The  $(2i+1)^{\text{st}}$  integral Stiefel-Whitney class  $w_{2i+1} \in H^{2i+1}BSO(n)$  is the unique class of order 2 which reduces mod 2 to  $w_{2i+1}$ .*

Note that in  $H^*BSO(2n+1)$ ,  $w_{2n+1}$  is the same as  $\chi$ . Since properties of the  $w_{2i+1}$  are given in the next proposition.

THEOREM 9.8. *The integral Stiefel-Whitney classes  $w_{2i+1} \in H^*BSO(n)$  satisfy:*

- (i)  *$w_{2i+1}$  reduces mod 2 to  $w_{2i+1}$  and has order 2. The class  $w_{2i+1}$  is characterized by these properties.*
- (ii)  *$i_n^*(w_{2i+1}) = w_{2i+1}$ .*
- (iii)  *$\mu_n^*(w_{2i+1}) = 0$ .*
- (iv)  *$\nu_n^*(w_{2i+1}) = w_{2i+1}^2$ .*
- (v)  *$H^*BSO(2n+\epsilon)$ ,  $\epsilon = 0$  or  $1$ , contains  $P_{\mathbb{Z}/2\mathbb{Z}}\{w_3, w_5, \dots, w_{2n-1}\}$  as a subalgebra. In  $H^*BSO(2n+1)$ ,  $w_{2n+1} = \chi$ .*

Note that in (v) we have omitted  $w_{2n+1}$  in the case  $\epsilon = 1$  because  $w_{2n+1}$  is  $\chi$ .

PROOF. (i) To say that  $w_{2i+1}$  has order 2 is equivalent to saying that it reduces to 0 in  $\Lambda$  coefficients. Hence this follows by Lemma 9.1.

(ii) This is true in  $\mathbb{Z}/2\mathbb{Z}$  and  $\Lambda$  coefficients. Apply Lemma 9.1.

(iii)  $H^*BU(n)$  is torsion free so this is forced on us.

(iv) Same as (ii).

(v) Mod 2, there are no nontrivial relations among the  $w_{2i+1}$ . Since these classes have order 2, this is sufficient to imply (as in Theorem ??(vi)]thm:8.6 that there are no nontrivial relations integrally.  $\square$

COROLLARY 9.9. *The integral Stiefel-Whitney classes of a bundle satisfy:*

- (i)  $w_{2i+1}(\xi)$  has order 2 and reduces mod 2 to  $w_{2i+1}(\xi)$ .
- (ii)  $w_{2i+1}(\xi \oplus 1) = w_{2i+1}(\xi)$ .
- (iii)  $w_{2i+1}(\xi^{\mathbb{R}}) = 0$  for a  $U(n)$  bundle  $\xi$ .
- (iv)  $c_{2i+1}(\xi_{\mathbb{C}}) = w_{2i+1}(\xi)^2$ .
- (v) If  $\xi$  is an  $BSO(2n+1)$  bundle, then  $w_{2n+1}(\xi) = \chi(\xi)$ .

Conspicuously absent is any result on integral Stiefel-Whitney classes of Whitney sums. This is because  $w_{2i+1}(\xi \oplus \eta)$  is difficult to describe.

The single most important fact about the integral Stiefel-Whitney classes is that the ordinary (mod 2) Stiefel-Whitney classes  $w_{2i+1}$  of an oriented bundle are reductions of integral classes which satisfy (i)-(v) above. For example, if  $B$  is a space such that  $H^*B$  has no 2-torsion then, for any  $SO(n)$ -bundle  $\xi$  over  $B$ ,  $w_{2i+1}(\xi) = 0$ .

In summary, we have examined the following subalgebras of  $H^*BO(n)$  and  $H^*BSO(n)$ :

$$\begin{aligned} P\{P_1, \dots, P_{n-1}, \xi\} \otimes P_{\mathbb{Z}/2\mathbb{Z}}\{w_3, w_5, \dots, w_{2n-1}\} &\subset H^*BSO(2n) \\ P\{P_1, \dots, P_n\} \otimes P_{\mathbb{Z}/2\mathbb{Z}}\{w_3, w_5, \dots, w_{2n+1} = \chi\} &\subset H^*BSO(2n+1) \\ P\{P_1, \dots, P_n\} &\subset H^*BO(2n) \text{ and } H^BO(2n+1). \end{aligned}$$

There are, of course, other classes in the integral cohomology of these spaces. For example,  $\nu_n^*(c_{2i+1}) = "w_{2i+1}^2"$  in  $H^*BO(n)$  (" $w_{2i+1}''$ " because there's no  $w_{2i+1}$  here). These form another polynomial subalgebra  $P_{\mathbb{Z}/2\mathbb{Z}}\{w_1^2, w_3^2, \dots, w_{2n-1}^2\}$  in  $H^*BO(2n-1)$  and  $H^*BO(2n)$ . Also, from the Bockstein spectral sequence it is apparent that  $w_1 w_{2i} + w_{2i+1} \in H^*(BO(n); \mathbb{Z}/2\mathbb{Z})$  is the reduction of a unique integral class of order 2. However, these classes are of no great importance in practice.





## Applications and Examples

We compute the characteristic classes of various bundles and apply them to the solution of typical problems.

The tangent bundle of a manifold  $M$  is denoted  $\tau_m$  or  $\tau$ .

The first space to consider is the sphere  $S^n$ . The usual imbedding  $S^n \subset \mathbb{R}^{n+1}$  has trivial normal bundle. Hence  $\tau_{S^n} \oplus \epsilon_1 = \epsilon_{n+1}$  where  $\epsilon_i$  is the trivial bundle of dimension  $i$ . That is,  $\tau_{S^n}$  is stably trivial.

REMARK 10.1.  $w(\tau_{S^n}) = 1$  and  $P(\tau_{S^n}) = 1$ .

PROOF. Note that  $w(\tau_{S^n}) = w(\tau_{S^n} \oplus \epsilon_1) = w(\epsilon_{n+1}) = 1$  and similarly for  $P$ . More generally, the Stiefel-Whitney and Pontrjagin classes of stably equivalent bundles are equal.  $\square$

Since  $\tau_{S^n}$  is orientable,  $\chi(\tau_{S^n})$  is defined. For odd  $n$ ,  $\chi(\tau_{S^n})$  is clearly zero since  $H^*S^n$  has no elements of order 2. For even  $n$ , the following lemma will enable us to determine  $\chi(S^n)$ . It is proved by studying the pair  $(M \times M, M \times M - \Delta)$  where  $\Delta$  is the diagonal. Recall that the Euler characteristic of a manifold is defined by

$$\chi_M = \sum_{i \geq 0} (-1)^i \text{rank } H^i(M).$$

LEMMA 10.2. *If  $(M, u)$  is an oriented compact  $n$ -manifold with orientation class (fundamental class)  $u \in H_n(M)$  then  $\langle \chi(\tau_M), u \rangle = \chi_M$ . If  $M$  is a compact  $n$ -manifold and  $u \in H_n(M; \mathbb{Z}/2\mathbb{Z})$  is the unique  $\mathbb{Z}/2\mathbb{Z}$ -orientation class then  $\langle w_n(\tau_M), u \rangle \equiv \chi_M \pmod{2}$ .*

PROOF. See [Spanier, p.348]. Recall that the mod 2 Euler class is  $w_n$ .  $\square$

Now  $\chi_{S^{2n}} = 2$  so  $\chi(\tau_{S^{2n}}) = 2u^*$  where  $u^*$  is dual to the fundamental class  $u \in H_{2n}(S^{2n})$ . Thus  $\chi$  distinguishes  $\tau_{S^{2n}}$  from the trivial bundle. Note that  $\chi$  is the only characteristic class we have defined which is not stably invariant. In fact  $\chi(\tau \oplus \epsilon_1) = 0$ . Of course we know that  $\tau_{S^{2n}}$  is nontrivial for geometric reasons also. If  $\tau_{S^{2n}}$  were trivial, it would have a nowhere zero cross section  $\nu : S^{2n} \rightarrow \tau_{S^{2n}}$  which would allow us to define a homotopy between the identity map and the antipodal map by  $h(x, t) = \cos(t)x + \sin(t)\nu(x)$ . Since the antipodal map of  $S^{2n}$  has degree  $-1$ , this is impossible.

The next simplest spaces are the projective spaces  $\mathbb{R}P^n$  and  $\mathbb{C}P^n$ . To avoid duplication, let  $K$  denote  $\mathbb{R}$  or  $\mathbb{C}$  and let  $KP^n$  be the associated projective space. To handle  $\tau_{KP^n}$  we must relate it to simpler bundles. Recall that  $BO(1) = O(\infty)/(O(1) \times O(\infty)) = \mathbb{R}P^\infty$  and  $BU(1) = U(\infty)/(U(1) \times U(\infty)) = \mathbb{C}P^\infty$ . The

universal  $O(1)$  and  $U(1)$  bundles are the Hopf bundles

$$\begin{array}{ccc} \frac{O(\infty)}{e \times O(\infty)} = S^\infty & \text{and} & \frac{U(\infty)}{e \times U(\infty)} = S^\infty \\ \downarrow & & \downarrow \\ \frac{O(\infty)}{O(1) \times O(\infty)} = \mathbb{R}P^\infty & & \frac{U(\infty)}{U(1) \times U(\infty)} = \mathbb{C}P^\infty. \end{array}$$

If we restrict to  $KP^n \subset KP^\infty$  we obtain the Hopf bundles

$$\begin{array}{ccc} \frac{O(n+1)}{e \times O(n)} = S^n & \text{and} & \frac{U(n+1)}{e \times U(n)} = S^{2n-1} \\ \downarrow & & \downarrow \\ \frac{O(n+1)}{O(1) \times O(n)} = \mathbb{R}P^\infty & & \frac{U(n+1)}{U(1) \times U(n)} = \mathbb{C}P^\infty \end{array}$$

The associated line bundles

$$\begin{array}{ccc} S^n \times_{O(1)} \mathbb{R} & \text{and} & S^{2n-1} \times_{U(1)} \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{R}P^n & & \mathbb{C}P^n \end{array}$$

admit a description which is more convenient for our purposes. Let  $E = \{([x], v) \in KP^n \times K^{n+1} \mid v \in [x]\}$  where we consider the point  $[x] \in KP^n$  as a line in  $K^{n+1}$ . Let  $\gamma_n : E \rightarrow KP^n$  be projection on the first factor. We then have obvious isomorphisms of bundles

$$\begin{array}{ccc} S^n \times_{O(1)} \mathbb{R} \xrightarrow{f} E & \text{and} & S^{2n-1} \times_{U(1)} \mathbb{C} \xrightarrow{f} E \\ \downarrow & & \downarrow \\ \mathbb{R}P^n \xlongequal{\quad} \mathbb{R}P^n & & \mathbb{C}P^n \xlongequal{\quad} \mathbb{C}P^n \end{array}$$

where  $f([x], \lambda) = ([x], x\lambda)$ . By abuse we will also call  $\gamma_n$  the Hopf bundle over  $KP^n$ . Recall that  $H^*\mathbb{C}P^\infty = H^*BU(1) = P\{c_1\}$  and  $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) = H^*(BO(1); \mathbb{Z}/2\mathbb{Z}) = P\{w_1\}$ . If  $i : KP^n \subset KP^\infty$  is the standard inclusion then  $H^*\mathbb{C}P^\infty = P\{x\}/(x^{n+1})$  where  $x = i^*(c_1)$  and  $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = P\{x\}/(x^{n+1})$  where  $x = i^*(w_1)$ .

**LEMMA 10.3.** *The total Chern class of  $\gamma_n : E \rightarrow \mathbb{C}P^n$  is  $1 + x$ . The total Stiefel-Whitney class of  $\gamma_n : E \rightarrow \mathbb{R}P^n$  is  $1 + x$ . The total Pontrjagin class of  $\gamma_n : E \rightarrow \mathbb{R}P^n$  is 1.*

**PROOF.** Since  $\gamma_n$  is the restriction of the universal bundle, these all follow immediately from the corresponding facts about the universal bundles over  $BO(1)$  and  $BU(1)$ .  $\square$

Now  $\gamma_n$  is a subbundle of the trivial bundle  $\epsilon_{n+1} : KP^n \times K^{n+1} \rightarrow KP^n$ . Its orthogonal complement is denoted  $\omega_n : \hat{E} \rightarrow KP^n$ . The total space  $\hat{E}$  may be described as  $\{([x], v) \in KP^n \times K^{n+1} \mid v \text{ is orthogonal to } x\}$ . By definition we have  $\gamma_n \oplus \omega_n = \epsilon_{n+1}$ . We can now give a convenient description of  $\tau_{KP^n}$ .

PROPOSITION 10.4.  $\tau_{KP^n} \cong \text{Hom}(\gamma_n, \omega_n)$ .

PROOF. Suppose  $f \in \text{Hom}(\gamma, \omega)$  is in the fiber over  $[x] \in KP^n$ . Then  $f$  is a  $K$ -linear map  $\gamma_{[x]} \rightarrow \omega_{[x]}$  (the subscript  $[x]$  means take the fiber over  $[x]$ ). If  $0 \neq y \in \gamma_{[x]} \subset K^{n+1}$  define a path in  $K^{n+1}$  by  $\alpha(t) = y + tf(y)$ . Letting  $\pi : K^{n+1} - 0 \rightarrow KP^n$  be the projection,  $\pi\alpha$  is a path in  $KP^n$  with  $\pi\alpha(0) = [x]$ . Then  $(\pi\alpha)'(0)$  is independent of the choice of  $y$ . Thus we have a map  $\text{Hom}(\gamma, \omega) \rightarrow \tau_{KP^n}$ . It is easy to verify that it is a bundle map and hence an isomorphism  $\square$

COROLLARY 10.5.  $\tau_{KP^n} \oplus \epsilon_1 \cong \text{Hom}(\gamma_n, \epsilon_{n+1}) = (n+1) \text{Hom}(\gamma_n, \epsilon_1)$

Note that if  $\xi$  is a bundle and  $m$  an integer, then  $m\xi$  denotes the sum  $\xi \oplus \cdots \oplus \xi$  with  $m$  factors.

PROOF. Since  $\text{Hom}(\gamma_n, \gamma_n) = \epsilon_1$  and  $\gamma_n \oplus \omega_n = \epsilon_{n+1}$ , we have

$$\begin{aligned} \tau_{KP^n} \oplus \epsilon_1 &= \text{Hom}(\gamma_n, \omega_n) \oplus \text{Hom}(\gamma_n, \gamma_n) \\ &= \text{Hom}(\gamma_n, \epsilon_{n+1}) \\ &= \text{Hom}(\gamma_n, (n+1)\epsilon_1) \\ &= (n+1) \text{Hom}(\gamma_n, \epsilon_1) \end{aligned} \quad \square$$

Now, a quick glance at the effect on transition functions shows that the operation which sends a bundle  $\xi$  to its dual bundle  $\text{Hom}(\xi, \epsilon_1)$  corresponds to the homomorphism  $O(n) \rightarrow O(n)$  or  $U(n) \rightarrow U(n)$  which sends  $A$  to  $(A^{-1})^t$ . Note that the inverse enters in because  $\text{Hom}(-, K)$  is contravariant.

LEMMA 10.6. *If  $\xi$  is an  $O(n)$  bundle then  $\text{Hom}(\xi, \epsilon_1) = \xi$ . If  $\xi$  is a  $U(n)$  bundle then  $\text{Hom}(\xi, \epsilon_1) = \bar{\xi}$  where  $\bar{\xi}$  denotes the conjugate bundle.*

PROOF. In  $O(n)$ ,  $A^{-1} = A^t$  so  $(A^{-1})^t = A$ . In  $U(n)$ ,  $A^{-1} = \bar{A}^t$  so  $(A^{-1})^t = \bar{A}$ .  $\square$

Now we have a grasp on  $\tau_{KP^n}$ . Briefly,

$$\begin{aligned} \tau_{\mathbb{R}P^n} \oplus \epsilon_1 &= (n+1)\gamma_n & (\gamma_n = \text{Hopf } O(1) \text{ bundle on } \mathbb{R}P^n) \\ \tau_{\mathbb{C}P^n} \oplus \epsilon_1 &= (n+1)\bar{\gamma}_n & (\gamma_n = \text{Hopf } U(1) \text{ bundle on } \mathbb{C}P^n). \end{aligned}$$

PROPOSITION 10.7.

$$\begin{aligned} c(\tau_{\mathbb{C}P^n}) &= (1-x)^{n+1} \in H^{**}\mathbb{C}P^n \\ w(\tau_{\mathbb{R}P^n}) &= (1+x)^{n+1} \in H^{**}(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

PROOF. By Corollary ??,  $c(\bar{\gamma}_n) = 1-x$ . Hence  $c(\tau_{\mathbb{C}P^n}) = c(\tau_{\mathbb{C}P^n} \oplus \epsilon_1) = c((n+1)\bar{\gamma}_n) = (1-x)^{n+1}$ . Similarly,  $w(\tau_{\mathbb{R}P^n}) = w((n+1)\gamma_n) = (1+x)^{n+1}$  since  $w(\gamma_n) = 1+x$ .  $\square$

We may also write Proposition 10.7 in the form

$$\begin{aligned} c_i(\tau_{\mathbb{C}P^n}) &= \binom{n+1}{i} (-x)^i \\ w_i(\tau_{\mathbb{R}P^n}) &= \binom{n+1}{i} x^i. \end{aligned}$$

Other characteristic classes of  $\tau_{\mathbb{R}P^n}$  and  $\tau_{\mathbb{C}P^n}$  can be computed from these. For example,

$$\chi((\tau_{\mathbb{C}P^n})^{\mathbb{R}}) = c_n(\tau_{\mathbb{C}P^n}) = \binom{n+1}{n} (-x)^n = (-1)^n (n+1)x^n.$$

For another example, let  $y \in H^2(\mathbb{R}P^n; \mathbb{Z})$  be the unique class which reduces to  $x^2$  in  $H^2(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ . Thus  $2y = 0$  and  $y^i$  is the unique class which reduces to  $x^{2i} \bmod 2$ . Now  $P_i(\xi)$  reduces to  $(w_{2i}(\xi))^2 \bmod 2$ . Since  $w_{2i}(\tau_{\mathbb{R}P^n}) = \binom{n+1}{2i} x^{2i}$ ,  $P_i(\tau_{\mathbb{R}P^n})$  must be  $\binom{n+1}{2i} y^{2i}$ . This gives an example of a case in which  $P(\xi \oplus \eta) \neq P(\xi)P(\eta)$ . Recall that  $P(\xi \oplus \eta)$  is congruent to  $P(\xi)P(\eta)$  modulo elements of order 2. We have  $P(\gamma_n) = 1$  (so  $P(\gamma_n)^{n+1} = 1$ ), while  $P((n+1)\gamma_n) = P(\tau_{\mathbb{R}P^n}) = 1 + \binom{n+1}{1} y^2 + \binom{n+1}{4} y^4 + \dots$ .

By Proposition 10.7  $w_1(\tau_{\mathbb{R}P^n}) = (n+1)x$ , that is

$$w_1(\mathbb{R}P^n) = \begin{cases} 0 & n \text{ odd} \\ x & n \text{ even.} \end{cases}$$

Hence  $\mathbb{R}P^{2n+1}$  is orientable and  $\mathbb{R}P^{2n}$  is not. The Euler class  $\chi(\tau_{\mathbb{R}P^{2n+1}})$  must be zero since the Euler characteristic of  $\mathbb{R}P^{2n+1}$  is  $1 + (-1)^{2n+1} = 0$ .

We now turn to non-immersion and non-embedding results. Let  $M^m$  and  $N^n$  be smooth manifolds and let  $f : M \rightarrow N$  be an immersion. Then  $\tau_M$  is a subbundle of  $f^*\tau_N$ . The normal bundle of  $f$ ,  $\nu_f$ , satisfies  $\tau_M \oplus \nu_f = f^*\tau_N$ . It may be obtained by choosing a Euclidean metric on  $\tau_N$  (which then induces one on  $f^*\tau_N$ ) and letting  $\nu_f$  be the orthogonal complement to  $\tau_M$  in  $f^*\tau_N$ . We may also take  $\tau_f$  to be the quotient bundle  $(f^*\tau_N)/\tau_M$ . The important feature of  $\nu_f$  is that

$$w(\tau_M)w(\nu_f) = f^*w(\tau_N).$$

The most important special case is when  $N = \mathbb{R}^n$ . Then  $\tau_N$  is trivial so we have  $\tau_M \oplus \nu_f = \epsilon_n$ . If  $g : M \rightarrow \mathbb{R}^k$  is also an immersion then  $\tau_M \oplus \nu_g = \epsilon_k$ . It follows that  $\nu_f$  and  $\nu_g$  are stably equivalent, for

$$\nu_f \oplus \epsilon_k = \nu_f \oplus \tau_M \oplus \nu_g = \epsilon_n \oplus \nu_g.$$

Hence we write  $\nu_m$  for the normal bundle of any immersion  $M \rightarrow \mathbb{R}^n$ . Since  $\tau_M \oplus \nu_M$  is trivial, we have

$$w(\tau_M)w(\nu_M) = 1.$$

Let us define  $\bar{w}(\tau_M)$  to be  $w(\nu_M) = w(\tau_M)^{-1}$ . Then we have the inductive formulas

$$\bar{w}_0 = 1$$

$$\bar{w}_i = \sum_{j=1}^i w_j \bar{w}_{i-j}$$

where we have dropped a sign since we are working mod 2.

**THEOREM 10.8.** *If  $M^m$  immerses in  $\mathbb{R}^{m+k}$ , then  $\bar{w}_i(\tau_M) = 0$  for  $i > k$ .*

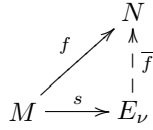
**PROOF.**  $\bar{w}_i(\tau_M) = w_i(\nu_M)$ . Since  $\nu_M$  is a  $k$  dimensional bundle,  $w_i(\nu_M) = 0$  for  $i > k$ .  $\square$

COROLLARY 10.9. *If  $M^m$  immerses in  $\mathbb{R}^{m+1}$  then  $w_i(\tau_M) = w_1(\tau_M)^i$ .*

PROOF. By the above inductive formulas and the proposition, we have  $\bar{w}_1 = w_1$  and, for  $i > 1$ ,  $w_i = w_{i_1}\bar{w}_1 = w_{i_1}w_1$ .  $\square$

For imbeddings, we can improve these results by using the *tubular neighborhood theorem*:

THEOREM 10.10. *Let  $M$  be a manifold without boundary and let  $f : M \rightarrow N$  be an imbedding with normal bundle  $\nu$ . Let  $E_\nu$  be the total space of  $\nu$  and let  $s : M \rightarrow E_\nu$  be the zero cross-section. Then we may extend  $f$  to an embedding  $\bar{f} : E_\nu \rightarrow N$  such that  $\bar{f}(E_\nu)$  is a neighborhood of  $f(M)$*



PROOF. See [Liulvicus, *On Characteristic Classes*, 5.7]  $\square$

Note that if  $M$  were a manifold with boundary, the imbedding  $\bar{f}$  would still exist by  $\bar{f}(E_\nu)$  would no longer be a neighborhood of  $f(M)$ .

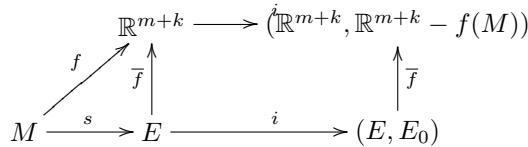
We apply this result in the following theorem.

THEOREM 10.11. *Let  $M^m$  be a manifold without boundary which imbeds as a closed subset of  $\mathbb{R}^{m+k}$  with normal bundle  $\nu$ . Assume  $\nu$  is orientable over the coefficient ring  $R$ . Then the Euler class  $\chi(\nu)$  in  $R$  cohomology is 0. In particular,  $\bar{w}_i(\tau_M) = w_i(\nu) = 0$  for  $i \geq k$ .*

PROOF. Let  $f : M \rightarrow \mathbb{R}^{m+k}$  be the imbedding and let  $s : M \rightarrow E$  be the zero cross-section of  $\nu$ . Extend  $f$  to  $\bar{f} : E \rightarrow \mathbb{R}^{m+k}$  by Theorem 10.10. Recall that if  $u_\nu \in H^*(E, E_0; R)$  is an  $R$ -orientation of  $\nu$  (where  $E_0 = E - s(M)$  is the complement of the zero cross-section), then  $\chi(\nu) = s^*i^*(u_\nu)$  where

$$H^*(E, E_0; R) \xrightarrow{i^*} H^*(E; R) \xrightarrow{s^*} H^*(M; R).$$

Consider the following diagram



Since  $f(M)$  is closed and  $M$  has no boundary

$$\bar{f} : (E, E_0) \rightarrow (\mathbb{R}^{m+k}, \mathbb{R}^{m+k} - f(M))$$

is a cohomology isomorphism by excision. (Note that this would fail if either of these hypotheses were dropped). It follows that  $u_\nu = \bar{f}^*(u)$  for some  $u \in H^*(\mathbb{R}^{m+k}, \mathbb{R}^{m+k} - f(M); R)$ . Then  $\chi(\nu) = s^*i^*\bar{f}^*(u) = f^*i^*(u)$ . But  $i^*(u) = 0$  since  $H^*(\mathbb{R}^{m+k}) = 0$ . Hence  $\chi(\nu) = 0$ . The last statement follows since every bundle is  $\mathbb{Z}/2\mathbb{Z}$ -orientable and the mod 2 Euler class is the top Stiefel-Whitney class  $w_k$ .  $\square$

Note that the proof also shows that  $\chi(\nu_f) = 0$  for an imbedding  $f : M \rightarrow N$  (as a closed subset) if  $H^k(N; R) = 0$  where  $k$  is the codimension of  $M$  in  $N$ .

**COROLLARY 10.12.** *If  $M^m$  embeds in  $\mathbb{R}^{n+1}$  as a closed subset then the normal bundle is trivial,  $\nu = \epsilon_1$ . Hence  $\tau_M$  is stably trivial:  $\tau_M \oplus \epsilon_1 = \epsilon_{n+1}$ .*

**PROOF.** The normal bundle  $n$  is a line bundle with  $w_1(\nu) = 0$ . Since  $BO(1) = K(\mathbb{Z}/2\mathbb{Z}, 1)$ ,  $w_1$  classifies line bundles. Thus,  $\nu$  is trivial.  $\square$

The requirement that  $M$  be embedded as a closed subset is necessary. For example, let  $M$  be the total space of the canonical line bundle over  $\mathbb{R}P^1 = S^1$ . Then  $M$  is just the open Moebius band which can be imbedded in  $\mathbb{R}^3$  (though not as a closed subset). The following proposition will enable us to calculate  $w(\tau_M)$ .

**PROPOSITION 10.13.** *Let  $\xi : E \rightarrow B$  be a smooth vector bundle ( $E$  and  $B$  are smooth manifolds and  $\xi$  is a smooth map). Then*

$$T_\xi \cong \xi^*(\tau_B) \oplus \xi^*(\xi).$$

**PROOF.** The tangent bundle  $\tau_\xi$  splits into the sum of the bundle of vectors tangent to the fiber and the bundle of vectors orthogonal to the fiber. These bundles are isomorphism to  $\xi^*(\xi)$  and  $\xi^*(\tau_B)$  respectively.  $\square$

Let  $\gamma : M \rightarrow S^1$  be the canonical line bundle as above. Now  $\tau_{S^1}$  is trivial so  $\gamma^*(\tau_{S^1}) = \epsilon_1$ . Also  $\gamma^*$  is a homotopy equivalence and  $w_1(\gamma) \neq 0$  so  $w_1(\gamma^*(\gamma)) \neq 0$ . Hence  $\bar{w}_1(\tau_M) = w_1(\tau_M) \neq 0$ . It follows that  $M$  cannot be imbedded as a closed subset of  $\mathbb{R}^3$ .

We can use the preceding results to obtain information on the immersion and imbedding of projective spaces in Euclidean space. We have shown that  $w(\tau_{\mathbb{R}P^n}) = (1+x)^{n+1}$ . Choose  $k$  so that  $n < 2^k \leq 2n$ . Then

$$(1+x)^{2^k} = 1 + x^{2^k} = 1.$$

Thus  $(1+x)^{n+1}(1+x)^{2^k-n-1} = 1$ . We conclude that

$$\bar{w}(\tau_{\mathbb{R}P^n}) = (1+x)^{2^k-n-1} \quad (n < 2^k \leq 2n).$$

**PROPOSITION 10.14.** *If  $\mathbb{R}P^n$  immerses in  $\mathbb{R}^m$  then  $m \geq 2^k - 1$ . If  $\mathbb{R}P^n$  embeds in  $\mathbb{R}^m$  then  $m \geq 2^k$ . ( $n < 2^k \leq 2n$ ).*

**PROOF.** Since  $\mathbb{R}P^n$  is compact any embedding has closed image. Now  $\bar{w}_{2^k-n-1}(\tau_{\mathbb{R}P^n}) \neq 0$  so if  $\mathbb{R}P^n$  embeds in  $\mathbb{R}^m$  then  $2^k - n - 1 < m - n$  so  $m \geq 2^k$ . If  $\mathbb{R}P^n$  immerses in  $\mathbb{R}^m$ , then  $2^k - n - 1 < m - n + 1$  so  $m \geq 2^k - 1$ .  $\square$

**REMARK 10.15.** (i) *By Whitney's embedding and immersion theorems  $\mathbb{R}P^n$  embeds in  $\mathbb{R}^{2n}$  and immerses in  $\mathbb{R}^{2n-1}$ . When  $n = 2^{k-1}$  the proposition shows this is the best possible.*

(ii) *At the other extreme, if  $\mathbb{R}P^n$  immerses in  $\mathbb{R}^{n+1}$  then  $n \geq 2^k - 2$  so either  $n = 2^k - 2$  or  $n = 2^k - 1$ . If  $\mathbb{R}P^n$  embeds in  $\mathbb{R}^{n+1}$  then  $n \geq 2^k - 1$  so  $n = 2^k - 1$ .*

Recall that a manifold is said to be parallelizable if its tangent bundle is trivial. This is equivalent to requiring the existence of  $n$  cross-sections of the tangent bundle, linearly independent at each point, where  $n$  is the dimension of the manifold.

**LEMMA 10.16.** *If  $\mathbb{R}P^n$  is parallelizable then  $n = 2^k - 1$ .*

**PROOF.** If  $\tau_{\mathbb{R}P^n}$  is trivial, then  $w(\tau_{\mathbb{R}P^n}) = 1$ . Let  $n+1 = 2^k m$  with  $m$  odd. Then  $w(\tau_{\mathbb{R}P^n}) = (1+x)^{2^k m} = (1+x^{2^k})^m = 1 + x^{2^k} + \dots \neq 1$  if  $m > 1$ . Hence  $m = 1$  and  $n = 2^k - 1$ .  $\square$

In fact  $\mathbb{R}P^1$ ,  $\mathbb{R}P^3$ , and  $\mathbb{R}P^7$  are the only projective spaces which are parallelizable.

The question of the existence of division algebras over  $\mathbb{R}$  is related to the parallelizability of  $S^{n-1}$  and  $\mathbb{R}P^{n-1}$ . A division algebra of dimension  $n$  over  $\mathbb{R}$  gives a nonsingular bilinear pairing  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . It is this which is easiest to work with.

**PROPOSITION 10.17.** *If there is a nonsingular bilinear pairing  $\mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  then  $\mathbb{R}P^{n-1}$  is parallelizable and hence  $n = 2^k$  for some  $k$ .*

**PROOF.** Write  $ab$  for  $\mu(a, b)$ . Let  $\{E_1, \dots, E_n\}$  be a basis of  $\mathbb{R}^n$ . For  $a \in \mathbb{R}^n$  the elements  $ae_1, ae_2, \dots, ae_n$  are linearly independent. Let  $v_i(ae_1)$  be the projection of  $ae_{i+1}$  on the plane perpendicular to  $ae_1$ . Then for  $x \in S^{n-1}$ ,  $\{v_1(x), \dots, v_{n-1}(x)\}$  is a basis for the tangent space of  $S^{n-1}$  at  $x$ . Thus  $v_1, \dots, v_{n-1}$  are linearly independent cross sections of  $\tau_{S^{n-1}}$ . Now  $v_i(-x) = -v_i(x)$  so the  $v_i$  also define  $n-1$  linearly independent cross sections of the tangent bundle of  $\mathbb{R}P^{n-1}$ . Thus  $S^{n-1}$  and  $\mathbb{R}P^{n-1}$  are parallelizable. By Lemma 10.16 this implies  $n-1 = 2^k - 1$ .  $\square$

For  $n = 1, 2, 4$ , and  $8$  such pairings exist; they are the real numbers, the complex numbers, the quaternions, and the Cayley numbers. It can be shown that these are the only possibilities. The existence of a structure of division algebra on  $\mathbb{R}^n$  is closely related to the existence of elements of Hopf invariant one in  $\pi_{2n-1}(S^n)$ . For references and a proof that  $n = 1, 2, 4$  and  $8$ , see [Adams, Hopf Invariant One, Annals v.72].

Regarding cross sections of general bundles, we have the following result.

**PROPOSITION 10.18.** *If the  $n$ -plane bundle  $\xi$  has  $k$  linearly independent cross sections then*

$$w_{n-k+1} = w_{n-k+2} = \dots = w_n = 0.$$

**PROOF.** If  $\xi$  has  $k$  linearly independent cross sections then they span a trivial  $k$ -dimensional subbundle of  $\xi$ . Thus we can split  $\xi$  as  $\epsilon_k \oplus \gamma$  where  $\gamma$  is an  $(n-k)$ -plane bundle. Then  $w_i(\xi) = w_i(\gamma)$  which is zero for  $i > n-k$ .  $\square$

Let us apply this to the projective space  $\mathbb{R}P^n$ .

**COROLLARY 10.19.** *Let  $n+1 = 2^r m$  with  $m$  odd. Then  $\tau_{\mathbb{R}P^n}$  has at most  $2^r - 1$  linearly independent cross sections.*

**PROOF.**  $w(\tau_{\mathbb{R}P^n}) = (1+x)^{n+1} = (1+x)^{2^r m} = (1+x^{2^r})^m = 1 + \dots + \binom{m}{1} x^{2^r(m-1)}$ . Since  $m$  is odd  $\binom{m}{1} \neq 0$  and hence  $w_{2^r(m-1)} \neq 0$ . If  $k$  is the maximum number of linearly independent cross sections then  $n-k \geq 2^r(m-1)$  so  $k \leq 2^r - 1$ .  $\square$

Obviously there are perfectly analogous results for complex bundles and Chern classes or for symplectic bundles and symplectic classes.

The Euler class will sometimes give more information about the non-existence of cross sections. Recall the following result.

**PROPOSITION 10.20.** *If  $2\chi(\xi) \neq 0$  then  $\xi$  has no odd-dimensional subbundles. In particular,  $\xi$  has no nowhere zero cross sections.*

**PROOF.** If  $\xi = \nu \oplus \eta$ , then  $\chi(\xi) = \chi(\nu)\chi(\eta)$ . If  $\nu$  is odd-dimensional then  $2\chi(\nu) = 0$ .  $\square$

If  $\xi = \tau_{S^{2n}}$  then  $w(\xi) = 1$  so the Stiefel Whitney classes give us no information. However,  $2\chi(\xi) \neq 0$  so we know there cannot be a nowhere zero cross section.

Characteristic classes has important applications in the theory of cobordism. We will only sketch these briefly.

Let  $M$  be a smooth, compact  $n$ -manifold and let  $[m] \in H_n(M; \mathbb{Z}/2\mathbb{Z})$  be the  $\mathbb{Z}/2\mathbb{Z}$ -fundamental class of  $M$ . For any partition of  $n$ ,  $n = j_1 + j_2 + \cdots + j_k$ , consider the class  $w_{j_1}w_{j_2} \cdots w_{j_k}$  in  $H^n(M; \mathbb{Z}/2\mathbb{Z})$  where  $w_i = w_i(\tau_M)$ . The Kronecker product

$$\langle w_{j_1}w_{j_2} \cdots w_{j_k}, [m] \rangle$$

is an element of  $\mathbb{Z}/2\mathbb{Z}$ . The *Stiefel Whitney characteristic numbers* of  $M$  are these elements of  $\mathbb{Z}/2\mathbb{Z}$  defined for each partition of  $n$ . Similarly, if  $M$  is a smooth, compact, orientable  $4m$  manifold with fundamental class  $[m] \in H_{4m}(M; \mathbb{Q})$  then we have *rational Pontrjagin characteristic numbers*

$$\langle P_{j_1}P_{j_2} \cdots P_{j_k}, [m] \rangle \in \mathbb{Q}$$

for each partition  $j_1 + \cdots + j_k = m$ . In the same manner we obtain the *Chern numbers* of a complex manifold. Characteristic numbers are important because they are invariants of cobordism. If  $M$  and  $N$  are smooth, compact  $n$ -manifolds (without boundary) then we say that  $M$  and  $N$  are (unoriented) *cobordant* if there exists an  $(n+1)$ -manifold with boundary,  $W$ , such that  $\partial W = M \cup N$ . If  $M$  and  $N$  are oriented then we say that  $M$  and  $N$  are *oriented cobordant* if there exists an oriented  $(n+1)$  manifold with boundary,  $W$ , such that  $\partial W = M \cup (-N)$ , where  $-N$  means  $N$  with the opposite orientation. The application of characteristic numbers in cobordism then comes from the following facts.

$M$  and  $N$  are unoriented cobordant if and only if they have the same Stiefel Whitney numbers.  $M$  and  $N$  oriented cobordant if and only if they have the same Stiefel Whitney numbers and the same rational Pontrjagin numbers. For an exposition of these results see [Liulevicos, On Char classes] or [Milnor, Char classes].



APPENDIX A

## Bott periodicity

### 1. Definition of the Maps

Our goal here is to prove the following theorem:

**THEOREM A.1.** *There exist maps of H-spaces, which are all weak homotopy equivalences (hence homotopy equivalences):*

$$\begin{aligned} \phi_0 : BU &\longrightarrow \Omega SU \\ \phi_1 : BO &\longrightarrow \Omega SU/SO & \phi_4 : BSp &\longrightarrow \Omega SU/Sp \\ \phi_2 : U/O &\longrightarrow \Omega Sp/U & \phi_5 : U/Sp &\longrightarrow \Omega SO/U \\ \phi_3 : Sp/U &\longrightarrow \Omega Sp & \phi_6 : SO/U &\longrightarrow \Omega Spin \end{aligned}$$

**COROLLARY A.2.**  $\pi_i(U) = \pi_{i+2}(U)$  for  $i \geq 0$ , and  $\pi_0 = 0$ ,  $\pi_1(U) = \mathbb{Z}$ .

**COROLLARY A.3.** *If  $G$  is any of  $O$ ,  $Sp$ ,  $U/Sp$ ,  $Sp/U$ ,  $O/U$ , or  $U/O$ , then  $\pi_i(G) = \pi_{i+8}(G)$  for all  $i \geq 0$ ; if  $G$  is  $BO$  or  $BSp$ , then  $\pi_i(G) = \pi_{i+8}(G)$  for  $i > 0$ . For  $i \geq 0$ ,*

$$\begin{aligned} \pi_i(O) &= \pi_{i+1}(BO) \stackrel{\phi_1}{\cong} \pi_{i+2}(U/O) \stackrel{\phi_2}{\cong} \pi_{i+3}(Sp/U) \stackrel{\phi_3}{\cong} \pi_{i+4}(Sp) \\ &= \pi_{i+5}(BSp) \stackrel{\phi_4}{\cong} \pi_{i+6}(U/Sp) \stackrel{\phi_5}{\cong} \pi_{i+7}(O/U) \stackrel{\phi_6}{\cong} \pi_{i+8}(O). \end{aligned}$$

Further,  $\pi_0(O) = \pi_1(O) = \mathbb{Z}_2$ ,  $\pi_3(O) = \pi_7(O) = \mathbb{Z}$ , and  $\pi_2(O) = \pi_4(O) = \pi_5(O) = \pi_6(O) = 0$ .

**PROOF OF COROLLARY A.3.**  $\pi_0(O) = \pi_1(O) = \mathbb{Z}_2$  is clear;  $\pi_2(O) = \pi_0(U/Sp) = 0$ ;  $\pi_3(O) = \pi_1(U/Sp) = \mathbb{Z}$ .  $\pi_i(O) = \pi_{i-4}(Sp) = 0$  for  $4 \leq i \leq 6$ , and  $\pi_7(O) = \pi_1(U/O) = \mathbb{Z}$ , by  $O \longrightarrow U \longrightarrow U/O$ .  $\square$

The method of proof is to produce maps  $\phi_i$  which induce isomorphisms on integral homology. When there is no torsion, we will work directly with integral homology or cohomology. In the other cases, we shall prove that  $\phi_i$  induces an isomorphism on mod  $p$  homology or cohomology for all primes  $p$  and shall then be able to conclude that  $\phi_i$  induces an isomorphism on integral homology by means of the following lemma.

**LEMMA A.4.** *Let  $f : X \longrightarrow Y$  be a continuous map between connected spaces having integral homology of finite type. If either  $f_*$  or  $f^*$  is an isomorphism for mod  $p$  coefficients and all primes  $p$ , then  $f_* : H_*(X; \mathbb{Z}) \longrightarrow H_*(Y; \mathbb{Z})$  is also an isomorphism.*

**PROOF.** Replacing  $Y$  by  $Mf$ , we may assume that  $f$  is an inclusion. Then  $H_*(Y, X; \mathbb{Z}_p) = 0$  for all  $p$ , hence  $H_*(Y, X; \mathbb{Z}) \otimes \mathbb{Z}_p \subset H_*(Y, X; \mathbb{Z}_p)$  is also zero and  $H_*(Y, X; \mathbb{Z}) = 0$ . The result follows.  $\square$

We next show that the  $\phi_i$  will induce isomorphisms on homotopy if they induce isomorphisms on integral homology. This conclusion follows from the fact that the  $\phi_i$  are  $H$ -maps (the point being that not all of our spaces are simply connected).

LEMMA A.5. *Let  $f : X \rightarrow Y$  be a fibration and a map of connected  $H$ -spaces. Then each  $\alpha \in f_* \pi_1(X, x_0)$  operates trivially on the homology of the fibre  $F = f^{-1}(y_0)$ , where  $x_0$  and  $y_0$  are the identities of  $X$  and  $Y$ .*

PROOF. Let  $q : I \rightarrow Y$  be a loop at  $y_0$  such that  $q = fr$  for a loop  $r : I \rightarrow X$  at  $x_0$ . Given  $s : \Delta_n \rightarrow F$ , define  $h(s) : I \times \Delta_n \rightarrow X$  by  $h(s)(t, u) = r(t)s(u)$ . Then  $h(s)$  satisfies the properties

- (1)  $h(s)(0, u) = x_0 s(u)$ ;
- (2)  $fh(s)(t, u) = q(t)y_0$ ;
- (3)  $h(s)(1 \times \delta_i) = h(\partial_i s)$ ,  $\delta_i : \Delta_{n-1} \rightarrow \Delta_n$ , the surjection to the  $i$ -th face;
- (4)  $h(s)(1, u) = x_0 s(u)$ .

$h_1 : SF \rightarrow SF$ ,  $h_1(s) = h(s)(1 \times \Delta_n)$ , induces the action of the class  $q$  on  $H_*(F)$ ;  $SF$  is the simplicial object corresponding to  $F$ . Since  $h_1$  is homotopic to the identity, this action is clearly trivial.  $\square$

PROPOSITION A.6. *Let  $f : X \rightarrow Y$  be a map of connected  $H$ -spaces which induces an isomorphism on integral homology. Then  $f$  is a homotopy equivalence (weak in general? heq if  $X, Y$  of htpy type of CW-cxs).*

PROOF. Let  $X' = \{(p, x) | x \in X, p : I \rightarrow Y, p(1) = f(x)\}$ , define  $f' : X' \rightarrow Y$  by  $f'(p, x) = f(x)$ , and define  $(p, x)(p', x') = (pp', xx')$ , where  $(pp')(t) = p(t)p'(t)$ . Then  $X$  is a deformation retract of  $X'$  and  $f'$  is a fibration and a map of  $H$ -spaces. Replacing  $X$  by  $X'$ , we may assume that  $f$  is a fibration. Since  $X$  and  $Y$  are  $H$ -spaces, their fundamental groups are Abelian, hence equal to their homology groups, hence equal to each other. By the lemma,  $\pi_1(Y)$  operates trivially on  $H_*(F)$ ,  $F = f^{-1}(y_0)$ .  $F$  is connected since  $\pi_1(f)$  is an isomorphism and, in the integral Serre spectral sequence of  $f$ ,  $E^r = H_*(Y; H_*(F))$  and  $E_{*,0}^2 = H_*(Y)$ . Since  $f_* : H_*(X) \cong H_*(Y)$ ,  $E_{*,0}^2 = E_{*,0}^\infty = E_{*,*}^\infty$ . By induction on  $q$ ,  $E_{p,q}^2 = 0$  for all  $q > 0$ , hence  $E_{0,q}^2 = H_q(F) = 0$  for  $q > 0$ . In detail, if  $E_{p,q}^2 = 0$  for  $q > 0$ , then  $E_{0,q+1}^2 = 0$ , since its elements are permanent cycles which cannot bound, and therefore  $E_{p,q+1}^2 = 0$  for all  $p$ , etc. Now  $H_*(F) = H_0(F) = \mathbb{Z}$ . Since  $F$  is an  $H$ -space,  $\pi_1(F)$  is Abelian, hence  $\pi_1(F) = H_1(F) = 0$ . By the Hurewicz theorem,  $\pi_q(F) = 0$  for all  $q$ , and the result follows from the long exact homotopy sequence of the fibration  $f$ .  $\square$

With these technicalities out of the way, we proceed to the definition of the maps  $\phi_i$ . In order to relate our various classical groups conveniently, we start with  $Z = Z' \oplus Z''^1$ , where  $Z'$  and  $Z''$  are right inner product spaces of countable dimension over  $\mathbb{H}$ . Regarding  $Z$  as a direct sum of copies of  $\mathbb{H}$ , we may also define a structure of a left quaternionic inner product space on  $Z$ . Define  $Y = \{z | zi = iz\} \subset Z$  and define  $X = \{y | yj = jy\} \subset Y$ . The left inner product on  $Z$  restricts to a complex inner product on  $Y$  and to a real inner product on  $X$ , and in fact we clearly have identifications of inner product spaces:

$$Z = Y_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} Y = X_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{R}} X$$

<sup>1</sup>This is easier to see starting with  $X$ ;  $Z = X \otimes_{\mathbb{R}} \mathbb{H}$ , and  $h(x \otimes h') = x \otimes hh'$  gives the left scalar multiplication

and

$$Y = X_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} X.$$

We shall write  $Z$  for  $Z$  regarded as a left inner product space.

Now  $O(X) = Sp(Z) \cap Sp(Z) = \{T | T \in Sp(Z), T(X) \subset X\}$ , and  $U(Y) = U(Z^{\mathbb{C}}) \cap Sp(Z) = \{T | T \in Sp(Z), T(Y) \subset Y\}$ . Thus, looking solely at right actions, we have the inclusions

$$(1) \quad Sp(Z') \times Sp(Z'') \subset Sp(Z) \subset U(Z^{\mathbb{C}}) \subset O(Z^{\mathbb{R}}),$$

and the intersection of this sequence with  $Sp(Z)$  is

$$(2) \quad O(X') \times O(X'') \subset O(X) \subset U(Y) \subset Sp(Z).$$

Now take  $BO(X) = O(X)/O(X') \times O(X'')$  and  $BSp(Z) = Sp(Z)/Sp(Z') \times Sp(Z'')$ . Then the inclusion of (2) in (1) induces the further inclusions

$$(3) \quad BO(X) \subset BSp(Z), U(Y)/O(X) \subset U(Z^{\mathbb{C}})/Sp(Z)$$

$$(4) \quad Sp(Z)/U(Y) \subset O(Z^{\mathbb{R}})/U(Z^{\mathbb{C}}), Sp(Z) \subset O(Z^{\mathbb{R}}).$$

Now, for  $0 \leq \theta \leq \pi$ , define  $\alpha_i(\theta) : Z \rightarrow Z$  by the formulas

$$(5) \quad \begin{aligned} \alpha_0(\theta)(z', z'') &= (z' e^{i\theta}, z'' e^{-i\theta}) \\ \alpha_1(\theta)(z', z'') &= \alpha_4(\theta)(z', z'') = (z' e^{i\theta/2}, z'' e^{-i\theta/2}) \\ \alpha_2(\theta)(z', z'') &= \alpha_5(\theta)(z', z'') = (z' e^{j\theta/2}, z'' e^{j\theta/2}) \\ \alpha_3(\theta)(z', z'') &= \alpha_6(\theta)(z', z'') = (z' e^{i\theta}, z'' e^{i\theta}) \end{aligned}$$

Using paths of length  $\pi$ , define  $\phi_i : O(Z^{\mathbb{R}}) \rightarrow PSO(Z^{\mathbb{R}})$ ,  $0 \leq i \leq 6$ , by

$$(6) \quad \phi_i(T)(\theta) = [T, \alpha_i(\theta)] = T \alpha_i(\theta) T^{-1} \alpha_i(\theta)^{-1}, 0 \leq \theta \leq \pi.$$

Since  $\alpha_i(0) = 1$ , our paths all start at the identity of  $SO(Z^{\mathbb{R}})$ . If  $i = 0, 3$ , or  $6$ , then  $\alpha_i(\pi) = 1$  and our paths are loops at the identity. Each  $\phi_i(T)(\theta)$  is a linear transformation of finite type, since  $T$  is, and is clearly a real isometry of determinant one. Thus  $\phi_i(T)$  is a path in  $SO(Z^{\mathbb{R}})$ , and  $\phi_i : O(Z^{\mathbb{R}}) \rightarrow PSO(Z^{\mathbb{R}})$  is continuous.

**PROPOSITION A.7.** *By restriction and passage to quotients, the maps  $\phi_i$  induce continuous functions, which are maps of  $H$ -spaces,*

$$(0) \quad \phi_0 : BU(Z) = U(Z)/U(Z') \times U(Z'') \rightarrow \Omega SU(Z)$$

$$(1) \quad \phi_1 : BO(X) = O(X)/O(X') \times O(X'') \rightarrow \Omega SU(Y)/SO(X)$$

$$(2) \quad \phi_2 : U(Y)/O(X) \rightarrow \Omega Sp(Z)/U(Y)$$

$$(3) \quad \phi_3 : Sp(Z)/U(Y) \rightarrow \Omega Sp(Z)$$

$$(4) \quad \phi_4 : BSp(Z) = Sp(Z)/Sp(Z') \times Sp(Z'') \rightarrow \Omega SU(Z)/Sp(Z)$$

$$(5) \quad \phi_5 : U(Z)/Sp(Z) \rightarrow \Omega SO(Z)/U(Z)$$

$$(6) \quad \phi_6 : SO(Z)/U(Z) \rightarrow \Omega_0 SO(Z) \simeq \Omega Spin(Z),$$

where  $\Omega_0$  denotes the component of the loop space containing the trivial loop.

PROOF. In each case, to prove that  $\phi_i : G_i/H_i \longrightarrow \Omega G'_i/H'_i$  is well-defined, we must verify the following three points

- (i):  $\phi_i(T)(\theta) \in G'_i$  if  $T \in G_i$
- (ii):  $\phi_i(T)(\theta) = 1$  if  $T \in H_i$  (so that  $[ST, \alpha_i(\theta)] = [S, \alpha_i(\theta)], S \in G_i$ )
- (iii):  $\phi_i(T)(\pi) \in H'_i$  for all  $T \in G_i$ .

Note that  $\phi_1, \phi_2$ , and  $\phi_3$  are obtained from  $\phi_4, \phi_5$ , and  $\phi_6$ , respectively, by restricting to linear transformations which commute with left multiplication by quaternions. Therefore  $\phi_{1,2,3}$  will be well-defined provided that  $\phi_{4,5,6}$  are well-defined. Now, (i) is satisfied for  $\phi_0$  and  $\phi_4$  since if  $T \in U(Z)$ , then  $[T, \alpha_i(\theta)]$  is a composite of linear transformations of  $Z$  regarded as a complex vector space (since  $\alpha_0(\theta)$  and  $\alpha_4(\theta)$  involve only  $i$  and not  $j$  or  $k$ ), and is easily seen to be isometric for the complex inner product; (i) is trivially satisfied for  $\phi_5$  and  $\phi_6$ . (ii) is satisfied for  $\phi_0$  and  $\phi_4$  since  $\alpha_0(\theta)$  and  $\alpha_4(\theta)$  are just scalar multiplications when restricted to  $Z'$  or to  $Z''$ . (ii) is satisfied for  $\phi_5$  since  $[T, \alpha_5(\theta)] = 1$  for  $T \in Sp(Z)$  follows from the fact that  $\alpha_5(\theta)$  is right multiplication by a quaternionic scalar. Similarly,  $[T, \alpha_6(\theta)] = 1$  for  $T \in U(Z)$  since  $\alpha_6(\theta)$  is right multiplication by a complex scalar, hence (ii) is satisfied for  $\phi_6$ . (iii) is trivially satisfied for  $\phi_0$  and  $\phi_6$ , since  $\alpha_0(\pi) = 1 = \alpha_6(\pi)$ , and it remains to verify (iii) for  $\phi_4$  and  $\phi_5$ .  $\alpha_4(z', z'') = (z'i, -z''i)$ ; let  $T \in Sp(Z)$ , with components  $T', T''$ . We must show that  $[T, \alpha_4(\pi)] \in Sp(Z)$ , and since  $T^{-1} \in Sp(Z)$ , it suffices to show that  $\alpha_4(\pi)T\alpha_4(-\pi) = ??$

$$\begin{aligned} \alpha_4(\pi)T^{-1}\alpha_4(-\pi)(z', z'') &= \alpha_4(\pi)(T'(-z'i, z''i), T''(-z'i, z''i)) \\ &= (T'(-z'i, z''i)i, -T''(-z'i, z''i)i) \\ &= (T'(z', -z''), T''(-z', z'')), \end{aligned}$$

from which it is clear that

$$\alpha_4(\pi)T\alpha_4(-\pi)(z'j, z''j) = \alpha_4(\pi)T\alpha_4(-\pi)(z', z'')j,$$

so that  $[T, \alpha_4(\pi)] \in Sp(Z)$ . Finally,  $\alpha_5(\pi)(z', z'') = (z'j, z''j)$ ; let  $T \in U(Z)$ . We must show that  $[T, \alpha_5(\pi)] \in U(Z)$ , and we have the formula

$$\begin{aligned} \alpha_5(\pi)T\alpha_5(-\pi)(z', z'') &= -\alpha_5(\pi)(T'(z'j, z''j), T''(z'j, z''j)) \\ &= -(T'(z'j, z''j)j, T''(z'j, z''j)j). \end{aligned}$$

Since  $T \in U(Z)$ , it follows easily that

$$\alpha_5(\pi)T\alpha_5(-\pi)(z'i, z''i) = \alpha_5(\pi)T\alpha_5(-\pi)(z', z'')i.$$

This proves (iii) for  $\phi_5$  and completes the proof that the  $\phi_i$  are well-defined. We must prove that the  $\phi_i$  are weak  $H$ -maps. We use the product induced from that on  $G'/H'$  on  $\Omega G'/H'$  (rather than the loop product). Let  $\{e'_i | i \geq 1\}, \{e''_i | i \geq 1\}$  be orthonormal symplectic bases for  $Z'$  and  $Z''$ . Define  $\mu : Z \longrightarrow Z$  by  $\mu(e'_i) = e'_{2i-1}$  and  $\mu(e''_i) = e''_{2i-1}$  and define  $\nu : Z \longrightarrow Z$  by  $\nu(e'_i) = e'_{2i}$  and  $\nu(e''_i) = e''_{2i}$ ;  $\mu$  and  $\nu$  induce maps  $O(Z_{\mathbb{R}}) \longrightarrow O(Z_{\mathbb{R}})$  by  $\mu(T)(e_{2i}) = e_{2i}$  and  $\mu(T)(e_{2i-1}) = \mu(T(e_i))$ ;  $\nu(T)(e_{2i-1}) = e_{2i-1}$  and  $\nu(T)(e_{2i}) = \nu(T(e_i))$ . The products are all induced from  $\phi : O(Z) \times O(Z) \longrightarrow O(Z)$  defined by  $\phi(S, T) = \mu(S) \cdot \nu(T)$  by restriction and projection. It suffices to show that  $\phi_i(ST)(\theta) = \phi_i(S)(\theta) \cdot \phi_i(T)(\theta)$  for all ??

$$[\mu(S)\nu(T), \alpha_i(\theta)] = \mu[S, \alpha_i(\theta)] \cdot \nu[T, \alpha_i(\theta)].$$

Visibly,  $\alpha_i(\theta) = \mu(\alpha_i(\theta))\nu(\alpha_i(\theta))$  and  $\mu(S)\nu(T) = \nu(T)\mu(S)$  for any  $S$  and  $T$ . Using these facts, we obtain the desired relation by a simple expansion of the relevant commutators.  $\square$

## 2. Commutative Diagrams in the $\phi_i$

Observe that  $\phi_0$  is defined for  $Y$  and that  $\phi_2, \phi_3, \phi_5,$  and  $\phi_6$  may be defined using  $Z'$  or  $Z''$  instead of  $Z$  (since classifying spaces are not involved). We take  $Z''$  to be a copy of  $Z'$ ,  $Z = Z' \oplus Z'$ , for convenience. We write  $\mu$  for the inclusions of (1) and any maps induced from them and  $\nu$  for the inclusions of (2) and any maps induced from them. Thus we have from (1) and (2) the commutative diagram

$$(1) \quad \begin{array}{ccccc} Sp(Z) & \xrightarrow{\mu} & U(Z) & \xrightarrow{\mu} & SO(Z) \\ \uparrow & & \uparrow & & \uparrow \\ O(X) & \xrightarrow{\nu} & U(Y) & \xrightarrow{\nu} & Sp(Z) \end{array}$$

We must next define maps  $\mu'$  and  $\nu'$  so as to have diagrams

$$(2) \quad \begin{array}{ccccc} Sp(Z') & \xrightarrow{\mu} & U'(Z') & \xrightarrow{\nu'} & Sp(Z) \\ & & \downarrow \mu & & \downarrow \mu \\ & & SO(Z') & \xrightarrow{\nu'} & SU(Z) \xrightarrow{\mu} SO(Z), \text{ and} \end{array}$$

$$(3) \quad \begin{array}{ccccc} O(X') & \xrightarrow{\nu} & U'(Y') & \xrightarrow{\mu'} & SO(X) \\ & & \downarrow \nu & & \downarrow \nu \\ & & Sp(Z') & \xrightarrow{\mu'} & SU(Y) \xrightarrow{\nu} Sp(Z). \end{array}$$

Here (3) will be obtained from (2) by restriction to transformations which commute with left multiplication by quaternions.  $U'(Z')$  denotes those elements of  $SO(Z')$  which commute with right multiplication by  $j$ , rather than  $i$  ( $\mathbb{C} \subset \mathbb{H}$  via  $i \mapsto j$  instead of  $i \mapsto i$ ), and  $U'(Y') = U'(Z') \cap Sp(Z')$ . Thus to obtain (2) and (3), it suffices to define  $\nu' : SO(Z') \rightarrow SU(Z)$  such that  $\nu'(U'(Z')) \subset Sp(Z)$ . For  $T : Z' \rightarrow Z', T \in SO(Z')$ , define

$$(4) \quad \begin{aligned} \nu'(T)(z', z'') &= \frac{1}{2}(T(z') - T(z''j) - T(z'i)i + T(z''k)i, \\ &T(z')j - T(z''j)j + T(z'i)k - T(z''k)k). \end{aligned}$$

Equivalently,  $\nu'(T) = \tau(T, T)\tau^{-1}$ , where  $\tau : Z \rightarrow Z$  is given by

$$(5) \quad \tau(z', z'') = \frac{1}{\sqrt{2}}(z' + z''i, z'j - z''k),$$

hence

$$\tau^{-1}(z', z'') = \frac{1}{\sqrt{2}}(z' - z''j, -z'i + z''k).$$

By a glance at formula (4),  $\nu'(T)(z'i, z''i) = \nu'(T)(z', z'')i$ ; and if  $T(z'j) = T(z')j$ , then  $\nu'(T)(z'j, z''j) = \nu'(T)(z', z'')j$ .  $\nu'(T)$  is easily verified to be a complex isometry of determinant one for  $T \in SO(Z')$ , and this gives diagrams (2) and (3). Note

that (4) reduces to

$$(6) \quad \nu'(T)(z', z'') = (T(z'), -T(z''j)j)$$

if  $T \in U(Z')$ ,  $T(z'i) = T(z')i$ . The  $\nu'$  and  $\mu'$  are equivalent to the standard inclusions and induce the same maps on homology and cohomology. In fact, if  $Z$  is given a new quaternionic structure  $\bar{Z}$  in such a manner that  $\tau : \bar{Z} \rightarrow Z$  is symplectic, then  $\bar{Z} = Z' \otimes_{\mathbb{C}} \mathbb{H}$  ( $\mathbb{C} = \{a + bj | a, b \in \mathbb{R}\}$ ) and  $\bar{Z}^{\mathbb{C}} = Z' \otimes_{\mathbb{R}} \mathbb{C}$  ( $\mathbb{C} = \{a + bi\}$ ), and  $\nu'$  is just symplectification (on  $U'(Z')$ ) or complexification (on  $SO(Z')$ ) followed by conjugation by  $\tau$ .

We shall also need the map  $\lambda : U(Z) \rightarrow U(Z)$  defined by

$$(7) \quad \lambda(T)(z) = [T, j_r](z) = -T(T^{-1}(zj)j), \quad \lambda : U(Z) \rightarrow U(Z),$$

where  $j_r$  denotes right multiplication by  $j$ .

PROPOSITION A.8. *The following diagrams are either commutative or homotopy commutative:*

$$\begin{array}{ccc} U'(Y')/O(X') & \xrightarrow{\phi'_2} & \Omega Sp(Z')/U'(Y') \\ \mu' \downarrow & & \downarrow \Omega(\mu') \\ BO(X) & \xrightarrow{\phi_1} & \Omega SU(Y)/SO(X) \\ \nu \downarrow & & \downarrow \Omega(\lambda) \\ BU(Y) & \xrightarrow{\phi_0} & \Omega SU(Y) \\ \mu' \uparrow & & \uparrow \Omega(\mu') \\ Sp(Z')/U(Y') & \xrightarrow{\phi_3} & \Omega Sp(Z') \end{array} \quad \begin{array}{ccc} U'(Z')/Sp(Z') & \xrightarrow{\phi'_5} & \Omega SO(Z')/U'(Z') \\ \nu' \downarrow & & \downarrow \Omega(\nu') \\ BSp(Z) & \xrightarrow{\phi_4} & \Omega SU(Z)/Sp(Z) \\ \mu \downarrow & & \downarrow \Omega(\lambda) \\ BU(Z) & \xrightarrow{\phi_0} & \Omega SU(Z) \\ \nu' \uparrow & & \uparrow \Omega(\nu') \\ SO(Z')/U(Z') & \xrightarrow{\phi_6} & \Omega Spin(Z') \end{array}$$

where the labels for the diagrams are given as written in: 2, 1, 3, 5, 4, 6 (topdown, then left to right).

PROOF. The left-hand diagrams are the restrictions of the right-hand diagrams to transformations which commute with left multiplication by quaternions, hence it suffices to prove that  $\Delta_4$ ,  $\Delta_5$ , and  $\Delta_6$  either commute or homotopy commute.

(i) *The diagram  $\Delta_6$  is commutative:*

*Proof:*  $\nu'$  is well-defined here in view of formula (6). By the definitions,

$$\begin{aligned} \Omega(\nu')\phi_6(T)(\theta) &= \nu'[T, \alpha_6(\theta)] \\ &= \tau([T, \alpha_6(\theta)], [T, \alpha_6(\theta)])\tau^{-1} \\ &= \tau([T, T], (\alpha_6(\theta), \alpha_6(\theta)))\tau^{-1} \\ &= [\tau(T, T)\tau^{-1}, \tau(\alpha_6(\theta), \alpha_6(\theta))\tau^{-1}], \\ \phi_0\nu'(T)(\theta) &= [\nu'(T), \alpha_0(\theta)] = [\tau(T, T)\tau^{-1}, \alpha_0(\theta)]. \end{aligned}$$

$\alpha_6(\theta) : Z' \rightarrow Z'$  is right multiplication by  $e^{i\theta}$ , hence commutes with right multiplication by  $i$ . By (6),  $\tau(\alpha_6(\theta), \alpha_6(\theta))\tau^{-1} = (\alpha_6(\theta), -j_r\alpha_6(\theta)j_r)$ . Since  $e^{i\theta}j = je^{-i\theta}$ ,  $\tau(\alpha_6(\theta), \alpha_6(\theta))\tau^{-1} = (e_r^{i\theta}, e_r^{-i\theta}) = \alpha_0(\theta)$ , as desired.

(ii) *The diagram  $\Delta_5$  is commutative:*

*Proof:*  $\phi'_5$  is defined from  $\alpha'_5(\theta) = e_r^{i\theta/2} : Z' \rightarrow Z'$ , since the roles of  $i$  and  $j$  must

be reversed by the definition of  $U'(Z')$ . Here  $\nu'$  is again well-defined in virtue of (6). Precisely, as in the previous proof,

$$\begin{aligned}\tau(\alpha'_5(\theta), \alpha'_5(\theta))\tau^{-1} &= (e_r^{i\theta/2}, -j_r e^{i\theta/2} j_r) \\ &= (e_r^{i\theta/2}, e_r^{-i\theta/2}) = \alpha_4(\theta),\end{aligned}$$

and

$$\begin{aligned}\Omega(\nu')\phi'_5(T)(\theta) &= [\tau(T, T)\tau^{-1}, \tau(\alpha'_5(\theta), \alpha'_5(\theta))\tau^{-1}] \\ &= [\nu'(T), \alpha_4(\theta)] = \phi_4\nu'(T)(\theta).\end{aligned}$$

(iii) *The diagram  $\Delta_4$  is homotopy commutative:*

*Proof:*  $\lambda(T) = [T, j_r] = 1$  if  $T \in Sp(Z)$ , hence  $\Omega(\lambda)$  is well-defined. Clearly  $\phi_0\mu(T)(\theta) = [T, \alpha_0(\theta)] = [T, \alpha_4(2\theta)]$ , since  $\alpha_0(\theta) = \alpha_4(2\theta)$ .  $j_r\alpha_4(\theta) = \alpha_4(-\theta)j_r$ ;

$$\begin{aligned}\Omega(\lambda)\phi_4(T)(\theta) &= [[T, \alpha_4(\theta)], j_r] \\ &= T\alpha_4(\theta)T^{-1}\alpha_4(-\theta)j_r\alpha_4(\theta)T\alpha_4(-\theta)T^{-1}j_r^{-1} \\ &= T\alpha_4(\theta)T^{-1}\alpha_4(-2\theta)T\alpha_4(\theta)T^{-1}\end{aligned}$$

(for  $T \in BSp(Z)$ ,  $[T, j_r] = 1$ ). If  $h : BSp(Z) \times I \rightarrow \Omega SU(Z)$  is defined by the formula  $h(T, t)(\theta) = T\alpha_4(\theta + t\theta)T^{-1}\alpha_4(-2\theta)T\alpha_4(\theta - t\theta)T^{-1}$ , then  $h$  is a homotopy from  $\Omega(\lambda)\phi_4$  to  $\phi_0\mu$ . This completes the proof of the proposition.  $\square$

We shall need certain auxiliary diagrams to study some of the  $\phi_i$ . Let  $P(G, H)$  denote the paths of length  $\pi$  in  $G$  which start at the identity and end in  $H$ . Let  $\pi : G \rightarrow G/H$  and let  $p : PG \rightarrow G$  and  $p : P(G, H) \rightarrow H$  denote the endpoint projection maps. We have  $\Omega(G/H) = P(G/H, H/H)$  and thus a commutative diagram

$$\begin{array}{ccccc} P(G, H) & \longrightarrow & P(G) & \xrightarrow{\pi p} & G/H \\ P(\pi) \downarrow & & P(\pi) \downarrow & & \parallel \\ \Omega(G/H) & \longrightarrow & P(G/H) & \xrightarrow{p} & G/H \end{array}$$

$P(\pi) : P(G, H) \rightarrow \Omega(G/H)$  is a weak homotopy equivalence, which, by abuse, we treat as an identification. Note that, by construction,  $\phi_1, \phi_2, \phi_4$ , and  $\phi_5$  all factor through the relevant  $P(\pi)$ .

LEMMA A.9. *The following are commutative diagrams of Serre fibrations:*

$$\begin{array}{ccc} P(SO(Z'), U(Z')) & & P(SU'(Y'), SO(X')) \\ \parallel & & \parallel \\ \Omega SO(Z')/U(Z') \xrightarrow{p} U(Z') & & \Omega SU'(Y')/SO(X') \xrightarrow{p} SO(X') \\ \downarrow \subset & & \downarrow \subset \\ PSO(Z') \xrightarrow{\nu' p} EU(Z) & & PSU'(Y') \xrightarrow{\mu' p} ESO(X) \\ \downarrow \pi p & & \downarrow \pi p \\ SO(Z')/U(Z')' \xrightarrow{\nu} BU(Z) & & SU'(Y')/SO(X')' \xrightarrow{\mu} BSO(X) \end{array}$$

Denote the lower-left square by  $\Gamma_2$  and the lower-right by  $\Gamma_1$ .

PROOF.  $EU(Z) = U(Z)/e \times U(Z')$  and  $ESO(X) = SO(X)/e \times SO(X')$ ,  $i(T) = T \times 1$ .  $\mu'$  and  $\nu'$  are well-defined in view of formula (6) above.  $\pi\nu'p = \nu'\pi p$  and  $\pi\mu'p = \mu'\pi$  since clearly  $\pi\nu' = \nu'\pi$  and  $\pi\mu' = \mu'\pi$ . If  $F \in P(SO(Z'), U(Z'))$ , then  $\nu'p(f) = \nu'f(\pi) = (f(\pi), j_r f(\pi) j_r^{-1})$  by (6), while  $ip(f) = (f(\pi), 1)$ . These are clearly equal in  $EU(Z)$ . The upper square of  $\Gamma_1$  commutes for the same reasons.  $\square$

LEMMA A.10. *With  $\lambda$  and  $\lambda'$  induced from (7), we have commutative diagrams*

$$\begin{array}{ccc} U(Z')/Sp(Z') & \xrightarrow{\phi_5} & \Omega SO(Z')/U(Z') \\ \uparrow \pi & \searrow \lambda & \downarrow p \\ U(Z') & \xrightarrow{\lambda'} & U(Z') \end{array} \qquad \begin{array}{ccc} U(Y')/O(X') & \xrightarrow{\phi_2} & \Omega Sp(Z')/U(Y') \\ \uparrow \pi & \searrow \lambda & \downarrow p \\ U(Y') & \xrightarrow{\lambda'} & U(Y') \end{array}$$

Write  $H_*(U; Z) = E\{x_{2i-1} | i \geq 1\}$ ,  $x_{2i-1} = \sigma^*(c_i)^*$ . Then  $\lambda'_*(x_{2i-1}) = [1 - (-1)^i]x_{2i-1}$ ,  $i \geq 1$ .

PROOF. The right-hand diagram is the restriction of the left-hand diagram to transformations which commute with left-multiplication by quaternions, hence it suffices to prove the result for the left-hand diagram. Note that  $\lambda(T) = p\phi_5(T) = [T, \alpha_5(\pi)] = [T, j_r]$ , hence  $\lambda'(T) = \lambda\pi(T) = [T, j_r] = T j_r T^{-1} j_r^{-1} = \phi(1 \times \alpha\gamma)\Delta(T)$ , where  $\Delta$  is the diagonal,  $\alpha(T) = j_r T j_r^{-1}$ ,  $\gamma(T) = T^{-1}$ , and  $\phi$  is the group product. Since  $\phi(\gamma \times 1)\Delta(T) = 1$ ,  $0 = \phi_*(\gamma_* \otimes 1)\Delta_*(x_i) = x_i + \gamma_*(x_i)$  and  $\gamma_*(x_i) = -x_i$ .  $\alpha$  is an automorphism, hence  $\alpha_*(x_i) = (-1)^{\varepsilon_i} x_i$ . In  $H^*(BU; Z)$ ,  $(-1)^{\varepsilon_i} c_i = (B\alpha)^*(c_i)$ , since  $\sigma^*(B\alpha)^* = \alpha^* \sigma^*$ , and  $\psi(B\alpha)^* = (B\alpha)^* \otimes (B\alpha)^* \psi$ , hence

$$(-1)^{\varepsilon_i} \sum_{j+k=i} c_j \otimes c_k = \sum_{j+k=i} (-1)^{\varepsilon_j + \varepsilon_k} c_j \otimes c_k.$$

Thus  $\varepsilon_j + \varepsilon_k = \varepsilon_i$ , and  $\varepsilon_i = i\varepsilon_1$ . To compute  $\varepsilon_1$ , observe that  $x_1$  is the image of the fundamental class of  $S^1$  under  $i : S^1 = U(1) \rightarrow U$ ,  $i(e^{i\theta}) = e_r^{i\theta}$  on the first coordinate of  $U = U(Z')$  and  $i(e^{i\theta})$  is the identity on the remaining coordinates. Since  $j_r e_r^{i\theta} j_r = e_r^{-i\theta}$ , it is clear that  $\alpha_*(x_1) = -x_1$ ,  $\varepsilon_1 = 1$ . Now the result follows since  $\lambda'_*(x_i) = \phi_*(1 \otimes \alpha_* \gamma_*)\Delta_*(x_i) = [1 - (-1)^i]x_i$ , because  $x_i$  is primitive.  $\square$

### 3. Proof of the Periodicity Theorem

We prove in turn that each  $\phi_{i*}$  or  $\phi_i^*$  is an isomorphism either integrally or mod  $p$  for all primes  $p$ . This will suffice.

LEMMA A.11.  $\phi_{0*} : H_*(BU; \mathbb{Z}) \rightarrow H_*(\Omega SU; \mathbb{Z})$  is an isomorphism.

PROOF.  $H^*(BU) = P\{c_i\}$ , hence  $H_*(BU) = P\{\gamma_i\}$ , where  $\langle c_1^i, \gamma^i \rangle = 1$ . Thus  $H_*(BU) = P(j_* H_*(BU(1)))$ ,  $j : BU(1) \rightarrow BU$ .

Consider  $\phi_0 : BU(Y) \rightarrow \Omega SU(Y)^2$ ,  $Y = Y' \oplus Y''$ , where  $Y'$  and  $Y''$  have complex orthonormal bases ...?  $\square$

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<sup>2</sup> $\phi_0(T)(\theta) = [T, \alpha_0(\theta)] = T\alpha_0(\theta)T^{-1}\alpha_0(\theta)^{-1}$ ,  $0 \leq \theta \leq \pi$ ,  $\alpha_0(\theta)(y', y'') = (e^{i\theta}y', e^{-i\theta}y'')$



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