NOTES ON 1- AND 2-GERBES

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These aim of these notes is to discuss in an informal manner the construction and some properties of 1- and 2-gerbes. They are for the most part based on the author's texts [1]-[4]. Our main aim is to describe the construction which associates to a gerbe or a 2-gerbe determines the corresponding degree 1 or 2 non-abelian cohomology class.

We begin by reviewing the well-known theory for principal bundles and show how to extend this to biprincipal bundles (a.k.a bitorsors). After reviewing the definition of stacks and gerbes, we construct the cohomology class associated to a gerbe. While the construction presented is equivalent to that in [4], it is clarified here by making use of diagram (5.1.9), a definite improvement over the corresponding diagram [4] (2.4.7), and of (5.2.7). After a short discussion regarding the role of gerbes in algebraic topology, we pass from 1− to 2−gerbes. The construction of the associated cohomology classes follows the same lines as for 1-gerbes, but with the additional degree of complication entailed by passing from 1- to 2-categories, so that it now it now involves diagrams reminiscent of those in [5]. Our emphasis will be on explaining how the fairly elaborate equations which define cocycles and coboundaries may be reduced to terms which can be described in the tradititional formalism of non-abelian cohomology.

Since the concepts discussed here are very general, we have at times not made explicit the mathematical objects to which they apply. For example, when we refer to "a space" this might mean a topological space, but also "a scheme" when one prefers to work in an algebro-geometric context, or even "a sheaf" and we place ourselves implicitly in the category of such spaces, schemes, or sheaves. Similarly, in computing cocycles, we will refer to spaces X endowed with a covering $\mathcal{U} := (U_i)_{i\in I}$ by open sets, but the discussion remains valid when the disjoint union $\coprod_{i\in I} U_i$ is replaced by an arbitrary covering morphism $Y \longrightarrow X$ for a given Grothendieck topology. The emphasis in vocabulary will be on spaces rather than schemes, and we have avoided any non-trivial result from algebraic geometry. In that sense, the text is implicitly directed towards topologists and category-theorists rather than algebraic geometers, even though we have not sought to make precise the category of spaces in which we work.

1. Torsors and bitorsors

1.1. Let G be a bundle of groups on a space X. The following definition of a principal space is standard, but note occurence of a structural bundle of groups, rather than simply a constant one. In other words, we give ourselves a family of groups G_x, parametrized by points $x \in X$, acting principally on the corresponding fibers P_x of P.

Definition 1.1. *A left principal G-bundle (or left G-torsor) on a topological space X is a space $P \xrightarrow{\pi} X$ above X, together with a left group action $G \times_X P \longrightarrow P$ such that the induced morphism*

$$\begin{array}{ccc} G \times_X P & \simeq & P \times_X P \\ (g, p) & \mapsto & (gp,\, p) \end{array} \qquad (1.1.1)$$

is an isomorphism. We require in addition that there exists a family of local sections $s_i : U_i \longrightarrow P$*, for some open cover* $\mathcal{U} = (U_i)_{i \in I}$ *of* X*. The groupoid of left* G*-torsors on* X *will be denoted* $\rightarrow (X, G)$*.*

The choice of a family of local sections $s_i : U_i \longrightarrow P$, determines a G-valued 1-cochain $g_{ij} : U_{ij} \longrightarrow G$, defined above $U_{ij} := U_i \cap U_j$ by the equations

$$s_i = g_{ij}\, s_j \qquad \forall\, i, j \in I\,, \tag{1.1.2}$$

and which therefore satisfies the 1-cocycle equation

$$g_{ik} = g_{ij}\, g_{jk} \tag{1.1.3}$$

above U_{ijk}. Two such families of local sections $(s_i)_{i\in I}$ and $(s'_i)_{i\in I}$ on the same open cover $\mathcal{U}$ differ by a G-valued 0-cochain $(g_i)_{i\in I}$ defined by

$$s'_i = g_i s_i \qquad\qquad \forall i \in I \tag{1.1.4}$$

and for which the corresponding 1-cocycles g_{ij} and g'_{ij} are related to each other by the coboundary relations

$$g'_{ij} = g_i\, g_{ij}\, g_j^{-1} \tag{1.1.5}$$

This equation determines an equivalence relation on the set of 1-cocycles $Z^1(\mathcal{U},\, G)$ (1.1.3), and the induced set of equivalence classes for this equivalence relation is denoted $H^1(\mathcal{U}, G)$. Passing to the limit over open covers $\mathcal{U}$ of X yields the Čech non-abelian cohomology set $\check{H}^1(X,\, G)$, which classifies isomorphism classes of G-torsors on X. This set is endowed with a distinguished element, the class of the trivial left G-torsor T_G.

Definition 1.2. *Let* X *be a space, and* G *and* H *a pair of bundles of groups on* X*. A* (G, H)*-bitorsor on* X *is a space* P *over* X*, together with fiber-preserving left and right actions of* G *and* H *on* P*, which commute with each other and which define both a left* G*-torsor and a right* H*-torsor structure on* P*. For any bundle of groups* G*, a* (G, G)*-bitorsor is simply called a* G*-bitorsor.*

A family of local sections s_i of a (G, H)-bitorsor P determines a local identification of P with both the trivial left G-torsor and the trivial right H-torsor. It therefore defines a family of local isomorphisms $u_i : H_{U_i} \longrightarrow G_{U_i}$ between the restrictions above U_i of the bundles H and G, which are explicitly given by the rule

$$s_i h = u_i(h) s_i \tag{1.1.6}$$

for all $h \in H_{U_i}$. This however does not imply that the bundles of groups H and G are globally isomorphic.

Example 1.3. *i*) The trivial G-bitorsor on X: the right action of G on the left G-torsor T_G is the trivial one, given by fibrewise right translation. This bitorsor will also be denoted T_G.

ii) The group $P^{\mathrm{ad}} := \mathrm{Aut}_G(P)$ of G-equivariant fibre-preserving automorphisms of a left G-torsor P acts on the right on P by the rule

$$pu := u^{-1}(p)$$

so that any left G-torsor P is actually a (G, P^{ad})-bitorsor. The group P^{ad} is know as the gauge group of P. In particular, a left G-torsor P is a (G, H)-bitorsor if and only if the bundle of groups P^{ad} is isomorphic to H.

iii) Let

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1 \tag{1.1.7}$$

be a short exact sequence of bundles of groups on X. Then H is a G_K-bitorsor on K, where the left and right actions above K of the bundle of groups $G_K := G \times_X K$ are given by left and right multiplication in H.

1.2. Let P be a (G,H)-bitorsor and Q be an (H,K)-bitorsor on X. The contracted product

$$P \wedge^H Q := \frac{P \times_X Q}{(ph,q) \sim (p,hq)} \tag{1.2.1}$$

of P and Q is a (G,K)-bitorsor on X. To any (G,H)-bitorsor P on X is associated the opposite (H,G)-bitorsor P^o, with same underlying space as P, and for which the right action of G (*resp.* left action of H) is induced by the given left G-action (*resp.* right H-action) on P. For a given bundle of groups G on X, the category of G-bitorsors is a group-like monoidal category on X, for which the tensor multiplication is the contracted product of G-bitorsors, the unit object is the trivial bitorsor T_G, and P^o is an inverse of P. Group-like monoidal categories are also known as *gr*-categories.

1.3. **Twisted objects:**

Let P be a left G-torsor on X, and E an X-object on which G acts on the right. We say that the X-object $E^P := E \wedge^G P$, defined as in (1.2.1), is the P-twisted form of E. The choice of a local section p of P above an open set U determines an isomorphism $\phi_p : E^P_{|U} \simeq E_{|U}$. Conversely, if E_1 is an X-object for which there exist a open cover $\mathcal{U}$ of X above which E_1 is locally isomorphic to E, then the space $\mathrm{Isom}_X(E_1,\, E)$ is a left torsor on X under the action of the bundle of groups $G := \mathrm{Aut}_X E$.

Proposition 1.4. *These two constructions are inverse to each other.*

Example 1.5. Let G be a bundle of groups on X and H a bundle of groups locally isomorphic to G and let $P := \mathrm{Isom}_X(H,\, G)$ be the left $\mathrm{Aut}(G)$-torsor of fiber-preserving isomorphisms from H to G. The map

$$\begin{array}{ccc} G \wedge^{\mathrm{Aut}(G)} P & \xrightarrow{\sim} & H \\ (g,\, u) & \mapsto & u^{-1}(g) \end{array}$$

identifies H with the P-twisted form of G, for the right action of $\mathrm{Aut}(G)$ on G induced by the standard left action. Conversely, for a fixed bundle of groups G on X, the giving of a G-torsor P determines a bundle of groups $H := G \wedge^{\mathrm{Aut}(G)} P$ locally isomorphic to G, and P is isomorphic to the left $\mathrm{Aut}(G)$-torsor $\mathrm{Isom}(H,\, G)$.

The next example is very well-known, but deserves to be spellt out in some detail.

Example 1.6. Any rank n vector bundle $\mathcal{V}$ on X is locally isomorphic to the trivial bundle $\mathbb{R}^n_X := X \times \mathbb{R}^n$, whose group of automorphisms is the trivial bundle of groups

$$GL(n,\mathbb{R})_X := GL(n,\mathbb{R}) \times X$$

on X. The left principal $GL(n,\, \mathbb{R})_X$-bundle associated to $\mathcal{V}$ is its bundle of frames $P_{\mathcal{V}} := \mathrm{Isom}(\mathcal{V},\, \mathbb{R}^n_X)$. The vector bundle $\mathcal{V}$ may be recovered from $P_{\mathcal{V}}$ via the isomorphism

$$\begin{array}{ccc} \mathbb{R}^n_X \wedge^{GL(n,\mathbb{R})_X} P_{\mathcal{V}} & \xrightarrow{\sim} & \mathcal{V} \\ (y,p) & \mapsto & p^{-1}(y) \end{array} \tag{1.3.1}$$

in other words as the $P_{\mathcal{V}}$-twist of the trivial vector bundle $\mathbb{R}^n_X$ on X. Conversely, for any principal $GL(n,\, \mathbb{R})_X$-bundle P on X, the twisted object $\mathcal{V} := \mathbb{R}^n_X \wedge^{GL(n,\mathbb{R})} P$ is known as the rank n vector bundle associated to P. Its frame bundle $P_{\mathcal{V}}$ is canonically isomorphic to P.

Remark 1.7. In (1.3.1), the right action on $\mathbb{R}^n_X$ of the linear group $\mathrm{GL}(n,\, \mathbb{R})_X$ is given by the rule

$$\begin{array}{ccc} \mathbb{R}^n \times \mathrm{GL}(n,\, \mathbb{R}) & \longrightarrow & \mathbb{R}^n \\ (Y,\, A) & \mapsto & A^{-1}Y \end{array}$$

where an element of $\mathbb{R}^n$ is viewed as a column matrix $Y = (\lambda_1, \dots, \lambda_n)^T$. A local section p of $P_{\mathcal{V}}$ determines a local basis $\mathcal{B} = \{p^{-1}(e_i)\}$ of $\mathcal{V}$ and the arrow (1.3.1) then identifies the column vector Y with the element of $\mathcal{V}$ with coordinates (λ_i) in the chosen basis p. The fact that the arrow (1.3.1) factors through the contracted product is a global version of the familiar linear algebra rule which in an n-dimensional vector space V describes the effect of a change of basis matrix A on the coordinates Y of a given vector $v \in V$.

1.4. **The cocyclic description of a bitorsor ([19], [1]):**

Consider a (G, H)-bitorsor P on X, with chosen local sections $s_i : U_i \longrightarrow P$ for some open cover $\mathcal{U} = (U_i)_{i \in I}$. Viewing P as a left G-torsor, we know by (1.1.2) that these sections define a family of G-valued 1-cochains g_{ij} satisfying the 1-cocycle condition (1.1.3). We have also seen that the right H-torsor structure on P is then described by the family of local isomorphisms $u_i : H_{U_i} \longrightarrow G_{U_i}$ defined by the equations (1.1.6) for all $h \in H_{U_i}$. It follows from (1.1.2) and (1.1.6) that the transition law for the restrictions of these isomorphisms above U_{ij} is

$$u_i = i_{g_{ij}}\, u_j \tag{1.4.1}$$

with i the inner conjugation homomorphism

$$\begin{array}{ccc} G & \stackrel{i}{\longrightarrow} & \mathrm{Aut}(G) \\ g & \mapsto & i_g \end{array} \tag{1.4.2}$$

defined by

$$i_g(\gamma) = g\gamma g^{-1}\,. \tag{1.4.3}$$

The pairs $(g_{ij},\, u_i)$ therefore satisfy the cocycle conditions

$$\begin{cases} g_{ik} = & g_{ij}\, g_{jk} \\ u_i = & i_{g_{ij}}\, u_j \end{cases} \tag{1.4.4}$$

A second family of local sections s'_i of P determines a corresponding cocycle pair $(u'_i,\, g'_{ij})$, These new cocycles differ from the previous ones by the coboundary relations

$$\begin{cases} g'_{ij} = g_i\, g_{ij}\, g_j^{-1} \\ u'_i = i_{g_i}\, u_i \end{cases} \tag{1.4.5}$$

where the 0-cochains g_i are defined by (1.1.4). Isomorphism classes of (G, H)-bitorsors on X with given local trivialization on an open covering $\mathcal{U}$ are classified by the quotient of the set of cocycles $(u_i,\, g_{ij})$ (1.4.4) by the equivalence relation (1.4.5). Note that when P is a G-bitorsor, the terms of the second equation in both (1.4.4) and (1.4.5) lives in the group $\mathrm{Aut}(G)$. In that case, the set of cocycle classes is the non-abelian hypercohomology set $H^0(\mathcal{U},\, G \longrightarrow \mathrm{Aut}(G))$, with values in the complex of groups (1.4.2). Passing to the limit over open covers, we obtain the Čech cohomology set $\check{H}^0(X,\, G \longrightarrow \mathrm{Aut}(G))$ which classifies isomorphism classes of G-bitorsors on X.

Let us see how the monoidal structure on the category of G-bitorsors is reflected at the cocyclic level. Let P and Q be a pair of G-bitorsors on X, with chosen local sections p_i and q_i. These determine corresponding cocycle pairs $(g_{ij},\, u_i)$ and $(\gamma_{ij},\, v_i)$ satisfying the corresponding equations (1.4.4). It is readily verified that the corresponding cocycle pair for the G-bitorsor $P \wedge^G Q$, locally trivialized by the family of local sections $p_i \wedge q_i$, is the pair

$$(g_{ij}\, u_i(\gamma_{ij}),\, u_i\, v_i) \tag{1.4.6}$$

so that the group law for cocycle pairs is simply the semi-direct product multiplication in the group $G \rtimes \mathrm{Aut}(G)$, for the standard left action of $\mathrm{Aut}(G)$ on G. The multiplication rule for cocycle pairs

$$(g_{ij},\, u_i) * (\gamma_{ij},\, v_i) = (g_{ij}\, u_i(\gamma_{ij}),\, u_i\, v_i)$$

passes to the set of equivalence classes, and therefore determines a group structure on the set $\check{H}^0(X,\, G \longrightarrow \mathrm{Aut}(G))$, which reflects the contracted product of bitorsors.

Remark 1.8. Let us choose once more a family of local sections s_i of a (G,H)-bitorsor P. The local isomorphisms u_i provide an identification of the restrictions H_{U_i} of H with the restrictions G_{U_i} of G. Under these identifications, the significance of equations (1.4.1) is the following. By (1.4.1), we may think of an element of H as given by a family of local elements $\gamma_i \in G_i$, glued to each other above the open sets U_{ij} according to the rule

$$\gamma_i = i_{g_{ij}}\,\gamma_j\,.$$

For this reason, a bundle of groups H which stands in such a relation to a given group G may be called an *inner form* of G. This is the terminology used in the context of Galois cohomology, *i.e* when X is a scheme $\mathrm{Spec}(k)$ endowed with the étale topology defined by the covering morphism $\mathrm{Spec}(k') \longrightarrow \mathrm{Spec}(k)$ associated to a Galois field extension k'/k ([19] III §1).

1.5. The previous discussion remains valid in a wider context, in which the inner conjugation homomorphism i is replaced by an arbitrary homomorphism of groups $\delta : G \longrightarrow \Pi$. The cocycle and coboundary conditions (1.4.4) and (1.4.5) are now respectively replaced by the rules

$$\begin{cases} g_{ik} &= g_{ij}\,g_{jk} \\ \pi_i &= \delta(g_{ij})\,\pi_j \end{cases} \tag{1.5.1}$$

and by

$$\begin{cases} g'_{ij} = g_i\,g_{ij}\,g_j^{-1} \\ \pi'_i = \delta(g_i)\,\pi_i \end{cases} \tag{1.5.2}$$

and the induced Čech hypercohomology set with values in the complex of groups $G \longrightarrow \Pi$ is denoted $\check{H}^0(\mathcal{U}, G \longrightarrow \Pi)$. In order to extend to $\check{H}^0(\mathcal{U}, G \longrightarrow \Pi)$ the multiplication (1.4.6), we require additional structure:

Definition 1.9. *A (left) crossed module is a group homomorphism $\delta : G \longrightarrow \Pi$, together with a left group action*

$$\begin{array}{ccc} \Pi \times G & \longrightarrow & G \\ (\pi,\, g) & \mapsto & {}^{\pi}g \end{array}$$

of Π on the group G, and such that the equations

$$\begin{cases} \delta({}^{\pi}g) = {}^{\pi}\delta(g) \\ {}^{\delta(\gamma)}g = {}^{\gamma}g \end{cases} \tag{1.5.3}$$

are satisfied, with G (resp. Π) acting on itself by the conjugation rule (1.4.3).

Crossed modules form a category, with a homomorphism of crossed modules

$$(G \xrightarrow{\delta} \pi) \longrightarrow ((K \xrightarrow{\delta'} \Gamma)$$

defined by a pair of homomorphisms (u, v) such that the diagram of groups

$$\begin{array}{ccc} G & \xrightarrow{u} & K \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ \Pi & \xrightarrow[v]{} & \Gamma \end{array} \tag{1.5.4}$$

commutes, and such that $u({}^{\pi}g) = {}^{v(\pi)}u(g)$ (in other words such that u is v-equivariant).

A left crossed module $G \xrightarrow{\delta} \Pi$ defines a group-like monoidal category $\mathcal{C}$ with a strict multiplication on objects, by setting

$$\mathrm{ob}\,\mathcal{C} := \Pi \qquad \mathrm{ar}\,\mathcal{C} := G \times \Pi \tag{1.5.5}$$

The source and target of an arrow $(g,\, \pi)$ are as follows:

$$\pi \xrightarrow{(g,\pi)} \delta(g)\pi$$

and the composite of two composable arrows

$$\pi \xrightarrow{(g,\pi)} \delta(g)\pi \xrightarrow{(g',\,\delta(g)\pi)} \delta(g'g), \pi \tag{1.5.6}$$

is the arrow $(g'g,\, \pi)$. The monoidal structure on this groupoid is given on the objects by the group multiplication in Π, and on the set $G \times \Pi$ of arrows by the semi-direct product group multiplication

$$(g,\, \pi) * (g'\, \pi') := (g\, {}^{\pi}g',\, \pi\, \pi') \tag{1.5.7}$$

for the given left action of Π on G. In particular the identity element of the group Π is the unit object I of this monoidal groupoid.

Conversely, to a monoidal category $\mathcal{M}$ with strict multiplication on objects is associated a crossed module $G \xrightarrow{\delta} \Pi$, where $\Pi := \mathrm{ob}\, \mathcal{M}$ and G is the set $\mathrm{Ar}_I\mathcal{M}$ of arrows of $\mathcal{M}$ sourced at the identity object, with δ the restriction to G of the target map. The group law on G is the restriction to this set of the multiplication of arrows in the monoidal category $\mathcal{M}$. The action of an object $\pi \in \Pi$ on an arrow $g : I \longrightarrow \delta(g)$ in G has the following categorical interpretation: the composite arrow

$$I \xrightarrow{\sim} \pi\, I\, \pi^{-1} \xrightarrow{\pi\, g\, \pi^{-1}} \pi\, \delta(g)\, \pi^{-1}$$

corresponds to the element ${}^{\pi}g$ in G. Finally, given a pair elements $g, g' \in \mathrm{Ar}_I\mathcal{M}$, it follows from the composition rule (1.5.6) for a pair of arrows that the composite arrow

$$I \xrightarrow{(g,I)} \delta(g) \xrightarrow{(g',\delta(g))} \delta(g'g)$$

(constructed by taking advantage of the monoidal structure on the category $\mathcal{M}$ in order to transform the arrow g' into an arrow $(g',\, \delta(g))$ composable with g) is simply given by the element $g'g$ of the group $\Pi = \mathrm{Ar}_I\mathcal{M}$.

A stronger concept than that of a homomorphism of crossed module is what could be termed a “crossed module of crossed modules”. This is the categorification of crossed modules and corresponds, when one extends the previous dictionary between strict monoidal categories and crossed modules, to strict monoidal bicategories. The most efficient description of such a concept is the notion of a crossed square, due to J.-L Loday. This consists of a homomorphism of crossed modules (1.5.4), together with a map

$$\begin{array}{ccc} K \times \Pi & \longrightarrow & G \\ (k,\, \pi) & \mapsto & \{k,\, \pi\} \end{array} \tag{1.5.8}$$

satisfying certain conditions for which we refer to [14] definition 5.1.

Remark 1.10. *i)* The definition (1.5.4) of a homomorphism of crossed modules is quite restrictive, and it is often preferable to relax it so that it defines a not necessarily strict monoidal functor between the associated (strict) monoidal groupoids. Such a definition of a weak homomorphism of crossed modules has recently been spellt out by B. Noohi in [16] definition 8.4.

ii) All these definitions obviously extends from groups to bundles of groups on X.

iii) The composition law (1.5.7) determines a multiplication

$$(g_{ij},\, \pi_i) * (g'_{ij}, \pi'_i) := (g_{ij}\, {}^{\pi_i}g'_{ij},\, \pi_i\, \pi'_i)$$

on $(G \longrightarrow \Pi)$-valued cocycle pairs, which generalizes (1.4.6), is compatible with the coboundary relations, and induces a group structure on the set $\check{H}^0(\mathcal{U},\, G \longrightarrow \Pi)$ of degree zero cohomology classes with values in the crossed module $G \longrightarrow \Pi$ on X.

1.6. The following proposition is known as the Morita theorem, by analogy with the corresponding characterization in terms of bimodules of equivalences between certain categories of modules.

Proposition 1.11. (Giraud [10]) *i) A (G, H)-bitorsor Q on X determines an equivalence*

$$\begin{array}{ccc} Tors(H) & \xrightarrow{\Phi_Q} & Tors(G) \\ M & \mapsto & Q \wedge^H M \end{array}$$

between the corresponding categories of left torsors on X. In addition, if P is an (H, K)-bitorsor on X, then there is a natural equivalence

$$\Phi_{Q \wedge^H P} \simeq \Phi_Q \circ \Phi_P$$

between functors from $\mathrm{Tors}(K)$ *to* $\mathrm{Tors}(G)$*. In particular, the equivalence Φ_{Q^o} in an inverse of Φ_Q.*

ii) Any such equivalence Φ between two categories of torsors is equivalent to one associated in this manner to an (H, G)-bitorsor.

Proof of *ii*) : To a given equivalence Φ is associated the left G-torsor $Q := \Phi(T_H)$. By functoriality of Φ, $H \simeq \mathrm{Aut}_H(T_H) \overset{\Phi}{\simeq} \mathrm{Aut}_G(Q)$, so that a section of H acts on the right on Q.

2. (1)-stacks

2.1. The concept of a stack is the categorical analog of a sheaf. Let us start by defining the analog of a presheaf.

Definition 2.1. *i): A category fibered in groupoids above a space X consists in a family of groupoids $\mathcal{C}_U$, for each open set U in X, together with an inverse image functor*

$$f^* : \mathcal{C}_U \longrightarrow \mathcal{C}_{U_1} \tag{2.1.1}$$

associated to every inclusion of open sets $f : U_1 \subset U$ (which is the identity whenever $f = 1_U$), and natural transfomations

$$\phi_{f,g} : (fg)^* \Longrightarrow g^* f^* \tag{2.1.2}$$

for every pair of composable inclusions

$$U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U\,. \tag{2.1.3}$$

For each triple of composable inclusions

$$U_3 \overset{h}{\hookrightarrow} U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U\,.$$

we also require that the composite natural transformations

$$\psi_{f,g,h} : (fgh)^* \Longrightarrow h^* (fg)^* \Longrightarrow h^* (g^* f^*)$$

and

$$\chi_{f,g,h} : (fgh)^* \Longrightarrow (gh)^* f^* \Longrightarrow (h^* g^*) f^*.$$

coincide.

ii) A cartesian functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a family of functors $F_U : \mathcal{C}_U \longrightarrow \mathcal{D}_U$ for all open sets $U \subset X$, together with natural transformations

$$\begin{array}{ccc} \mathcal{C}_U & \longrightarrow & \mathcal{C}_{U_1} \\ {\scriptstyle F_U}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle F_{U_1}} \\ \mathcal{D}_U & \longrightarrow & \mathcal{D}_{U_1} \end{array} \tag{2.1.4}$$

for all inclusion $f : U_1 \subset U$ compatible via the natural transformations (2.1.2) *for a pair of composable inclusions* (2.1.3)

iii) A natural transformation $\Psi : F \Longrightarrow G$ between a pair of cartesian functors consists of a family of natural transformations $\Psi_U : F_U \Longrightarrow G_U$ compatible via the 2-arrows (2.1.4) *under the inverse images functors* (2.1.1).

The following is the analogue for fibered groupoids of the notion of a sheaf of sets, formulated here in an informal style:

Definition 2.2. *A stack in groupoids above a space X is a fibered category in groupoids above X such that*

- *("Arrows glue") For every pair of objects $x, y \in \mathcal{C}_U$, the presheaf $\mathrm{Ar}_{\mathcal{C}_U}(x,\, y)$ is a sheaf on U.*
- *("Objects glue") Descent is effective for objects in $\mathcal{C}$.*

The term descent comes from algebraic geometry. A descent condition is the giving, for any open cover $\mathcal{U} = (U_\alpha)$ of an open set $U \subset X$, of a family of objects $x_\alpha \in \mathcal{C}_{U_\alpha}$ and a family of isomorphisms $\phi_{\alpha\beta} : x_{\beta\,|U_{\alpha\beta}} \longrightarrow x_{\alpha\,|U_{\alpha\beta}}$ such that

$$\phi_{\alpha\beta}\ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$$

above $U_{\alpha\beta\gamma}$. The descent is said to be effective if for any such pairs $(x_\alpha,\, \phi_{\alpha\beta})$ there exists an object $x \in \mathcal{C}_U$ together with isomorphisms $x_{|U_\alpha} \simeq x_\alpha$ compatible with the morphisms $\phi_{\alpha\beta}$. When the objects of $\mathcal{C}$ satisfy the less categorical requirement that the presheaf of objects of $\mathcal{C}$ form a sheaf, then one has a fibered category which only has partial gluing properties, since while arrows still glue, descent is only effective under very stringent conditions. One then says that $\mathcal{C}$ is a prestack. A sheafification process, analogous to the one which transforms a presheaf into a sheaf, associates a stack to a given prestack.

3. 1-gerbes

3.1. We begin with the global description of the 2-category of gerbes, due to Giraud [10]. For another early discussion of gerbes, see [9].

Definition 3.1. *i) A (1)-gerbe on a space X is a stack in groupoids $\mathcal{G}$ on X which is locally* **non-empty** *and locally* **connected**.

*ii) A morphism of gerbes (*resp. *a natural transformation between a pair of such morphisms) is a cartesian functor between the underlying stacks (*resp. *a natural transformation between this pair of cartesian functors).*

Example 3.2. Let G be a bundle of groups on X. The stack $\mathcal{C} := \mathrm{Tors}(G)$ of left G-torsors on X is a gerbe on X: first of all, it is non-empty, since the category $\mathcal{C}_U$ always has at least one object, the trivial torsor T_{G_U}. In addition, every G-torsor on U is locally isomorphic to the trivial one, so the objects in the category $\mathcal{C}_U$ are locally connected.

A gerbe $\mathcal{P}$ on X is said to be *neutral* (or trivial) when the fiber category $\mathcal{P}_X$ is non empty. In particular, a gerbe $\mathrm{Tors}(G)$ is neutral with distinguished object the trivial G-torsor T_G on X. Conversely, the choice of a global object $x \in \mathcal{P}_X$ in a neutral gerbe $\mathcal{P}$ determines an equivalence of gerbes

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\sim} & \mathrm{Tors}(G) \\ y & \mapsto & \mathrm{Isom}_{\mathcal{P}}(y,\, x) \end{array} \tag{3.1.1}$$

on X, where $G := \mathrm{Aut}_{\mathcal{P}}(x)$, acting on $\mathrm{Isom}_{\mathcal{P}}(x,\, y)$ by composition of arrows.

Let $\mathcal{P}$ be a gerbe on X and $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. We now **choose** objects $x_i \in \mathrm{ob}\ \mathcal{P}_{U_i}$ for each $i \in I$. These objects determine corresponding bundles of groups $G_i := \mathrm{Aut}_{\mathcal{P}_{U_i}}(x_i)$ above

U_i. When in addition there exists a bundle of groups G above X, together with U_i-isomorphisms $G_{|\,U_i} \simeq G_i$, for all $i \in I$, we say that $\mathcal{P}$ is a G-gerbe on X.

4. Semi-local description of a gerbe

4.1. Let $\mathcal{P}$ be a G-gerbe on X, and let us choose a family of local objects $x_i \in \mathcal{P}_{U_i}$. These determine as in (3.1.1) equivalences

$$\Phi_i : \mathcal{P}_{U_i} \longrightarrow \mathrm{Tors}(G)_{|\,U_i}$$

above U_i. Chosing inverses for the Φ_i we get an induced family of equivalences

$$\Phi_{ij} := \Phi_{i\,|U_{ij}} \circ \Phi^{-1}_{j\,|U_{ij}} : \mathrm{Tors}(G)_{U_{ij}} \longrightarrow \mathcal{P}_{|U_{ij}} \longrightarrow \mathrm{Tors}(G)_{U_{ij}}$$

above U_{ij}, which corresponds by proposition 1.11 to a family of G-bitorsors P_{ij} above U_{ij}. By construction of the Φ_{ij}, there are also natural transformations

$$\Psi_{ijk} : \Phi_{ij}\,\Phi_{jk} \Longrightarrow \Phi_{ik}$$

above U_{ijk}, satisfying a coherence condition on U_{ijkl}. These define isomorphisms of G-bitorsors

$$\psi_{ijk} : P_{ij} \wedge^G P_{jk} \longrightarrow P_{ik} \tag{4.1.1}$$

above U_{ijk} for which this coherence condition is described by the commutativity of the diagram of bitorsors

$$\begin{array}{ccc} P_{ij} \wedge P_{jk} \wedge P_{kl} & \xrightarrow{\psi_{ijk} \wedge P_{kl}} & P_{ik} \wedge P_{kl} \\ {\scriptstyle P_{ij} \wedge \psi_{ijk}} \downarrow & & \downarrow {\scriptstyle \psi_{ikl}} \\ P_{ij} \wedge P_{jl} & \xrightarrow[\psi_{ijl}]{} & P_{il} \end{array} \tag{4.1.2}$$

above U_{ijkl}

4.2. **Additional comments**:

i) The isomorphism (4.1.1), satisfying the coherence condition (4.1.2), may be viewed as a 1-cocycle condition on X with values in the monoidal stack of G-bitorsors on X. We say that a family of such bitorsors P_{ij} constitutes a bitorsor cocycle on X.

ii) In the case of *abelian* G-gerbes[1] ([4] definition 2.9), the monoidal stack of bitorsors on U_{ij} may be replaced by the symmetric monoidal stack of G-torsors on U_{ij}. In particular, for the multiplicative group $G = GL(1)$, the $GL(1)$-torsors P_{ij} correspond to line bundles L_{ij}. This the point of view regarding abelian $GL(1)$ gerbes set forth by N. Hitchin in [11].

iii) The semi-local construction extends from G-gerbes to general gerbes. In that case a local group $G_i := \mathrm{Aut}_{\mathcal{P}}(x_i)$ above U_i is associated to each of the chosen objects x_i. The previous discussion remains valid, with the proviso that the P_{ij} are now $(G_j,\, G_i)$-bitorsors rather than simply G-bitorsors, and the ψ_{ijk} (4.1.1) are isomorphisms of $(G_k,\, G_i)$-bitorsors.

iv) If we replace the chosen trivializing open cover $\mathcal{U}$ of X by a single covering morphism $Y \longrightarrow X$ in some Grothendieck topology, the theory remains unchanged, but takes on a somewhat different flavor. The giving of an object $x \in \mathcal{P}_Y$ determines a bundle of groups group $G := \mathrm{Aut}_{\mathcal{P}_Y}$ over Y, together with a (p_2^*G, p_1^*G)-bitorsor P above $Y \times_X Y$ satisfying the coherence condition analogous to (4.1.2) above $Y \times_X Y \times_X Y$. A bitorsor P on Y satisfying this coherence condition has been called cocycle bitorsor by K.-H. Ulbrich [21], and a bundle gerbe by M.K. Murray [15]. It corresponds to

[1] which are not simply G-gerbes for which the structure group G is abelian !

a bouquet in Duskin's theory (see [20]). It therefore equivalent[2] to the giving of a gerbe $\mathcal{P}$ on X, together with a trivialization of its pullback to Y.

5. **Cocycles and coboundaries for gerbes**

5.1. Let us keep the notations of section 3.1. In addition to choosing local objects $x_i \in \mathcal{P}_{U_i}$ in a gerbe $\mathcal{P}$ on X, we now **choose** arrows

$$x_j \xrightarrow{\phi_{ij}} x_i \tag{5.1.1}$$

in $\mathcal{P}_{U_{ij}}$[3]. Since $G_i := \mathrm{Aut}_{\mathcal{P}}(x_i)$, a chosen arrow ϕ_{ij} induces by conjugation a homomorphism of group bundles

$$\begin{array}{ccc} G_{j\,|U_{ij}} & \xrightarrow{\lambda_{ij}} & G_{i\,|U_{ij}} \\ \gamma & \longmapsto & \phi_{ij}\,\gamma\,\phi_{ij}^{-1} \end{array} \tag{5.1.2}$$

above the open sets U_{ij}. To state this slightly differently, such a homomorphism λ_{ij} is characterized by the commutativity of the diagrams

$$\begin{array}{ccc} x_j & \xrightarrow{\gamma} & x_j \\ {\scriptstyle \phi_{ij}}\downarrow & & \downarrow{\scriptstyle \phi_{ij}} \\ x_i & \xrightarrow[\lambda_{ij}(\gamma)]{} & x_i \end{array} \tag{5.1.3}$$

for every $\gamma \in G_{|U_{ij}}$. The choice of objects x_i and arrows ϕ_{ij} in $\mathcal{P}$ determines, in addition to the morphisms λ_{ij} (5.1.2), a family of elements $g_{ijk} \in G_{i\,|U_{ijk}}$ for all (i, j, k), defined by the commutativity of the diagrams

$$\begin{array}{ccc} x_k & \xrightarrow{\phi_{jk}} & x_j \\ {\scriptstyle \phi_{ik}}\downarrow & & \downarrow{\scriptstyle \phi_{ij}} \\ x_i & \xrightarrow[g_{ijk}]{} & x_i \end{array} \tag{5.1.4}$$

above U_{ijk}. These in turn induce by conjugation the following commutative diagrams of bundles of groups

$$\begin{array}{ccc} G_k & \xrightarrow{\lambda_{jk}} & G_j \\ {\scriptstyle \lambda_{ik}}\downarrow & & \downarrow{\scriptstyle \lambda_{ij}} \\ G_i & \xrightarrow[i_{g_{ijk}}]{} & G_i \end{array} \tag{5.1.5}$$

above U_{ijk}. The commutativity of diagram (5.1.5) may be stated algebraically as the cocycle equation

$$\lambda_{ij}\,\lambda_{jk} = i_{g_{ijk}}\,\lambda_{ik} \tag{5.1.6}$$

with i the inner conjugation arrow (1.4.2). The following equation is the second cocycle equation satisfied by the pair $(\lambda_{ij},\, g_{ijk})$. While the proof of lemma 1 given here is essentially the same as the one in [4], the present cubical diagram (5.1.9) is much more intelligible than diagram (2.4.7) of [4].

---

[2]For a more detailed discussion of this when the covering morphism $Y \longrightarrow X$ is the morphism of schemes associated as in remark 1.8 to a Galois field extension k'/k, see [2] §5.

[3] Actually, this is a simplification, since the gerbe axioms only allow us to choose such an arrow locally, above each element U^{α}_{ij} of an open cover of U_{ij}. Such families of open sets $(U_i,\, U^{\alpha}_{ij})$, and so on, form what is known as a hypercover of X. For simplicity, we assume from now on that our topological space X is paracompact. In that case, we may carry out the entire discussion without hypercovers.

Lemma 1. *The elements g_{ijk} satisfy the λ_{ij}-twisted 2-cocycle equation*

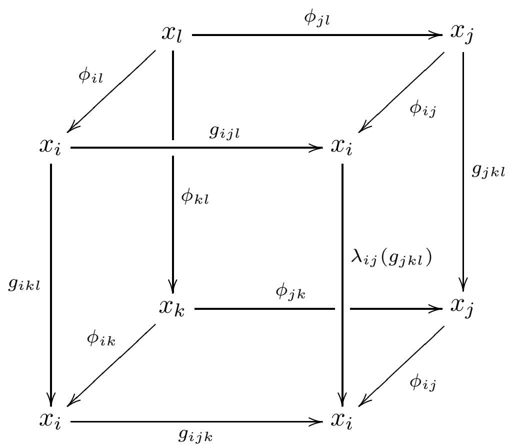
$$\lambda_{ij}(g_{jkl})\, g_{ijl} = g_{ijk} g_{ikl} \tag{5.1.7}$$

in $G_{i\,|U_{ijkl}}$.

Proof of lemma 1: Note that equation (5.1.7) is equivalent to the commutativity of the diagram of groups

$$\begin{array}{ccc} x_i & \xrightarrow{g_{ijl}} & x_i \\ {\scriptstyle g_{ikl}}\downarrow & & \downarrow{\scriptstyle \lambda_{ij}(g_{jkl})} \\ x_i & \xrightarrow[g_{ijk}]{} & x_i \end{array} \tag{5.1.8}$$

above U_{ijkl}. Let us now consider the following cubical diagram:



(5.1.9)

in which the left, back, top and bottom squares are of type (5.1.4), and the right-hand one of type (5.1.3). Since these five faces are commutative squares, and all the arrows in the diagram are invertible, the sixth (front) face is also commutative. Since the latter is simply the square (5.1.8), the lemma is proved

□

A pair $(\lambda_{ij},\, g_{ijk})$ satisfying the equations (5.1.6) and (5.1.7) :

$$\begin{cases} \lambda_{ij}\, \lambda_{jk} &= i_{g_{ijk}}\, \lambda_{ik} \\ \lambda_{ij}(g_{jkl})\, g_{ijl} &= g_{ijk}\, g_{ikl} \end{cases} \tag{5.1.10}$$

is called a G_i-valued cocycle pair. It may be viewed as consisting of a 2-cocycle equation for the elements g_{ijk}, together with auxiliary data attached to the isomorphisms λ_{ij}. However, in contrast with the abelian case in which the inner conjugation term $i_{g_{ijk}}$ is trivial, these two equations cannot in general be uncoupled. When such a pair is associated to a G-gerbe $\mathcal{P}$ for a fixed bundle of groups G, the term λ_{ij} is a section above U_{ij} of the bundle of groups $\mathrm{Aut}_X(G)$, and g_{ijk} is a section of G above U_{ijk}. Such pairs $(\lambda_{ij},\, g_{ijk})$ will be called G-valued cocycle pairs.

5.2. The corresponding coboundary relations will now be worked out by a similar diagrammatic process. Let us give ourselves a second family of local objects x'_i in $\mathcal{P}_{U_i}$, and of arrows

$$x'_j \xrightarrow{\phi'_{ij}} x'_i \tag{5.2.1}$$

above U_{ij}. To these correspond by the constructions (5.1.3) and (5.1.4) a new cocycle pair $(\lambda'_{ij},\, g'_{ijk})$ satisfying the cocycle relations (5.1.6) and (5.1.7). In order to compare the previous trivializing data

(x_i, ϕ_{ij}) with the new one, we also choose a family of arrows

$$x_i \xrightarrow{\chi_i} x'_i \tag{5.2.2}$$

in $\mathcal{P}_{U_i}$ for all i. The lack of compatibility between these arrows and the previously chosen arrows (5.1.1) and (5.2.1) is measured by the family of arrows $\delta_{ij} : x_i \longrightarrow x_i$ in $\mathcal{P}_{U_{ij}}$ determined by the commutativity of the following diagram:

$$\begin{array}{ccc} x_j & \xrightarrow{\phi_{ij}} & x_i \\ & & \downarrow{\scriptstyle \chi_i} \\ {\scriptstyle \chi_j}\downarrow & & x'_i \\ & & \downarrow{\scriptstyle \delta_{ij}} \\ x'_j & \xrightarrow[\phi'_{ij}]{} & x'_i \end{array} \tag{5.2.3}$$

The arrow $\chi_i : x_i \longrightarrow x'_i$ induces by conjugation an isomorphism $r_i : G_i \longrightarrow G'_i$, characterized by the commutativity of the square

$$\begin{array}{ccc} x_i & \xrightarrow{u} & x_i \\ {\scriptstyle \chi_i}\downarrow & & \downarrow{\scriptstyle \chi_i} \\ x'_i & \xrightarrow[r_i(u)]{} & x'_i \end{array} \tag{5.2.4}$$

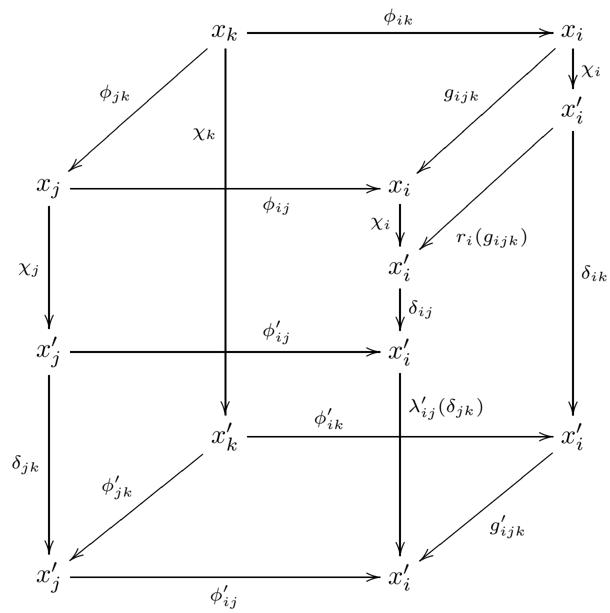
for all $u \in G_i$. The diagram (5.2.3) therefore conjugates to a diagram

$$\begin{array}{ccc} G_j & \xrightarrow{\lambda_{ij}} & G_i \\ & & \downarrow{\scriptstyle r_i} \\ {\scriptstyle r_j}\downarrow & & G'_i \\ & & \downarrow{\scriptstyle i_{\delta_{ij}}} \\ G'_j & \xrightarrow[\lambda'_{ij}]{} & G'_i \end{array} \tag{5.2.5}$$

above U_{ij} whose commutativity is expressed by the equation

$$\lambda'_{ij} = i_{\delta_{ij}}\, r_i\, \lambda_{ij}\, r_j^{-1}\,. \tag{5.2.6}$$

Consider now the diagram



(5.2.7)

Both the top and the bottom squares commute, since these squares are of type (5.1.4). So do the back, the left and the top front vertical squares, since all three are of type (5.2.3). The same is true of the lower front square, and the upper right vertical square, since these two are respectively of the form (5.1.3) and (5.2.4). It follows that the remaining lower right square in the diagram is also commutative, since all the arrows in diagram (5.2.7) are invertible. The commutativity of this final square is expressed algebraically by the equation

$$g'_{ijk}\,\delta_{ik} = \lambda'_{ij}(\delta_{jk})\,\delta_{ij}\,r_i(g_{ijk}), \tag{5.2.8}$$

an equation equivalent to [4] (2.4.17).

Let us say that two cocycle pairs $(\lambda_{ij},\, g_{ijk})$ and $(\lambda'_{ij} g'_{ijk})$ are cohomologous if we are given a pair (r_i, δ_{ij}), with $r_i \in \mathrm{Isom}(G_i,\, G'_i)$ and $\delta_{ij} \in G_{i\,|U_{ij}}$ satisfying the equations

$$\begin{cases} \lambda'_{ij} &= i_{\delta_{ij}}\; r_i\, \lambda_{ij}\, r_j^{-1} \\ g'_{ijk} &= \lambda'_{ij}(\delta_{jk})\,\delta_{ij}\, r_i(g_{ijk})\,\delta_{ik}^{-1} \end{cases} \tag{5.2.9}$$

Suppose now that $\mathcal{P}$ is a G-gerbe. All the terms in the first equations in both (5.1.10) and (5.2.9) are then elements of $\mathrm{Aut}(G)$, while the terms in the corresponding second equations live in G. The set of equivalence classes of cocycle pairs (5.1.10), for the equivalence relation defined by equations (5.2.9), is then denoted $H^1(\mathcal{U},\, G \longrightarrow \mathrm{Aut}(G))$, a notation consistent with that introduced in §1.4 The limit over the open covers $\mathcal{U}$ is the Čech hypercohomology set $\check{H}^1(X,\, G \longrightarrow \mathrm{Aut}(G))$. We refer to [4] §2.6 for the inverse construction, starting from a Čech cocycle pair, of the corresponding G-gerbe[4]. This hypercohomology set therefore classifies G-gerbes on X up to equivalence.

In geometric terms, this can be understood once we introduce the following definition, a categorification of the definition (1.1.1) of a G-torsor.:

[4]In [4] §2.7, we explain how this inverse construction extends to the more elaborate context of hypercovers, where a beautiful interplay between the Čech and the descent formalisms arises. This is also discussed, in more simplicial terms, in [1] §6.3-6.6.

Definition 5.1. *Let $\mathcal{G}$ be a monoidal stack on X. A left $\mathcal{G}$-torsor on X is a stack $\mathcal{Q}$ on X together with a left action functor*

$$\mathcal{G} \times \mathcal{Q} \longrightarrow \mathcal{Q}$$

which is coherently associative and satisfies the unit condition, and for which the induced functor

$$\mathcal{G} \times \mathcal{Q} \longrightarrow \mathcal{Q} \times \mathcal{Q}$$

defined as in (1.1.1) *is an equivalence. In addition, we require that $\mathcal{Q}$ be locally non-empty.*

The following three observations, when put together, explain in more global term why G-gerbes are classified by the set $H^1(X, G \longrightarrow \mathrm{Aut}(G))$.

- To a G-gerbe $\mathcal{P}$ on X is associated its "bundle of frames" $\mathcal{E}q(\mathcal{P}, \mathrm{Tors}(G))$, and the latter is a left torsor under the monoidal stack $\mathcal{E}q(\mathrm{Tors}(G), \mathrm{Tors}(G))$.
- By the Morita theorem, this monoidal stack is equivalent to the monoidal stack $\mathrm{Bitors}(G)$ of G-bitorsors on X.
- The cocycle computations leading up to (1.4.4) imply that the monoidal stack $\mathrm{Bitors}(G)$ is the stack associated to the monoidal prestack defined by the crossed module $G \longrightarrow \mathrm{Aut}(G)$.

Remark 5.2. For a related discussion of non-abelian cocycles in a homotopy-theoretic context, see the recent preprints of J. F. Jardine [12] and [13], where a classification of gerbes equivalent to ours is given, including in the case in which hypercovers are required.

5.3. A topological interpretation of a G-gerbe ([3] 4.2)

Let G be a bundle of groups G above a space X and BG its classifying space, whose fiber at a point $x \in X$ is the classifying space BG_x of the group G_x. We attach to G the group-like topological monoid $\mathrm{Eq}_X(BG)$ of self-homotopy equivalences over X of BG. The homotopy fibre of the evaluation map

$$\mathrm{ev}_{X,*} : \mathrm{Eq}_X(BG) \longrightarrow BG$$

of an equivalence at the distinguished point $*$ of BG is the submonoid $\mathrm{Eq}_{X,*}(BG)$ of pointed equivalences, and the latter is homotopy equivalent, by the functor $\pi_1(-,*)$, to the discrete bundle of groups $\mathrm{Aut}(G)$. The induced fibration sequence

$$\mathrm{Aut}(G) \longrightarrow \mathrm{Eq}_X(BG) \longrightarrow BG$$

deloops to a fibration sequence of topological monoids (the first two of which are of course discrete groups)

$$i : G \xrightarrow{i} \mathrm{Aut}(G) \longrightarrow \mathrm{Eq}_X(BG) \tag{5.3.1}$$

This yields an identification of $\mathrm{Eq}_X(BG)$ with the Borel construction $EG \wedge^G \mathrm{Aut}(G)$. Our discussion in §1.4 asserts that this identification preserves the multiplications, so long as the multiplication on the Borel construction is given by the semi-direct product construction (1.5.7). We refer to [3] for a somewhat more detailed discussion of this assertion, and to [8] §4 for a discussion of the fibration sequence

$$BG \longrightarrow B\,\mathrm{Aut}(G) \longrightarrow B\mathrm{Eq}(BG)$$

obtained by applying the classifying space construction to the fibration sequence (5.3.1) (or rather to its generalization with BG replaced by an arbitrary topological space Y). This proves:

Proposition 5.3. *The simplicial group associated to the crossed module $G \longrightarrow Aut(G)$ is a model for the group-like topological monoid $Eq_X(BG)$.*

In this context, the set of classes of 1-cocycles $H^1(X, G \longrightarrow \mathrm{Aut}(G))$ classifies the fibrations on X which are locally homotopy equivalent to BG, and the corresponding assertion when G is a bundle of groups is also true. We refer to the recent preprint of J. Wirth and J. Stasheff [22] for a related discussion of fiber homotopy equivalence classes of locally homotopy trivial fibrations, also from a cocyclic point of view.

Example 5.4. Let us sketch here a modernized proof of O. Schreier's cocyclic classification (in 1926 !) of (non-abelian, non-central) group extensions [18], which is much less well-known than the special case in which the extensions are central. It is in fact just a strengthened version of the discussion carried out above in (1.1.7):

Consider a short exact sequence of groups (1.1.7). Applying the classifying space functor B, this induces a fibration

$$BG \longrightarrow BH \xrightarrow{\pi} BK$$

of pointed spaces above BK, and all the fibers of π are homotopically equivalent to BG. It follows that this fibration determines an element in the pointed set $H^1(BK,\, G \longrightarrow \mathrm{Aut}(G))$. Conversely, such a cohomology class determines a fibration of pointed spaces of this type above BK, and therefore, by applying the loop functor, a sequence of A_∞-spaces

$$G \longrightarrow H \longrightarrow K\,.$$

Since G and K are discrete groups, so is the middle term H.

□

6. **2-stacks and 2-gerbes**

6.1. We will now extend the discussion of section 5 from 1- to 2-categories. A 2-groupoid is defined here as a 2-category whose 1-arrows are invertible up to a 2-arrow, and whose 2-arrows are strictly invertible.

Definition 6.1. *A fibered 2-category in 2-groupoids above a space X consists in a family of 2-groupoids $\mathcal{C}_U$, for each open set U in X, together with an inverse image 2-functor*

$$f^* : \mathcal{C}_U \longrightarrow \mathcal{C}_{U_1}$$

associated to every inclusion of open sets $f : U_1 \subset U$ (which is the identity whenever $f = 1_U$), and a natural transfomation

$$\phi_{f,g} : (fg)^* \Longrightarrow g^*\, f^*$$

for every pair of composable inclusions

$$U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U\,.$$

For each triple of composable inclusions

$$U_3 \overset{h}{\hookrightarrow} U_2 \overset{g}{\hookrightarrow} U_1 \overset{f}{\hookrightarrow} U,$$

we require a modification

$$(fgh)^* \underset{\chi_{f,g,h}}{\overset{\psi_{f,g,h}}{\rightrightarrows}} h^*g^*f^* \qquad \Downarrow \alpha_{f,g,h}$$

betweeen the composite natural transfomations

$$\psi_{f,g,h} : (fgh)^* \Longrightarrow h^*\,(fg)^* \Longrightarrow h^*\,(g^*f^*)$$

and

$$\chi_{f,g,h} : (fgh)^* \Longrightarrow (gh)^*\, f^* \Longrightarrow (h^* g^*)\, f^*.$$

Finally, for any $U_4 \overset{k}{\hookrightarrow} U_3$*, the two methods by which the modifications* α *compare the composite 2-arrows*

$$(fghk)^* \Longrightarrow (ghk)^* f^* \Longrightarrow ((hk)^* g^* f^* \Longrightarrow k^* h^* g^* f^*$$

and

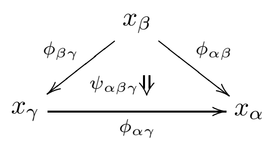
$$(fghk)^* \Longrightarrow k^*(fgh)^* \Longrightarrow k^*(h^*(fg)^*) \Longrightarrow k^* h^* g^* f^*$$

must coincide.

Definition 6.2. *A 2-stack in 2-groupoids above a space* X *is a fibered 2-category in 2-groupoids above* X *such that*

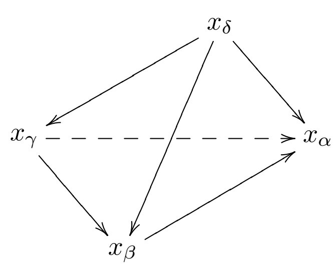
- *For every pair of objects* $X, Y \in \mathcal{C}_U$*, the fibered category* $\mathrm{Ar}_{\mathcal{C}_U}(X,\, Y)$ *is a stack on* U*.*
- *2-descent is effective for objects in* $\mathcal{C}$*.*

The 2-descent condition asserts that we are given, for an open covering $(U_\alpha)_{\alpha \in J}$ of an open set $U \subset X$, a family of objects $x_i \in \mathcal{C}_{U_i}$, of 1-arrows $\phi_{\alpha\beta} : x_\alpha \longrightarrow x_\beta$ between the restrictions to $\mathcal{C}_{U_{\alpha\beta}}$ of the objects x_α and x_β and a family of 2-arrow



(6.1.1)

for which the tetrahedral diagram of 2-arrows whose four faces are the restrictions of the requisite 2-arrows ψ (6.1.1) to $\mathcal{C}_{U_{\alpha\beta\gamma\delta}}$ commutes:



The descent condition $(x_i,\, \phi_{\alpha\beta},\, \psi_{\alpha\beta\gamma})$ is effective if there exists an object $x \in \mathcal{C}_U$ together with isomorphisms $x_{|U_\alpha} \simeq x_\alpha$ compatible with the given 1- and 2-arrows $\phi_{\alpha,\beta}$ and $\psi_{\alpha\beta\gamma}$.

Definition 6.3. *A 2-gerbe* $\mathcal{P}$ *is a 2-stack in 2-groupoids on* X *which is locally non-empty and locally connected.*

To each object x in $\mathcal{P}_U$ is associated a group like monoidal stack (or *gr*-stack) $\mathcal{G}_x := \mathcal{A}r_U(x,\, x)$ above U.

Definition 6.4. *Let* $\mathcal{G}$ *be a group-like monoidal stack on* X*. We say that a 2-gerbe* $\mathcal{P}$ *is a* $\mathcal{G}$*-2-gerbe if there exists an open covering* $\mathcal{U} := (U_i)_{i \in I}$ *of* X*, a family of objects* $x_i \in \mathcal{P}_{U_i}$*, and* U_i*-equivalences* $\mathcal{G}_{U_i} \simeq \mathcal{G}_{x_i}$*.*

6.2. Cocycles for 2-gerbes :

In order to obtain a cocyclic description of a $\mathcal{G}$-2-gerbe $\mathcal{P}$, we will now categorify the constructions in §5. We choose paths

$$\phi_{ij} : x_j \longrightarrow x_i \tag{6.2.1}$$

in the 2-groupoid $\mathcal{P}_{U_{ij}}$, together with quasi-inverses $x_i \longrightarrow x_j$ and pairs of 2-arrows

$$\phi_{ij}\,\phi_{ij}^{-1} \overset{r}{\Longrightarrow} 1_{x_i} \qquad\qquad \phi_{ij}^{-1}\,\phi_{ij} \overset{s}{\Longrightarrow} 1_{x_j}\ . \tag{6.2.2}$$

These determine a monoidal equivalence

$$\lambda_{ij} : \mathcal{G}_{|U_{ij}} \longrightarrow \mathcal{G}_{|U_{ij}} \tag{6.2.3}$$

as well as, functorially each object $\gamma \in \mathcal{G}_{|U_{ij}}$, a 2-arrow

$$\begin{array}{ccc} x_j & \xrightarrow{\ \gamma\ } & x_j \\ {\scriptstyle \phi_{ij}}\Big\downarrow & \Nearrow & \Big\downarrow{\scriptstyle \phi_{ij}} \\ x_i & \xrightarrow[\ \lambda_{ij}(\gamma)\ ]{} & x_i \end{array} \tag{6.2.4}$$

which categorifies diagram (5.1.3) and which we will denote by $M_{ij}(\gamma)$. In fact, the 2-arrows r and s (6.2.2) can be chosen coherently, and the induced 2-arrow (6.2.4) therefore does not carry any significant cohomological information. We will acknowledge this fact by not giving this 2-arrow a name and will in the sequel treat diagram (5.1.3), and similar ones which define a 1-arrow as the conjugate of another one, as commutative squares.

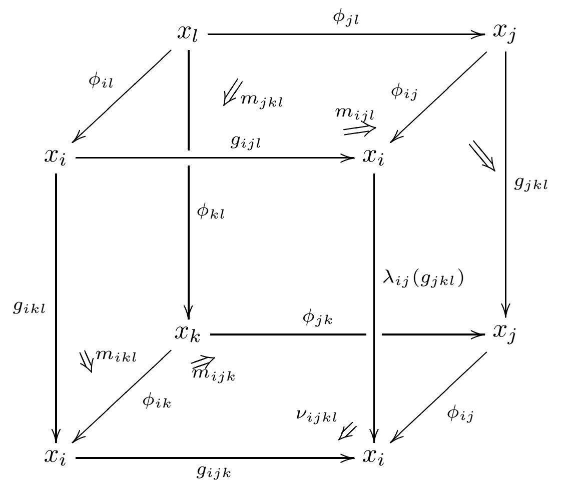
The paths ϕ_{ij} and their inverses also give us objects $g_{ijk} \in \mathcal{G}_{U_{ijk}}$ and 2-arrows m_{ijk}:

$$\begin{array}{ccc} x_k & \xrightarrow{\ \phi_{jk}\ } & x_j \\ {\scriptstyle \phi_{ik}}\Big\downarrow & \overset{m_{ijk}}{\Nearrow} & \Big\downarrow{\scriptstyle \phi_{ij}} \\ x_i & \xrightarrow[\ g_{ijk}\ ]{} & x_i \end{array} \tag{6.2.5}$$

These in turn determine a 2-arrow ν_{ijkl} above U_{ijkl}

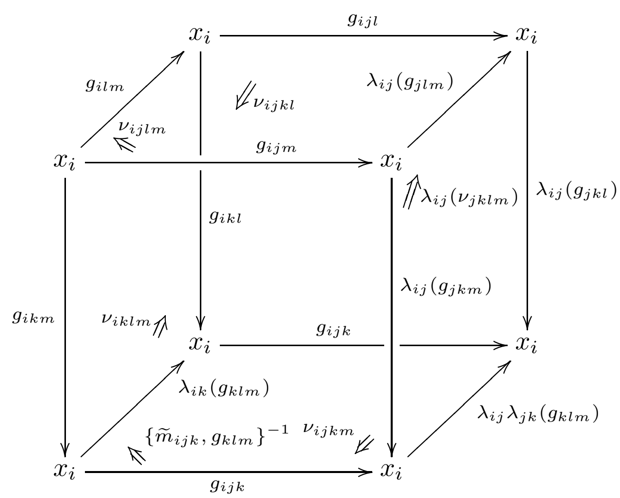
$$\begin{array}{ccc} x_i & \xrightarrow{\ g_{ijl}\ } & x_i \\ {\scriptstyle g_{ikl}}\Big\downarrow & \overset{\nu_{ijkl}}{\Swarrow} & \Big\downarrow{\scriptstyle \lambda_{ij}(g_{jkl})} \\ x_i & \xrightarrow[\ g_{ijk}\ ]{} & x_i \end{array} \tag{6.2.6}$$

as the unique 2-arrow such that the following diagram of 2-arrows with right-hand face (6.2.4) and front face ν_{ijkl} commutes:



(6.2.7)

This cube in $\mathcal{P}_{U_{ijkl}}$ will be denoted C_{ijkl}. Consider now the following diagram:



(6.2.8)

In order to avoid any possible ambiguity, we spell out in the following table the names of the faces of the cube (6.2.8):

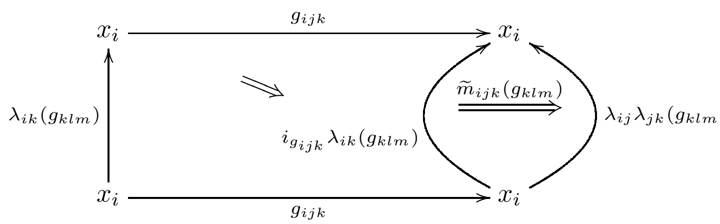
left	right	top	bottom	front	back
ν_{iklm}	$\lambda_{ij}(\nu_{jklm})$	ν_{ijlm}	$\{\widetilde{m}_{ijk},\, g_{klm}\}^{-1}$	ν_{ijkm}	ν_{ijkl}

TABLE 1. The faces of cube (6.2.8)

As we see from this table, five of its faces are defined by arrows ν (6.2.6). The remaining bottom 2-arrow $\{\widetilde{m}_{ijk},\, g_{klm}\}^{-1}$ is essentially the inverse of the 2-arrow $\widetilde{m}_{ijk}(g_{klm})$ obtained by evaluating the natural transformation

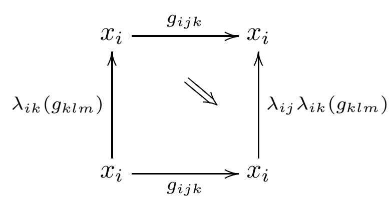
$$\widetilde{m}_{ijk} : i_{g_{ijk}} \lambda_{ik} \Rightarrow \lambda_{ij}\, \lambda_{jk} \tag{6.2.9}$$

induced by conjugation from the 2-arrow m_{ijk} (6.2.5) on the object $g_{klm} \in G := \mathrm{Aut}_{\mathcal{P}}(x_k)$. More precisely, if we compose the latter 2-arrow as follows with the unlabelled 2-arrow $M_{g_{ijk}}(\lambda_{ik}(g_{klm}))$ associated to $i_{g_{ijk}}$:



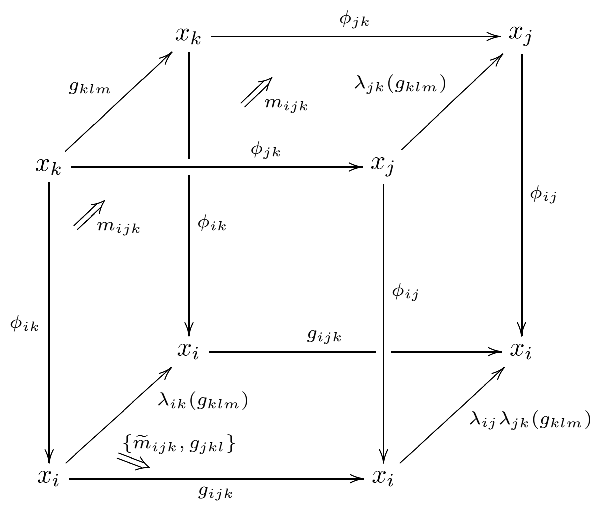
(6.2.10)

we obtain a 2-arrow



(6.2.11)

which we denote by $\{\widetilde{m}_{ijk},\, g_{klm}\}$. It may be characterized as the unique 2-arrow such that the cube

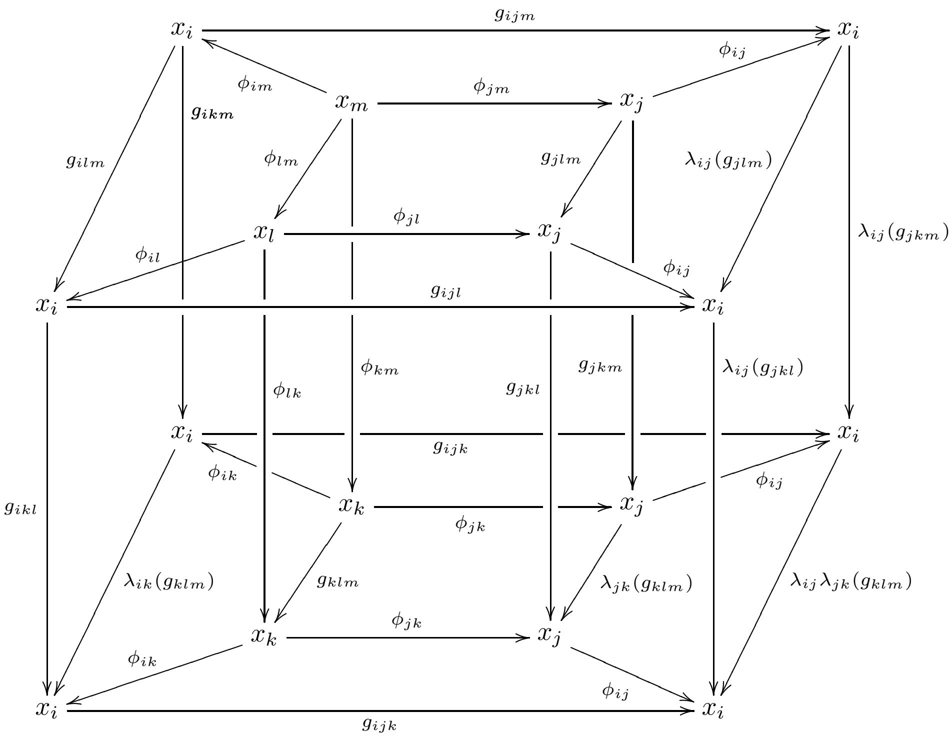


(6.2.12)

(with three unlabelled faces of type (6.2.4)) is commutative. For that reason, this cube will be denoted $\{\,,\,\}$. The following proposition provides a geometric interpretation for the cocycle equation which the 2-arrows ν_{ijkl} satisfy.

Proposition 6.5. *The diagram of 2-arrows* (6.2.8) *is commutative.*

Proof: Consider the following hypercubic diagram, from which the 2-arrows have all been omitted for greater legibility.



(6.2.13)

The following table is provided as a help in understanding diagram (6.2.13). The first line describes the position in the hypercube of each of the eight cubes from which it has been constructed, and the middle line gives each of these a name. Finally, the last line describes the face by which it is attached to the inner cube C_{jklm}.

inner	left	right	top	bottom	front	back	outer
C_{jklm}	C_{iklm}	Conj(ϕ_{ij})	C_{ijlm}	$\{\ ,\ \}$	C_{ijkl}	C_{ijkm}	(6.2.8)
	m_{klm}	ν_{jklm}	m_{jlm}	$M_{jk}(m_{klm})$	m_{jkl}	m_{jkm}	

Table 2. The constituent cubes of diagram (6.2.13)

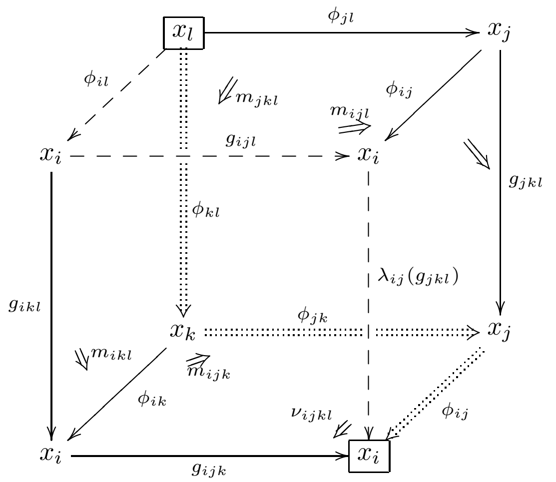
Only one cube in this table has not yet been described. It is the cube Conj(ϕ_{ij}) which appears on the right in diagram (6.2.13). It describes the construction of the 2-arrow $\lambda_{ij}(\nu_{jklm})$ starting from ν_{jklm}, by conjugation of its source and target arrows by the 1-arrows ϕ_{ij}.

Now that diagram (6.2.13) has been properly described, the proof of proposition 6.5 is immediate, and goes along the same lines as the proof of lemma 1. One simply observes that each of the first seven cubes in table 2 is a commutative diagram of 2-arrows. Since all the constituent 2-arrows are invertible, the remaining outer cube is also a commutative diagram of 2-arrows. Since the latter cube is simply (6.2.8), though with a different orientation, the proof of the proposition is now complete.

□

6.3. **Algebraic description of the 3-cocycle condition:**

In order to obtain a genuinely cocyclic description of a $\mathcal{G}$-2-gerbe, it is necessary to translate proposition 6.5 into an algebraic statement. As a preliminary step, we implement such a translation for the cubical diagram C_{ijkl} (6.2.7) by which we defined the 2-arrow ν_{ijkl}. We reproduce this cube as

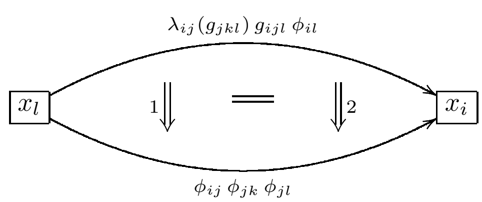


(6.3.1)

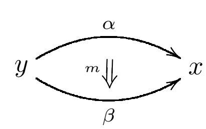
and consider the two composite paths of 1-arrows from the framed vertex x_l to the framed vertex x_i respectively displayed by arrows of type $- - \succ$ and $\cdots\triangleright$. The commutativity of our cube is equivalent to the assertion that the two possible composite 2-arrows from the path $- - \succ$ to the path $\cdots\triangleright$ coincide. This assertion translates, when taking into account the whiskerings which arise whenever one considers a face of the cube which does not contain the framed vertex x_i, to the equation

$$m_{ijk}\,(g_{ijk} * m_{ikl})\,\nu_{ijkl} = (\phi_{ij} * m_{jkl})\,(\lambda_{ij}(g_{jkl}) * m_{ijl}) \tag{6.3.2}$$

which algebraically defines the 2-arrow ν_{ijkl} in terms of the 2-arrows of type m_{ijk} (6.2.5). For reasons which will appear later on, we have neglected here the whiskerings by 1-arrows on the right, for faces of the cube which do not contain the framed vertex x_l from which all paths considered originate. With the left-hand side of this equality labelled "1" and the right-hand side "2", the two sides are compared according to the following scheme in the 2-category $\mathcal{P}_{U_{ijkl}}$:



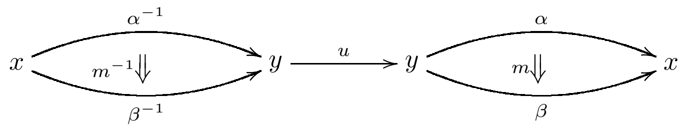
Consider now a 2-arrow



(6.3.3)

in $\mathcal{P}_U$, and denote by α_* and β_* the functors $\mathcal{G}_U \longrightarrow \mathcal{G}_U$ which conjugation by α and β respectively define. The conjugate of any 1-arrow $u \in \mathrm{ob}\,\mathcal{G}_U = \mathrm{Ar}_{\mathcal{P}_U}(y, y)$ by the 2-arrow m is the composite

2-arrow

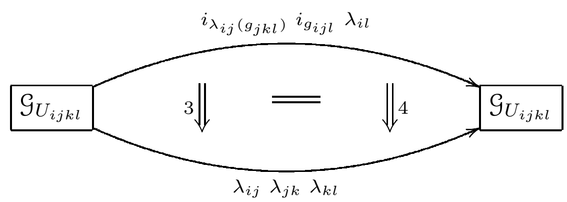


(6.3.4)

where m^{-1} is the horizontal inverse of the 2-arrow m. We denote by $\widetilde{m} : \alpha_* \Longrightarrow \beta_*$ the natural transformation which m defines in this way. It is therefore an arrow

$$\widetilde{m} : \alpha_* \longrightarrow \beta_*$$

in the monoidal category $\mathcal{E}q(\mathcal{G})_U$. With this notation, it follows that equation (6.3.2) conjugates according to the scheme



to the following equation between the arrows “3” and “4” in the category $\mathcal{E}q(\mathcal{G})_{U_{ijkl}}$:

$$\widetilde{m}_{ijk}\ {}^{g_{ijk}}\widetilde{m}_{ikl}\ i(\nu_{ijkl}) = (\lambda_{ij}\ \widetilde{m}_{jkl})\ {}^{\lambda_{ij}(g_{jkl})}\widetilde{m}_{ijl} \tag{6.3.5}$$

In such an equation, i the inner conjugation functor[5]

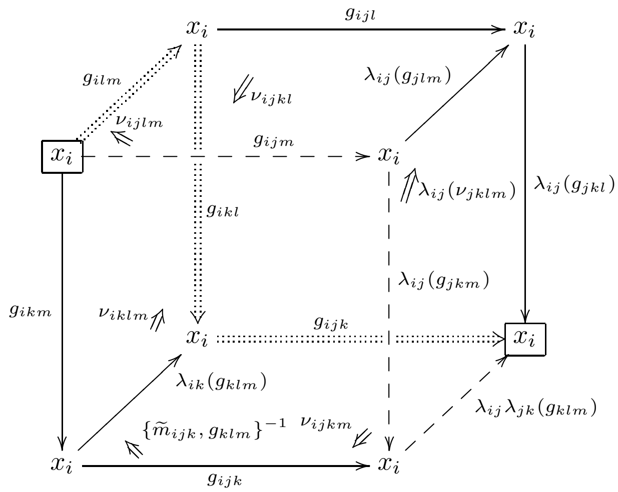
$$\mathcal{G} \xrightarrow{i} \mathcal{E}q(\mathcal{G}) \tag{6.3.6}$$

associated to the monoidal stack $\mathcal{G}$. By an expression such as ${}^{g_{ijk}}\widetilde{m}_{ikl}$, we mean the conjugate of the 1-arrow $\widetilde{m}_{ikl}$ by the object $i(g_{ijk})$ in the monoidal category $\mathcal{E}q(\mathcal{G})_{U_{ijkl}}$. We observe here that the right whiskerings of a 2-arrow m or ν (*i.e.* the composition a 2-arrow with a 1-arrow which precedes it) have no significant effect upon the conjugation operation which associates to a 2-arrow m (*resp.* ν in $\mathcal{P}$ the corresponding natural transformation $\widetilde{m}$ (*resp.* $i(\nu)$), an arrow in $\mathcal{E}q(\mathcal{G})$. It was for this reason harmless to ignore the right whiskerings in formula (6.3.2) and we will do so in similar contexts in the sequel.

---

[5] which should not be confused with the inner conjugation homomorphism (1.4.2) which arises when $\mathcal{G}$ is the stack Bitors(G) associated to a bundle of groups G.

Let us display once more the cube (6.2.8), but now decorated according to the same conventions as in (6.3.1):



(6.3.7)

The commutativity of this diagram of 2-arrows translates (according to the recipe which produced the algebraic equation (6.3.2) from the cube (6.3.1)) to the following very twisted algebraic 3-cocycle condition for ν [6]:

$$\nu_{ijkl}\,({}^{\lambda_{ij}(g_{jkl})}\nu_{ijlm})\;\lambda_{ij}(\nu_{jklm}) \;=\; {}^{g_{ijk}}\nu_{iklm}\;\{\widetilde{m}_{ijk}, g_{klm}\}^{-1}\;({}^{\lambda_{ij}\lambda_{jk}(g_{klm})}\nu_{ijkm}) \tag{6.3.8}$$

This is an equation satisfied by elements with values in $\mathrm{Ar}\,(\mathcal{G}_{U_{ijklm}})$. Note the occurrence here of the term $\{\widetilde{m}_{ijk}, g_{klm}\}^{-1}$, corresponding to the lower face of (6.2.8). While such a term does not exist in the standard definition of an abelian Čech 3-cocycle equation, non-abelian 3-cocycle relations of this type goes back to the work of P. Dedecker [7]. They arise there in the context of group rather than Čech cohomology, with his cocycles taking their values in an unnecessarily restrictive precursor of a crossed square, which he calls a super-crossed group.

The following definition, which summarizes the previous discussion, may be also viewed as a categorification of the notion of a G-valued cocycle pair, as defined by equations (5.1.10):

Definition 6.6. *Let $\mathcal{G}$ be a group-like monoidal stack on a space X, and $\mathcal{U}$ an open covering of X. A $\mathcal{G}$-valued Čech 1-cocycle quadruple is a quadruple of elements*

$$(\lambda_{ij},\, \widetilde{m}_{ijk},\, g_{ijk},\, \nu_{ijkl}) \tag{6.3.9}$$

satisfying the following conditions. The term λ_{ij} is an object in the monoidal category $\mathcal{E}q_{U_{ij}}(\mathcal{G}_{|U_{ij}})$ and $\widetilde{m}_{ijk}$ is an arrow

$$\widetilde{m}_{ijk} : i_{g_{ijk}}\lambda_{ik} \Rightarrow \lambda_{ij}\,\lambda_{jk} \tag{6.3.10}$$

in the corresponding monoidal category $\mathcal{E}q_{U_{ijk}}(\mathcal{G}_{|U_{ijk}})$. Similarly, g_{ijk} is an object in the monoidal category $\mathcal{G}_{U_{ijk}}$ and

$$\nu_{ijkl} : \lambda_{ij}(g_{jkl})\,g_{ijl} \longrightarrow g_{ijk}\,g_{ikl}$$

an arrow (6.2.6) in the corresponding monoidal category $\mathcal{G}_{U_{ijkl}}$. Finally, we require that the two equations (6.3.5) and (6.3.8), which we reproduce here for the reader's convenience, be satisfied:

$$\left\{\begin{array}{lcl} \widetilde{m}_{ijl}\;{}^{g_{ijk}}\widetilde{m}_{ikl}\;i(\nu_{ijkl}) & = & (\lambda_{ij}\;\widetilde{m}_{jkl})\;{}^{\lambda_{ij}(g_{jkl})}\,\widetilde{m}_{ijl} \\ \nu_{ijkl}\,({}^{\lambda_{ij}(g_{jkl})}\nu_{ijlm})\;\lambda_{ij}(\nu_{jklm}) & = & {}^{g_{ijk}}\nu_{iklm}\;\{\widetilde{m}_{ijk}, g_{klm}\}^{-1}\;({}^{\lambda_{ij}\lambda_{jk}(g_{klm})}\nu_{ijkm}) \end{array}\right. \tag{6.3.11}$$

[6] This is essentially the 3-cocycle equation (4.2.17) of [4], but with the terms in opposite order due to the fact that the somewhat imprecise definition of a 2-arrow ν given on page 71 of [4] yields the inverse of the 2-arrow ν defined here by equation (6.3.2).

Returning to our discussion, let us consider such a $\mathcal{G}$-valued Čech 1-cocycle quadruple

$$(\lambda_{ij},\, \widetilde{m}_{ijk},\, g_{ijk},\, \nu_{ijkl})\,. \tag{6.3.12}$$

In order to produce from the weak crossed module of crossed modules (6.3.6) something which looks like a crossed square, it is expedient for us to restrict ourselves, in both the categories $\mathcal{G}$ and $\mathcal{E}q(\mathcal{G})$, to those arrows whose source is the identity object. Diagram (6.3.6) then becomes

$$\begin{array}{ccc} \mathrm{Ar}_I\, \mathcal{G} & \xrightarrow{\;i\;} & \mathrm{Ar}_I\, \mathcal{E}q(\mathcal{G}) \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ \mathrm{Ob}\, \mathcal{G} & \xrightarrow[\;i\;]{} & \mathrm{Ob}\, \mathcal{E}q(\mathcal{G}) \end{array} \tag{6.3.13}$$

where t is the target map. Recall that one can assign to any arrow $u : X \longrightarrow Y$ in a group-like monoidal category the arrow $uX^{-1} : I \longrightarrow YX^{-1}$ sourced at the identity, without loosing any significant information. In particular, the arrow $\widetilde{m}_{ijk}$ (6.3.10) may be replaced by an arrow

$$I \longrightarrow \lambda_{ij}\, \lambda_{jk}\, \lambda_{ik}^{-1}\, (i_{g_{ijk}})^{-1}$$

in $(\mathrm{Ar}_I\, \mathcal{E}q(\mathcal{G}))_{U_{ijk}}$ and the arrow ν_{ijkl} (6.2.6) by an arrow

$$I \longrightarrow g_{ijk}\;\, g_{ikl}\;\, g_{ijl}^{-1}\;\, (\lambda_{ij}(g_{ijk}))^{-1}\,,$$

in $(\mathrm{Ar}_I\, \mathcal{G})_{U_{ijkl}}$ which we again respectively denote by $\widetilde{m}_{ijk}$ and ν_{ijkl}. Our quadruple (6.3.9) then takes its values in the square

$$\begin{array}{ccc} (\mathrm{Ar}_I\, \mathcal{G})_{U_{ijkl}} & \xrightarrow{\;i\;} & (\mathrm{Ar}_I\, \mathcal{E}q(\mathcal{G}))_{U_{ijk}} \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ (\mathrm{Ob}\, \mathcal{G})_{U_{ijk}} & \xrightarrow[\;i\;]{} & (\mathrm{Ob}\, \mathcal{E}q(\mathcal{G}))_{U_{ij}} \end{array} \tag{6.3.14}$$

in the positions

$$\begin{pmatrix} \nu_{ijkl} & \widetilde{m}_{ijk} \\ g_{ijk} & \lambda_{ij} \end{pmatrix} \tag{6.3.15}$$

Since the evaluation action of $\mathcal{E}q(\mathcal{G})$ on $\mathcal{G}$ produces a map

$$\mathrm{Ar}_I\, \mathcal{E}q(\mathcal{G}) \times \mathrm{Ob}\, \mathcal{G} \longrightarrow \mathrm{Ar}_I\, \mathcal{G}$$

which is the analog of the morphism (1.5.8), the quadruple (6.3.9) may now be viewed as a cocycle with values in the (total complex associated to the) weak crossed square (6.3.13). We will say that this modified quadruple (6.3.12) is a Čech 1-cocycle for the covering $\mathcal{U}$ on X with values in the (weak) crossed square (6.3.13). The discussion in paragraph 6.2 will now be summarized as follows in purely algebraic terms:

Proposition 6.7. *To a $\mathcal{G}$-2-gerbe $\mathcal{P}$ on X, locally trivialized by the choice of objects x_i in $\mathcal{P}_{U_i}$ and local paths ϕ_{ij}* (6.2.1)*, is associated 1-cocycle* (6.3.9) *with values in the weak crossed square* (6.3.13).

Remark 6.8. When $\mathcal{G}$ is the *gr*-stack associated to a crossed module $\delta : G \longrightarrow \Pi$, this coefficient crossed module of *gr*-stacks is a stackified version of the following crossed square associated by K.J.Norrie (see [17], [6]) to the crossed module $G \longrightarrow \pi$:

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Der}^*(\pi, G) \\ {\scriptstyle \delta}\big\downarrow & & \big\downarrow \\ \pi & \longrightarrow & \mathrm{Aut}(G \to \pi) \end{array} \tag{6.3.16}$$

It is however less restrictive than Norrie's version, since the latter corresponds to the diagram of gr-stacks

$$\mathcal{G} \longrightarrow \mathrm{Isom}(\mathcal{G})$$

whereas we really need to consider, as in (6.3.6), self-equivalences of the monoidal stack $\mathcal{G}$, rather than automorphisms. To phrase it differently, we need to replace the term $\mathrm{Aut}(G \longrightarrow \pi)$ in the square (6.3.16) by the weak automorphisms of the crossed module $G \longrightarrow \pi$, as discussed in remark 1.10, and modify the term $\mathrm{Der}^*(\pi, G)$ accordingly.

6.4. **Coboundary relations**

We now choose a second set of local objects $x'_i \in \mathcal{P}_{U_i}$, and of local arrows (6.2.1)

$$\phi'_{ij} : x'_j \longrightarrow x'_i$$

By proposition 6.7, these determine a second crossed square valued 1-cocycle

$$(\lambda'_{ij},\, g'_{ijk},\, \widetilde{m}'_{ijk},\, \nu'_{ijkl})\,. \tag{6.4.1}$$

In order to compare it with the 1-cocycle (6.3.9), we proceed as we did in section 5.2 above, but now in a 2-categorical setting. We choose once more an arrow χ_i (5.2.2). There now exist 1-arrows δ_{ij}, and 2-arrows ζ_{ij} in $\mathcal{P}_{U_{ij}}$.

$$\begin{array}{ccc} x_j & \xrightarrow{\phi_{ij}} & x_i \\ & & \big\downarrow \chi_i \\ \chi_j \big\downarrow & \overset{\zeta_{ij}}{\nearrow\!\!\!\nearrow} & x'_i \\ & & \big\downarrow \delta_{ij} \\ x'_j & \xrightarrow[\phi'_{ij}]{} & x'_i \end{array} \tag{6.4.2}$$

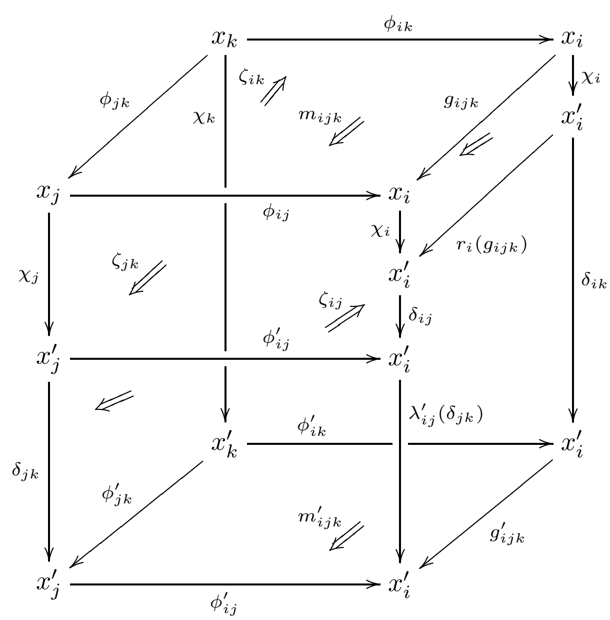
The arrow χ_i induces by conjugation a self-equivalence $r_i : \mathcal{G} \longrightarrow \mathcal{G}$ and 2-arrows

$$\begin{array}{ccc} x_i & \xrightarrow{u} & x_i \\ \chi_i \big\downarrow & \swarrow\!\!\!\swarrow & \big\downarrow \chi_i \\ x'_i & \xrightarrow[r_i(u)]{} & x'_i \end{array} \tag{6.4.3}$$

which are functorial in u. Furthermore, the diagram (6.4.2) induces by conjugation a diagram in $\mathcal{G}_{U_{ij}}$:

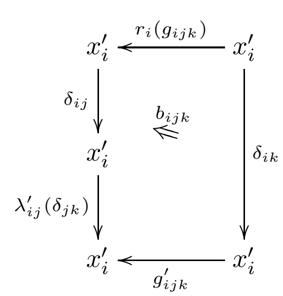
$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\lambda_{ij}} & \mathcal{G} \\ & & \big\downarrow r_i \\ r_j \big\downarrow & \overset{\widetilde{\zeta}_{ij}}{\nearrow\!\!\!\nearrow} & \mathcal{G} \\ & & \big\downarrow i_{\delta_{ij}} \\ \mathcal{G} & \xrightarrow[\lambda'_{ij}]{} & \mathcal{G} \end{array} \tag{6.4.4}$$

with $\widetilde{\zeta}_{ij}$ the natural transformation induced by ζ_{ij}. Consider now the diagram of 2-arrows



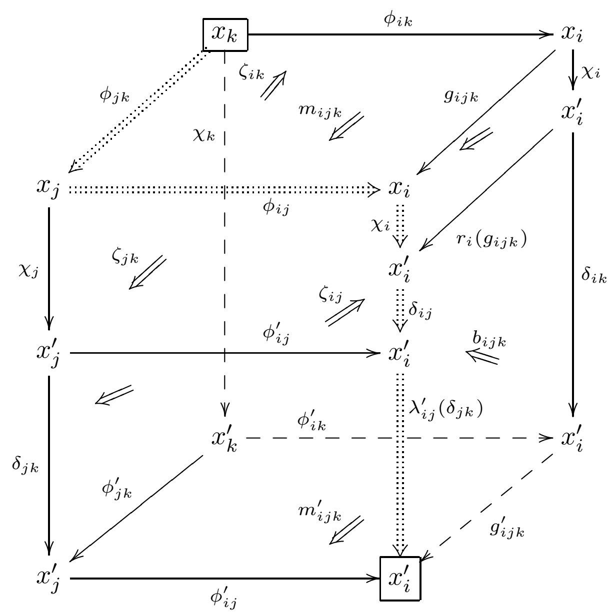
(6.4.5)

which extends (5.2.7). Three of its 2-arrows are of the form ζ_{ij}, the top and the bottom ones are of the form m_{ijk} (6.2.5). The unlabelled lower front 2-arrow and the right-hand upper are respectively part of the definitions of $\lambda'_{ij}(\delta_{jk})$ and of $r_i(g_{ijk})$. Since these seven 2-arrows are invertible, diagram (6.4.5) uniquely defines a 2-arrow b_{ijk} filling in the remaining lower right-hand square:



(6.4.6)

so that diagram (6.4.5) becomes the following commutative diagram of 2-arrows, which we directly display in decorated form, according to the conventions of (6.3.1):



(6.4.7)

We derive from this diagram the algebraic equation

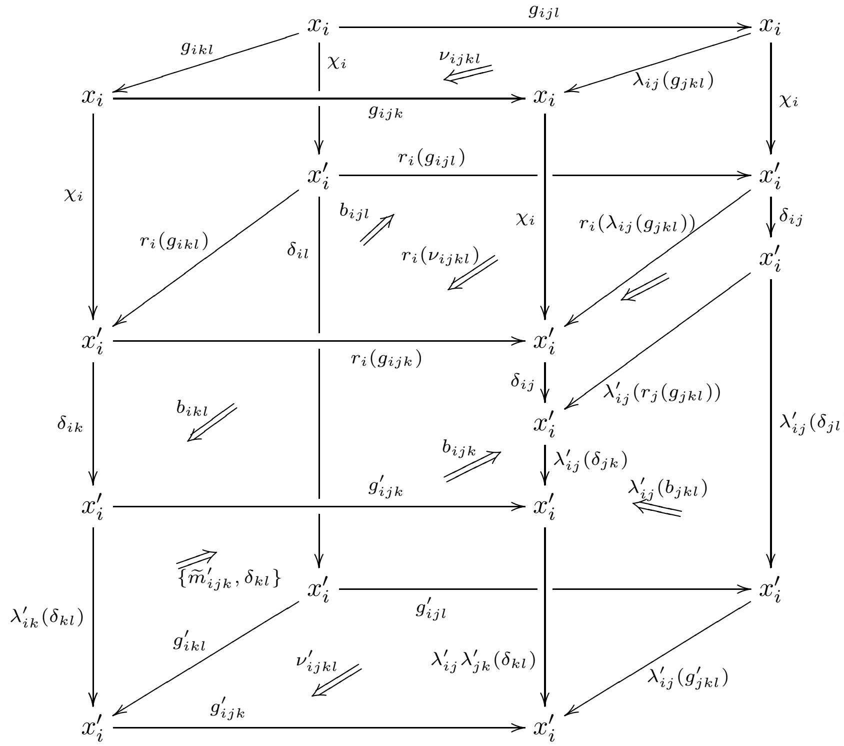
$$(\lambda'_{ij}(\delta_{kl}) * \zeta_{ij})\ (\phi'_{ij} * \zeta_{jk})\ m'_{ijk} = ((\lambda'_{ij}(\delta_{jk})\, \delta_{ij}\, \chi_i) * m_{ijk})\ b_{ijk}\ (g'_{ijk} * \zeta_{ik})$$

for the equality between the two corresponding 2-arrows between the decorated paths. With the same notations as for equation (6.3.5), the conjugated version of equation is

$$^{\lambda'_{ij}(\delta_{jk})}\widetilde{\zeta}_{ij}\ \lambda'_{ij}(\widetilde{\zeta}_{jk})\, \widetilde{m}'_{ijk} = {}^{\lambda'_{ij}(\delta_{jk})\, \delta_{ij}\, r_i}\, \widetilde{m}_{ijk}\ i(b_{ijk})\ ({}^{g'_{ijk}}\widetilde{\zeta}_{ik}) \tag{6.4.8}$$

This equation is the analog, with the present conventions, of equation [4] (4.4.12).

A second coboundary condition relates the cocycle quadruples (6.3.9) and (6.4.1). In geometric terms, it asserts the commutativity of the following diagram of 2-arrows, in which the unlabelled 2-arrow in the middle of the right vertical face is $\{\widetilde{\zeta}_{ij},\, g_{klm}\}^{-1}$ defined in the same way as the 2-arrow which we denoted $\{\widetilde{m}_{ijk},\, g_{klm}\}$ (6.2.11):



(6.4.9)

This cubic diagram compares the 2-arrows ν_{ijkl} and ν'_{ijkl}, which are respectively its top and bottom faces.It actually consists of two separate cubes. The upper one is trivially commutative, as it simply defines the 2-arrow $r_i(\nu_{ijkl})$, which is the common face between the two cubes considered.

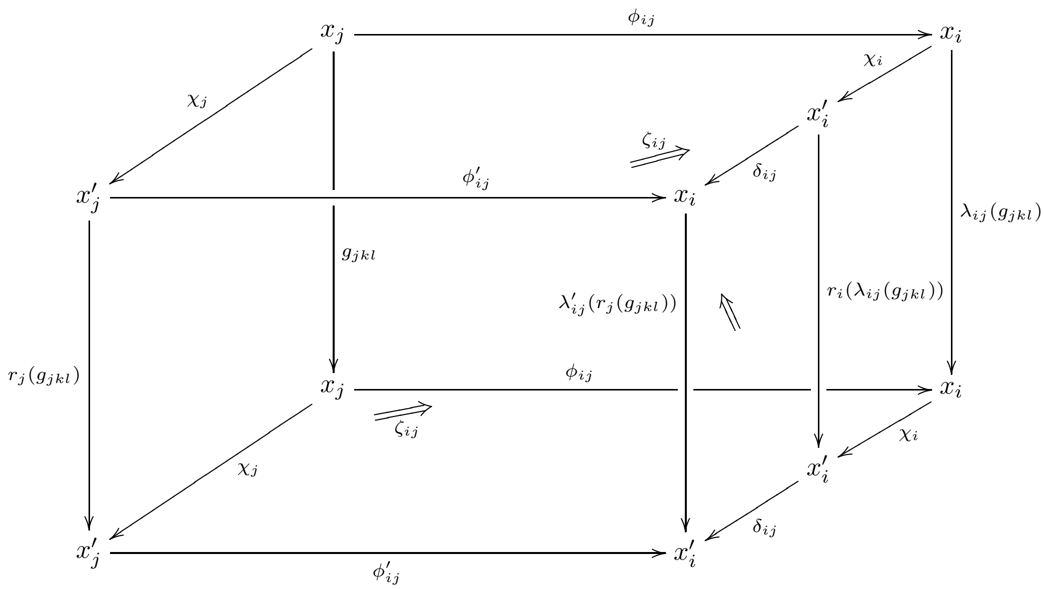
Lemma 2. *The cube of 2-arrows* (6.4.9) *is commutative.*

Proof: The proof that the full diagram (6.4.9) commutes is very similar to the proof of proposition 6.5. We consider a hypercube analogous to diagram (6.2.13), and which therefore consists of eight cubes called left, right, top, bottom, front, back, inner and outer. The outer cube in this diagram is the cube (6.4.9). We will now describe the seven other cubes. Since these seven are commutative, this will suffice in order to prove that the outer one also is, so that the lemma will be proved. As this hypercubic diagram is somewhat more complicated than (6.2.13), we will now describe it in words, instead of displaying it.

The top cube is a copy of cube (6.2.7), oriented so that its face ν_{ijkl} is on top, consistently with the top face of (6.4.9). The bottom cube is a cube of similar type which defines the 2-arrow ν'_{ijkl}. Since it is built from objects x', arrows ϕ' and g' and 2-arrows m' and ν', we will refer to it as the primed version of (6.2.7) . It is time oriented so that ν'_{ijkl} is the bottom face.

We now describe the six other cubes. Four of these are of the type (6.4.7). If we denote the latter by the symbol P_{ijk} determined by its indices, these are respectively the left cube P_{ikl}, the back cube P_{ijl}, the inner cube P_{jkl} and the front cube P_{ijk}. Each of the first three rests on the corresponding face m'_{ikl}, m'_{ijl}, and m'_{jkl} of the bottom cube, and is attached at the top to the similar face m of the top cube. The cube P_{ijk} is attached to the corresponding face m'_{ijk} of the top cube, but it does not constitute the full front hypercube. Below it is a copy of the primed version of the cube (6.2.12), resting on the face m'_{ijk} of the bottom cube. Finally, the right cube is itself constituted of two cubes.

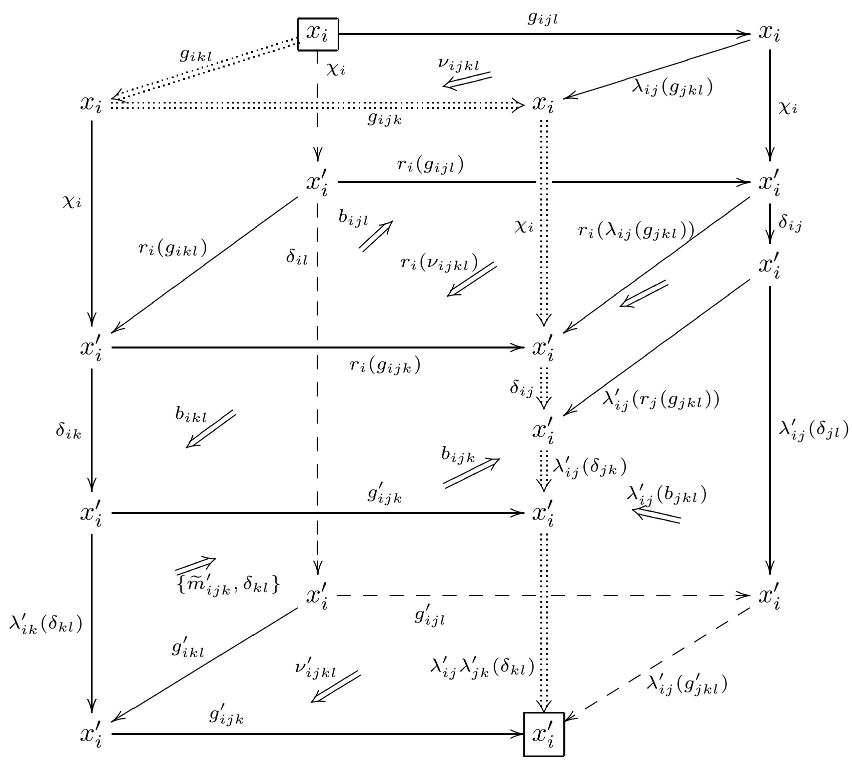
The lower one constructs the 2-arrow $\lambda'_{ij}(b_{jkl})$, starting from the 2-arrow b_{jkl} (6.4.6). The upper one is a commutative cube of same type as (6.2.12), but this time associated to the face $\{\widetilde{\zeta}_{ij},\, g_{jkl}\}$ rather than to $\{\widetilde{m}'_{ijk},\, g'_{jkl}\}$. More precisely it is the commutative cube with unlabelled 2-arrow $\{\widetilde{\zeta}_{ij},\, g_{jkl}\}$ at the front of the right-hand vertical face, and whose four missing 2-arrows are the obvious ones .



(6.4.10)

□

In order to translate the commutativity of the cube (6.4.9) into an algebraic expression, we decorate it as follows, invoking once more the conventions of diagram (6.3.1):



(6.4.11)

Reading off the two composite 2-arrows between the decorated 1-arrows (6.4.11), and taking into account the appropriate whiskerings, we see that the commutativity of diagram (6.4.11) is equivalent to the following algebraic equation.

$$\begin{aligned}({}^{\lambda'_{ij}\lambda'_{jk}(\delta_{kl})\lambda'_{ij}(\delta_{jk})\delta_{ij}r_i}\,\nu_{ijkl})\,({}^{\lambda'_{ij}\lambda'_{jk}(\delta_{kl})\lambda'_{ij}(\delta_{jk})}\{\widetilde{\zeta}_{ij},\,g_{klm}\}^{-1})\;\lambda'_{ij}(b_{jkl})\;({}^{\lambda'_{ij}(g'_{jkl})}b_{ijl})\\ = ({}^{\lambda'_{ij}\lambda'_{jk}(\delta_{kl})}b_{ijk})\,\{\widetilde{m}'_{ijk},\,\delta_{kl}\}\,({}^{g'_{ijk}}b_{ikl})\;\nu'_{ijkl}\end{aligned} \tag{6.4.12}$$

This equation is the analog, under our present conventions, of equation (4.4.15) of [4]. It describes the manner in which the various terms of type b_{ijk} determine a coboundary relation between the non-abelian cocycle terms ν_{ijkl} and ν'_{ijkl}. A certain amount of twisting takes place, however, and the extra terms $\{\widetilde{\zeta}_{ij},\,g_{klm}\}^{-1}$ and $\{\widetilde{m}'_{ijk},\,\delta_{kl}\}$ need to be inserted in their proper locations, just as the factor $\{\widetilde{m}_{ijk}, g_{klm}\}^{-1}$ was necessary in order to formulate equation (6.3.8). Once more, an equation such as (6.4.12) cannot be viewed in isolation from its companion equation (6.4.8). In addition, any arrow in either of the monoidal categories $\mathrm{Ar}(\mathcal{G})$ or $\mathcal{G}$ must be replaced by the corresponding one sourced at the identity, without changing its name. The following definition summarizes the previous discussion.

Definition 6.9. *Let* $(\lambda_{ij},\,\widetilde{m}_{ijk},\,g_{ijk},\,\nu_{ijkl})$ *and* $(\lambda'_{ij},\,g'_{ijk},\,\widetilde{m}'_{ijk},\,\nu'_{ijkl})$ *be a pair of 1-cocycles with values in the weak crossed square* (6.3.13)*. A 1-coboundary relation between this pair of 1-cocycles is a quadruple* $(r_i,\,\widetilde{\zeta}_{ij},\,\delta_{ij},\,b_{ijk})$ *with values in the weak crossed cube* (6.3.13)*. More precisely, these elements take their values in the square*

$$\begin{array}{ccc}(\mathrm{Ar}_I\,\mathcal{G})_{U_{ijk}} & \xrightarrow{\;i\;} & (\mathrm{Ar}_I\,\mathcal{E}q(\mathcal{G}))_{U_{ij}}\\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t}\\ (\mathrm{Ob}\,\mathcal{G})_{U_{ij}} & \xrightarrow[\;i\;]{} & (\mathrm{Ob}\,\mathcal{E}q(\mathcal{G}))_{U_i}\end{array} \tag{6.4.13}$$

in the positions

$$\begin{pmatrix} b_{ijk} & \widetilde{\zeta}_{ij}\\ \delta_{ij} & r_i\end{pmatrix} \tag{6.4.14}$$

The arrows b_{ijk} *and* $\widetilde{\zeta}_{ij}$ *are respectively of the form*

$$I \xrightarrow{\;b_{ijk}\;} \lambda'_{ij}(\delta_{jk})\,\delta_{ij}\,r_i(g_{ijk})\,\delta_{ik}^{-1}\,(g'_{ijk})^{-1}$$

and

$$I \xrightarrow{\;\widetilde{\zeta}_{ij}\;} i_{\delta_{ij}}\,r_i\,\lambda_{ij}\,r_j^{-1}\,{\lambda'_{ij}}^{-1}$$

and satisfy the equations (6.4.8) *and* (6.4.12)*. The set of equivalence classes of 1-cocycle quadruples* (6.3.15)*, for the equivalence defined by these coboundary relations will be called the Čech degree 1 cohomology set for the open covering* $\mathcal{U}$ *of* X *with values in the weak crossed square* (6.3.13)*. Passing to the limit over the families of such open coverings of* X*, one obtains the Čech degree 1 cohomology set of* X *with values in this square.*

The discussion in paragraphs 6.2-6.4 can now be entirely summarized as follows:

Proposition 6.10. *The previous constructions associate to a* $\mathcal{G}$*-2-gerbe* $\mathcal{P}$ *on a space* X *an element of the Čech degree 1 cohomology set of* X *with values in the square* (6.3.13)*, and this element is independent of the choice of local objects and arrows in* $\mathcal{P}$*.*

We refer to chapter 5 of [4] for the converse to this proposition, which asserts that to each such 1-cohomology class corresponds a $\mathcal{G}$-2-gerbe, uniquely defined up to equivalence.

Remark 6.11. As we observed in footnote 3, the proposition is only true as stated when the space X satisfies an additional assumption such as paracompactness. The general case is discussed in [4], where the open covering $\mathcal{U}$ of X is replaced by a hypercover.

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