Quasi-categories vs Simplicial categories

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Abstract

We show that the coherent nerve functor from simplicial categories to simplicial sets is the right adjoint in a Quillen equivalence between the model category for simplicial categories and the model category for quasi-categories.

Introduction

A quasi-category is a simplicial set which satisfies a set of conditions introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. A quasi-category is often called a weak Kan complex in the literature. The category of simplicial sets $\mathbf{S}$ admits a Quillen model structure in which the cofibrations are the monomorphisms and the fibrant objects are the quasi-categories [J2]. We call it *the model structure for quasi-categories.* The resulting model category is Quillen equivalent to the model category for complete Segal spaces and also to the model category for Segal categories [JT2]. The goal of this paper is to show that it is also Quillen equivalent to the model category for simplicial categories via the coherent nerve functor of Cordier.

We recall that a *simplicial category* is a category enriched over the category of simplicial sets $\mathbf{S}$. To every simplicial category X we can associate a category X' enriched over the homotopy category of simplicial sets $Ho(\mathbf{S})$. A simplicial functor $f : X \to Y$ is called a *Dwyer-Kan equivalence* if the functor $f' : X' \to Y'$ is an equivalence of $Ho(\mathbf{S})$-categories. It was proved by Bergner, that the category of (small) simplicial categories $\mathbf{SCat}$ admits a Quillen model structure in which the weak equivalences are the Dwyer-Kan equivalences [B1].

If X is a simplicial category and A is a category, a *homotopy coherent diagram* $A \to X$ is defined to be a simplicial functor $C_*(A) \to X$, where $C_*(A)$ is a certain simplicial resolution of the category A. This notion was introduced by Vogt in [V]. The *coherent nerve* of a simplicial category X is the simplicial set $C^! X$ obtained by putting

$$(C^! X)_n = \mathbf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. This notion was introduced by Cordier in [C]. The functor $C^!$ has a left adjoint $C_!$,

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

We shall prove that the pair $(C_!, C^!)$ is a Quillen equivalence between the model structure for quasi-categories and the model structure for simplicial categories. The proof uses the results in [B2], [JT2] and [J3]. See Lurie [Lu] for a different proof.

The theory of Segal categories was developed by Hirschowitz and Simpson for its applications to algebraic geometry. We recall that a *simplicial space* is defined to be a simplicial object $X : \Delta^o \to \mathbf{S}$. A simplicial space X is called a *pre-category* if the simplicial set X_0 is discrete. In this case, we have a decomposition

$$X_n = \bigsqcup_{a \in X_0^{n+1}} X(a)$$

for every $n \geq 0$, where $X(a) = X(a_0, a_1, \ldots, a_n)$ denotes the fiber of the vertex map $X_n \to X_0^{n+1}$ at $a = (a_0, a_1, \cdots, a_n)$. A precategory X is called a *Segal category* if the Segal map

$$X_n(a_0, a_1, \cdots, a_n) \to X(a_0, a_1) \times \cdots \times X(a_{n-1}, a_n)$$

is a weak homotopy equivalence for every $(a_0, a_1, \cdots, a_n)$. To every Segal category X we can associate a category X' enriched over the homotopy category of simplicial sets. A map of Segal categories $f : X \to Y$ is called a *categorical equivalence* if the functor $f' : X' \to Y'$ is an equivalence of enriched categories. More generally, Hirschowitz and Simpson introduces a notion of weak categorical equivalence between pre-categories. They show that the category of pre-categories **PCat** admits a model structure in which the cofibrations are the monomorphisms and the weak equivalences are the weak categorical equivalences. We call the model structure the *injective* model structure for Segal categories and we denote the resulting model category by **PCat**.

A simplicial space can be regarded as a bisimplicial set by putting $X_{mn} = (X_m)_n$ for every $m, n \geq 0$. Consider the functor j^* which associates to pre-category X its first row $X_{\star 0}$. The functor has a left adjoint q^*,

$$q^* : \mathbf{S} \leftrightarrow \mathbf{PCat} : j^*.$$

It turns out that the pair (q^*, j_*) is Quillen equivalence between the model category for quasi-categories and the model category for Segal categories [JT2].

If X is a simplicial category, then we can construct a simplicial category $X^{(n)}$ for every $n \geq 0$. By definition, $ObX^{(n)} = ObX$ and

$$X^{(n)}(a, b) = X(a, b)^{\Delta[n]}$$

for every pair $a, b \in ObX$. To a pair of simplicial categories X and Y, we can attach a simplicial set $hom(X, Y)$ by putting

$$Hom(X, Y)_n = \mathbf{SCat}(X, Y^{(n)})$$

for every $n \geq 0$. This defines a simplical enrichement of the category **SCat**. The *strong coherent nerve* a simplicial category X is the pre-category $K^! X$ obtained by putting

$$(K^! X)_m = Hom(C_\star[m], X)$$

for every $m \geq 0$. The first row of $K^! X$ is the coherent nerve $C^! X$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathbf{SCat} & \xrightarrow{C^!} & \mathbf{S} \\ & \searrow^{K^!} & \uparrow{\scriptstyle j^*} \\ & & \mathbf{PCat} \end{array}$$

The functor $K^!$ has a left adjoint $K_!$,

$$K_! : \mathbf{PCat} \leftrightarrow \mathbf{SCat} : K^!.$$

In order to prove that the pair $(C_!, C^!)$ is Quillen equivalence, it suffices to show that the pair $(K_!, K^!)$ is a Quillen equivalence.

We shall say that a map of pre-categories $f : X \to Y$ is a *locally trivial fibration* if the map $f_0 : X_0 \to Y_0$ is surjective and the map $X(a) \to Y(fa)$ induced by f is a trivial fibration for every $a \in X_0^{n+1}$ and $n \geq 1$. It was proved by Bergner [B2] that the category **PCat** admits a model structure in which the the weak equivalences are the weak categorical equivalences and the acyclic fibrations are the locally trivial fibrations. We call the model structure the *projective* model structure for Segal categories and we denote the resulting model category by $\mathbf{PCat}'$. By a theorem of Bergner, the pair of identity functors

$$Id : \mathbf{PCat}' \leftrightarrow \mathbf{PCat} : Id.$$

is a Quillen equivalence between the projective model structure and the injective model structures [B2]. It follows that the pair

$$K_! : \mathbf{PCat} \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen equivalence iff the pair

$$K_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen equivalence.

The nerve of a simplicial category X is the precategory $N^! X$ defined by putting

$$(N^! X)_n = Hom([n], X)$$

for every $n \geq 0$. The functor $N^! : \mathbf{SCat} \to \mathbf{PCat}$ has a left adjoint $N_!$. The pair

$$N_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : N^!,$$

is a Quillen equivalence by a theorem of Bergner [B2]. The augmentation $\epsilon : C_*[n] \to [n]$ induces a natural transformation $\phi : K_! \to N_!$. We show that the map $\phi_X : K_!(X) \to N_!(X)$ is a Dwyer-Kan equivalence for every cofibrant object $X \in \mathbf{PCat}'$. It follows that the pair

$$K_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen equivalence, since the pair $(N_!, N^!)$ is a Quillen equivalence. It follows that the pair

$$K_! : \mathbf{PCat} \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen equivalence.

The paper is organised as follows. In the first section we show that the pair of adjoint functors $(C_!, C^!)$ is a Quillen pair. In the second section we show that the pair of adjoint functors $(K_!, K^!)$ is a Quillen pair. In the third section we show that the the pairs $(K_!, K^!)$ and $(C_!, C^!)$ are Quillen equivalences.

In order to establish the properties of the pair $(K_!, K^!)$ we use the *fibered model structures* sudied in [J3]. There is a fibrered model structure associated to each of the three model structure considered above. The fibered model structures are simplicial. They are described in appendix.

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1 The nerve functors

1.1 The strict nerve functors

The category Δ is a full subcategory of **Cat**. The *nerve* of a category $C \in \mathbf{Cat}$ is the simplicial set NC obtained by putting $(NC)_n = \mathbf{Cat}([n], C)$ for every $n \geq 0$. The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ is full and faithful and we shall regard it as an inclusion $N : \mathbf{Cat} \subset \mathbf{S}$ by adopting the same notation for a category and its nerve. The functor N has a left adjoint

$$\tau_1 : \mathbf{S} \to \mathbf{Cat}.$$

We say that $\tau_1 X$ is the *fundamental category* of a simplicial set X. The fundamental groupoid $\pi_1 X$ is obtained by inverting the arrows of $\tau_1 X$.

The category of small simplicial categories **SCat** is enriched over **S** by Proposition 5.8. The nerve of a simplicial category X is the simplicial space $N^! X$ defined by putting

$$(N^! X)_n = Hom([n], X)$$

for every $n \geq 0$, where the poset $[n]$ is viewed as a simplicial category with discrete hom sets. We have $(N^! X)_0 = Hom([0], X) = X_0$. Hence the simplicial space is a precategory, since X_0 is discrete. If X is a simplicial category we shall denote by X_n the ordinary category obtained by putting $X_n(a, b) = X(a, b)_n$ for every $n \geq 0$. Then we have

$$(N^! X)_{mn} = \mathbf{Cat}([m], X_n)$$

for every $m, n \geq 0$. The simplicial set $(N^! X)_n$ is the coproduct of the simplicial sets

$$X(a_0, a_1) \times \cdots \times X(a_{n-1}, a_n)$$

for $(a_0, \ldots, a_n) \in ObX^{n+1}$. The functor $N^!$ has a left adjoint

$$N_! : \mathbf{PCat} \to \mathbf{SCat}.$$

If Y is a pre-category, then we have $(N_! Y)_n = \tau_1(Y_{\star n})$ for every $n \geq 0$.

1.2 The unreduced coherent nerve functor

Recall that a *graph* X is a map $(s,t) : X^a \to X^{ver} \times X^v$. An element of X^{ar} is called an *arrow* and an element of X^v a *vertex*. If $f \in X^a$, then $s(f)$ is called the *source* of the arrow f and $t(f)$ the *target*. We shall denote by **Grph** the category of graphs (it is a presheaf category). The obvious forgetful functor $U : \mathbf{Cat} \to \mathbf{Grph}$ has a left adjoint

$$F : \mathbf{Grph} \to \mathbf{Cat},$$

where FX is the category freely generated by a graph X. By construction, the objects of FX are the vertices of X. If $a,b \in X^v$, then a *morphism* $f \in (FX)(a,b)$ is a path of length $n \geq 0$ in the graph X:

$$a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \cdots \qquad \cdots a_{n-1} \xrightarrow{f_n} a_n = b\ .$$

The composite of $f \in (FX)(a,b)$ with $g \in (FX)(b,c)$ is their concatenation $g \star f : (FX)(a,c)$. If A is a category and $a,b \in ObA$, then a morphism $f \in (FUA)(a,b)$ can be represented by a map $f : [n] \to A$ in Cat such that $f(0) = a$ and $f(n) = b$. Hence the arrows of the category FUA are the simplices of the nerve of the category A.

It follows from the adjointness $F \vdash U$, that the functor $P = FU : \mathbf{Cat} \to \mathbf{Cat}$ has the structure of a comonad. We shall denote its comultiplication by $\delta : P \to P^2$ and its counit by $\epsilon : P \to Id$. If A is a category, the objects of the category $PA = FUA$ are the objects of A. If $a,b \in ObA$, then a path $f : a \to b$ of length n can be represented as a functor $f : [n] \to A$ such that $f(0) = a$ and $f(n) = b$. It is standard [] that the family $P_nA = P^{n+1}(A)$ for $n \geq 0$ has the structure of a simplicial object. The simplicial set $n \mapsto Ob(P_nA)$ is constant with value $Ob(A)$. It follows that $P_\star A$ can be viewed as a simplicial category instead of a category object in $\mathbf{S}$. This defines a functor

$$P_\star : \mathbf{Cat} \to \mathbf{SCat}.$$

The *unreduced coherent nerve* of a simplicial category X is the simplicial set $P^! X$ defined by putting

$$(P^! X)_n = \mathbf{SCat}(P_\star[n], X)$$

for every $n \geq 0$. The functor $P^! : \mathbf{SCat} \to \mathbf{S}$ has a left adjoint $P_!$ which is the left Kan extension of the functor $[n] \mapsto P_\star[n]$ along the Yoneda functor $\Delta \to \mathbf{S}$. By construction we have

$$P_! X = \varinjlim_{\Delta[n] \to X} P_\star[n]$$

where the colimit is taken over the category of elements of X. The objects of the simplicial category $P_!(X)$ are the vertices of X. We shall see in 1.4 that we have $P_! A = P_\star A$ for every category A.

We shall denote by Δ' the subcategory of Δ whose arrows are the maps $u : [m] \to [n]$ such that $u(0) = 0$. If $u(0) = 0$ and $u(m) = n$, we shall say that u is *wide*. The wide maps form a subcategory $\Delta" \subset \Delta'$. The *concatenation* of a wide map $u : [m] \to [n]$ with a wide map $v : [p] \to [q]$ is the wide map $v \star u : [p + m] \to [q + n]$ obtained by putting

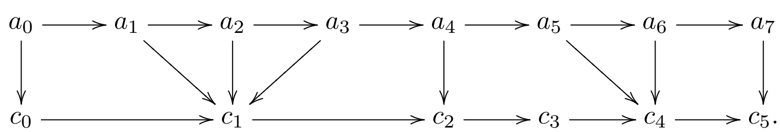
$$(v \star u)(i) = \begin{cases} u(i) & \text{if } 0 \leq i \leq m \\ n + v(i - m) & \text{if } m \leq i \leq m + p. \end{cases}$$

The functor of two variables

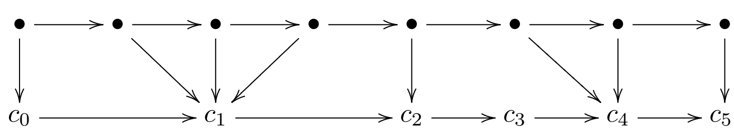
$$\star : \Delta" \times \Delta" \to \Delta".$$

is the tensor product of a (strict) monoidal structure on the category Δ''.

Recall that a *2-category* is a category enriched over **Cat**. If A is a category, let us denote by QA the 2-category constructed as follows. By definition, the 1-skeleton of QA (obtained by forgetting the 2-cells of QA) is the category FUA. Thus, a *0-cell* of QA is an object of A; if $a, b \in ObA$, then a *1-cell* $a \to b$ is a path $f : [m] \to A$ such that $f(0) = a$ and $f(n) = b$. If $f : [m] \to A$ and $g : [n] \to A$ are paths $a \to b$, then a 2-cell $u : f \Rightarrow g$ is a wide map $u : [m] \to [n]$ such that $gu = f$. The 2-cell is a commutative diagram in the category A,



The top and bottom paths of this diagram are defining 1-cells of $Q(A)$. The configuration of downward arrows is representing a wide map defining a 2-cell. Every downward arrow of the diagram is a unit in A. For example, we have $a_0 = c_0$ and $a_1 = a_2 = a_3 = c_1$. Observe that a 2-cell $u : f \Rightarrow g$ is determined by the pair (u, g), where $g : [m] \to A$ is a path and $u : [m] \to [n]$ is a wide map. It can thus be represented by a diagram



The vertical composition of a 2-cell $u : f \Rightarrow g : a \to b$ defined by a wide map $u : [m] \to [n]$ with a 2-cell $v : g \Rightarrow h : a \to b$ defined by a wide map $v : [n] \to [p]$ is the 2-cell $vu : f \Rightarrow h : a \to b$ defined by the wide map $vu : [m] \to [p]$. The horizontal composition of a 2-cell $u : f \Rightarrow g : a \to b$ with a 2-cell $v : k \Rightarrow l : b \to c$ is the concatenation of wide maps $v \star u : k \star f \to l \star g : a \to c$.

The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ preserves products. Hence a 2-category X becomes a simplicial category SX if we put

$$(SX)(a, b) = N(X(a, b))$$

for every pair of objects $a, b \in X$.

Theorem 1.1 *The simplicial category $P_\star A$ is isomorphic to the category SQA.*

Proof This follows from **??**, since the monad U_0F_0 : **Graph** $\to$ **Graph** is cartesian. ∎

We shall say that a graph object $(s,t) : X^a \to X^v \times X^v$ in **Cat** is a *categorical graph* if X^v is a discrete category. Similarly, we shall say that a graph object $(s,t) : X^a \to X^v \times X^v$ in **S** is a *simplicial graph* if X^v is a discrete simplicial set. The nerve of a categorical graph is a simplicial graph. We shall denote by **SGraph** the category of simplicial graphs.

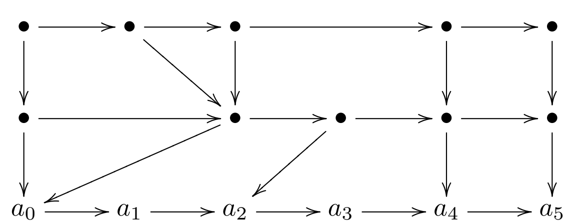
A simplicial set is a contravariant functor $X : \Delta \to$ **Set**. Hence it has category of elements $el(X) = X/\Delta$. We shall denote by $wel(X)$ the category of elements of the contravariant functor $X \mid \Delta$" obtained by restricting X to the subcategory of wide maps $\Delta" \subset \Delta$. The map $X_n \to X_0$ which associates to a simplex $x : \Delta[n] \to X$ the vertex $x(0) \in X_0$ induces a functor $s : wel(X) \to X_0$, where the set X_0 is viewed as a discrete category. Similarly, the map $X_n \to X_0$ which associates to a simplex $x : \Delta[n] \to X$ the vertex $x(n) \in X_0$ induces a functor $t : wel(X) \to X_0$. We thus obtain a categorical graph $(s,t) : wel(X) \to X_0 \times X_0$. The nerve of this categorical graph is a simplicial graph $(s,t) = (Ns, Nt) : Nwel(X) \to X_0 \times X_0$ that we shall denote by $G(X)$. This defines a functor

$$G : \mathbf{S} \to \mathbf{SGraph}.$$

Lemma 1.2 *The functor $G :$ **S** $\to$ **SGraph** is cocontinuous.*

Proof: It suffices to show that the functor $X \mapsto Nwel(X)$, from the category **S** to itself, is cocontinuous. If C is a small category, then the functor El : $[C^o, \mathbf{Set}] \to \mathbf{S}$ defined by putting $El(X) = N(el(X))$ is cocontinuous by **??**. The result follows, since we have $Nwel(X) = El(X \mid \Delta")$ and since the functor $X \mapsto X \mid \Delta"$ is cocontinuous. ∎

If A is a category and $n > 0$, then the category $P_n(A) = P^{n+1}(A)$ is freely generated by the graph $UP_{n-1}(A) = UP^n(A)$. We shall say that an element of $P_n(A)(a,b)$ is a *carpet* of *height* n from a to b. For example, he following picture represents a carpet of heght 2:



It follows from Theorem 1.1 that a carpet of *height* n from a to b is a path $\alpha : [n] \to QA(a,b)$ of length n in the category $QA(a,b)$,

$$f_0 \xrightarrow{u_1} f_1 \xrightarrow{u_2} f_2 \cdots \qquad \cdots f_{n-1} \xrightarrow{u_n} f_n \;.$$

The 1-cell f_i is a path $f_i : [r_i] \to A$ and the the 2-cell $u_i : f_{i-1} \Rightarrow f_i$ is a wide map $u_i : [r_{i-1}] \Rightarrow [r_i]$. It follows from these observations, that the carpet α is determined by the sequence

$$[r_0] \xrightarrow{u_1} [r_1] \xrightarrow{u_2} [r_2] \cdots \qquad \cdots [r_{n-1}] \xrightarrow{u_n} [r_n] \to^{f_n} A\ .$$

Hence there is a natural bijection between the set of carpets of length n in A and the set of n-simplices of the nerve of the category $welN(A)$.

Lemma 1.3 *The simplicial graphs $UP_*(A) = USQ(A)$ and $G(A)$ are naturally isomorphic.*

Proof: This follows from the discussion above. ∎

Recall that the functor

$$P_! : \mathbf{S} \to \mathbf{SCat}$$

is defined to be the left Kan extension of the functor $[k] \mapsto P_\star([k])$.

Theorem 1.4 *If A is a category, then $P_!(A) = P_\star(A)$*

Proof: It suffices to show that functor $P_\star : \mathbf{Cat} \to \mathbf{SCat}$ admits a cocontinuous extension to the category $\mathbf{S}$. For this, it suffices to show that the functor $P_n = P^{n+1} : \mathbf{Cat} \to \mathbf{Cat}$ admits a cocontinuous extension to $\mathbf{S}$ for every $n \geq 0$. But we have $P^{n+1}(A) = FUP^n(A)$. Hence it suffices to show that the functor $UP^n : \mathbf{Cat} \to \mathbf{Grph}$ admits a cocontinuous extension to $\mathbf{S}$, since the functor $F : \mathbf{Grph} \to \mathbf{Cat}$ is cocontinuous. The functor $\mathbf{S} \to \mathbf{Grph}$ which takes a simplicial set X to the graph $(\partial_1, \partial_0) : X_1 \to X_0 \times X_0$ is cocontinuous and it extends the functor $U : \mathbf{Cat} \to \mathbf{Grph}$. Hence the result is true for $n = 0$. The graph $UP^{n+1}(A) = UP_n(A)$ is isomorphic to the graph $G_n(A) = G(A)_n$ by Lemma 1.3. The result then follows by Lemma 1.2. ∎

1.3 The coherent nerve functor

If X is a simplicial category and A is a category, a *homotopy coherent diagram* $A \to X$ is defined to be a simplicial functor $C_*(A) \to X$, where $C_*(A)$ is a certain simplicial resolution of the category A. This notion was introduced by Vogt in [V]. The *coherent nerve* of a simplicial category is the adjoint construction $C^! X$ introduced by Cordier in [C]. There is a natural bijection between the simplicial functors $C_*(A) \to X$ and the maps of simplicial sets $A \to C^! X$. The simplicial set $C^! X$ is a quasi-category when the simplicial category X is enriched over Kan complexes [CP]. We shall prove that the pair of adjoint functors

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen pair between the model category for simplicial categories and the model category for quasi-categories. We shall later prove that the pair $(C_!, C^!)$ is actually a Quillen equivalence.

Recall that a *reflexive graph* X is a graph equipped with a map $u : X^{ver} \to X^{ar}$ satisfying $su = id = tu$. If $a \in X^{ver}$ the arrow $u(a) : a \to a$ is said to be the *unit* of the vertex a. We shall denote by $\mathbf{Grph}_0$ the category of reflexive graphs (it is a presheaf category). The obvious forgetful functor $U_0 : \mathbf{Cat} \to \mathbf{Grph}_0$ has a left adjoint

$$F_0 : \mathbf{Grph}_0 \to \mathbf{Cat},$$

where F_0X is the category freely generated by a reflexive graph X.

By construction, the objects of F_0X are the vertices of X. We shall say that a path $f : a \to b$ of length b is *reduced* if the arrow f_i is a non-unit for every $1 \leq i \leq n$. A concatenation of reduced paths is reduced. The category of reduced paths F_0X is a subcategory of the catagory FX. If A is a category, then a path $f : [n] \to A$ is reduced iff it is a non-degenerate simplex of the nerve of A. Hence the arrows of the category F_0U_0A are the non-degenerate simplices of the nerve of the category A.

It follows from the adjointness $F_0 \vdash U_0$, that the functor $C = F_0U_0 : \mathbf{Cat} \to \mathbf{Cat}$ has the structure of a comonad. We shall denote its comultiplication by $\delta : C \to C^2$ and its counit by $\epsilon : C \to Id$. Hence the sequence of categories $C_nA = C^{n+1}(A)$ for $n \geq 0$ has the structure of a simplicial object in $\mathbf{Cat}$. The simplicial set $n \mapsto Ob(C_nA)$ is constant with value $Ob(A)$. It follows that $C_\star A$ can be viewed as a simplicial category instead of a simplicial object in $\mathbf{Cat}$. This defines a functor

$$C_\star : \mathbf{Cat} \to \mathbf{SCat}.$$

The *coherent nerve* of a simplicial category X is the simplicial set $C^!X$ defined by putting

$$(C^!X)_n = \mathbf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. The functor $C^! : \mathbf{SCat} \to \mathbf{S}$ has a left adjoint $C_!$ which is the left Kan extension of the functor $[n] \mapsto C_\star[n]$ along the Yoneda functor $\Delta \to \mathbf{S}$. By construction, we have

$$C_!X = \varinjlim_{\Delta[n] \to X} C_\star[n]$$

where the colimit is taken over the category of elements of X. The objects of the simplicial category $C_!(X)$ are the vertices of X. We shall see in 1.11 that we have $C_!A = C_\star A$ for every category A.

The following theorem is the main result of the section. It was anticipated by Cordier and Poter in [CP].

Theorem *The adjoint pair of functors*

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen pair between the model category for quasi-categories and the model category for simplicial categories.

The proof is given in 1.21.

If $n \geq 0$, let us denote by $B[n]$ the (nerve of) the poset of non-empty subsets of $[n]$ ordered by the inclusion relation. From a map $f : [m] \to [n]$, we obtain a map $B(f) : B[m] \to B[n]$ by putting $B(f)(S) = f(S)$ for $S \subseteq [m]$. This defines a functor $B : \Delta \to \mathbf{S}$. We shall denote again by

$$B : \mathbf{S} \to \mathbf{S}$$

the left Kan extension of the functor $B : \Delta \to \mathbf{S}$ along the Yoneda functor $\Delta \subset \mathbf{S}$. The simplicial set $B(X)$ is the *barycentric subdivision* of a simplicial set X. By definition we have

$$B(X) = \varinjlim_{\Delta[n] \to X} B[n]$$

where the colimit is taken over the category Δ/X of elements of X. We need an explicit description of the simplicies of X. Let us denote by $\mid X \mid$ the set of non-degenerated simplices of X.

Lemma 1.5 *Let $U : \Delta^s \to$ **Set** be a functor defined on the subcategory of surjections $\Delta^s \subset \Delta$. Then the left Kan extension $V : \mathbf{S} \to$ **Set** of the functor U along the inclusion $\Delta^s \subset \mathbf{S}$ is cocontinuous and for any simplicial set X we have*

$$V(X) = \bigsqcup_{x \in \mid X \mid} U[dx],$$

where dx denotes the dimension of x.

Proof: Let us denote by U' the left Kan extension of the functor U along the inclusion $\Delta' \subset \Delta$. Then the functor V is the left Kan extension of the functor U' along the Yoneda functor $\Delta \subset \mathbf{S}$. Thus, V is cocontinuous by a general result of category theory [Mac]. Let us prove the second statement. By a general formula for left Kan extension, we have

$$V(X) = \varinjlim_{\Delta'/X} Up$$

where the colimit is taken over the category of elements of the functor $X \mid \Delta'$ and where p is the canonical functor $X \mid \Delta' \to \Delta'$. It is easy to verify that the set $\mid X \mid$ of non-degenerated simplicies of X is final in the category Δ'/X. Thus,

$$\varinjlim_{\Delta'/X} Up = \bigsqcup_{x \in \mid X \mid} U[dx].$$

■

Proposition 1.6 *Let $B(X)$ be the barycentric subdivision of a simplicial set X. Then a n-simplex of $B(X)$ is a pair (T,x), where $x : [k] \to X$ is a* non-degenerated *simplex of X and where $T = (T_0, \dots, T_n)$ is a chain of non-empty subsets of $[k]$ ending with $T_n = [k]$,*

$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n = [k] \to X.$$

In particular, we have $B(X)_0 = \mid X \mid$.

Proof: We shall use lemma 1.5. Let us put $B_n X = B(X)_n$ for every $n \geq 0$. This defines a cocontinuous functor $B_n : \mathbf{S} \to \mathbf{Set}$, since the functor B is cocontinuous. If $k \geq 0$, let us denote by $U_n[k]$ the set of chains $T = (T_0, \dots, T_n)$ of subsets of $[k]$ ending with $T_n = [k]$. If $u : [k] \to [r]$ is a surjection and $T \in U_n[k]$, then $u(T) = (u(T_0), \dots, u(T_n)) \in U_n[r]$. This defines a functor $U_n : \Delta^s \to \mathbf{Set}$. Let us show that the functor B_n is a left Kan extension of the functor U_n along the (composite) inclusion $\Delta^s \subset \mathbf{S}$. By 1.5, it suffices to show that we have

$$B_n \Delta[k] = \bigsqcup_{[r] \rightarrowtail [k]} U_n[r],$$

where the coproduct is taken over the set of monomorphisms with codomain $[k]$. An element S of $B_n \Delta[k]$ is a chain of non-empty subsets

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq [k]$$

For each $S \in B_n \Delta[k]$, there is a unique monomorphism $x : [r] \rightarrowtail [k]$ such that $Im(x) = S_n$ and a unique $T \in U_n[r]$ and such that $x(T_i) = S_i$ for every $0 \leq i \leq n$. This defines a bijection between $B_n \Delta[k]$ and the disjoint union of the sets $U_n[dx]$, where x runs over the set of monomorphisms with codomain $[k]$. ∎

Let us give an explicit description of the face and degeneracy operators of the simplicial set $B(X)$. If $T = (T_0, \dots, T_n)$ and $0 \leq i \leq n$, let us denote by $\partial_i T$ the sequence $(T_0, \dots, \hat{T}_i, \dots, T_n)$ obtained from T by omiting the entry in position i. If $(T,x) \in B(X)_n$ then $\partial_i(T,x) = (\partial_i T, x)$ for every $0 \leq i < n$. If $i = n$, let us denote by x' the restriction of $x : T_n \to X$ to T_{n-1} and let $x' = yp : T_{n-1} \to [r] \to X$ be the factorisation of x' as a surjection followed by a non-degenerated simplex. Then we have $\partial_n(T,x) = (p(\partial_n T), y)$,

$$\begin{array}{ccccccccc}
T_0 & \hookrightarrow & \cdots & T_{n-2} & \hookrightarrow & T_{n-1} & \hookrightarrow & & T_n = [k] \\
\downarrow & & & \downarrow & & \downarrow p & & & \downarrow x \\
p(T_0) & \hookrightarrow & \cdots & p(T_{n-1}) & \hookrightarrow & p(T_{n-1}) = [r] & \xrightarrow{y} & & X.
\end{array}$$

Finally, we have $\sigma_i(T,x) = (\sigma_i T, x)$ where the sequence

$$\sigma_i T = (T_0, \dots, T_i, T_i, \dots, T_n)$$

is obtained from the sequence $T = (T_0, \ldots, T_n)$ by repeating the entry in position i to position $i+1$.

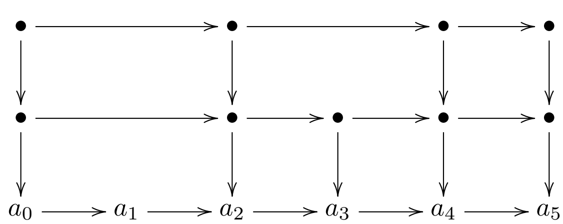
We shall say that a subset $T \subseteq [n]$ is *wide* if $\{0, n\} \subseteq T$. Let $B"(X)$ be the simplicial subset of $B(X)$ whose simplices are the pairs (T, x), where $x : [k] \to X$ is a non-degenerated simplex of X and where $T = (T_0, \ldots, T_n)$ is a chain of wide subsets of $[k]$ ending with $T_n = [k]$. The fact that $B"(X)$ is a simplicial subset of $B(X)$ follows from the description of the face and degeneracy operators given above. If $x : \Delta[n] \to X$ is a non-degenerated simplex, let us put $s(x) = x(0)$ and $t(x) = x(n)$; if $(T, x) \in B"(X)_n$, let us put $s(T, x) = s(x)$ and $t(T, x) = t(x)$. This defines a simplicial graph $(s, t) : B"(X) \to X_0 \times X_0$. If $a \in X_0$, let us put $u(a) = a \in\mid X \mid= B"(X)_0$. The map u is a unit for the graph $(s, t) : B"(X) \to X_0 \times X_0$. We shall denote the resulting reflexive graph by $H(X)$. Let us denote by $\mathbf{SGraph}_0$ the category of reflexive simplicial graphs. A map of simplicial sets $f : X \to Y$ induces a map of simplicial sets $B"(f) : B"(X) \to B"(Y)$ and a map of simplicial graphs $H(f) : H(X) \to H(Y)$. We thus obtain a functor

$$H : \mathbf{S} \to \mathbf{SGraph}_0.$$

If A is a category and $a, b \in ObA$, then a n-simplex of $H(A)(a, b)$ is represented by a pair (u, f), where $u = (u_1, \ldots, u_n)$ is a chain of wide monomorphisms

$$[r_0] \xrightarrow{u_1} [r_1] \xrightarrow{u_2} [r_2] \cdots \qquad \cdots [r_{n-1}] \xrightarrow{u_n} [r_n] \xrightarrow{f} A$$

and $f : [r_n] \to A$ is a reduced path, We shall say that (u, f) is a *reduced* carpet of height n and *degree* r_0. For example, the following diagram represents a reduced carpet of height 2 of and degree 3.



Notice that the units of the graph $H(A)$ are the carpets of degree 0. We now define a category structure on the simplicial graph $H(A)$. The horizontal composition of a reduced carpet $(u, f) \in H(A)(a, b)_n$ with a reduced carpet $(v, g) \in H(A)(b, c)_n$ is defined to be the reduced carpet $(v \star u, g \star f)$, where $v \star u = (v_1 \star u_1, \ldots, v_n \star u_n)$. This gives the reflexive graph $H(A)$ the structure of a simplicial category $H'(A)$.

Lemma 1.7 *If A is a category, then the simplicial graph $H(A)$ has the structure of a simplicial category $H'(A)$ isomorphic to $C_*(A)$.*

Proof: Let us show that we have $H'(A) = C_\star(A)$. Obviously, $H'(A)_0 = C(A) = C_0(A)$. Let us shows that the category $H'(A)_n$ is free on the reflexive graph $U_0H(A)_{n-1} = H(A)_{n-1}$ for every $n > 0$. Every carpet of height n can be expressed as a composite of carpets of degree 1 and height n and this decomposition is unique. Hence we have $K(A)_n = F_0J_n(A)$, where $J_n(A) \subseteq U_0K(A)_n$ denotes the (reflexive) subgraph of carpets of degree 1 and height n. For any map $v : [k] \to [r]$ in Δ, let us denote by v^+ is the unique wide map $[1] \to [k]$. If (u, f) is a reduced carpet of degree > 0 and height $n-1$, let us put $\sigma(u, f) = (u', f)$, where $u = (u_1, \ldots, u_{n-1})$ and $u' = (u_1^+, u_1, \ldots, u_n)$. The carpet $\sigma(u, f)$ has degree 1 and height n. Every reduced carpet of degree 1 and height n is of the form $\sigma(u, f)$ for a unique reduced carpet (u, f) of degree > 0 and height $n-1$. Thus, σ induces an isomorphism of reflexive graphs $\sigma : H(A)_{n-1} \to J_n(A)$. It follows that we have canonical isomorphism $H'(A)_n = F_0H(A)_{n-1}$ for $n > 0$. It follows by recursion that we have canonical isomorphism $H'(A)_n = C^{n+1}(A)$ for every $n \geq 0$. It can then be proved by induction on $n \geq 0$ that these canonical isomorphisms are compatible with the face an degeneracy operators of the the simplicial categories $H'(A)$ and $C_*(A)$. ∎

Proposition 1.8 *Let A be a poset. If $a, b \in A$, then the simplicial set $C_\star(A)(a, b)$ is (the nerve of) the poset of finite non-empty chains $T \subseteq A$ such that $\min T = a$ and $\max T = b$. The composition*

$$C_\star(A)(c, d) \times C_\star(A)(a, b) \to C_\star(A)(a, d)$$

is the union $(U, T) \mapsto U \cup T$.

Proof: This follows from 1.7. ∎

We can now describe the simplicial category $C_\star[n]$. It follows from 1.8 that if $a, b \in [n]$ and $a \leq b$, then $C_\star[n](a, b)$ is (the nerve of) the poset of wide subsets of the interval $[a, b]$ Every wide subset $T \subseteq [a, b]$ is of the form $T = \{a, b\} \cup V$ for a subset V of the interval $(a, b) = \{x \in [n] : a < x < b\}$. It follows that $C_\star[n](a, b)$ is isomorphic to the cube $I^{(a,b)}$.

Lemma 1.9 *If A is a category, then we have a canonical isomorphism of simplicial graphs $H(A) = U_0C_*(A)$.*

Proof: By lemma 1.7, we have a canonical isomorphism $H'(A) = C_*(A)$. Hence we obtain a canonical isomorphism $H(A) = U_0C_*(A)$, since we have $U_0H'(A) = H(A)$. ∎

Proposition 1.10 *The functor $H : \mathbf{S} \to \mathbf{SGraph}_0$ is cocontinuous.*

Proof: It suffices to show that the functor $B" : \mathbf{S} \to \mathbf{S}$ is cocontinuous. But for this, it suffices to show that the functor $X \mapsto B"(X)_n$ is cocontinuous for every $n \geq 0$. If $k \geq 0$, let us denote by $U_n[k]$ the set of chains $T = (T_0, \ldots, T_n)$ of wide subsets of $[k]$ ending with $T_n = [k]$. This defines a functor $U_n : \Delta^s \to \mathbf{Set}$, where $\Delta^s \subset \Delta$ is the subcategory of surjections. Then for any simplicial set X we have

$$B"(X)_n = \bigsqcup_{x \in |X|} U_n[dx],$$

where dx denotes the dimension of a simplex x. This shows by lemma 1.5 that the functor $X \mapsto B"(X)_n$ is cocontinuous. ∎

Recall that the functor

$$C_! : \mathbf{S} \to \mathbf{SCat}$$

is defined to be the left Kan extension of the functor $[k] \mapsto C_\star([k])$.

Theorem 1.11 *If A is a category, then $C_!(A) = C_\star(A)$*

Proof: It suffices to show that functor $C_\star : \mathbf{Cat} \to \mathbf{SCat}$ admits a cocontinuous extension to the category $\mathbf{S}$. For this, it suffices to show that the functor $C_n = C^{n+1} : \mathbf{Cat} \to \mathbf{Cat}$ admits a cocontinuous extension to $\mathbf{S}$ for every $n \geq 0$. But we have $C^{n+1}(A) = F_0 U_0 C^n(A)$. Hence it suffices to show that the functor $UC^n : \mathbf{Cat} \to \mathbf{Grph}$ admits a cocontinuous extension to $\mathbf{S}$, since the functor $F_0 : \mathbf{Grph_0} \to \mathbf{Cat}$ is cocontinuous. The functor $\mathbf{S} \to \mathbf{Grph}$ which takes a simplicial set X to the reflexive graph $(X_1, X_0, \partial_1, \partial_0, \sigma_0)$ is cocontinuous and it extends the functor $U_0 : \mathbf{Cat} \to \mathbf{Grph}_0$. Hence the result is true for $n = 0$. The reflexive graph $U_0 C^{n+1}(A) = UC_n(A)$ is isomorphic to the graph $H_n(A) = H(A)_n$ by Lemma 1.9. The result then follows by Lemma 1.10. ∎

The *coherent nerve* of a simplicial category X is the simplicial set $C^! X$ defined by putting

$$(C^! X)_n = \mathbf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. The functor $C^! : \mathbf{SCat} \to \mathbf{S}$ is right adjoint to the functor $C_!$.

The notion of homotopy coherent diagram was introduced by Vogt [V]. If A is a category and X is a simplicial category we shall say that a map of simplicial categories $C_\star(A) \to X$ is a *homotopy coherent diagram* $A \to X$.

Corollary 1.12 [C] *If A is a category and X is a simplicial category, then there is a natural bijection between the homotopy coherent diagrams $A \to X$ and the maps of simplicial sets $A \to C^! X$.*

Proof: There is a natural bijection between the maps of simplicial categories $C_!(A) \to X$ and the map of simplicial sets $A \to C^! X$, since the functor $C_!$ is left adjoint to the functor $C^!$. But we have $C_!(A) = C_\star(A)$ by 1.11, since A is a category. ∎

Theorem 1.13 *If X is a simplicial set, then the category $C_!(X)_n$ is freely generated by the reflexive graph $H(X)_{n-1}$ if $n > 0$ and by the reflexive graph $Sk^1(X)$ if $n = 0$.*

Proof: Let us suppose $n > 0$. We have to show that $C_!(X)_n = F_0 H(X)_{n-1}$ for every simplicial set X. The functor $X \mapsto C_!(X)_n$ is cocontinuous since the functor $C_!$ is cocontinuous. The functor $X \mapsto F_0 H(X)_{n-1}$ is cocontinuous, since the functor H is cocontinuous by 1.10 and since the functor F_0 is cocontinuous. Hence it suffices to prove the result in the case where $X = [k]$. But if A is category, we have $C_!(A) = C_\star(A)$ by 1.11 and we haved $H(A) = U_0 C_\star(A)$ by 1.7. Thus,

$$C_!(A)_n = C_n(A) = CC_{n-1}(A) = F_0 U_0 C_{n-1}(A) = F_0 H(A)_{n-1}.$$

This proves the result in the case $n > 0$. In the case $n = 0$ we have to show that $C_!(X)_0 = F_0 Sk^1(X)$ for every simplicial set X. It suffices to prove the result in the case where $X = [k]$ since the functors $X \mapsto C_!(X)_0$ and $X \mapsto F_0 Sk^1(X)$ are cocontinuous. But if A is category, we have

$$C_!(A)_0 = C_\star(A)_0 = C_0(A) = C(A) = F_0 U_0(A).$$

This proves the result since $U_0(A) = Sk^1(A)$. ∎

Corollary 1.14 *The functor $C_! : \mathbf{S} \to \mathbf{SCat}$ takes a monomorphism to a monomorphism.*

Proof: It suffices to show that the functor $X \mapsto C_!(X)_n$ takes a monomorphism to a monomorphism for each $n \geq 0$. If $n > 0$, we have $C_!(X)_n = F_0 H(X)_{n-1}$ by 1.13. The barycentric subdivision functor B takes a monomorphism to a monomorphism by 1.6. Hence also the subfunctor $B^w \subseteq B$. It follows that the functor H takes a monomorphism to a monomorphism. Hence also the functor $X \mapsto H(X)_{n-1}$. It is easy to verify that the functor $F_0 : \mathbf{Cat} \to \mathbf{Cat}$ takes a monomorphism to a monomorphism. Hence the functor $X \mapsto C_!(X)_n = F_0 H(X)_{n-1}$ takes a monomorphism to a monomorphism. It remains to consider the case $n = 0$. But we have $C_!(X)_0 = F_0 Sk^1 X$ by 1.13. The result then follows from the fact that the functor $Sk^1 : \mathbf{S} \to \mathbf{Graph}_0$ takes a monomorphism to a monomorphism. ∎

Corollary 1.15 *Let X be a simplicial set, and $A = \bigcup_i A_i$ be a union of a family of simplicial subsets $A_i \subseteq X$. Then the simplicial subcategory $C_! A \subseteq C_! X$ is generated by the simplicial subcategories $C_! A_i \subseteq C_! X$.*

Recall that if A is a subset of $[n]$, then the generalised horn $\Lambda^A[n] \subset \Delta[n]$ is defined by putting

$$\Lambda^A[n] = \bigcup_{i \notin A} \partial_i \Delta[n].$$

In particular, $\Lambda^{\{k\}}[n] = \Lambda^k[n]$ and $\Lambda^{\emptyset}[n] = \partial\Delta[n]$. By 1.14, we have

$$C_!\Lambda^A[n] \subseteq C_!\Delta[n] = C_*[n].$$

We shall comupute $C_!\Lambda^A[n]$ explicitly in the case where $A \subsetneq (0,n)$.

Lemma 1.16 *Suppose that $n > 0$ and that $A \subsetneq (0,n)$. Then for every $(a,b) \neq (0,n)$ we have*

$$C_!\Lambda^A[n](a,b) = C_*[n](a,b).$$

Proof: By 1.14 we have $C_!\Lambda^A[n](a,b) \subseteq C_*[n](a,b)$. Without lost of generality we can suppose that $a \le b$ since we have $C_*[n](a,b) = \emptyset$ otherwise. Let us first consider the case where $b < n$. We have $\partial_n\Delta[n] \subset \Lambda^A[n]$ since $A \subsetneq (0,n)$. Hence the map $C_*(d_n) : C_*[n-1] \to C_*[n]$ factors through the subcategory $C_!\Lambda^A[n] \subseteq C_*[n]$. But $C_*(d_n)$ induces an isomorphism between $C_*[n-1]$ and the full simplicial subcategory of $C_*[n]$ spanned by the objects $0, \ldots, n-1$. Thus, $C_!\Lambda^A[n](a,b) = C_\star[n](a,b)$ for every $a \le b < n$. The proof is similar in the case where $0 < a \le b$. ∎

The cube of dimension n is (the nerve of) the poset

$$I^n = \{(x_1, \ldots, x_n) : x_i \in \{0,1\}\}$$

If $1 \le i \le n$, let us put

$$\begin{aligned} \partial_i^0 I^n &= \{(x_1, \ldots, x_n) \in I^n : x_i = 0\} \\ \text{and} \quad \partial_i^1 I^n &= \{(x_1, \ldots, x_n) \in I^n : x_i = 1\}. \end{aligned}$$

We call $\partial_i^0 I^n$ a *bottom face* and $\partial_i^1 I^n$ a *top face*. We have $\partial I^n = \partial^0 I^n \cup \partial^1 I^n$ where

$$\partial^0 I^n = \bigcup_i \partial_i^0 I^n \quad \text{and} \quad \partial^1 I^n = \bigcup_i \partial_i^1 I^n.$$

We call $\partial^1 I^n$ the *upper cap* of the cube and $\partial^0 I^n$ the *lower cap*. Notice that the $\partial^1 I^n$ is the nerve of the poset $I^n \setminus \{0\}$ obtained by removing the smallest element 0 from I^n. If A is a subset of $\{1, \ldots, n\}$, we shall put

$$\Lambda^A I^n = \partial^1 I^n \cup \bigcup_{i \notin A} \partial_i^0 I^n.$$

The simplicial set $\Lambda^k I^n = \Lambda^{\{k\}} I^n$ is the *open box* obtained by removing the bottom face $\partial_k^0 I^n$ from the boundary ∂I^n. Notice that $\Lambda^{\emptyset} I^n = \partial I^n$.

If $n > 0$, the simplicial set $C_\star[n](0,n)$ is a cube $I^{(0,n)}$ of dimension $n-1$. The cube has two faces for each $0 < i < n$,

$$\partial_i^0 C_\star[n](0,n) = \{F \subseteq [n] : \{0,n\} \subseteq F \quad \text{and} \quad i \notin F\}$$

$$\partial_i^1 C_\star[n](0,n) = \{F \subseteq [n] : \{0,n\} \subseteq F \quad \text{and} \quad i \in F\}.$$

If A is a subset of $(0,n)$, we shall denote by $\lambda^A C_\star[n]$ the simplical subgraph of $C_\star[n]$ obtained by putting

$$\lambda^A C_\star[n](a,b) = \begin{cases} C_\star[n](a,b) & \text{if } (a,b) \neq (0,n) \\ \Lambda^A C_\star[n](0,n) & \text{if } (a,b) = (0,n). \end{cases}$$

We shall put $\delta C_\star[n] = \lambda^\emptyset C_\star[n]$.

Recall from 1.14 that we have $C_! \Lambda^A[n] \subseteq C_\star[n]$.

Theorem 1.17 *Let $n > 0$ and $A \subsetneq (0,n)$. Then $\lambda^A C_\star[n]$ is a simplicial subcategory of $C_\star[n]$ and we have $C_! \Lambda^A[n] = \lambda^A C_\star[n]$.*

Proof Let us show that $\lambda^A C_\star[n]$ is a simplicial subcategory of $C_\star[n]$. For this, it suffices to show that the image of the composition map

$$\gamma_b : C_*[n](b,n) \times C_*[n](0,b) \to C_*[n](0,n)$$

is contained is contained in $\partial^1 P[n](0,n)$ for every $0 < b < n$, since we have $\partial^1 P[n](0,n) \subseteq \lambda^A C_*[n](0,n)$. But if $T \in C_*[n](0,b)$ and $V \in C_*[n](b,n)$ then $\gamma_b(V,T) = V \cup T \in \partial^1_b P[n](0,n)$ since $b \in V \subseteq V \cup T$. Hence the image of γ_b is contained in $\partial^1_b P[n](0,n)$. It follows that the image of γ_b is contained in $\partial^1 P[n](0,n)$. Let us now show that $C_! \Lambda^A[n] \subseteq \lambda^A C_*[n]$. It follows from lemma 1.15 that the subcategory $C_! \Lambda^A[n] \subseteq C_*[n]$ is generated by the subcategories $C_* \partial_k \Delta[n] \subseteq C_*[n]$, where k runs in the complement of the subset $A \subseteq [n]$. Hence it suffices to show that the functor $C_*(d_i) : C_*[n-1] \to C_*[n]$ factors through the inclusion $\lambda^A C_*[n] \subset C_*[n]$ for every $i \notin A$. This is clear if $i = n$ since the functor $C_*(d_n) : C_*[n-1] \to C_*[n]$ induces an isomlorphism between $C_*[n-1]$ and the full simplicial subcategory of $C_*[n]$ spanned by the objects $\{0, \ldots, n-1\}$. Similarly in the case $i = 0$. Let us suppose that $0 < i < n$ (and that $i \notin A$). In this case it suffices to show that the map $f_i : C_*[n-1](0,n-1) \to C_*[n](0,n)$ induced by $C_*(d_i)$ factors through the inclusion $\lambda^A C_*(0,n) \subset C_*[n](0,n)$. But f_i takes a wide subset $T \subseteq [n-1]$ to the subset $d_i(T) \subseteq [n]$. It follows from this description that f_i induces an isomorphism between $C_*[n-1](0,n-1)$ and (the nerve of) the poset of wide subsets $V \subset [n]$ such that $i \notin V$. This shows that the image of f_i is equal to the bottom face $\partial^0_i C_*[n](0,n)$. But we have $\partial^0_i C_*[n](0,n) \subseteq \lambda^A C_*[n](0,n)$ since $i \notin A$ by hypothesis. The inclusion $C_! \Lambda^A[n] \subseteq \lambda^A C_*[n]$ is proved. It remains to prove the opposite inclusion $\lambda^A C_*[n] \subseteq C_! \Lambda^A[n]$. If $(a,b) \neq (0,n)$, then we have $C_! \Lambda^A[n](a,b) = C_*[n](a,b)$ by 1.16. The inclusion $\lambda^A C_*[n](a,b) \subseteq C_! \Lambda^A[n](a,b)$ follows in this case. Hence it remains to prove the inclusion

$$\lambda^A C_*[n](0,n) \subseteq C_! \Lambda^A[n](0,n).$$

Let us first show that we have

$$\partial^1 P[n](0,n) \subseteq C_! \Lambda^A[n](0,n).$$

For this, it suffices to show that we have $\partial_r^1 P[n](0,n) \subseteq C_!\Lambda^A[n](0,n)$ for every $0 < r < n$. But every k-simplex $T = (T_0, \dots, T_k) \in \partial_r^1 P[n](0,n)_k$ admits a decomposition

$$T = V \star U = (V_0 \cup U_0, \dots, V_k \cup U_k),$$

with $U_i = T_i \cap [r]$ and $V_i = T_i \cap [r,n]$. We have $U \in C_!\Lambda^A[n](0,r)_k$ by 1.16 since $r < n$ and we have $V \in C_!\Lambda^A[n](r,n)_k$ since $0 < r$. Thus, $T = V \star U \in C_!\Lambda^A[n](0,n)_k$. It remains to show that we have

$$\partial_i^0 C_*[n](0,n) \subseteq C_!\Lambda^A[n](0,n).$$

for every $i \in (0,n) \setminus A$. But $C_!\Lambda^A[n]$ contains the image of the map $C_*(d_i) : C_*[n-1] \to C_*[n]$. We saw above that this image contains the bottom face $\partial_i^0 C_*[n](0,n)$. ■

Recall the supension functor

$$S : \mathbf{S} \leftrightarrow \{0,1\}\backslash\mathbf{SCat} : hom$$

defined in 3.9.

Lemma 1.18 *If $n > 0$ we have a pushout square of simplicial categories*

$$\begin{array}{ccc} S(\partial I^{(0,n)}) & \longrightarrow & \delta C_*[n] \\ \downarrow & & \downarrow \\ S(I^{(0,n)}) & \xrightarrow{i} & C_*[n] \end{array}$$

where $i(0) = 0$ and $i(1) = n$. And if $0 < k < n$ we have another pushout square of simplicial categories

$$\begin{array}{ccc} S(\Lambda^k I^{(0,n)}) & \longrightarrow & \lambda^k C_*[n] \\ \downarrow & & \downarrow \\ S(I^{(0,n)}) & \xrightarrow{i} & C_*[n] \end{array}$$

where $i(0) = 0$ and $i(1) = n$.

Lemma 1.19 *The inclusion $C_!\partial\Delta[n] \subset C_*[n]$ is a cofibration for every $n \geq 0$. The inclusion $C_!\Lambda^k[n] \subset C_*[n]$ is an acyclic cofibration for every $0 < k < n$.*

Proof: Let us prove the first statement. If $n = 0$, we have $C_!\partial\Delta[0] = C_!\emptyset = \emptyset$ since the functor $C_!$ is cocontinuous. The inclusion $\emptyset \subset C_*[0] = 1$ is a cofibration in **SCat** (since every acyclic fibration is surjective on objects by theorem 3.8). This proves the result in the case $n = 0$. Let us suppose $n > 0$. We have $\partial\Delta[n] = \Lambda^\emptyset[n]$. Thus, $C_!\partial\Delta[n] = \lambda^\emptyset C_*[n] = \delta C_*[n]$ by 1.17. Hence it suffices to prove that the inclusion $\delta C_*[n] \subset C_*[n]$ is a cofibration. For this, it suffices to show that the map $S(\partial I^{(0,n)}) \to S(I^{(0,n)})$ is a cofibration by 1.18. But this is clear since S is a left Quillen functor by 3.9. Let us now prove that the inclusion $C_! \Lambda^k[n] \subset C_*[n]$ is an acyclic cofibration if $0 < k < n$. We have $C_! \Lambda^k[n] = \lambda^k C_*[n]$ by 1.17. Hence it suffices to prove that the inclusion $\lambda^k C_*[n] \subset C_*[n]$ is an acyclic cofibration. For this, it suffices to show that the map $S(\Lambda^k I^{(0,n)}) \to S(I^{(0,n)})$ is an acyclic cofibration by 1.18. But this is clear since S is a left Quillen functor by 3.9 and since the inclusion $\Lambda^k I^{(0,n)} \subset I^{(0,n)}$ is a weak homotopy equivalence. ∎

If A is a category, then the canonical map $\epsilon : C(A) \to A$ defines an augmentation $\epsilon : C_\star(A) \to A$.

Lemma 1.20 *The augmentation $\epsilon : C_\star(A) \to A$ is a Dwyer-Kan equivalence for any category A. The functor $C_* :$ **Cat** $\to$ **SCat** takes an equivalence of categories to a Dwyer-Kan equivalence.*

Proof: The map $U_0(\epsilon) : U_0 C_\star(A) \to U_0 A$ is a homotopy equivalence of simplicial graphs by 6.1. Hence the map $C_\star(A)(a,b) \to A(a,b)$ is a homotopy equivalence for every pair of objects $a, b \in A$. This shows that the augmentation $\epsilon : C_\star(A) \to A$ is a Dwyer-Kan equivalence. If $u : A \to B$ is an equivalence of categories, then the horizontal maps of the commutative square

$$\begin{array}{ccc} C_\star(A) & \xrightarrow{\epsilon} & A \\ {\scriptstyle C_\star(u)}\downarrow & & \downarrow{\scriptstyle u} \\ C_\star(B) & \xrightarrow{\epsilon} & B \end{array}$$

are Dwyer-Kan equivalences by what we just proved. The functor u is a Dwyer-Kan equivalence since it is an equivalence of quasi-categories. Hence also the map $C_\star(u)$ by three-for-two. ∎

Theorem 1.21 *The adjoint pair of functors*

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

is a Quillen pair between the model category for quasi-categories and the model category for simplicial categories.

Proof: We shall use the criterions of proposition 6.18. For this we need to show: (i) that the functor $C_!$ takes a cofibration to a cofibration; (ii) that the functor $C^!$ takes a fibration between fibrant objects to a fibration between fibrant objects. In order to show (i) it suffices to show that the functor $C^!$ takes an acyclic fibration to an acyclic fibration. Let $f : X \to Y$ be an acyclic fibration between simplicial categories. Let us show that we have $\delta_n \pitchfork C^!(f)$ for every $n \geq 0$. But the condition $\delta_n \pitchfork C^!(f)$ is equivalent to the condition $C_!(\delta_n) \pitchfork f$ by the adjointness $C_! \vdash C^!$. But we have $C_!(\delta_n) \pitchfork f$ since the map $C_!(\delta_n)$ is a cofibration by 1.19 and since f is an acyclic fibration by hypothesis. This proves that functor $C^!$ takes an acyclic fibration to an acyclic fibration. It follows by the adjointness that the functor $C_!$ takes a cofibration to a cofibration. Let us now show that the functor $C^!$ a fibration between fibrant objects to a fibration between fibrant objects. For this, let us first show that it takes a fibration to a mid fibration. Let $f : X \to Y$ be a Dwyer-Kan fibration between simplicial categories. Let us show that we have $h_n^k \pitchfork C^!(f)$ for every $0 < k < n$. The condition $h_n^k \pitchfork C^!(f)$ is equivalent to the condition $C_!(h_n^k) \pitchfork f$. But we have $C_!(h_n^k) \pitchfork f$ since the map $C_!(h_n^k)$ is an acyclic cofibration by 1.19 and since f is a Dwyer-Kan fibration by hypothesis. This proves that the functor $C^!$ takes a fibration to a mid fibration. In particular, it takes a fibrant simplicial categories to a quasi-categories. Let us now show that the functor $C^!$ takes a fibration between fibrant objects to a fibration between fibrant objects. Let $f : X \to Y$ be a Dwyer-Kan fibration between fibrant simplicial categories. The map $C^!(f) : C^!X \to C^!Y$ is a mid fibration between quasi-categories by what we just proved. Let us show that $C^!(f)$ is a quasi-fibration. By 3.5, it suffices to show that we have $j_0 \pitchfork C^!(f)$, where j_0 is the inclusion $\{0\} \subset J$. But the condition $j_0 \pitchfork C^!(f)$ is equivalent to the condition $C_!(j_0) \pitchfork f$. Hence it suffices to show that $C_!(j_0)$ is an acyclic cofibration. It is a cofibation since j_0 is monic and the functor $C_!$ takes a cofibration to a cofibration. It is also an acyclic map by lemma 1.20, since j_0 is an equivalence of categories. ■

The obvious forgetful functor $U_0 : \mathbf{SCat} \to \mathbf{SGrph}_0$ has a left adjoint

$$F_0 : \mathbf{Grph} \to \mathbf{Cat},$$

where F_0X is the category freely generated by a reflexive simplicial graph X. We shall use the following elementary result in the next section.

Corollary 1.22 *If $u : A \to B$ is a monomorphism of reflexive simplicial graphs, then the map $F_0(u) : F_0A \to F_0B$ is a cofibration in* $\mathbf{SCat}$.

1.4 The strong coherent nerve functor

Recall from Proposition 5.8 that the category $\mathbf{SCat}$ is simplicial. The *strong coherent nerve* a simplicial category X is the precategory $K^!X$ obtained by putting

$$(K^!X)_n = Hom(C_\star[n], X)$$

for every $n \geq 0$. We have

$$(K^!X)_0 = Hom(1, X) = ObX.$$

This defines a functor

$$K^! : \mathbf{SCat} \to \mathbf{PCat}.$$

Recall from Proposition 5.2 that the category $\mathbf{PCat}$ is simplicial and that it admits tensor and cotensor products.

Proposition 1.23 *The functor $K^!$ is simplicial. It admits a simplicial left adjoint*

$$K_! : \mathbf{PCat} \to \mathbf{SCat}$$

and the adjunction $K_! \dashv K^!$ is strong.

Proof: Let us first show that the functor $K^!$ admits a left adjoint. We shall use the canonical functor $\pi : \Delta \times \Delta \to \Delta^{02}$ of Proposition 3.14. Let $\rho : \Delta \times \Delta \to \mathbf{SCat}$ be the functor defined by putting

$$\rho([m],[n]) = C_\star[m] \otimes \Delta[n]$$

for every $m, n \geq 0$. We have $\rho([0],[n]) = C_\star[0] \otimes \Delta[n] = 1 \otimes \Delta[n] = 1$ by 5.14. It follows that there is a unique functor

$$k : \Delta^{02} \to \mathbf{SCat}$$

such that $k\pi = \rho$, where π is the canonical functor $\Delta \times \Delta \to \Delta^{02}$. See Proposition 3.14. We shall denote by $k_!$ the left Kan extension of the functor k along the Yoneda functor $\Delta^{02} \to \mathbf{PCat}$. The functor $k_!$ has a right adjoint $k^!$. By definition, for every $X \in \mathbf{SCat}$ we have

$$(k^! X)_{mn} = \mathbf{SCat}(C_\star[m] \otimes \Delta[n], X)$$

for every $m, n \geq 0$. But we have

$$\mathbf{SCat}(C_\star[m] \otimes \Delta[n], X) = \mathbf{SCat}(C_\star[m], X^{[\Delta[n]]}) = \mathbf{SCat}(C_\star[m], X^{(n)}).$$

This shows $(k^! X)_m = Hom(C_\star[m], X)$ and hence that $k^! = K^!$. Thus, $K^!$ has a left adjoint $K_! = k_!$. Let us now show that the functor $K^!$ preserves cotensor products. If A is a simplicial set, then

$$\begin{aligned} K^!(X^{[A]})_m &= Hom(C_\star[m], X^{[A]}) \\ &= Hom(C_\star[m], X)^A \\ &= ((K^! X)_m)^A = ((K^! X)^{[A]})_m \end{aligned}$$

by Proposition 5.2. Thus, $K^!(X^{[A]}) = (K^! X)^{[A]}$. This shows that the functor $K^!$ preserves cotensor products. It follows by adjointness that the functor $K_!$ preserves tensor products. It follows also that the functors $K^!$ and $K_!$ are simplicial and that the adjunction $K_! \dashv K^!$ is strong. ∎

Proposition 1.24 *The first row of the pre-category $K^! X$ is the coherent nerve $C^! X$. Hence we have $j^* K^! = C^!$, where $j^* : \mathbf{PCat} \to \mathbf{S}$ is the first row functor.*

Proof: By definition, if $X \in \mathbf{SCat}$ then we have

$$(K^! X)_{m0} = \mathbf{SCat}(C_\star[m] \otimes \Delta[0], X) = \mathbf{SCat}(C_\star[m], X) = (C^! X)_m.$$

for every $m \geq 0$ This shows that $C^! X$ is the first row of $K^! X$. ∎

Recall from Proposition 5.3 that if A and B are simplicial sets, then we have $A \odot B = (A\Box 1) \otimes_2 B$.

Lemma 1.25 *If A and B are simplicial sets, then we have a canonical isomorphism*

$$K_!(A \odot B) = C_!(A) \otimes B.$$

Proof: Let us first show that we have a canonical isomorphism $K_!(A\Box 1) = C_!(A)$. The functor $A \mapsto A\Box 1 = q^*(A)$ is left adjoint to the first row functor $j^* : \mathbf{PCat} \to \mathbf{S}$ by Proposition 3.21. Hence the functor $A \mapsto K_!(A\Box 1)$ is left adjoint to the functor $j^*K^! = C^!$ by Proposition 1.24. It follows by uniqueness of adjoint that we have a canonical isomorphism $K_!q^* = C_!$. This proves that we have a canonical isomorphism $K_!(A\Box 1) = C_!(A)$. The functor $K_!$ preserves tensor products by Proposition 1.23. But we have $A \odot B = (A\Box 1) \otimes_2 B$ by Proposition 5.3. Thus,

$$K_!(A \odot B) = K_!(A\Box 1) \otimes B = C_!(A) \otimes B.$$

■

Proposition 1.26 *The pair of adjoint functors $(K_!, K^!)$ is a Quillen pair*

$$K_! : \mathbf{PCat} \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen pair between the injective model structure for Segal categories and the model structure for simplicial categories.

Proof: For this it suffices to show: (i) that the functor $K_!$ takes a cofibration to a cofibration; (ii) that the functor $K^!$ takes a fibration between fibrant objects to a fibration between fibrant objects. In order to show (i) it suffices to show that $K_!$ takes a generator of the saturated class of cofibrations to a cofibration. By Proposition 5.5, the class of monomorphisms in **PCat** is generated by the maps $\delta_m \odot' \delta_n$, for $m > 0$ and $n \geq 0$, together with the map $\delta_0 : \emptyset \subset 1$. We have $K_!(\delta_m \odot' \delta_n) = C_!(\delta_m) \otimes' \delta_n$ by Proposition 1.25. But $C_!(\delta_m)$ is a cofibration in **SCat** by Theorem 1.21. Moreover, $C_!(\delta_m)$ is biunivoque, since δ_m is biunivoque. Thus, $C_!(\delta_m) \otimes' \delta_n$ is a cofibration in **SCat** by Proposition 5.15. We have $K_!(\delta_0) = K_!(\delta_0 \odot 1) = C_!(\delta_0) \otimes 1 = C_!(\delta_0)$ by Proposition 1.25. This shows that $K_!(\delta_0)$ is a is a cofibration in **SCat** by Theorem 1.21. Let us now show that the functor $K^!$ takes a fibration between fibrant objects to a fibration between fibrant objects. We shall first prove that $K^!$ takes a fibration to a mid fibration. See Definition 3.19 for this notion. By Proposition 5.7, it suffices to show that its left adjoint $K_!$ takes a mid cofibration to an acyclic cofibration. By Definition 5.6, the class of mid cofibrations is generated by the following two sets of maps: (a) the maps $\delta_m \odot' h^k_n$ for $m > 0$, $n > 0$ and $0 \leq k \leq n$; (b) the maps $h^k_m \odot' \delta_n$ for $0 < k < m$ and $n \geq 0$. Hence it suffices to show that the functor $K_!$ takes these generators to acyclic cofibrations. But we have $K_!(\delta_m \odot' h^k_n) = C_!(\delta_m) \otimes' h^k_n$ and $K_!(h^k_m \odot' \delta_n) = C_!(h^k_m) \otimes' \delta_n$ by 1.24. The map $C_!(\delta_m)$ is a cofibration by Theorem 1.21. Hence the map $C_!(\delta_m) \otimes' h_n^k$ is an acyclic cofibration by 5.15, since h_n^k is a weak homotopy equivalence. The map $C_!(h_m^k)$ is an acyclic cofibration by Theorem 1.21, since $0 < k < m$. It is also biunivoque, since h_m^k is biunivoque. Hence the map $C_!(h_m^k) \otimes' \delta_n$ is an acyclic cofibration by 5.15. We have proved that the functor $K^!$ takes a fibration to a mid fibration. In particular, it takes a fibrant simplicial category to a fibrant object by Proposition 3.20. If $f : X \to Y$ is a fibration between fibrant simplicial categories, then $K^!(f)$ is a Reedy fibration between fibrant objects by the same proposition. Let us show that $K^!(f)$ is a quasi-fibration. But for this, it suffices to show by 3.18 that $K^!(f)$ has the right lifting property with respect to the map $j_0 \Box 1 = j_0 \odot 1$, where j_0 denotes the inclusion $\{0\} \subset J$. By adjointness, it suffices to show that f has the right lifting property with respect to the map $K_!(j_0 \Box 1) = C_!(j_0)$. But $C_!(j_0)$ is an acyclic cofibration by Theorem 1.21, since j_0 is a monic weak categorical equivalence. This shows that we have $C_!(j_0) \pitchfork f$. ■

Corollary 1.27 *The pair of adjoint functors* $(K_!, K^!)$ *is a Quillen pair*

$$K_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : K^!$$

betwen the projective model structure for Segal categories and the model structure for simplicial categories.

Proof: This follows from Proposition 1.26 and Proposition 3.25. ■

Corollary 1.28 *The pair of adjoint functors* $(K_!, K^!)$ *is a Quillen pair*

$$K_! : f\mathbf{PCat} \leftrightarrow f\mathbf{SCat} : K^!$$

betwen the fibered injective model structure for Segal categories and the fibered model structure for simplicial categories

Proof: Let us show that $K_!$ is a left Quillen functor. The functor $K_!$ takes a weak categorical equivalence to a Dwyer-Kan equivalence by Proposition 1.26 and by Ken Brown's lemma. It takes a biunivoque map to a biunivoque map, since we have $ObK_!(X) = X_0$. Let us show that it takes an immersion in **PCat** to an immersion in **SCat**. The saturated class of immersions in **PCat** is generated the class of monomorphisms together with the map $1 \sqcup 1 \to 1$ by Proposition 4.8 and Proposition 4.7. Hence it suffices to show that $K_!$ takes a monomorphism to an immersion and the map $1 \sqcup 1 \to 1$ to an immersion. The functor $K_!$ takes a monomorphism to a cofibration by Proposition 1.26. Hence it takes a monomorphism to an immersion, since a cofibration is an immersion by Proposition 4.5. The functor $K_!$ takes the map $1 \sqcup 1 \to 1$ to the map $1 \sqcup 1 \to 1$, since it preserves coproducts and since $K_!(1) = 1$. But the map $1 \sqcup 1 \to 1$ is an immersion in **SCat** by Proposition 4.4. ■

Corollary 1.29 *The pair of adjoint functors $(K_!, K^!)$ is a Quillen pair*

$$K_! : f\mathbf{PCat}' \leftrightarrow f\mathbf{SCat} : K^!$$

betwen the fibered projective model structure for Segal categories and the fibered model structure for simplicial categories.

Proof: This follows from Corollary 1.28 and Proposition 4.14. ∎

2 The equivalence

For a category A, the augmentation $\epsilon_A : C_*A \to A$ is a natural transformation between two functors with values in **SCat**. There is a unique natural transformation

$$\epsilon : C_! \to \tau_1$$

which extends the augmentation $C_*[n] \to [n]$ for every $n \geq 0$, since the functor $C_!$ is cocontinuous. If A is a category, the map $\epsilon_A : C_!(A) \to A$ coincide with the augmentation $C_*(A) \to A$, since in this case we have $C_!(A) = C_*(A)$ by Proposition 1.11.

Recall from lemmas 1.25 and 5.13 that if A and B are simplicial sets, then we have $K_!(A \odot B) = C_!(A) \otimes B$ and $N_!(A \odot B) = \tau_1(A) \otimes B$. In particular, for every $m, n \geq 0$, we have

$$K_!(\Delta[m] \odot \Delta[n]) = C_*[m] \otimes \Delta[n] \quad \text{and} \quad N_!(\Delta[m] \odot \Delta[n]) = [m] \otimes \Delta[n].$$

Hence we can put

$$\phi_{mn} = \epsilon_m \otimes \Delta[n] : K_!(\Delta[m] \odot \Delta[n]) \to N_!(\Delta[m] \odot \Delta[n]).$$

There is then a unique natural transformation

$$\phi : K_! \to N_!$$

which extends the map ϕ_{mn} for every $m, n \geq 0$, since the functor $K_!$ is cocontinuous.

Lemma 2.1 *If $X = A \odot B$, then the map $\phi_X : K_!(X) \to N_!(X)$ is isomorphic to the map*

$$\epsilon_A \otimes B : C_!(A) \otimes B \to \tau_1(A) \otimes B.$$

Proof: We shall prove that the following square commutes

$$\begin{array}{ccc} C_!(A) \otimes B & \xrightarrow{\epsilon_A \otimes B} & \tau_1(A) \otimes B \\ \downarrow & & \downarrow \\ K_!(A \odot B) & \xrightarrow{\phi} & N_!(A \odot B), \end{array}$$

where the vertical maps are the canonical isomorphisms of lemmas 1.25 and 5.13. For this it suffices to show that the square commutes in the case where $A = \Delta[m]$ and $B = \Delta[n]$, since the functor $(A, B) \mapsto K_!(A \odot B)$ is cocontinuous in each variable. But the result is obvious in this case. ∎

If $X \in \mathbf{SCat}$, then we have a map of simplicial sets

$$Hom(\epsilon_n, X) : Hom([n], X) \to Hom(C_*[n], X)$$

for each $n \geq 0$. This defines a map of pre-categories

$$\epsilon^!_X = Hom(\epsilon, X) : N^!(X) \to K^!(X)$$

for each $X \in \mathbf{SCat}$ and hence a (strong) natural transformation $\epsilon^! : N^! \to K^!$.

Lemma 2.2 *The natural transformation $\phi : K_! \to N_!$ is strong and it is the left transpose of the natural transformation $\epsilon^! : N^! \to K^!$.*

Proof: Let us first prove that the natural transformation $\epsilon^! : N^! \to K^!$ is the right transpose of the natural transformation $\phi : K_! \to N_!$. The transformation ϕ has a right transpose $\phi' : N^! \to K^!$ (not necessarily strong). If $X \in \mathbf{SCat}$, then the map $\phi'_X : N^!(X) \to K^!(X)$ takes an element $x : [m] \otimes \Delta[n] \to X$ to the element $x\phi_{mn} : C_*[m] \otimes \Delta[n] \to X$. But we have $\phi_{mn} = \epsilon_m \otimes \Delta[n]$. Hence the map ϕ'_X takes an element $x : [m] \to X^{(n)}$ to the element $x\epsilon_m : C_*[m] \to X^{(n)}$. This shows that $\phi'_X = \epsilon^!_X$. The second statement of the proposition is proved. Let us prove the first statement. The adjunctions $N_! \vdash N^!$ and $K_! \vdash K^!$ are strong by 5.9 and 1.23. Hence the left transpose of a strong natural transformation $N^! \to K^!$ is a strong natural transformation $K_! \to N_!$. ∎

Lemma 2.3 *If $u : A \to B$ is a monomorphism of simplicial sets, then the map of pre-categories $i_m \odot' u$ is a strong cofibration for every $m \geq 0$.*

Proof: By Theorem 3.24, it suffices to show that we have $(i_m \odot' u) \pitchfork f$ for every locally trivial fibration f. But the condition $(i_m \odot' u) \pitchfork f$ is equivalent to the condition $u \pitchfork \langle i_m, f \rangle$ by 6.3. Hence it suffices to show that the map $\langle i_m, f \rangle$ is a trivial fibration. But for this, it suffices to show that we have $\delta_n \pitchfork \langle i_m, f \rangle$ for every $n \geq 0$ by Proposition 6.6. But the condition $\delta_n \pitchfork \langle i_m, f \rangle$ is equivalent to the condition $(i_m \odot' \delta_n) \pitchfork f$ by 6.3. This proves the result, since the map $i_m \odot' \delta_n$ is a strong cofibration. ∎

If A and B are simplicial sets, we shall denote by $A \circledast B$ the pre-category defined by the pushout square of bisimplicial sets

$$\begin{array}{ccc} A_0 \Box B & \longrightarrow & A \Box B \\ \downarrow & & \downarrow \\ A_0 \Box 1 & \longrightarrow & A \circledast B. \end{array}$$

Notice that $A \circledast B = A \odot B$, when B is connected. If we apply the functor $\pi_! : \mathbf{S}^{(2)} \to \mathbf{PCat}$ to the pushout square above, we obtain a pushout square of pre-categories

$$\begin{array}{ccc} A_0 \odot B & \longrightarrow & A \odot B \\ \downarrow & & \downarrow \\ A_0 \odot 1 & \longrightarrow & A \circledast B. \end{array}$$

Lemma 2.4 *The domain of the map $i_m \odot' \delta_n$ is the pre-category $\Delta[m] \circledast \partial\Delta[n]$ and its codomain is the pre-category $\Delta[m] \odot \Delta[n]$.*

Proof: By definition, we have a pushout square

$$\begin{array}{ccc} [m]_0 \odot \partial\Delta[n] & \longrightarrow & \Delta[m] \odot \partial\Delta[n] \\ \downarrow & & \downarrow \\ [m]_0 \odot 1 & \longrightarrow & \Delta[m] \circledast \partial\Delta[n], \end{array}$$

The first statement follows, since $[m]_0 \odot 1 = [m]_0 \odot \Delta[n]$. The second statement is obvious. ∎

Lemma 2.5 *If A is a simplicial set, then square*

$$\begin{array}{ccc} [m]_0 \odot A & \longrightarrow & \Delta[m] \odot A \\ \downarrow & & \downarrow \\ [m]_0 \odot 1 & \longrightarrow & \Delta[m] \circledast A. \end{array}$$

is a homotopy pushout of cofibrant objects in the model category $\mathbf{PCat}'$.

Proof: The map $i_m \odot A : [m]_0 \odot A \to \Delta[m] \odot A$ is isomorphic to the map $i_m \odot' u_A$, where u_A is the inclusion $\emptyset \subseteq A$. It is thus a strong cofibration by 2.3. The pre-category $[m]_0 \odot A$ is a coproduct of terminal objects, since we have $[m]_0 \odot A = [m]_0 \Box \pi_0 A$. It is thus strongly cofibrant. In particular the pre-category $[m]_0 \odot 1$ is strongly cofibrant. ∎

Proposition 2.6 *If $X \in \mathbf{PCat}$ is strongly cofibrant, then the map*

$$\phi_X : K_!(X) \to N_!(X)$$

is a biunivoque Dwyer-Kan equivalence between cofibrant simplicial categories.

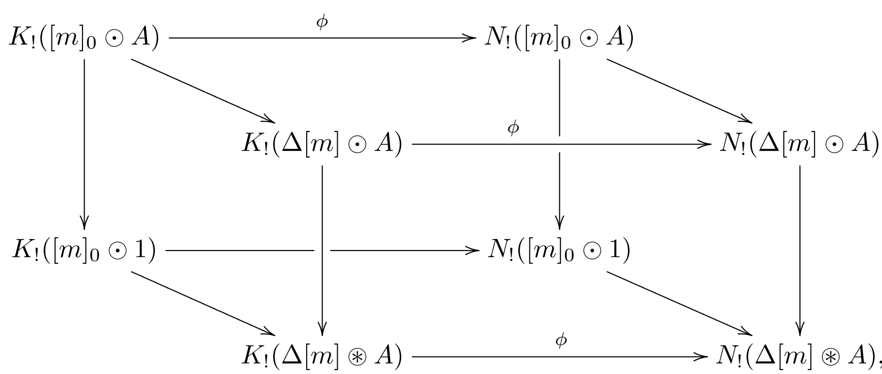
Proof: For this, we shall apply Proposition 6.36 to the natural transformation $phi : K_! \to N_!$ and to the Quillen pairs

$$K_! : f\mathbf{PCat}' \leftrightarrow f\mathbf{SCat} : K^! \quad \text{and} \quad N_! : f\mathbf{PCat}' \leftrightarrow f\mathbf{SCat} : N^!$$

of Corollary 1.29 and Theorem 4.15. The transformation ϕ has a right transpose $\epsilon^! : N^! \to K^!$ by 2.2. The class $\mathcal{A}$ is the set of maps described in Proposition 4.12. Clearly, ϕ_X is an isomorphism when X is the domain or the codomain of of one of the maps $\emptyset \to 1$ and $1 \sqcup 1 \to 1$. Let us show that ϕ_X is an isomorphism when X is the domain or the codomain of a map $i_m \odot' \delta_n$. By Lemma 2.4, the domain of $i_m \odot' \delta_n$ is the pre-category $\Delta[m] \circledast \partial\Delta[n]$ and its codomain is the pre-category $\Delta[m] \odot \Delta[n]$. If A is a simplicial set and $X = [m]_0 \odot A$, then the map ϕ_X is an isomorphism, since $K_!([m]_0 \odot A) = [m]_0 \otimes A = N_!([m]_0 \odot A)$ by Lemma 2.1. Let us show that ϕ_X is a biunivoque Dwyer-Kan equivalence if $X = \Delta[m] \odot A$ or $X = \Delta[m] \circledast A$. The square

$$\begin{array}{ccc} [m]_0 \odot A & \longrightarrow & \Delta[m] \odot A \\ \downarrow & & \downarrow \\ [m]_0 \odot 1 & \longrightarrow & \Delta[m] \circledast A. \end{array}$$

is a homotopy pushout of cofibrant objects in the model category $\mathbf{PCat}'$ by Lemma 2.5. It is thus a homotopy pushout of cofibrant objects in the model category $f\mathbf{PCat}'$ by 4.11. Hence the left and the right hand faces of the following cube



are homotopy pushout squares of cofibrant objects in the model category $f\mathbf{SCat}$. since the functor $K_!$ and $N_!$ are left Quillen functor $f\mathbf{PCat}' \to f\mathbf{SCat}$. We saw above that the horizontal maps of the back face are isomorphisms. Hence the result will be proved by the cube lemma if we show that the map ϕ_X is a biunivoque Dwyer-Kan equivalence in the case where $X = \Delta[m] \odot A$. But in this case, ϕ_X is isomorphic to the map

$$\epsilon_m \odot A : C_*[m] \otimes A \to [m] \otimes A$$

by Lemma 2.1. But ϵ_m is a (biunivoque) Dwyer-Kan equivalence by 1.20. Hence the map $\epsilon_m \odot A$ is a biunivoque Dwyer-Kan equivalence by Ken Brown's lemma, since the fibered model structure $f\mathbf{SCat}$ is simplicial by 4.2. ∎

Corollary 2.7 *If $X \in \mathbf{SCat}$ is fibrant, then the map*

$$\epsilon^!_X : K^!(X) \to N^!(X)$$

is a biunivoque equivalence of Segal categories.

Theorem 2.8 *The adjoint pair of functors*

$$K_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen equivalence between the projective model category for Segal categories and the model category for simplicial categories.

Proof: By Corollary 1.27, the pair of adjoint functors $(K_!, K^!)$ is a Quillen pair

$$K_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : K^!$$

betwen the projective model structure for Segal categories and the model structure for simplicial categories. By Theorem 3.26, the pair of adjoint functors $(N_!, N^!)$ is a Quillen equivalence

$$N_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : N^!$$

between the projective model structure for Segal categories and the model structure for simplicial categories. The natural transformation $\phi : K_! \to N_!$ induces a natural transformation of left derived functors

$$\phi^L : K_!^L \to N_!^L : Ho(\mathbf{PCat}') \to Ho(\mathbf{SCat}).$$

Let us show that ϕ^L_X is invertible for every every object X. It suffices to consider the case where X is cofibrant, since X is isomorphic in the homotopy category to a cofibrant object. In this case the map $\phi^L_X : K_!^L(X) \to N_!^L(X)$ is induced by the map $\phi_X : K_!(X) \to N_!(X)$. But if X is cofibrant, ϕ_X is invertible in the homopy category $Ho(\mathbf{SCat})$, since ϕ_X is a Dwyer-Kan equivalence in this case by Proposition 2.6. We have proved that ϕ^L is invertible. The functor $N_!^L$ is an equivalence of categories, since the pair $(N_!, N^!)$ is a Quillen equivalence. Hence the functor $K_!^L$ is an equivalence of categories, since ϕ^L is invertible. This shows that the pair $(K_!, K^!)$ is a Quillen equivalence. ∎

Corollary 2.9 *The adjoint pair of functors*

$$K_! : \mathbf{PCat} \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen equivalence between the injective model structure for Segal categories and the model structure for simplicial categories.

Proof: This follows from Theorem 2.8 and Theorem 3.25 if we use Proposition 6.26. ∎

Theorem 2.10 *The adjoint pair of functors* $(C_!, C^!)$ *is a Quillen equivalence*

$$C_! : \mathbf{S} \leftrightarrow \mathbf{SCat} : C^!$$

between the model structure for quasi-categories and the model structure for simplicial categories.

Proof: We shall use the Quillen equivalence

$$q^* : \mathbf{S} \leftrightarrow \mathbf{PCat} : j^*$$

of Theorem 3.22. The pair

$$K_! : \mathbf{PCat} \leftrightarrow \mathbf{SCat} : K^!$$

is a Quillen equivalence by Corollary 2.9. But we have $j^*K^! = C^!$ by Proposition 1.24. This shows by Proposition 6.26 that the pair $(C_!, C^!)$ is a Quillen equivalence. ∎

3 Appendix on 5 model categories

3.1 The model structure for quasi-categories

Quasi-categories were introduced by Boardman and Vogt in their work on homotopy invariant algebraic structures [BV]. The category $\mathbf{S}$ admits a model structure in which the fibrant objects are the quasi-categories [J2]. We call it the *model structure for quasi-categories*. Before describing this model structure it is good to describe a related model structure on **Cat**. We shall say that a functor $p : X \to Y$ is a *quasi-fibration* if for every object $a \in X$ and every isomorphism $g \in Y$ with source $p(a)$ there exists an isomorphism $f \in X$ with source a such that $p(f) = g$. A functor $p : X \to Y$ is a quasi-fibration iff it has the right lifting property with respect to the inclusion $\{0\} \subset J$, where J is the groupoid generated by one isomorphism $0 \to 1$. We say that a functor $A \to B$ is *monic on objects* if the induced map $Ob(A) \to Ob(B)$ is injective.

Theorem 3.1 [JT1] *The category* **Cat** *admits a model structure in which a cofibration is a functor monic on objects, a weak equivalence is an equivalence of categories and a fibration is a quasi-fibration. The model structure is proper and cartesian closed. Every object is fibrant and cofibrant.*

We shall call this model structure the *natural model structrure* on **Cat**. A functor $A \to B$ is an acyclic fibration iff it is an equivalence surjective on objects.

The category Δ is a full subcategory of **Cat**. The *nerve* of a category $C \in \mathbf{Cat}$ is the simplicial set NC obtained by putting $(NC)_n = \mathbf{Cat}([n], C)$ for every $n \geq 0$. The nerve functor $N : \mathbf{Cat} \to \mathbf{S}$ is full and faithful and we shall regard it as an inclusion $N : \mathbf{Cat} \subset \mathbf{S}$ by adopting the same notation for a category and its nerve. The functor N has a left adjoint

$$\tau_1 : \mathbf{S} \to \mathbf{Cat}.$$

We say that $\tau_1 X$ is the *fundamental category* of a simplicial set X. The fundamental groupoid $\pi_1 X$ is obtained by inverting the arrows of $\tau_1 X$.

We shall say that a horn $\Lambda^k[n] \subset \Delta[n]$ is *inner* if $0 < k < n$. A simplicial set X is called a *quasi-category* if every inner horn $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$. The nerve of a category and a Kan complex are examples. We shall denote by **QCat** the category of quasi-categories; it is a full subcategory of $\mathbf{S}$.

The next step is to introduce an appropriate notion of equivalence for quasi-categories. If A is a simplicial set, we shall denote by $\tau_0 A$ the set of isomorphism classes of objects of the fundamental category $\tau_1 A$. The functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ preserves finite products [GZ], hence also the functor $\tau_0 : \mathbf{S} \to \mathbf{Set}$. If $A, B \in \mathbf{S}$ let us put

$$\tau_0(A, B) = \tau_0(B^A).$$

If we apply the functor τ_0 to the composition map $C^B \times B^A \to C^A$ we obtain a composition law

$$\tau_0(B, C) \times \tau_0(A, B) \to \tau_0(A, C)$$

for a category $\mathbf{S}^{\tau_0}$, where $\mathbf{S}^{\tau_0}(A, B) = \tau_0(A, B)$. We shall say that a map of simplicial sets is a *categorical equivalence* if it is invertible in the category $\mathbf{S}^{\tau_0}$. If X and Y are quasi-categories, a categorical equivalence $X \to Y$ is called an *equivalence of quasi-categories*. We shall say that a map of simplicial sets $u : A \to B$ is a *weak categorical equivalence* if the map

$$\tau_0(u, X) : \tau_0(B, X) \to \tau_0(A, X)$$

is bijective for every quasi-category X.

Definition 3.2 *A map of simplicial sets is called a* quasi-fibration *if it has the right lifting property with respect to every monic weak categorical equivalence.*

Theorem 3.3 [J2] *The category of simplicial sets* $\mathbf{S}$ *admits a model structure in which a cofibration is a monomorphism, a weak equivalence is a weak categorical equivalence and a fibration is a quasi-fibration. The acyclic fibrations are the trivial fibrations. The fibrant objects are the quasi-categories. The model structure is cartesian and left proper.*

We shall say that it is the *model structure for quasi-categories*. The quasi-fibrations between quasi-categories have a simple description. To see this we introduce the following notion:

Definition 3.4 *We shall say that a map of simplicial sets is a* mid fibration *if it has the right lifting property with respect to every inner horn* $\Lambda^k[n] \subset \Delta[n]$.

Let us regard the groupoid J as a simplicial set via the nerve functor.

Proposition 3.5 *Every quasi-fibration is a mid fibration. Conversely, a mid fibration between quasi-categories $p : X \to Y$ is a quasi-fibration iff the following equivalent conditions are satisfied:*

- *p has the right lifting property with respect to the inclusion $\{0\} \subset J$*
- *the functor $\tau_1 p : \tau_1 X \to \tau_1 Y$ is a quasi-fibration.*

Proposition 3.6 *The pair of adjoint functors*

$$\tau_1 : \mathbf{S} \leftrightarrow \mathbf{Cat} : N$$

is a Quillen pair between the model structure for quasi-categories and the natural model structure on **Cat**. *A functor $u : A \to B$ in* **Cat** *is a quasi-fibration (resp. an equivalence) iff the map $Nu : NA \to NB$ is a quasi-fibration (a weak categorical equivalence) in* **S**.

It follows that the functor $\tau_1 : \mathbf{S} \to \mathbf{Cat}$ takes a weak categorical equivalence to an equivalence of categories.

Proposition 3.7 *The classical model structure on* **S** *is a Bousfield localisation of the model structure for quasi-categories.*

Thus, every weak categorical equivalence is a weak homotopy equivalence and every Kan fibration is a quasi-fibration. Conversely, a map between Kan complexes is a weak homotopy equivalence (resp, a Kan fibration) iff it is a weak categorical equivalence (resp. a quasi-fibration).

3.2 The model structures for simplicial categories

We recall that a *simplicial category* is a category enriched over **S**. For the basic notion of enriched category theory, see [K]. A *map* of simplical categories is a strong functor, also called a *simplicial functor*. We shall denote by **SCat** the category of small simplicial categories.

To every simplicial category X we can associate a category X' enriched over the homotopy category of simplicial sets $Ho(\mathbf{S})$. A simplicial functor $f : X \to Y$ is called a *Dwyer-Kan equivalence* if the functor $f' : X' \to Y'$ is an equivalence of $Ho(\mathbf{S})$-categories.

An ordinary category can be viewed as a simplicial category with discrete hom sets. The inclusion functor $\mathbf{Cat} \subset \mathbf{SCat}$ has a left adjoint

$$ho : \mathbf{SCat} \to \mathbf{Cat}.$$

We shall say that hoX is the *homotopy category* of a simplicial category X. We have $Ob(hoX) = Ob(X)$ and

$$(hoX)(a,b) = \pi_0 X(a,b)$$

for every $a, b \in Ob(X)$. We shall say that a map of simplicial categories $f : X \to Y$ is *essentially surjective* if the functor $ho(f) : hoX \to hoY$ is essentially surjective. A map $f : X \to Y$ is a Dwyer-Kan equivalence iff it is essentially surjective and the map $X(a,b) \to Y(fa, fb)$ induced by f is a weak homotopy equivalence for every pair $a, b \in Obx$.

We call a map of simplicial categories $f : X \to Y$ a *Dwyer-Kan fibration* if the map $X(a,b) \to Y(fa, fb)$ is a Kan fibration for every pair of objects $a, b \in X$ and the functor $ho(f) : hoX \to hoY$ is a quasi-fibration. See 3.1 for the notion of quasi-fibration in **Cat**.

Theorem 3.8 [B1] *The category of simplicial categories* **SCat** *admits a model structure in which a weak equivalence is a Dwyer-Kan equivalence and a fibration is a Dwyer-Kan fibration. The fibrant objects are the categories enriched over Kan complexes. A map $f : X \to Y$ is an acyclic fibration iff it is surjective on objects and the map $X(a,b) \to Y(fa, fb)$ is a trivial fibration for every pair of objects $a, b \in X$.*

We call this model structure the *model structure for simplicial categories.*

Let $\{0,1\}$ be the discrete category with two objects. If X is a simplicial category, then a map $\{0,1\} \to X$ is pair (a,b) of objects of X. Consider the functor

$$hom : \{0,1\}\backslash\mathbf{SCat} \to \mathbf{S}$$

which associates to (X, a, b) the simplicial set $X(a,b)$. The functor hom has a left adjoint

$$S : \mathbf{S} \to \{0,1\}\backslash\mathbf{SCat}$$

which associates to a simplicial set A a simplicial category $S(A)$. The simplical category $S(A)$ has two objects 0 and 1 and

$$S(A)(a,b) = \begin{cases} A & \text{if } a = 0 \text{ and } b = 1, \\ \{id\} & \text{if } a = b, \\ \emptyset & \text{if } a = 1 \text{ and } b = 0. \end{cases}$$

We shall say that the simplicial category $S(A)$ is the *algebraic suspension* of the simplicial set A.

Proposition 3.9 *The adjoint pair of functors*

$$S : \mathbf{S} \leftrightarrow \{0,1\}\backslash\mathbf{SCat} : hom$$

is a Quillen pair between the classical model structure on simplicial sets and the model category for simplicial categories.

Proof: It suffices to show that the functor *hom* is a right Quillen functor. But if $f : X \to Y$ is a fibration (resp. an acyclic fibration) then the map $X(a,b) \to Y(fa, fb)$ induced by f is a Kan fibration (resp. trivial fibration) for any pair of objects $a, b \in X$. ∎

Proposition 3.10 *A map of simplicial categories is cofibration iff it belongs to the saturated class generated by the following maps:*

- *the maps $S(\delta_n)$ for $n \geq 0$;*
- *the map $\emptyset \to 1$.*

Proof: Let us denote this set of maps by U. By 3.8, a map of simplicial categories $f : X \to Y$ is an acyclic fibration iff it is surjective on objects and the map $X(a,b) \to Y(fa, fb)$ is a trivial fibration for every pair of objects $a, b \in X$. It follows from this description that f is an acyclic fibration iff it has the right lifting property with respect to the maps in U. Thus, $U^{\pitchfork}$ is the class of acyclic fibrations. But if $\overline{U}$ denotes the saturated class generated by U, then the pair $(\overline{U}, U^{\pitchfork})$ is a weak factorisation system by 6.9. It then follows from 3.8 that $\overline{U}$ is the class of cofibrations. ∎

3.3 The model structure for Segal spaces

Segal spaces were introduced by Rezk in [Rez]. They are the fibrant objects of a model structure on the category of bisimplicial sets. The model structure is a Bousfield localisation of a Reedy model structure.

We say that a map of simplicial spaces $f : X \to Y$ is a *column-wise weak homotopy equivalence* if the map $f_m : X_m \to Y_m$ is a weak homotopy equivalence for every $m \geq 0$. Let us denote by δ_n the inclusion $\partial\Delta[n] \subset \Delta[n]$. We say that a map $f : X \to Y$ is a *Reedy fibration*, if the map $\langle \delta_m \backslash f \rangle$ is a Kan fibration for every $m \geq 0$.

Theorem 3.11 [Ree] *The simplicial category $(\mathbf{S}^{(2)}, Hom_2)$ admits a simplicial model structure in which a cofibration is a monomorphism, a weak equivalence is a column-wise weak homotopy equivalence and a fibration is a Reedy fibration.*

For $n > 0$, the *n-chain* $I_n \subseteq \Delta[n]$ is defined to be the union of the edges $(i, i+1) \subseteq \Delta[n]$ for $0 \leq i \leq n-1$. Let us put $I_0 = 1$. For any simplicial space X we have a canonical bijection

$$I_n \backslash X = X_1 \times_{\partial_0, \partial_1} X_1 \times \cdots \times_{\partial_0, \partial_1} X_1,$$

where the successive fiber products are calculated by using the face maps $\partial_0, \partial_1 : X_1 \to X_0$. The map

$$i_n \backslash X : \Delta[n] \backslash X \longrightarrow I_n \backslash X$$

obtained from the inclusion $i_n : I_n \subseteq \Delta[n]$ is called the *Segal map*.

Definition 3.12 [Rez] *We shall say that a simplicial space X satisfies the* Segal condition *if the Segal map*

$$X_n \longrightarrow I_n \backslash X$$

is a weak homotopy equivalence for every $n \geq 2$. A Segal space is a Reedy fibrant simplicial space which satisfies the Segal condition,

We shall say that a map of simplicial spaces $u : A \to B$ is a *Segal equivalence* if the map

$$Hom_2(u, X) : Hom_2(B, X) \to Hom_2(A, X)$$

is a weak homotopy equivalence for every Segal space X.

Theorem 3.13 [Rez] *The simplicial category $(\mathbf{S}^{(2)}, Hom_2)$ admits a simplicial model structure in which a cofibration is a monomorphism and a weak equivalence is a Segal equivalence. The fibrant objects are the Segal spaces. A map between Segal spaces is a fibration iff it is a Reedy fibration.*

3.4 The injective model structure for Segal categories

A simplicial space X is called a *pre-category* if the simplicial set X_0 is discrete. We shall denote by **PCat** the full subcategory of $\mathbf{S}^{(2)}$ spanned by the pre-categories. If X is a simplicial space, then for each map $i : [0] \to [n]$ we have a map $v_i : X_n \to X_0$. The map $(v_0, \ldots, v_n) : X_n \to X_0^{n+1}$ is called the *vertex map*. When X is a pre-category, we have a decomposition

$$X_n = \bigsqcup_{a \in X_0^{n+1}} X(a),$$

where $X(a) = X(a_0, \ldots, a_n)$ denotes the fiber at $a = (a_0, \cdots, a_n)$ of the vertex map $X_n \to X_0^{n+1}$.

Observe that a simplicial set X is discrete iff the contravariant functor $X : \Delta \to \mathbf{Set}$ takes every map in Δ to a bijection. Similarly, a simplicial space X is a pre-category iff the contravariant functor $X : \Delta \times \Delta \to \mathbf{Set}$ takes every map in $[0] \times \Delta$ to a bijection. Let us put

$$\Delta^{02} = ([0] \times \Delta)^{-1}(\Delta \times \Delta)$$

and let π be the canonical functor $\Delta^2 \to \Delta^{02}$.

Proposition 3.14 *The functor π^* induces an equivalence between the category of presheaves on Δ^{02} and the subcategory $\mathbf{PCat} \subset \mathbf{S}^{(2)}$. It has a left adjoint $\pi_!$ and a right adjoint π_*.*

We shall regard the functor π^* as an inclusion by adopting the same notation for a presheaf $X : (\Delta^{02})^o \to \mathbf{Set}$ and the pre-category $\pi^*(X)$. We shall need a concrete description of the functors $\pi_!$. For any simplicial space X, we have a canonical map $1 \Box X_0 \to X$.

Proposition 3.15 *For any simplicial space X we have a pushout square of simplicial spaces*

$$\begin{array}{ccc} 1\Box X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ 1\Box \pi_0 X_0 & \longrightarrow & \pi_!(X). \end{array}$$

We shall say that a map in **PCat** is a *trivial fibration* if it has the right lifting property with respect to every monomorphism.

Proposition 3.16 *A map of pre-categories $f : X \to Y$ is a trivial fibration iff the map $f_0 : X_0 \to Y_0$ is surjective and the map $\langle \delta_m \backslash f \rangle$ is a trivial fibration for every $m > 0$.*

A *Segal category* is defined to be a pre-category which satisfies the Segal condition. If X is a pre-category, then the Segal map $X_n \to I_n \backslash X$ induces a map

$$\psi_a : X_n(a_0, \cdots, a_n) \to X(a_0, a_1) \times \cdots \times X(a_{n-1}, a_n)$$

for every $a = (a_0, a_1, \cdots, a_n)$. A pre-category X is a Segal category iff ψ_a is a weak homotopy equivalence for every $a \in X_0^{n+1}$ and every $n \geq 2$.

To every Segal category X corresponds naturally a category X' enriched over the homotopy category of simplicial sets $Ho(\mathbf{S})$. By construction, we have $Ob(X') = X_0$ and $X'(a, b) = X(a, b)$ for every $a, b \in X_0$. The composition law

$$X'(b, c) \times X'(a, b) \to X'(a, c)$$

is obtained by composing the flip map $X(b, c) \times X(a, b) \simeq X(a, b) \times X(b, c)$ with the inverse in $Ho(\mathbf{S})$ of the Segal map

$$X(a, b, c) \to X(a, b) \times X(b, c).$$

A map of Segal categories $f : X \to Y$ is called a *categorical equivalence* if the functor $f' : X' \to Y'$ is an equivalence of $Ho(\mathbf{S})$-enriched categories. Hirschowitz and Simpson construct a functor $S : PCat \to PCat$ which associates to a pre-category X a Segal category $S(X)$ "generated" by X. A map of pre-categories $f : X \to Y$ is called a *weak categorical equivalence* if the map $S(f) : S(X) \to S(Y)$ is a categorical equivalence.

Theorem 3.17 [HS] *The category* **PCat** *admits a model structure in which a cofibration is a monomorphisms and a weak equivalence is a weak categorical equivalence. A pre-category X is fibrant iff it is a Segal space. A map of pre-categories is an acyclic fibration iff it is a trivial fibration.*

We call the model structure, the *Hirschowitz-Simpson model structure* or the *injective model structure for Segal categories.* We call a fibration of this model structure a *quasi-fibration*. The fibrant pre-categories were characterised by Bergner in [B3].

If C is a category, then the simplicial space $N(C) = C\Box 1$ is a Segal category. This defines a functor $N : \mathbf{Cat} \to \mathbf{PCat}$. The functor N has a left adjoint

$$\tau_1 : \mathbf{PCat} \to \mathbf{Cat}.$$

If X is a pre-category, then $\tau_1 X$ is the fundamental category of the simplicial set $n \mapsto \pi_0 X_n$.

Let J be the groupoid generated by one isomorphism $0 \to 1$.

Proposition 3.18 *Every quasi-fibration in* $\mathbf{PCat}$ *is a Reedy fibration. Conversely, a Reedy fibration between pre-categories* $f : X \to Y$ *is a quasi-fibration iff the following equivalent conditions are satisfied:*

- *f has the right lifting property with respect to the inclusion* $\{0\}C\Box 1 \subset J\Box 1$
- *the functor* $\tau_1(f) : \tau_1 X \to \tau_1 Y$ *is a quasi-fibration in* $\mathbf{Cat}$.

The fibrant pre-category can be characterised by filling conditions. For this, we introduce the notion of mid fibration.

Definition 3.19 *We shall say that a map between pre-categories* $f : X \to Y$ *is a* mid fibration *if it satisfies the following two conditions:*

- *The map* $\langle \delta_m \backslash f \rangle$ *is a Kan fibration for every* $m \geq 0$*;*
- *The map* $\langle f/\delta_n \rangle$ *is a mid fibration for every* $n \geq 0$.

Proposition 3.20 *A pre-category* X *is fibrant (ie is a Segal space) iff the map* $X \to 1$ *is a mid fibration. A map between fibrant pre-categories is a Reedy fibration iff it is a mid fibration.*

The first projection $p_1 : \Delta \times \Delta \to \Delta$ inverts every arrow in $[0] \times \Delta$. Hence there is a unique functor $q : \Delta^{02} \to \Delta$ such that $q\pi = p_1$. The functor p_1 is left adjoint to the functor $i_1 : \Delta \to \Delta \times \Delta$ obtained by putting $i_1([n]) = ([n], 0)$. It follows that the functor q is left adjoint to the functor $j = \pi i_1 : \Delta \to \Delta^{02}$.

Lemma 3.21 *The functor* $q^* : \mathbf{PCat} \to \mathbf{S}$ *is left adjoint to the functor* j^**. If* X *is a pre-category, then* $j^*(X)$ *is the first row of* X*. If* A *is a simplicial set, then* $q^*(A) = A\Box 1$.

Theorem 3.22 [JT2] *The pair of adjoint functors* (q^*, j^*) *is a Quillen equivalence*

$$q^* : \mathbf{S} \leftrightarrow \mathbf{PCat} : j^*$$

between the model structure for quasi-categories and the model structure for Segal categories.

3.5 The projective model structure for Segal categories

We shall say that a map of pre-categories $f : X \to Y$ is a *locally trivial fibration* if the map $f_0 : X_0 \to Y_0$ is surjective and the map $X(a) \to Y(fa)$ induced by f is a trivial fibration for every $a \in X_0^{n+1}$ and $n \geq 1$.

Let i_n be the inclusion $[n]_0 \subseteq \Delta[n]$.

Definition 3.23 *We shall say that a map of pre-categories is strong cofibration if it belongs to the saturated class generated by the following maps:*

- *the map* $i_m \odot' \delta_n$ *for* $m, n \geq 0$*;*
- *the map* $\emptyset \to 1$*.*

Theorem 3.24 [B2] *The category* **PCat** *admits a model structure in which the cofibration are the strong cofibrations, the weak equivalences are the weak categorical equivalences and the acyclic fibrations are the locally trivial fibrations.*

We call the model structure, the *projective model structure for Segal categories*. We shall denote the resulting model category by $\mathbf{PCat}'$. Every fibrant object is a Segal category. See [B3] for a characterisation of the fibrant objects.

Theorem 3.25 [B2] *The pair* (Id, Id) *is a Quillen equivalence*

$$Id : \mathbf{PCat}' \leftrightarrow f\mathbf{PCat} : Id$$

betwen the projective model structure and the injective model structure for Segal categories.

The category **SCat** is enriched over simplicial sets by 5.8 The *nerve* of a simplicial category X is the pre-category $N^! X$ defined by putting

$$(N^! X)_n = Hom([n], X)$$

for every $n \geq 0$, where the poset $[n]$ is viewed as a simplicial category with discrete hom set. The functor N has a left adjoint

$$N_! : \mathbf{PCat} \to \mathbf{SCat}.$$

If X is a pre-category, then we have $(N_! X)_n = \tau_1(X_{\star n})$ for every $n \geq 0$.

Theorem 3.26 [B2] *The pair of adjoint functors* $(N_!, N^!)$ *is a Quillen equivalence*

$$N_! : \mathbf{PCat}' \leftrightarrow \mathbf{SCat} : N^!$$

between the projective model structure for Segal categories and the model structure for simplicial categories.

4 Appendix on 3 fibered model categories

Consider the functor $Ob : \mathbf{Cat} \to \mathbf{Set}$ which associates to a category A its set of objects ObA. It is easy to verify that the functor Ob is a Grothendieck fibration. A functor $u : A \to B$ in **Cat** is cartesian with respect to the functor Ob iff it is fully faithful. We shall say that a functor $u : A \to B$ is *biunivoque* if the map $Ob(u) : ObA \to ObB$ is bijective. Every functor $u : A \to B$ admits a factorisation $u = pq$ with q a biunivoque functor and p a fully faithful functor. The factorisation is unique up to unique isomorphism. We shall say that it is the *Gabriel factorisation* of the functor u.

4.1 The fibered model structure for simplicial categories

Consider the functor $Ob : \mathbf{SCat} \to \mathbf{Set}$ which associates to a simplical category X its set of objects ObX. It is easy to verify that the functor Ob is a Grothendieck fibration. A simplicial functor $f : X \to Y$ is cartesian with respect to the functor Ob iff it is fully faithful. We shall say that a simplicial functor $f : X \to Y$ is *biunivoque* if the map $Ob(f) : ObX \to ObY$ is bijective. Every simplicial functor $f : X \to Y$ admits a factorisation $f = pq$ with q a biunivoque functor and p a fully faithful simplicial functor. The factorisation is unique up to unique isomorphism. We shall say that it is the *Gabriel factorisation* of the functor f.

Definition 4.1 *We shall say that a map of simplicial categories $f : X \to Y$ is a* local fibration *if the induced map $X(a, b) \to Y(fa, fb)$ is a Kan fibration for every pair of objects $a, b \in X$.*

A biunivoque simplicial functor $f : X \to Y$ is a Dwyer-Kan equivalence iff the induced map $X(a, b) \to Y(fa, fb)$ is a weak homotopy equivalence for every pair of objects $a, b \in X$.

Theorem 4.2 *The simplical category* **SCat** *admits a simplicial model structure in which a fibration is a local fibration and a weak equivalence is a biunivoque Dwyer-Kan equivalence.*

We call this model structure the *fibered model structure for simplicial categories.* We shall denote the resulting model category by $\mathbf{SCat}_f$.

Definition 4.3 *We shall say that a map of simplicial categories $f : X \to Y$ is an* immersion *if it is a cofibration in the fibered model structure on* **SCat**.

Proposition 4.4 *A map of simplicial categories is an immersion if it belongs to the saturated class generated by the cofibrations and the map $1 \sqcup 1 \to 1$.*

Proposition 4.5 *Every cofibration in* **SCat** *is an immersion. An immersion $f : X \to Y$ is a cofibration iff the map $Ob(f) : ObX \to ObY$ is monic.*

4.2 The fibered injective model structure for Segal categories

Consider the functor $Ob : \mathbf{PCat} \to \mathbf{Set}$ which associates to a pre-category X its set of vertices $ObX = X_0 = X_{00}$. A map of pre-categories $f : X \to Y$ is cartesian with respect to Ob iff the map $X(a) \to Y(fa)$ induced by f is bijective for every $a \in X_0^{n+1}$ and every $n > 0$. We shall say that a cartesian map is *fully faithful*. We shall say that a map of pre-categories $f : X \to Y$ is *biunivoque* if the map $f_0 : X_0 \to Y_0$ is bijective. Every map of pre-categories $f : X \to Y$ admits a factorisation $f = pq : X \to Z \to Y$ where $q : X \to Z$ is a biunivoque map and with $p : Z \to Y$ a fully faithful map. The factorisation is unique up to unique isomorphism. We shall say that it is the *Gabriel factorisation* of the map f.

We shall say that a pre-category is *reduced* if it has exactly one object. Let $\mathbf{PCat}_\bullet \subset \mathbf{PCat}$ be the full subcategory spanned by the reduced pre-categories. The subcategory $\mathbf{PCat}_\bullet$ is reflective. Let

$$\rho : \mathbf{PCat} \to \mathbf{PCat}_\bullet$$

be the reflection functor. If X is a pre-category, then we have a pushout square

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \rho X. \end{array}$$

Definition 4.6 *We shall say that a map of pre-categories $u : X \to Y$ is a immersion if the map $\rho(u) : \rho X \to \rho Y$ is monic.*

Proposition 4.7 *In* $\mathbf{PCat}$*, every monomorphism in* $\mathbf{PCat}$ *is an immersion. An immersion $f : X \to Y$ is monic iff the map $f_0 : X_0 \to Y_0$ is monic.*

Proposition 4.8 *The class of immersions in* $\mathbf{PCat}$ *is saturated and generated by the following maps*

- *the inclusions $\partial(\Delta[m] \odot \Delta[n]) \subset \Delta[m] \odot \Delta[n]$, for $m > 0$ and $n \geq 0$;*
- *the maps $\emptyset \to 1$ and $1 \sqcup 1 \to 1$.*

Theorem 4.9 *The simplicial category* $\mathbf{PCat}$ *admits a simplicial model structure in which a cofibration is an immersion and a weak equivalence is a biunivoque weak categorical equivalence. A pre-category is fibrant iff it is a Segal space. A map between fibrant pre-categories is a fibration iff it is a Reedy fibration. A map is an acyclic fibration iff it is a biunivoque trivial fibration.*

We call this model structure the *fibered injective model structure for Segal category*. We shall denote the resulting model category by $f\mathbf{PCat}$.

4.3 The fibered projective model structure for Segal categories

Definition 4.10 *We shall say that a map of pre-categories $u : X \to Y$ is a strong immersion if the map $\rho(u) : \rho X \to \rho Y$ is a strong cofibration.*

Proposition 4.11 *Every strong cofibration in* **PCat** *is a strong immersion. A strong immersion $f : X \to Y$ is strong cofibration iff the map $f_0 : X_0 \to Y_0$ is monic.*

Proposition 4.12 *The class of strong immersions in* **PCat** *is saturated and generated by the following maps*

- *the maps $i_m \odot' \delta_n$ for $m, n \geq 0$;*
- *the maps $\emptyset \to 1$ and $1 \sqcup 1 \to 1$.*

A biunivoque map is a locally trivial fibration iff it is a trivial fibration.

Theorem 4.13 [B2] *The simplicial category* **PCat** *admits a simplicial model structure in which the cofibration are the strong immersions the weak equivalences are the biunivoque weak categorical equivalences and the acyclic fibrations are the biunivoque trivial fibrations.*

We call this model structure the *projective fibered model structure for Segal category*. We shall denote the resulting model category by $f\mathbf{PCat}'$.

Theorem 4.14 [J3] *The pair (Id, Id) is a Quillen equivalence*

$$Id : \mathbf{PCat}' \leftrightarrow f\mathbf{PCat} : Id$$

betwen the fibered projective model structure and the fibered injective model structure for Segal categories.

Theorem 4.15 [J3] *The pair of adjoint functors $(N_!, N^!)$ is a Quillen equivalence*

$$N_! : f\mathbf{PCat}' \leftrightarrow f\mathbf{SCat} : N^!$$

between the fibered projective model structure for Segal categories and the fibered model structure for simplicial categories.

5 Appendix on 3 simplicial enrichements

5.1 Enrichement of the category of simplicial spaces

A *bisimplicial set* is defined to be a contravariant functor $\Delta \times \Delta \to \mathbf{Set}$. We shall denote the category of bisimplicial sets by $\mathbf{S}^{(2)}$. A *simplicial space* is a contravariant functor $\Delta \to \mathbf{S}$. We regard a simplicial space X as a bisimplicial set by putting $X_{mn} = (X_m)_n$ for every $m, n \geq 0$. Conversely, we regard a bisimplicial set X as a simplicial space by putting $X_m = X_{m\star}$ for every $m \geq 0$. The *box product* $A \Box B$ of two simplicial sets A and B is the bisimplicial set obtained by putting

$$(A \Box B)_{mn} = A_m \times B_n$$

for every $m, n \geq 0$. This defines a functor of two variables $\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}^{(2)}$. The functor is divisible on both sides. This means that the functor $A \Box (-) : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $A \backslash (-) : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set A. If $X \in \mathbf{S}^{(2)}$, then a simplex $\Delta[n] \to A \backslash X$ is a map $A \Box \Delta[n] \to X$. The simplicial set $\Delta[m] \backslash X$ is the mth column X_{m*} of X. Dually, the functor $(-) \Box B : \mathbf{S} \to \mathbf{S}^{(2)}$ admits a right adjoint $(-)/B : \mathbf{S}^{(2)} \to \mathbf{S}$ for every simplicial set B. If $X \in \mathbf{S}^{(2)}$, then a simplex $\Delta[m] \to X/B$ is a map $\Delta[m] \Box B \to X$. The simplicial set $X/\Delta[n]$ is the nth row X_{*n} of X. If $X \in \mathbf{S}^{(2)}$ and $A, B \in \mathbf{S}$, there is a bijection between the following three kinds of maps

$$A \Box B \to X, \qquad B \to A \backslash X, \qquad A \to X/B.$$

Hence the contravariant functors $A \mapsto A \backslash X$ and $B \mapsto B \backslash X$ are mutually right adjoint.

The second projection $p_2 : \Delta^2 \to \Delta$ is left adjoint to the functor i_2 defined by putting $i_2([n]) = ([0], [n])$. It follows that the functor p_2^* is left adjoint to the functor i_2^*. If X is a simplicial space, then $i_2^*(X) = X_0$ is the first column of X. If A is a simplicial set, then $p_2^*(A) = 1 \Box A$. The category $\mathbf{S}^{(2)}$ is cartesian closed. For any pair $X, Y \in \mathbf{S}^{(2)}$, let us put

$$Hom_2(X, Y) = i_2^*(Y^X).$$

This defines a simplicial enrichement of $\mathbf{S}^{(2)}$.

Proposition 5.1 *The simplicial category $(\mathbf{S}^{(2)}, Hom_2)$ admits tensor and cotensor products. The tensor product of a simplicial space X by a simplicial set A is the simplicial space*

$$X \otimes_2 A = X \times p_2^*(A)$$

and the cotensor product is the simplicial space

$$X^{[A]} = X^{p_2^*(A)}.$$

We have $(X \otimes_2 A)_n = X_n \times A$ and $(X^{[A]})_n = X_n^A$ for every $n \geq 0$.

5.2 Enrichement of the category of pre-categories

We saw in 5.1 that the simplicial category $(\mathbf{S}^{(2)}, Hom_2)$ admits tensor and cotensor products. The simplicial enrichement Hom_2 of $\mathbf{S}^{(2)}$ induces a simplicial enrichement of its full subcategory **PCat**. If X is a pre-category, then so is the simplicial space $X^{[A]}$, since the simplicial set $(X^{[A]})_0 = X_0^A$ is discrete in this case. It follows that the simplicial category $(\mathbf{PCat}, Hom_2)$ admits cotensor products. Recall from Proposition 3.14 that the inclusion functor $\pi^* : \mathbf{PCat} \subset \mathbf{S}^{(2)}$ admits a left adjoint $\pi_!$.

Proposition 5.2 *The simplicial category* $(\mathbf{PCat}, Hom_2)$ *admits tensor and cotensor products. The cotensor product of a pre-category* X *by a simplicial set* A *is the precategory* $X^{[A]}$ *and its tensor product is the pre-category* $X \otimes_2 A = \pi_!(X \times_2 A)$*. We have a pushout square of bisimplicial sets*

$$\begin{array}{ccc} X_0 \Box A & \longrightarrow & X \times_2 A \\ \downarrow & & \downarrow \\ X_0 \Box \pi_0 A & \longrightarrow & X \otimes_2 A. \end{array}$$

The functors $\pi_!$ *and* π^* *are strong and the adjunction* $\pi_! \vdash \pi^*$ *is strong.*

If A and B are simplicial sets, we shall put

$$A \odot B = \pi_!(A \Box B).$$

Notice that $A \odot 1 = A \Box 1$ for every simplicial set A.

Proposition 5.3 *We have* $A \odot B = (A \Box 1) \otimes_2 B$*. Moreover, we have a pushout square of bisimplicial sets*

$$\begin{array}{ccc} A_0 \Box B & \longrightarrow & A \Box B \\ \downarrow & & \downarrow \\ A_0 \Box \pi_0 B & \longrightarrow & A \odot B. \end{array}$$

If $v : S \to T$ and $u : A \to B$ are maps of simplicial sets, we shall denote by $v \odot' u$ the map

$$S \odot B \sqcup_{S \odot A} T \odot A \longrightarrow T \odot B$$

obtained from the commutative square of pre-categories

$$\begin{array}{ccc} S \odot A & \longrightarrow & T \odot A \\ \downarrow & & \downarrow \\ S \odot B & \longrightarrow & T \odot B. \end{array}$$

If $f : X \to Y$ is a map of pre-categories, then we have

$$v \pitchfork \langle f/u \rangle \quad \Leftrightarrow \quad (v \odot' u) \pitchfork f \quad \Leftrightarrow u \pitchfork \langle u \backslash f \rangle.$$

Lemma 5.4 *If* $v : S \to T$ *and* $u : A \to B$ *are monomorphisms of simplicial sets and* v *is biunivoque, then* $v \odot' u$ *is monic and biunivoque.*

Proposition 5.5 *The saturated class of monomorphisms in* **PCat** *is generated by the maps* $\delta_m \odot' \delta_n$*, for* $m > 0$ *and* $n \geq 0$*, together with the map* $\emptyset \subset 1$*.*

Definition 5.6 *We shall say that a map of pre-categories is a* mid cofibration *if it belongs to the saturated class generated by the following two sets of maps:*

- *the maps* $\delta_m \odot' h_n^k$ *for* $m > 0$, $n > 0$ *and* $0 \leq k \leq n$*;*
- *the maps* $h_m^k \odot' \delta_n$ *for* $0 < k < m$ *and* $n \geq 0$*.*

Proposition 5.7 *If* $\mathcal{A}$ *is the class of mid cofibrations in* **PCat** *and* $\mathcal{B}$ *is the class of mid fibrations, then the pair* $(\mathcal{A}, \mathcal{B})$ *is a weak factorisation system.*

5.3 Enrichement of the category of simplicial categories

For any simplicial category X there is a simplicial category $X^{(n)}$ for every $n \geq 0$. By definition, $ObX^{(n)} = ObX$ and

$$X^{(n)}(a, b) = X(a, b)^{\Delta[n]}$$

for every pair $a, b \in ObX$. The functor $X \mapsto X^{(n)}$ has the structure of a monad with the unit $X \to X^{(n)}$ defined by the map $\Delta[n] \to 1$ and the multiplication $X^{(n)(n)} \to X^{(n)}$ defined by the diagonal $\Delta[n] \to \Delta[n] \times \Delta[n]$. If X and Y are simplicial categories, let us put

$$Hom(X, Y)_n = \mathbf{SCat}(X, Y^{(n)})$$

for every $n \geq 0$. This defines a simplicial set $Hom(X, Y)$. By using the monad structure of $X^{(n)}$ we can define a Kleisly composition

$$Hom(Y, Z)_n \times Hom(X, Y)_n \to Hom(X, Z)_n$$

for each $n \geq 0$. This defines a map of simplicial sets

$$Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z).$$

Proposition 5.8 *The map above is the composition law of a simplical enrichement of* **SCat**. *It admits tensor product and cotensor product.*

We shall denote by $X \otimes A$ the tensor product of a simplicial category X by a simplicial set A and by $X^{[A]}$ their cotensor product.

An ordinary category can be viewed as a simplicial category with discrete hom sets. The nerve of a simplicial category X is the precategory $N^! X$ defined by putting

$$(N^! X)_n = Hom([n], X)$$

for every $n \geq 0$, where the poset $[n]$ is viewed as a simplicial category with discrete hom set. If X is a simplicial category we shall denote by X_n the ordinary category obtained by putting $X_n(a, b) = X(a, b)_n$ for every $n \geq 0$. Then we have

$$(N^! X)_{mn} = \mathbf{Cat}([m], X_n)$$

for every $m, n \geq 0$. The simplicial set $(N^! X)_n$ is the coproduct of the simplicial sets

$$X(a_0, a_1) \times \cdots \times X(a_{n-1}, a_n)$$

for $(a_0, \ldots, a_n) \in ObX^{n+1}$. The functor $N^!$ has a left adjoint

$$N_! : \mathbf{PCat} \to \mathbf{SCat}.$$

If Y is a pre-category, then we have $(N_!Y)_n = \tau_1(Y_{\star n})$ for every $n \geq 0$.

We shall say that a pre-category Y satisfies the Segal condition *strictly* if the Segal map

$$Y_n \to Y_1 \times_{\partial_0, \partial_1} \cdots \times_{\partial_0, \partial_1} Y_1$$

is an isomophism for every $n \geq 2$.

Proposition 5.9 *The functors $N_!$ and $N^!$ are simplicial and the adjunction $N_! \vdash N^!$ is strong. The functor $N^!$ is fully faithful. A pre-category belongs to the essential image of $N^!$ iff it satisfies the Segal condition strictly.*

Corollary 5.10 *If X is a simplicial category and A is a simplicial set, we have the formulas*

$$N^!(X^{[A]}) = N^!(X)^{[A]} \quad \text{and} \quad X \otimes A = N_!(N^!(X) \otimes_2 A).$$

The following proposition is useful for computing the cotensor product of a simplicial category by a simplicial set.

Proposition 5.11 *If A is a connected simplicial set and X is a simplicial category, then we have $ObX^{[A]} = ObX$ and*

$$X^{[A]}(a,b) = X(a,b)^A$$

for every pair of objects $a, b \in X$. In general, we have $Ob(X^{[A]}) = Ob(X)^{\pi_0 A}$ and

$$X^{[A]} = \prod_{i \in \pi_0 A} X^{[A_i]},$$

where A_i denotes a connected component of A.

Corollary 5.12 *If X is a simplicial category, then we have*

$$X^{(n)} = X^{[\Delta[n]]}$$

for every $n \geq 0$.

Proposition 5.13 *If A and B are simplicial sets, then we have a canonical isomorphism*

$$N_!(A \odot B) = \tau_1(A) \otimes B.$$

Proof: If Y is a pre-category, then we have $(N_!Y)_n = \tau_1(Y_{\star n})$ for every $n \geq 0$. It follows that we have $N_!(A \Box 1)_n = \tau_1(A)$ for every $n \geq 0$. This shows that $N_!(A \Box 1) = \tau_1(A)$, a simplicial category with discrete hom sets. The functor $N_!$ preserves tensor products by Proposition 5.9. But we have $A \odot B = (A \Box 1) \otimes_2 B$ by Proposition 5.3. Thus,

$$N_!(A \odot B) = N_!(A \Box 1) \otimes B = \tau_1(A) \otimes B.$$

∎

Proposition 5.14 *If X is a simplicial category and A is a simplicial set, then*

$$Ob(X \otimes A) = Ob(X) \times \pi_0(A).$$

Moreover, $1 \otimes A = \pi_0(A)$.

Proposition 5.15 *Let $v : S \to T$ be a cofibration in* **SCat** *and $u : A \to B$ be a monomorphism of simplicial sets. If v is binivoque then the map $v \otimes' u$ is a biunivoque cofibration. Moreover, $v \otimes' u$ is a biunivoque weak categorical equivalence if v is a biunivoque weak categorical equivalence or if u is a weak homotopy equivalence.*

6 Appendix on category theory and homotopical algebra

6.1 Adjunctions, monads and resolutions

Recall that an *adjunction* $\theta : F \dashv G$ between two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ is defined to be a natural isomorphism

$$\theta : \mathcal{B}(FA, B) \simeq \mathcal{A}(A, GB).$$

The functor F is said to be the *left adjoint* and the functor G to be the *right adjoint* . We shall write $F \dashv G$, or write

$$F : \mathcal{A} \leftrightarrow \mathcal{B} : G,$$

to indicate that F is left adjoint to G. The image by θ of the identity of FA is a map $\eta_A : A \to GFA$. This defines a natural transformation $\eta : Id \to GF$ called the *unit* of the adjunction. The image by θ^{-1} of the identity of GB is a map $FGB \to B$.This defines a natural transformation $\epsilon : FG \to Id$ called the *counit* of the adjunction. The natural transformations satisfy the following *adjuntion identities*,

$$(\epsilon \circ F)(F \circ \eta) = 1_F \quad \text{and} \quad (G \circ \eta)(\eta \circ G) = 1_G.$$

Conversely, if a pair of of natural transformations $\eta : Id \to GF$ and $\epsilon : FG \to Id$ satisfies these identities, then they are defined by a unique adjunction $\theta : F \dashv G$. In this case, the pair (η, ϵ) is said to be an *adjunction* and we write $(\eta, \epsilon) : F \dashv G$. We shall often write $F \dashv G$ or write

$$F : \mathcal{A} \leftrightarrow \mathcal{B} : G$$

to indicate that the functor F is left adjoint to the functor G. We shall say that (F, G) is an *adjoint pair*.

Suppose that we have two adjunctions $(\eta_0, \epsilon_0) : F_0 \dashv G_0$ and $(\eta_1, \epsilon_1) : F_1 \dashv G_1$, where

$$F_0 : \mathcal{A} \leftrightarrow \mathcal{B} : G_0 \quad \text{and} \quad F_1 : \mathcal{A} \leftrightarrow \mathcal{B} : G_1.$$

Then the *right transpose* of a natural transformation $\alpha : F_0 \to F_1$ is the natural transformation $\alpha^t : G_1 \to G_0$ obtained by composing

$$G_1 \xrightarrow{\eta_0 \circ G_1} G_0 F_0 G_1 \xrightarrow{G_0 \circ \alpha \circ G_1} G_0 F_1 G_1 \xrightarrow{G_0 \circ \epsilon_1} G_0$$

The map $\alpha \mapsto \alpha^t$ is a bijection between the natural transformations $F_0 \to F_1$ and the natural transformations $G_1 \to G_0$. The inverse bijection associates to a natural transformation $\beta : G_1 \to G_0$ its *left transpose* ${}^t\beta : F_0 \to G_1$ obtained by composing

$$F_0 \xrightarrow{F_0 \circ \eta_1} F_0 G_1 F_1 \xrightarrow{F_0 \circ \beta \circ F_1} F_0 G_0 F_1 \xrightarrow{\epsilon_0 \circ F_1} F_1$$

We shall say that a functor of two variables

$$\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$$

is *divisible on the left* if the functor $A \odot (-) : \mathcal{E}_2 \to \mathcal{E}_3$ admits a right adjoint $A\backslash(-) : \mathcal{E}_3 \to \mathcal{E}_2$ for every object $A \in \mathcal{E}_1$. In this case we obtain a functor of two variables $(A, X) \mapsto A\backslash X$,

$$\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2,$$

called the *left division functor*. Dually, we shall say that $\odot$ is *divisible on the right* if the functor $(-) \odot B : \mathcal{E}_1 \to \mathcal{E}_3$ admits a right adjoint $(-)/B : \mathcal{E}_3 \to \mathcal{E}_1$ for every object $B \in \mathcal{E}_2$. In this case we obtain a functor of two variables $(X, B) \mapsto X/B$,

$$\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1,$$

called the *right division functor*. When the functor $\odot$ is divisible on both sides, there is a bijection between the following three kinds of maps

$$A \odot B \to X, \qquad B \to A\backslash X, \qquad A \to X/B.$$

Hence the contravariant functors $A \mapsto A\backslash X$ and $B \mapsto B\backslash X$ are mutually right adjoint.

Remark: If a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on both sides, then so are the left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ and the right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$. This is called a tensor-hom-cotensor situation in [G].

Recall that a monoidal category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *closed* if the tensor product $\otimes$ is divisible on each side. Let $\mathcal{E} = (\mathcal{E}, \otimes, \sigma)$ be a *symmetric* monoidal closed category, with symmetry $\sigma : A \otimes B \simeq B \otimes A$. Then the objects X/A and $A\backslash X$ are canonicaly isomorphic; we can identify them by adopting a common notation, for example $[A, X]$.

Recall that a category with finite products $\mathcal{E}$ is said to be *cartesian closed* if the functor $A \times - : \mathcal{E} \to \mathcal{E}$ admits a right adjoint $(-)^A$ for every object $A \in \mathcal{E}$.

A cartesian closed category $\mathcal{E}$ is symmetric monoidal closed. Every presheaf category and more generally every topos is cartesian closed.

Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables with values in a finitely cocomplete category $\mathcal{E}_3$. If $u : A \to B$ is map in $\mathcal{E}_1$ and $v : S \to T$ is a map in $\mathcal{E}_2$, we shall denote by $u \odot' v$ the map

$$A \odot T \sqcup_{A \odot S} B \odot S \longrightarrow B \odot T$$

obtained from the commutative square

$$\begin{array}{ccc} A \odot S & \longrightarrow & B \odot S \\ \downarrow & & \downarrow \\ A \odot T & \longrightarrow & B \odot T. \end{array}$$

This defines a functor of two variables

$$\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I,$$

where $\mathcal{E}^I$ denotes the category of arrows of a category $\mathcal{E}$.

We remark here that if $u : A \subseteq B$ and $v : S \subseteq T$ are inclusions of sub-presheaves in a presheaf category, then the map $u \times' v$ is the inclusion of presheaves

$$(A \times T) \cup (B \times S) \subseteq B \times T.$$

We remark here that if $u : A \subseteq B$ and $v : S \subseteq T$ are inclusions of sub-objects in a topos, then the map $u \times' v$ is the inclusion of sub-objects

$$(A \times T) \cup (B \times S) \subseteq B \times T.$$

Suppose now that a functor $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is divisible on both sides, that $\mathcal{E}_1$ and $\mathcal{E}_2$ are finitely complete and that $\mathcal{E}_3$ is finitely cocomplete. Then the functor $\odot' : \mathcal{E}_1^I \times \mathcal{E}_2^I \to \mathcal{E}_3^I$ is divisible on both sides. If $u : A \to B$ is map in $\mathcal{E}_1$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, let us denote by $\langle u \backslash\ f \rangle$ the map

$$B \backslash X \to B \backslash Y \times_{A \backslash Y} A \backslash X$$

obtained from the commutative square

$$\begin{array}{ccc} B \backslash X & \longrightarrow & A \backslash X \\ \downarrow & & \downarrow \\ B \backslash Y & \longrightarrow & A \backslash Y. \end{array}$$

Then the functor $f \mapsto \langle u \backslash f \rangle$ is right adjoint to the functor $v \mapsto u \odot' v$. Dually, if $v : S \to T$ is map in $\mathcal{E}_2$ and $f : X \to Y$ is a map in $\mathcal{E}_3$, we shall denote by $\langle f / v \rangle$ the map

$$X/T \to Y/T \times_{Y/S} X/S$$

obtained from the commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & X/S \\ \downarrow & & \downarrow \\ Y/T & \longrightarrow & Y/S. \end{array}$$

The functor $f \mapsto \langle f \backslash v \rangle$ is right adjoint to the functor $u \mapsto u \odot' v$.

Let Δ_+ be the category of finite ordinals and order preserving maps. We shall denote an ordinal of cardinality n by n, so that $[n] = n + 1$. We shall put $[-1] = 0$. We recall that an *augmented simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\Delta_+ \to \mathcal{C}$. If $X : \Delta_+ \to \mathcal{C}$ is an augmented simplicial object, then the restriction $X \mid \Delta$ is a simplicial object. We shall denote the object $X(n)$ by X_{n-1} for every $n \geq 0$, so that $X_{-1} = X(0)$. From the map $d_0 : [-1] \to [0]$ we obtain a map $\partial_0 : X_0 \to X_{-1}$ called the *augmentation*. The augmentation can be viewed as a map from the simplicial object $X^+ = X \mid \Delta$ to the constant simplicial object with value X_{-1}. The ordinal sum $(m, n) \mapsto m+n$ gives the category Δ_+ the structure of a monoidal category with $0 = [-1]$ for unit object. The map $1 + 1 \to 1$ gives the object $[0] = 1$ the structure of a monoid in this category, with the map $0 \to 1$ as the unit map. The monoidal category Δ_+ is freely generated by this monoid by a classical result []. Let $S : \Delta_+ \to \Delta_+$ be the successor functor $n \mapsto 1 + n$. The functor S has the structure of a monad: the unit $\eta : Id \to S$ is obtained from the map $0 \to 1$ and the multiplication $\mu : S^2 \to S$ from the map $1 + 1 \to 1$. If $\mathcal{E}$ is a category, then the endo-functor $X \mapsto XS$ of the category of augmented simplicial objects $[\Delta_+^o, \mathcal{E}]$ has the structure of a comonad. If $C = (C, \epsilon, \delta)$ is a comonad on $\mathcal{E}$, then there is a unique functor

$$C^\bullet : \mathcal{E} \to [\Delta_+^o, \mathcal{E}]$$

such that $C^0(A) = A$, $C^\bullet S = CC^\bullet$, $C^\bullet \circ \eta = \epsilon \circ C^\bullet$ and $C^\bullet \circ \mu = \delta \circ C^\bullet$. Let us put $C_n(A) = C^{n+1}(A)$ for every $n \geq 0$. The sequence $C_\star(A) = (C_n(A) : n \geq 0)$ has then the structure of a simplicial object equipped with an augmentation $C_0(A) \to A$ given by the counit $\epsilon : C(A) \to A$. The face and degeneracy operators of $C_\star(A)$ are given by

$$\begin{aligned} \partial_i &= C^i \circ \epsilon \circ C^{n-i} : C^{n+1} \to C^n \quad \text{and} \\ \sigma_i &= C^i \circ \delta \circ C^{n-i} : C^{n+1} \to C^{n+2} \end{aligned}$$

for every $i \in [n]$.

Let Δ' be the subcategory of Δ whose arrows are the maps $f : [m] \to [n]$ such that $f(0) = 0$. The obvious forgetful functor $V : \Delta' \to \Delta_+$ has a left adjoint $E : \Delta_+ \to \Delta'$ which associates to an ordinal n, the ordinal $1 + n = [n]$. We recall that a *split augmented simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\Delta' \to \mathcal{C}$. If $X : \Delta' \to \mathcal{C}$ is a split augmented simplicial object, then $XE : \Delta_+ \to \mathcal{C}$ is an augmented simplicial object. We shall denote the object

$XE(n) = X([n])$ by X_{n-1} for every $n \geq 0$, so that $X_{-1} = X([0])$. The face operator $\partial_i : X_n \to X_{n-1}$ is the map $X(d_{i+1}) : X([n+1]) \to X([n])$ for every $n > 0$ and $0 \leq i \leq n$. The face operator $\sigma_i : X_{n-1} \to X_n$ is the map $X(s_{i+1}) : X([n]) \to X([n+1])$ for every $n > 0$ and $0 \leq i < n$. The augmentation $\epsilon : X_0 \to X_{-1}$ is the map $X(d_1) : X([1]) \to X([0])$. We shall denote the map $X(s_0) : X([n]) \to X([n+1])$ by $\sigma = \sigma_{-1} : X_{n-1} \to X_n$. The sequence of maps

$$X_{-1} \xrightarrow{\sigma} X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\sigma} \cdots$$

determines the spliting of the augmented simplicial object XE. Recall that the augmentation ϵ can be viewed as a map from the simplicial object $X^+ = XE \mid \Delta$ to the constant simplicial object with value X_{-1}. The following result is classical:

Lemma 6.1 *Let $X : \Delta' \to \mathcal{C}$ be a split augmented simplicial object. Then the augmentation $\epsilon : X^+ \to X_{-1}$ is a homotopy equivalence.*

If $\mathcal{C}$ is a category, then the product

$$E = E \odot \mathcal{C} : \Delta_+ \odot \mathcal{C} \leftrightarrow \Delta' \odot \mathcal{C} : V \odot \mathcal{C} = V$$

is an adjoint pair. The functor $i : \mathcal{C} \to \Delta_+ \odot \mathcal{C}$ is the universal example of a functor from $\mathcal{C}$ to the left term of an adjoint pair. More precisely, for every adjoint pair $F : \mathcal{G} \to \mathcal{E} : U$ and every functor $K : \mathcal{C} \to \mathcal{G}$ there exists a unique pair of functors $K_0 : \Delta_+ \odot \mathcal{C} \to \mathcal{G}$ and $K_\bullet : \Delta' \odot \mathcal{C} \to \mathcal{E}$ such that the following two squares commute,

$$\begin{array}{ccc} \Delta' \odot \mathcal{C} & \xrightarrow{K_\bullet} & \mathcal{E} \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle U} \\ \Delta_+ \odot \mathcal{C} & \xrightarrow{K_0} & \mathcal{G} \end{array} \qquad \begin{array}{ccc} \Delta' \odot \mathcal{C} & \xrightarrow{K_\bullet} & \mathcal{E} \\ {\scriptstyle E}\uparrow & & \uparrow{\scriptstyle F} \\ \Delta_+ \odot \mathcal{C} & \xrightarrow{K_0} & \mathcal{G}, \end{array}$$

and for which $K_\bullet \circ \eta = \eta \circ K_\bullet$, $K_0 \circ \epsilon = \epsilon \circ K_0$ and $K_0 i = K$. Dually, the product

$$V^o = V^o \odot \mathcal{C} : \Delta'^o \odot \mathcal{C} \leftrightarrow \Delta_0^o \odot \mathcal{C} : E^o \odot \mathcal{C} = E^o$$

is an adjoint pair. The functor $i : \mathcal{C} \to \Delta_+^o \odot \mathcal{C}$ is the universal example of a functor from $\mathcal{C}$ to the right term of an adjoint pair. More precisely, for every adjoint pair $F : \mathcal{G} \to \mathcal{E} : U$ and every functor $K : \mathcal{C} \to \mathcal{E}$ there exists a unique pair of functors $K_0 : \Delta_+^o \odot \mathcal{C} \to \mathcal{E}$ and $K_\bullet : \Delta'^o \odot \mathcal{C} \to \mathcal{G}$ such that the following two squares commute,

$$\begin{array}{ccc} \Delta_+^o \odot \mathcal{C} & \xrightarrow{K_0} & \mathcal{E} \\ {\scriptstyle E^o}\downarrow & & \downarrow{\scriptstyle U} \\ \Delta'^o \odot \mathcal{C} & \xrightarrow{K_\bullet} & \mathcal{G} \end{array} \qquad \begin{array}{ccc} \Delta_+^o \odot \mathcal{C} & \xrightarrow{K_0} & \mathcal{E} \\ {\scriptstyle V^o}\uparrow & & \uparrow{\scriptstyle F} \\ \Delta'^o \odot \mathcal{C} & \xrightarrow{K_\bullet} & \mathcal{G}, \end{array}$$

and for which $K_0 \circ \eta = \eta \circ K_0$, $K_\bullet \circ \epsilon = \epsilon \circ K_\bullet$ and $K_0 i = K$. In paticular, if $K = Id$, we have two squares,

$$\begin{array}{ccc} \Delta_+^o \odot \mathcal{E} & \xrightarrow{I_0} & \mathcal{E} \\ {\scriptstyle E^o}\downarrow & & \downarrow{\scriptstyle U} \\ \Delta'^o \odot \mathcal{E} & \xrightarrow{I_\bullet} & \mathcal{G} \end{array} \qquad \begin{array}{ccc} \Delta_+^o \odot \mathcal{E} & \xrightarrow{I_0} & \mathcal{E} \\ {\scriptstyle V^o}\uparrow & & \uparrow{\scriptstyle F} \\ \Delta'^o \odot \mathcal{E} & \xrightarrow{I_\bullet} & \mathcal{G}, \end{array}$$

By exponential adjointness we obtain two commutative squares,

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{I^0} & [\Delta_+^o, \mathcal{E}] \\ {\scriptstyle I^\bullet}\downarrow & & \downarrow{\scriptstyle [\Delta_+^o, U]} \\ [\Delta'^o, \mathcal{G}] & \xrightarrow{[E, \mathcal{G}]} & [\Delta_+^o, \mathcal{G}] \end{array} \qquad \begin{array}{ccc} \mathcal{E} & \xrightarrow{I^0} & [\Delta_+^o, \mathcal{E}] \\ {\scriptstyle I^\bullet}\downarrow & & \downarrow{\scriptstyle [V, \mathcal{E}]} \\ [\Delta'^o, \mathcal{G}] & \xrightarrow{[\Delta'^o, F]} & [\Delta'^o, \mathcal{E}] \end{array}$$

The functor I^0 associates to an object $A \in \mathcal{E}$ the augmented simplicial object $(C^n(A) : n \geq 0)$, where C is the comonad $FU : \mathcal{E} \to \mathcal{E}$. The functor $I^\bullet$ associates to A a split augmented simplicial object $(UC^n(A) : n \geq 0)$. The splitting is given by the sequence of maps

$$UA \xrightarrow{\eta \circ U} UCA \xrightarrow{\eta \circ UC} UC^2A \xrightarrow{\eta \circ UC^2} \cdots .$$

Let us denote by $C_\star(A)$ the simplicial object obtained by putting $C_n(A) = C^{n+1}(A)$ for every $n \geq 0$. The augmentation $\epsilon : C_\star(A) \to A$ is given by the map $\epsilon : C_0(A) \to A$. It can be viewed as a map from $C_\star(A)$ to the constant simplicial object A.

Lemma 6.2 *The map $U(\epsilon) : UC_\star(A) \to UA$ is a homotopy equivalence.*

Proof: This follows from 6.1. ∎

6.2 Homotopical algebra

The goal of this appendix is to review the basic homotopical algebra needed in the paper and to introduce some notation.

We shall denote by $Ob\mathcal{C}$ the class of objects of a category $\mathcal{C}$ and by $\mathcal{C}(A, B)$ the set of arrows between two objects of $\mathcal{C}$.

We shall denote by **Set** the category of sets and by **Cat** the category of small categories. If A is a small category and $\mathcal{E}$ is a category (possibly large) we shall denote the category of functors $A \to \mathcal{E}$ by $[A, \mathcal{E}]$ or by $\mathcal{E}^A$. Recall that a *presheaf* on a small category A is a contravariant functors $A \to \mathbf{Set}$. We shall denote by $\hat{A}$ the category $[A^o, \mathbf{Set}]$ of presheaves on A. We shall regard the Yoneda functor $y : A \to \hat{A}$ as an inclusion by adopting the same notation for an object $a \in A$ and the representable functor $y(a) = A(-, a)$. For every functor $u : A \to \mathcal{E}$, we shall denote by $u^! : \mathcal{E} \to \hat{A}$ the functor obtained by putting $u^!(X)(a) = \mathcal{E}(u(a), X)$ for every object $X \in \mathcal{E}$ and every object $a \in A$. If the category $\mathcal{E}$ is cocomplete, the functor $u^!$ has a left adjoint $u_!$. The functor $u_! : \hat{A} \to \mathcal{E}$ is the left Kan extension of the functor u along the Yoneda functor $A \to \hat{A}$. If B is a small category and $u : A \to B$, we shall denote by $u^* : \hat{B} \to \hat{A}$ the functor obtained by putting $u^*(X) = Xu$ for every presheaf $X \in \hat{B}$. The functor u^* has a left adjoint denoted $u_!$ and a right adjoint denoted u_*.

We denote by Δ the category whose objects are the finite non-empty ordinals and whose arrows are the order preserving maps. The ordinal $n+1$ is represented by the ordered set $[n] = \{0, \ldots, n\}$, so that $Ob\Delta = \{[n] : n \geq 0\}$. A *simplicial set* is a presheaf on Δ. If X is a simplicial set, the set $X([n])$ is denoted by X_n for every $n \geq 0$. We denote the category of simplicial sets $\hat{\Delta}$ by $\mathbf{S}$. Recall that the simplex $\Delta[n]$ is defined to be the representable functor $\Delta(-, [n])$. We shall denote its boundary by $\partial\Delta[n]$. A category enriched over $\mathbf{S}$ is called a *simplicial category*.

Let $c : \mathbf{Set} \to \mathbf{S}$ be the functor which associate to a set S the constant simplicial set cS obtained by putting $(cS)_n = S$ for every $n \geq 0$. The functor c is full and faithful and we shall regard it as an inclusion $\mathbf{Set} \subset \mathbf{S}$ by adopting the same notation for S and cS. The functor c has a left adjoint

$$\pi_0 : \mathbf{S} \to \mathbf{Set},$$

where $\pi_0(X)$ is the set of connected components of a simplicial set X.

If $u : A \to B$ and $f : X \to Y$ are two maps in a category $\mathcal{E}$, we write $u \pitchfork f$ to indicate that f has the right lifting property with respect to u. We recall that this means that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & X \\ {\scriptstyle u}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & Y \end{array}$$

has a *diagonal filler* $d : B \to X$ (that is, $du = x$ and $fd = y$). If S is an object of $\mathcal{E}$, we write $u \pitchfork S$ to indicate that the map $\mathcal{E}(u, S) : \mathcal{E}(B, S) \to \mathcal{E}(A, S)$ is surjective and we write $S \pitchfork f$ to indicate that the map $\mathcal{E}(S, f) : \mathcal{E}(S, A) \to \mathcal{E}(S, B)$ is surjective. If $\mathcal{E}$ has a terminal object $\top$, the condition $u \pitchfork S$ is equivalent to the condition $u \pitchfork t_S$, where t_S is the map $S \to \top$. If $\mathcal{E}$ has an initial object $\bot$, the condition $S \pitchfork f$ is equivalent to the condition $i_S \pitchfork f$, where i_S is the map $\bot \to S$.

For any class of maps $\mathcal{M} \subseteq \mathcal{E}$, we denote by ${}^{\pitchfork}\mathcal{M}$ (resp. $\mathcal{M}^{\pitchfork}$) the class of maps having the left (resp. right) lifting property with respect to every map in $\mathcal{M}$. If $\mathcal{A}$ and $\mathcal{B}$ are two classes of maps in $\mathcal{E}$, we write $\mathcal{A} \pitchfork \mathcal{B}$ to indicate that we have $a \pitchfork b$ for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then

$$\mathcal{A} \subseteq {}^{\pitchfork}\mathcal{B} \iff \mathcal{A} \pitchfork \mathcal{B} \iff \mathcal{B} \subseteq \mathcal{A}^{\pitchfork}.$$

If $F : \mathcal{D} \leftrightarrow \mathcal{E} : G$ is a pair of adjoint functors, then for an arrow $u \in \mathcal{D}$ and an arrow $f \in \mathcal{V}$ we have

$$u \pitchfork G(f) \quad \Longleftrightarrow \quad F(u) \pitchfork f.$$

The verification of the following result is left to the reader. See [J2].

Proposition 6.3 *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of two variables divisible on both sides, where $\mathcal{E}_i$ is a finitely bicomplete category for $i = 1, 2, 3$. If $u \in \mathcal{E}_1$, $v \in \mathcal{E}_2$ and $f \in \mathcal{E}_3$, then*

$$(u \odot' v) \pitchfork f \quad \Longleftrightarrow \quad u \pitchfork \langle f/v \rangle \quad \Longleftrightarrow \quad v \pitchfork \langle u \backslash f \rangle.$$

Definition 6.4 *We shall say that a pair $(\mathcal{A}, \mathcal{B})$ of classes of maps in a category $\mathcal{E}$ is a weak factorisation system if the following conditions are satisfied:*

- *every map $f \in \mathcal{E}$ admits a factorisation $f = pi$ with $i \in \mathcal{A}$ and $p \in \mathcal{B}$;*
- *$\mathcal{B} = \mathcal{A}^{\pitchfork}$ and $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$.*

We call $\mathcal{A}$ is the left class and $\mathcal{B}$ the right class of the weak factorisation system.

Definition 6.5 *We shall say that a map in a topos is a trivial fibration if it has the right lifting property with respect to every monomorphism.*

This terminology is non-standard but it is useful. The trivial fibrations often coincide with the acyclic fibrations (which can be defined in any model category).

Proposition 6.6 [GZ] *A map of of simplicial sets is a trivial fibration iff it has the right lifting property with respect to the inclusion $\delta_n : \partial\Delta[n] \subset \Delta[n]$ for every $n \geq 0$.*

Proposition 6.7 [J2] *If $\mathcal{A}$ is the class of monomorphisms in a topos and $\mathcal{B}$ is the class of trivial fibrations, then the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorisation system.*

Recall that a map $u : A \to B$ in a category $\mathcal{E}$ is said to be a *retract* of a map $f : X \to Y$ if u is a retract of f as objects of the category of arrows $\mathcal{E}^I$. Recall that a map $u : A \to B$ is called a *domain retract* of a map $v : C \to B$, if u is a retract of v as objects of the category $\mathcal{E}/B$. There is a dual notion of codomain retract. The two classes of a weak factorisation system are closed under retracts.

Definition 6.8 *We shall say that a class $\mathcal{A}$ of maps in a cocomplete category $\mathcal{E}$ is* saturated *if it contains the isomorphisms and is closed under composition, transfinite composition, cobase change and codomain retracts.*

The class ${}^{\pitchfork}\mathcal{M}$ is saturated for any class $\mathcal{M} \subseteq \mathcal{E}$. In particular, the class $\mathcal{A}$ of a weak factorisation system $(\mathcal{A}, \mathcal{B})$ in $\mathcal{E}$ is saturated. Every class of maps $\mathcal{M} \subseteq \mathcal{E}$ is contained in a smallest saturated class $\overline{\mathcal{M}} \subseteq \mathcal{E}$ called the *saturated class generated* by $\mathcal{M}$.

The following proposition is a special case of a more general result, see [J2]:

Proposition 6.9 *If Σ is a set of maps in a presheaf category, then the pair $(\overline{\Sigma}, \Sigma^{\pitchfork})$ is a weak factorisation system.*

Recall that a monoidal category $\mathcal{E} = (\mathcal{E}, \otimes)$ is said to be *closed* if the tensor product $\otimes$ is divisible on each side. Let $\mathcal{E} = (\mathcal{E}, \otimes, \sigma)$ be a *symmetric* monoidal closed category, with symmetry $\sigma : A \otimes B \simeq B \otimes A$. Then the objects X/A and $A \backslash X$ are canonicaly isomorphic; we can identify them by adopting a common notation, for example $[A, X]$.

Recall that a category with finite products $\mathcal{E}$ is said to be *cartesian closed* if the functor $A \times - : \mathcal{E} \to \mathcal{E}$ admits a right adjoint $(-)^A$ for every object $A \in \mathcal{E}$. A cartesian closed category $\mathcal{E}$ is symmetric monoidal closed. Every presheaf category and more generally every topos is cartesian closed.

We now recall the notion of a Quillen model category:

Definition 6.10 [Q] *Let $\mathcal{E}$ be a finitely bicomplete category. A* model structure *on $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes of maps in $\mathcal{E}$ satisfying the following conditions:*

- *("three-for-two") if two of the three maps $u : A \to B$, $v : B \to C$ and $vu : A \to C$ belong to $\mathcal{W}$, then so does the third;*
- *the pair $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system;*
- *the pair $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a weak factorisation system.*

These conditions imply that $\mathcal{W}$ is closed under retracts by 6.11 below. A category $\mathcal{E}$ equipped with a model structure is called a *model category*. A map in $\mathcal{C}$ is called a *cofibration*, a map in $\mathcal{F}$ a *fibration* and a map in $\mathcal{W}$ a *weak equivalence*. A map in $\mathcal{W}$ is also said to be *acyclic*. An object $X \in \mathcal{E}$ is *fibrant* if the map $X \to 1$ is a fibration, where 1 is the terminal object of $\mathcal{E}$. Dually, an object $A \in \mathcal{E}$ is *cofibrant* if the map $0 \to A$ is a cofibration, where 0 is the initial object of $\mathcal{E}$.

Any two of the three classes of a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ determine the third.

A model structure is said to be *left proper* if the cobase change of an acyclic map along a cofibration is acyclic. Dually, a model structure is said to be *right proper* if the base change of an acyclic map along a fibration is acyclic. A model structure is *proper* if it is both left and right proper.

Proposition 6.11 [JT2] *The class of weak equivalence of a model category is closed under retracts.*

Lemma 6.12 *In a model category, a cofibration is acyclic iff it has the left lifting property with respect to every fibration between fibrant objects.*

Proof: The necessity is clear. Conversely, let us suppose that a cofibration $u : A \to B$ has the left lifting property with respect to every fibration between fibrant objects. We shall prove that u is acyclic. For this, let us choose a fibrant replacement $j : B \to B'$ of the object B together with a factorisation of the composite $ju : A \to B'$ as a weak equivalence $i : A \to A'$ followed by a fibration $p : A' \to B$. The square

$$\begin{array}{ccc} A & \xrightarrow{i} & A' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{j} & B' \end{array}$$

has a diagonal filler $d : B \to A'$ since p is a fibration between fibrant objects. The arrows i and j are invertible in the homotopy category since they are acyclic. The relations $pd = j$ and $du = i$ then implies that d is invertible in the homotopy category. It thus acyclic [Q]. It follows by three-for-two that u is acyclic. ∎

The *homotopy category* of a model category $\mathcal{E}$ is defined to be the category of fractions $Ho(\mathcal{E}) = \mathcal{W}^{-1}\mathcal{E}$. We shall denote by $[u]$ the image of an arrow $u \in \mathcal{E}$ by the canonical functor $\mathcal{E} \to Ho(\mathcal{E})$. The arrows $[u]$ is invertible iff u is a weak equivalence by a result in [Q].

Let $\mathcal{E}_f$ (resp. $\mathcal{E}_c$) be the full sub-category of fibrant (resp. cofibrant) objects of $\mathcal{E}$ and let us put $\mathcal{E}_{fc} = \mathcal{E}_f \cap \mathcal{E}_c$. Let us put $Ho(\mathcal{E}_f) = \mathcal{W}_f^{-1}\mathcal{E}_f$ where $\mathcal{W}_f = \mathcal{W} \cap \mathcal{E}_f$ and similarly for $Ho(\mathcal{E}_c)$ and $Ho(\mathcal{E}_{fc})$. . Then the diagram of inclusions

$$\begin{array}{ccc} \mathcal{E}_{fc} & \longrightarrow & \mathcal{E}_f \\ \downarrow & & \downarrow \\ \mathcal{E}_c & \longrightarrow & \mathcal{E} \end{array}$$

induces a diagram of equivalences of categories

$$\begin{array}{ccc} Ho(\mathcal{E}_{fc}) & \longrightarrow & Ho(\mathcal{E}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{E}_c) & \longrightarrow & Ho(\mathcal{E}). \end{array}$$

A *fibrant replacement* of an object $X \in \mathcal{E}$ is a weak equivalence $X \to RX$ with codomain a fibrant object. Dually, a *cofibrant replacement* of X is a weak equivalence $LX \to X$ with domain a cofibrant object.

Recall from [Ho] that a cocontinuous functor $F : \mathcal{U} \to \mathcal{V}$ between two model categories is said to be a *left Quillen functor* if it takes a cofibration to a cofibration and an acyclic cofibration to and acyclic cofibration. Dually, a continuous functor $G : \mathcal{V} \to \mathcal{U}$ between two model categories is said to be a *right Quillen functor* if it takes a fibration to a fibration and an acyclic fibration to an acyclic fibration.

Proposition 6.13 [Q] *Let $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ be an adjoint pair of functors between two model categories. Then F is a left Quillen functor iff G is a right Quillen functor.*

The adjoint pair (F, G) is said to be a *Quillen pair* if the conditions of 6.13 are satisfied.

The following lemma is due to Ken Brown, see [Ho] and [JT2].

Lemma 6.14 *Let $\mathcal{E}$ be a model category and $F : \mathcal{E} \to \mathcal{D}$ be a functor with values in a category equipped with a class of a weak equivalences $\mathcal{W}'$ which satisfies three-for-two. If F takes an acyclic cofibration between cofibrant objects to a weak equivalence, then it takes a weak equivalence between cofibrant objects to a weak equivalence.*

Corollary 6.15 *A left Quillen functor takes a weak equivalence between cofibrant objects to a weak equivalence.*

The following result is due to Reedy [Ree]. See [Hi] and [JT2].

Proposition 6.16 *The cobase change along a cofibration of a weak equivalence between cofibrant objects is a weak equivalence.*

Corollary 6.17 *If every object of model category is cofibrant then the model structure is left proper.*

Proposition 6.18 *An adjoint pair of functors $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ between two model categories is a Quillen pair iff the following two conditions are satisfied:*

- *F takes a cofibration to a cofibration;*
- *G takes a fibration between fibrant objects to a fibration.*

Proof: The necessity is obvious. Let us prove the sufficiency. For this it suffices to show that F is a left Quillen functor by 6.13. Thus we show that F takes an acyclic cofibration $u : A \to B$ to an acyclic cofibration $F(u) : F(A) \to F(B)$. But $F(u)$ is acyclic iff it has the left lifting property with respect to every fibration between fibrant objects $f : X \to Y$ by Lemma 6.12. But the condition $F(u) \pitchfork f$ is equivalent to the condition $u \pitchfork G(f)$ by the adjointness $F \dashv G$. We have $u \pitchfork G(f)$ since $G(f)$ is a fibration by (ii). This proves that we have $F(u) \pitchfork f$. Thus, $F(u)$ is acyclic. ∎

A left Quillen functor $F : \mathcal{U} \to \mathcal{V}$ induces a functor $F_c : \mathcal{U}_c \to \mathcal{V}_c$ hence also a functor $Ho(F_c) : Ho(\mathcal{U}_c) \to Ho(\mathcal{V}_c)$ by Proposition 6.15. A *left derived functor* is a functor

$$F^L : Ho(\mathcal{U}) \to Ho(\mathcal{V})$$

for which the following diagram of functors commutes up to isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{U}_c) & \xrightarrow{Ho(F_c)} & Ho(\mathcal{V}_c) \\ \downarrow & & \downarrow \\ Ho(\mathcal{U}) & \xrightarrow{F^L} & Ho(\mathcal{V}), \end{array}$$

The functor F^L is unique up to a canonical isomorphism. It can be computed as follows. For each object $A \in \mathcal{U}$, we can choose a cofibrant replacement $\lambda_A : LA \to A$, with λ_A an acyclic fibration. We can then choose for each arrow $u : A \to B$ an arrow $L(u) : LA \to LB$ such that $u\lambda_A = \lambda_B L(u)$,

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_A} & A \\ {\scriptstyle L(u)}\downarrow & & \downarrow{\scriptstyle u} \\ LB & \xrightarrow{\lambda_B} & B. \end{array}$$

Then

$$F^L([u]) = [F(L(u))] : FLA \to FLB.$$

A right Quillen functor $G : \mathcal{V} \to \mathcal{U}$ induces a functor $G_f : \mathcal{V}_f \to \mathcal{U}_f$ hence also a functor $Ho(G_f) : Ho(\mathcal{V}_f) \to Ho(\mathcal{U}_f)$ by Proposition 6.15. The *right derived functor* is a functor

$$G^R : Ho(\mathcal{V}) \to Ho(\mathcal{U})$$

for which the following diagram of functors commutes up to a canonical isomorphism,

$$\begin{array}{ccc} Ho(\mathcal{V}_f) & \xrightarrow{Ho(G_f)} & Ho(\mathcal{U}_f) \\ \downarrow & & \downarrow \\ Ho(\mathcal{V}) & \xrightarrow{G^R} & Ho(\mathcal{U}). \end{array}$$

The functor G^R is unique up to a canonical isomorphism. It can be computed as follows. For each object $X \in \mathcal{V}$ let us choose a fibrant replacement $\rho_X : X \to RX$, with ρ_X an acyclic cofibration. We can then choose for each arrow $u : X \to Y$ an arrow $R(u) : RX \to RY$ such that $R(u)\rho_X = \rho_Y u$,

$$\begin{array}{ccc} X & \xrightarrow{\rho_X} & RX \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle R(u)} \\ Y & \xrightarrow{\rho_Y} & RY. \end{array}$$

Then

$$G^R([u]) = [G(R(u))] : GRX \to GRY.$$

A Quillen pair of adjoint functors $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ induces a pair of adjoint functors

$$F^L : Ho(\mathcal{U}) \leftrightarrow Ho(\mathcal{V}) : G^R.$$

If $A \in \mathcal{U}$ is cofibrant, the adjunction unit $A \to G^R F^L(A)$ is obtained by composing the maps $A \to GFA \to GRFA$, where $FA \to RFA$ is a fibrant replacement of FA. If $X \in \mathcal{V}$ is fibrant, the adjunction counit $F^L G^R(X) \to X$ is obtained by composing the maps $FLGX \to FGX \to X$, where $LGX \to GX$ is a cofibrant replacement of GX.

Definition 6.19 *We shall say that a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a* homotopy reflection *$\mathcal{U} \to \mathcal{V}$ if the right derived functor G^R is full and faithful. Dually, we shall say that the pair (F, G) is a homotopy coreflection $\mathcal{V} \to \mathcal{U}$ if the left derived functor F^L is full and faithful.*

Proposition 6.20 *The following conditions on a Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ are equivalent:*

- *The pair (F, G) is a homotopy reflection $\mathcal{U} \to \mathcal{V}$;*
- *The map $FLGX \to X$ is a weak equivalence for every fibrant object $X \in \mathcal{V}$, where $LGX \to GX$ denotes a cofibrant replacement of GX;*
- *The map $FLGX \to X$ is a weak equivalence for every fibrant-cofibrant object $X \in \mathcal{V}$, where $LGX \to GX$ denotes a cofibrant replacement of GX.*

Proof: The functor G^R is full and faithful iff the counit of the adjunction $F^L \dashv G^R$ is an isomorphism. But if $X \in \mathcal{V}$ is fibrant, this counit is obtained by composing the maps $FLGX \to FGX \to X$, where $LGX \to GX$ is a cofibrant replacement of GX. This proves the equivalence (i)⇔(ii). The implication (ii)⇒(iii) is obvious. Let us prove the implication (iii)⇒(ii). For every fibrant objet X, there is a an acyclic fibration $p : Y \to X$ with domain a cofibrant object Y. The map $Gp : GY \to GX$ is an acyclic fibration, since G is a right Quillen functor. Let $q : LGY \to GY$ be a cofibrant replacement of GY. Then the map $FLGY \to FGY \to Y$ is a weak equivalence by assumption, since Y is fibrant-cofibrant. But the composite $G(p)q : LGY \to GY \to GX$ is a cofibrant replacement of GX, since $G(p)$ is a weak equivalence. Moreover, the composite $FLGY \to FGX \to X$ is a weak equivalence, since p is a weak equivalence and the following diagram commutes

$$\begin{array}{ccccc} FLGY & \longrightarrow & FGY & \longrightarrow & Y \\ & \searrow & \downarrow & & \downarrow p \\ & & FGX & \longrightarrow & X. \end{array}$$

This proves that condition (ii) is satisfied for a cofibrant replacement of GX. ∎

Proposition 6.21 *If $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a homotopy reflection, then the right adjoint G preserves and reflects weak equivalences between fibrant objects.*

Proof: The functor G^R is equivalent to the functor $Ho(G_f) : Ho(\mathcal{V}_f) \to Ho(\mathcal{U}_f)$ induced by the functor G. Thus, $Ho(G_f)$ is full and faithful since G^R is full and faithful. This proves the result since a full and faithful functor is conservative. ∎

Proposition 6.22 *Let $M_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. Suppose that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Then the identity functor $\mathcal{E} \to \mathcal{E}$ is a homotopy reflection $M_1 \to M_2$.*

Definition 6.23 *Let $M_i = (\mathcal{C}_i, \mathcal{W}_i, \mathcal{F}_i)$ $(i = 1, 2)$ be two model structures on a category $\mathcal{E}$. If $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$, we shall say that M_2 is a* Bousfield localisation *of M_1.*

Proposition 6.24 *Let $M_2 = (\mathcal{C}_2, \mathcal{W}_2, \mathcal{F}_2)$ be a Bousfield localisation of a model structure $M_1 = (\mathcal{C}_1, \mathcal{W}_1, \mathcal{F}_1)$ on a category $\mathcal{E}$. Then a map between M_2-fibrant objects is a M_2-fibration iff it is a M_1-fibration.*

Proof: By hypothesis, we have $\mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{W}_1 \subseteq \mathcal{W}_2$. It follows that we have $\mathcal{F}_2 \cap \mathcal{W}_2 = \mathcal{F}_1 \cap \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Let $f : X \to Y$ be a map between two M_2-fibrant objects. Let us show that f is a M_2-fibration iff it is a M_1-fibration. The implication ($\Rightarrow$) is clear, since $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Conversely, if $f \in \mathcal{F}_1$, let us show that $f \in \mathcal{F}_2$. Let us choose a factorisation $f = pi : X \to Z \to Y$ with $i \in \mathcal{C}_2 \cap \mathcal{W}_2$ and $p \in \mathcal{F}_2$. We have $i \in \mathcal{W}_1$ by Proposition 6.15, since the identity functor is a right Quillen functor $M_2 \to M_1$ and since i is a map between M_2-fibrant objects. Thus, $i \in \mathcal{W}_1 \cap \mathcal{C}_1$, since $\mathcal{C}_1 = \mathcal{C}_2$. Hence the square

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[p]{} & Y \end{array}$$

has a diagonal filler, making f a retract of p and therefore $f \in \mathcal{F}_2$. ∎

A Quillen pair (F, G) is said to be a *Quillen equivalence* if the adjoint pair (F^L, G^R) is an equivalence of categories.

Proposition 6.25 *A Quillen pair $F : \mathcal{U} \leftrightarrow \mathcal{V} : G$ is a Quillen equivalence iff the following equivalent conditions are satisfied:*

- *The pair (F, G) is a both a homotopy reflection and coreflection;*
- *The pair (F, G) is a homotopy reflection and the functor F reflects weak equivalences between cofibrant objects;*
- *The pair (F, G) is a homotopy coreflection and the functor G reflects weak equivalences between fibrant objects;*

The composite of two adjoint pairs

$$F_1 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_2 : G_1 \quad \text{and} \quad F_2 : \mathcal{E}_2 \leftrightarrow \mathcal{E}_3 : G_2$$

is an adjoint pair $F_2F_1 : \mathcal{E}_1 \leftrightarrow \mathcal{E}_3 : G_1G_2$.

Proposition 6.26 (Three-for-two) [Ho] *The composite of two Quillen pairs (F_1, G_1) and (F_2, G_2) is a Quillen pair (F_2F_1, G_1G_2). Moreover, if two of the pairs (F_1, G_1), (F_2, G_2) and (F_2F_1, G_1G_2) are Quillen equivalences, then so is the third.*

Definition 6.27 Ho *We shall say that a functor of two variables between three model categories*

$$\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$$

is a left Quillen functor *if it is cocontinuous in each variable and the following conditions are satisfied:*

- *$u \odot' v$ is a cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations;*
- *$u \odot' v$ is an acyclic cofibration if $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations and if u or v is acyclic.*

Dually, we shall say that $\odot$ is a right Quillen functor *if the opposite functor $\odot^o : \mathcal{E}_1^o \times \mathcal{E}_2^o \to \mathcal{E}_3^o$ is a left Quillen functor.*

Proposition 6.28 *Let $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a left Quillen functor of two variables between three model categories. If $A \in \mathcal{E}_1$ is cofibrant, then the functor $B \mapsto A \odot B$ is a left Quillen functor $\mathcal{E}_2 \to \mathcal{E}_3$.*

Proof: If $A \in \mathcal{E}_1$ is cofibrant, then the map $i_A : \bot \to A$ is a cofibration, where $\bot$ is the initial object. If $v : S \to T$ is a map in $\mathcal{E}_2$, then we have $A \odot v = i_A \odot' v$. Thus, $A \odot v$ is a cofibration if v is a cofibration and $A \odot v$ is acyclic if moreover v is acyclic. ∎

Proposition 6.29 *Let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ be three model categories. A functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ divisible on the left is a left Quillen functor iff the corresponding left division functor $\mathcal{E}_1^o \times \mathcal{E}_3 \to \mathcal{E}_2$ is a right Quillen functor. Dually, a functor of two variables $\odot$ divisible on the right is a left Quillen functor iff the corresponding right division functor $\mathcal{E}_3 \times \mathcal{E}_2^o \to \mathcal{E}_1$ is a right Quillen functor.*

Proposition 6.30 *Let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ be three model categories. Suppose that a functor of two variables $\odot : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ satisfies the following conditions:*

- *it is cocontinuous in each variable;*
- *If $u \in \mathcal{E}_1$ and $v \in \mathcal{E}_2$ are cofibrations, then so is $u \odot' v$;*
- *the functor $(-) \odot B$ preserves acyclic cofibrations for every object $B \in \mathcal{E}_2$;*
- *the functor $A \odot (-)$ preserves acyclic cofibrations for every object $A \in \mathcal{E}_1$.*

Then $\odot$ is a left Quillen functor.

Proof: Let $u : A \to B$ be a cofibration in $\mathcal{E}_1$ and $v : S \to T$ be a cofibration in $\mathcal{E}_2$. Let us show that $u \odot' v$ is acyclic if u or v is acyclic. We only consider the case where v is acyclic. Consider the commutative diagram

$$\begin{array}{ccccc} A \odot S & \xrightarrow{u \odot S} & B \odot S & & \\ {\scriptstyle A \odot v}\downarrow & & \downarrow{\scriptstyle i_2} & \searrow^{B \odot v} & \\ A \odot T & \xrightarrow{i_1} & Z & \xrightarrow{u \odot' v} & B \odot T \end{array}$$

where $Z = A \odot T \sqcup_{A \odot S} B \odot S$ and where $(u \odot' v) i_1 = u \odot T$. The map $A \odot v$ is an acyclic cofibration since v is an acyclic cofibration. Similarly for the map $B \odot v$. It follows that i_2 is an acyclic cofibration by cobase change. Thus, $u \odot' v$ is acyclic by three-for-two since $(u \odot' v) i_2 = B \odot v$ is acyclic.

Definition 6.31 [Ho] *A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a monoidal closed category $\mathcal{E} = (\mathcal{E}, \otimes)$ with unit object U is said to be* monoidal *if the tensor product $\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the object U is cofibrant.*

In a monoidal closed model category, if f is a fibration then so are the maps $\langle u \backslash f \rangle$ and $\langle f / u \rangle$ for any cofibration u. Moreover, the maps $\langle u \backslash f \rangle$ and $\langle f / u \rangle$ are acyclic if in addition u or f is acyclic.

Definition 6.32 *We shall say that a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a cartesian closed category $\mathcal{E}$ is* cartesian closed *if the cartesian product $\times : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen functor of two variables and if the terminal object 1 is cofibrant.*

We recall a few notions of enriched category theory [K]. Let $\mathcal{V} = (\mathcal{V}, \otimes, \sigma)$ a bicomplete symmetric monoidal closed category. A category enriched over $\mathcal{V}$ is called a *$\mathcal{V}$-category*. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, there is a notion of a *strong* functor $F : \mathcal{A} \to \mathcal{B}$; it is an ordinary functor equipped with a *strength* which is a natural transformation $\mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$ preserving composition and units. A natural transformation $\alpha : F \to G$ between strong functors $F, G : \mathcal{A} \to \mathcal{B}$ is said to be *strong* if the following square commutes

$$\begin{array}{ccc} \mathcal{A}(X, Y) & \longrightarrow & \mathcal{B}(GX, GY) \\ \downarrow & & \downarrow{\scriptstyle \mathcal{B}(\alpha_X, GY)} \\ \mathcal{B}(FX, FY) & \xrightarrow{\mathcal{B}(FX, \alpha_Y)} & \mathcal{B}(FX, GY). \end{array}$$

for every pair of objects $X, Y \in \mathcal{A}$. A *strong adjunction* $\theta : F \dashv G$ between strong functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ is a strong natural isomorphism

$$\theta_{XY} : \mathcal{A}(FX, Y) \to \mathcal{B}(X, GY).$$

A strong functor $G : \mathcal{B} \to \mathcal{A}$ has a strong left adjoint iff it has an ordinary left adjoint $F : \mathcal{A} \to \mathcal{B}$ and the map

$$\mathcal{B}(FX, Y) \longrightarrow \mathcal{A}(GFX, GY) \xrightarrow{\mathcal{A}(\eta_X, GY)} \mathcal{A}(X, GY)$$

obtained by composing with the unit η_X of the adjunction is an isomorphism for every pair of objects $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. Recall that a $\mathcal{V}$-category $\mathcal{E}$ is said to admit *tensor products* if the functor $Y \mapsto \mathcal{E}(X, Y)$ admits a strong left adjoint $A \mapsto A \otimes X$ for every object $X \in \mathcal{E}$. A $\mathcal{V}$-category is said to be (strongly) *cocomplete* if it cocomplete as an ordinary category and if it admits tensor products. These notions can be dualised. A $\mathcal{V}$-category $\mathcal{E}$ is said to admit *cotensor products* if the opposite $\mathcal{V}$-category $\mathcal{E}^o$ admits tensor products. This means that the (contravariant) functor $X \mapsto \mathcal{E}(X, Y)$ admits a strong right adjoint $A \mapsto Y^{[A]}$ for every object $Y \in \mathcal{E}$. A $\mathcal{V}$-category is said to be (strongly) *complete* if it complete as an ordinary category and if it admits cotensor products. We shall say that a $\mathcal{V}$-category is (strongly) *bicomplete* if it is both $\mathcal{V}$-complete and cocomplete.

Definition 6.33 [Q]*Let $\mathcal{E}$ be a strongly bicomplete simplicial category. We shall say that a model structure on $\mathcal{E}$ is* simplicial *if the hom functor*

$$\mathcal{E}^o \times \mathcal{E} \to \mathbf{S}$$

is a right Quillen functor of two variables, where $\mathbf{S}$ is equipped with its classical model structure.

A simplicial category equipped with a simplicial model structure is called a *simplicial model category.*

Proposition 6.34 *Let $\mathcal{E}$ be a simplicial model category. Then a map between cofibrant objects $u : A \to B$ is acyclic iff the map of simplicial sets*

$$\mathcal{E}(u, X) : \mathcal{E}(B, X) \to \mathcal{E}(A, X)$$

is a weak homotopy equivalence for every fibrant object X.

Proof: When X is fibrant, the functor $A \mapsto \mathcal{E}(A, X)$ takes an (acyclic) cofibration to an (acyclic) Kan fibration. It follows by Proposition 6.15 that it takes an acyclic map between cofibrant objects to an acyclic map. Conversely, let $u : A \to B$ be a map between cofibrant objects in $\mathcal{E}$. If the map $\mathcal{E}(u, X) : \mathcal{E}(B, X) \to \mathcal{E}(A, X)$ is a weak homotopy equivalence for every fibrant object X, let us show that u is acyclic. Let us first suppose that A and B are fibrant. Let $\mathcal{E}_{cf}$ be the full subcategory of fibrant and cofibrant objects of $\mathcal{E}$. We shall prove that u is acyclic by showing that u is invertible in the homotopy category $Ho(\mathcal{E}_{cf})$. But if $S, X \in \mathcal{E}_{cf}$, then we have $Ho(\mathcal{E}_{cf})(S,X) = \pi_0\mathcal{E}(S,X)$ by [Q]. Hence the map $Ho(\mathcal{E}_{cf})(u,X) : Ho(\mathcal{E}_{cf})(B,X) \to Ho(\mathcal{E}_{cf})(A,X)$ is equal to the map $\pi_0\mathcal{E}(u,X) : \pi_0\mathcal{E}(B,X) \to \pi_0\mathcal{E}(A,X)$. But the map $\pi_0\mathcal{E}(u,X)$ is bijective since the map $\mathcal{E}(u,X)$ is a weak homotopy equivalence. This shows that the map $Ho(\mathcal{E}_{cf})(u,X)$ is bijective for every $X \in \mathcal{E}_{cf}$. It follows by the Yoneda lemma that u is invertible in $Ho(\mathcal{E}_{cf})$. Thus, u is acyclic by [Q]. In the general case, let us choose a fibrant replacement $i_A : A \to A'$ with i_A an acyclic cofibration. Similarly, let us choose a fibrant replacement $i_B : B \to B'$ with i_B an acyclic cofibration. Then there exists a map $u' : A' \to B'$ such that $u'i_A = i_B u$. We then have a commutative square

$$\begin{array}{ccc} \mathcal{E}(B',X) & \longrightarrow & \mathcal{E}(A',X) \\ \downarrow & & \downarrow \\ \mathcal{E}(B,X) & \longrightarrow & \mathcal{E}(A,X) \end{array}$$

for every object X. If X is fibrant, then the vertical maps of the square are weak homotopy equivalences by the first part of the proof. Hence also the map $\mathcal{E}(u',X) : \mathcal{E}(B',X) \to \mathcal{E}(A',X)$ by three-for-two. This shows that u' is an acyclic map since A' and B' are fibrant. It follows by three-for-two that u is acyclic. ■

Proposition 6.35 *Let $\mathcal{E}$ be a simplicial model category in which the class of cofibrations is generated by a subclass $\mathcal{A}$ of cofibrations between cofibrant objects. Then a map between fibrant objects $f : X \to Y$ is acyclic iff the following square of simplicial sets*

$$\begin{array}{ccc} \mathcal{E}(B,X) & \longrightarrow & \mathcal{E}(A,X) \\ \downarrow & & \downarrow \\ \mathcal{E}(B,Y) & \longrightarrow & \mathcal{E}(A,Y) \end{array}$$

is homotopy cartesian for every map $u : A \to B$ in the class $\mathcal{A}$.

Proof: The necessity is clear. Let us prove the sufficiency. By factoring the map $f : X \to Y$ as a weak equivalence followed by a fibration, we can suppose that f is a fibration. If $u \in \mathcal{A}$, then the map

$$\langle f, u\rangle : \mathcal{E}(B,X) \to \mathcal{E}(B,Y) \times_{\mathcal{E}(A,Y)} \mathcal{E}(A,X)$$

is a Kan fibration, since u is a cofibration. It is thus a trivial fibration, since it is a weak homotopy equivalence by the assumption on f. Hence we have $u \pitchfork f$, since a trivial fibration is surjective. It follows that we have $u \pitchfork f$ for every cofibration, since the class of cofibrations is generated by the class $\mathcal{A}$. ■

Proposition 6.36 *Let $\mathcal{D}$ and $\mathcal{E}$ be two simplicial model categories, let*

$$F_0 : \mathcal{D} \to \mathcal{E} : G_0 \quad \text{and} \quad F_1 : \mathcal{D} \to \mathcal{E} : G_1$$

be two adjoint pairs of Quillen functors and let

$$\alpha : F_0 \to F_1 \quad \text{and} \quad \beta : G_1 \to G_0$$

be a transposed pair of natural transformations. We suppose that the functors are simplicial, that the adjunctions are strong and that the natural transformations are strong. We suppose that the class of cofibrations in $\mathcal{D}$ is generated by a subclass $\mathcal{A}$ of cofibrations between cofibrant objects. If the map $\alpha_A : F_0(A) \to G_0(A)$ is a weak equivalence when A is the domain or the codomain of a map in $\mathcal{A}$, then the map α_A is a weak equivalence for every cofibrant object $A \in \mathcal{D}$; moreover, the map $\beta_X : G_1(X) \to G_0(X)$ is a weak equivalence for every fibrant object $X \in \mathcal{E}$.

Proof: Let us show that the map $\beta_X : G_1(X) \to G_0(X)$ is a weak equivalence for every fibrant object $X \in \mathcal{E}$. By Proposition 6.35, it suffices to show that square

$$\begin{array}{ccc} \mathcal{D}(B, G_1X) & \longrightarrow & \mathcal{D}(A, G_1X) \\ \downarrow & & \downarrow \\ \mathcal{D}(B, G_0X) & \longrightarrow & \mathcal{D}(A, G_0X) \end{array}$$

is homotopy cartesian for every map $u : A \to B$ in the class $\mathcal{A}$. But the square is isomorphic to the square

$$\begin{array}{ccc} \mathcal{E}(F_1B, X) & \longrightarrow & \mathcal{E}(F_1A, X) \\ \downarrow & & \downarrow \\ \mathcal{E}(F_0B, X) & \longrightarrow & \mathcal{E}(F_0A, X). \end{array}$$

since we have $\beta = \alpha^t$. The maps $\alpha_A : F_0A \to F_1A$ and $\alpha_B : F_0B \to F_1B$ are weak equivalence, since A is the domain and B the codomain of a map in $\mathcal{A}$. Hence the vertical maps of the square are homotopy equivalences by Ken Brown's lemma, since X is fibrant. This show that square is homotopy cartesian. Thus, β_X is a weak equivalence by Proposition 6.35. Let us now show that the map $\beta_X : G_1(X) \to G_0(X)$ is a weak equivalence for every fibrant object $X \in \mathcal{E}$. By Proposition 6.34 it suffices to show that the map

$$\mathcal{E}(\alpha_A, X) : \mathcal{E}(F_1A, X) \to \mathcal{E}(F_0A, X)$$

is a homotopy equivalence for every fibrant object X. But the map is isomorphic to the map

$$\mathcal{D}(A, \beta_X) : \mathcal{D}(A, G_1X) \to \mathcal{D}(A, G_0X),$$

since $\beta = \alpha^t$. But $\mathcal{D}(A, \beta_X)$ is a homotpy equivalences by Ken Brown's lemma, since β_X is a weak homotopy equivalence and A is cofibrant. ∎

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