AUSTRALIAN CATEGORY THEORY

TALKS BY STEVE LACK IN CHICAGO, 2006

Contents

0.1. **Introduction.** There are bicategories, 2-categories, and **Cat**-categories. The latter two are exactly the same, but two different ways of thinking about things. The first two are nominally different, although not really up to biequivalence, but also have a very different flavor.

Anything you take from category theory can be enriched more or less smoothly. Took some years and don't want to play down what was done, but now we can call that easy.

Bicategories are sort of easy also, in some sense. Never say that two arrows are equal, only isomorphic, and sort of obvious what you should do to **Cat**-category theory to get bicategory theory: make equations into isomorphisms, and then worry about coherence. In principle that's easy, but getting the details right can be really hard. People tend not to bother; if they're really good, it may be true (although not always), but hard to understand for mortals.

In 2-category theory, using enriched category theory, but not applying it in a simple-minded way. There is some thought in passing from **Cat**-categories to 2-categories. Over in bicategories, everything is true as long as you make it suitably weak, although it can be hard to state and prove, but in 2-category theory it's not always obvious what the theorems are.

I guess a preference for 2-categories is more Steve than Australian. All of these are Australian.

Also have $s\mathscr{S}$-enriched categories, which are also closely related. Can take nerves to get one of these, which are good things to look at. Not really going to talk about them.

And double categories, internal categories in **Cat**. Not going to say much about these either. 2-categories live inside of double categories, but in a variety of different ways. Some sorts of things work better in the internal world than in the enriched world, so there can be advantages to using double categories instead of 2-categories sometimes.

0.2. **Examples of 2-categories. Cat** is the mother of all 2-categories, just as **Set**is the mother of all categories. From many points of view, it has all the best properties.

Two directions of generalization of this.

- $\mathscr{V}$-**Cat** is one possible generalization of **Cat**, where $\mathscr{V}$ is generally a symmetric monoidal closed category, complete and cocomplete.
- Can also consider $\mathscr{W}$-**Cat**, where $\mathscr{W}$ is a bicategory. Not a monoidal bicategory, but a 'many-object monoidal category'. Going to talk about this much later on.
- **Cat**($\mathbb{E}$): internal categories in a category $\mathbb{E}$ with finite limits (though can get away with less, like pullbacks or even less, and usually want more, such as a topos, for good things to be true).

There is another class of examples consisting of 'categories with structure':

- Finite products
- finite limits
- monoidal
- toposes
- etc.

At each stage need to ask what morphisms you want. Normally don't want strictly algebraic, i.e. preserving structure on the nose; that can be technically useful, but not fundamentally interesting. Generally use functors preserving structure 'up to iso'.

Then we have **MonCat**, which means monoidal categories, monoidal functors (by which we mean *lax* monoidal functors), and monoidal natural transformations. Our monoidal functors have transformations $FA \otimes FB \to F(A \otimes B)$; the other direction we will call 'opmonoidal', although there is some controversy here. Two justifications for this:

- The forgetful functor U from $\mathscr{A}b$ to **Set**, for example, definitely doesn't preserve structure up to iso, only up to the universal bilinear map $UG \times UH \to U(G \otimes H)$
- These functors take monoids to monoids
- Suppose $\mathscr{A} \rightleftarrows \mathscr{B}$ is an adjunction with $\mathscr{A}$, $\mathscr{B}$ monoidal. Typically the left adjoint F is strong monoidal (i.e. the transformation is invertible) but U is just monoidal. (Think of the tensor product as a type of colimit, so the left adjoint preserves it, but the right adjoint doesn't necessarily.)

Moreover, an internal adunction in **MonCat** is always in this case; this is called *doctrinal adjunction*. If you're Australian, one thing you do is throw the words 'doctrinal adjunction' around a lot, and people don't know what you're talking about. In general, given an adjunction between categories which are algebras for some doctrine, to make the right adjoint a lax morphism is equivalent to making the left adjoint an oplax morphism, and to make the whole thing into an adjunction in the 2-category of algebras and lax morphisms is equivalent to making the left adjoint a strong morphism.

0.3. **Examples of bicategories. Rel** consists of sets and relations. Relations from X to Y, written $X \nrightarrow Y$, are monomorphic functions $R \rightarrowtail X \times Y$. This bicategory is 'locally posetal', in other words, given two parallel 1-cells, there is at most one 2-cell between them. We get a 2-category biequivalent to this one by identifying isomorphic 1-cells; this works for any locally posetal 2-category.

Par consists of sets and partial functions. A partial function from X to Y is a diagram $X \hookleftarrow \rightarrow Y$ in **Set**. Again, we get a biequivalent 2-category by identifying isomorphic 1-cells.

Span consists of sets and 'spans' $X \leftarrow E \rightarrow Y$ in **Set**, with composition by pullback. Unlike the previous two, this one is no longer locally posetal, so to get a biequivalent 2-category we need to do more. We know we can always do it in a formal way, but in fact naturally occurring concrete bicategories are equivalent to naturally defined 2-categories as well. In this case, we can take the 2-category whose objects are sets and whose morphisms are adjunctions $\mathbf{Set}/X \leftrightarrows \mathbf{Set}/Y$. This is equivalent via pulling back along one of the arrows in a span, $X' \mapsto E'$.

$$\begin{tikzcd}
X' \arrow[d] & E' \arrow[l] \arrow[d] & \\
X & E \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Mat has objects sets, and 1-cells funct(ors) $X \times Y \to \mathbf{Set}$. These are $X \times Y$ 'matrices' of sets. Composition is matrix multiplication. This is essentially the same as **Span**. A biequivalent 2-category consists of sets and adjunctions $\mathbf{Set}^X \rightleftarrows \mathbf{Set}^Y$, since $\mathbf{Set}^X$ is the free cocompletion of X; which of course is the same as what we did for **Span**. These differences will become important when we start to enrich and internalize.

Mod has (for today) objects rings R, 1-cells $R \nrightarrow S$ are 'modules'. One thing about being Australian is that you never call it a bimodule, just a module. Modules are understood to go from one ring to another, although one of them may happen to be the integers. A biequivalent 2-category involves adjunctions $R\mathbf{Mod} \rightleftarrows S\mathbf{Mod}$.

This is really '$\mathscr{A}b - \mathbf{Mod}$' since it's enriched over abelian groups. Another Australian thing is to work over any base; if we didn't write the base we'd generally assume **Set**. Also no reason to have only one objects. Thus $\mathscr{V}$-**Mod** (or $\mathscr{W}$-**Mod**) consists of $\mathscr{V}$-categories and $\mathscr{V}$-modules $\mathscr{A} \nrightarrow \mathscr{B}$ are functors $\mathscr{A} \to [\mathscr{B}^{op}, V]$, or equivalently an adjunction $[\mathscr{A}^{op}, \mathscr{V}] \rightleftarrows [\mathscr{B}^{op}, V]$.

Now let's internalize and enrich the other examples. If $\mathbb{E}$ is a 'regular category', i.e. morphisms factor as a strong epi followed by a monomorphism and strong epis are stable under pullback (could really be any factorization as long as stable), then we can form $\mathbf{Rel}(\mathbb{E})$ whose objects are those of $\mathbb{E}$ and whose morphisms $X \nrightarrow Y$ are monomorphisms $R \rightarrowtail X \times Y$. To compose these, we need a factorization system, and to make it associative, we need the factorization system to be stable under pullback.

Similarly, if $\mathscr{C}$ is a category and $\mathscr{M}$ is a class of monomorphisms in $\mathscr{C}$, then we can look at $\mathbf{Par}(\mathscr{C}, \mathscr{M})$, defined as above where the given monomorphism is in $\mathscr{M}$. There are conditions on $\mathscr{M}$ you need to make this work well.

If $\mathbb{E}$ has finite limits, we can look at $\mathbf{Span}(\mathbb{E})$ defined in an obvious way. Need the pullbacks for composition to work. No exactness is involved, although if you wanted to get a nice biequivalent 2-category, you are going to need to start making more assumptions on $\mathbb{E}$. It turns out that $\mathbf{Span}(\mathbb{E})$ is crucial to building up internal categories in $\mathbb{E}$ (an internal category in $\mathbb{E}$ is the same as a monoid in $\mathbf{Span}(\mathbb{E})$).

Mat, on the other hand, gets enriched rather than internalized. Then $\mathscr{V}$-**Mat** (traditionally, things enriched over go in front, while things internalized in go on the right) has objects *sets* and morphisms $\mathscr{V}$-valued matrices $X \times Y \to \mathscr{V}$. These matrices will be the internal homs of $\mathscr{V}$-categories.

0.4. **Duality.** $\mathscr{B}$ a bicategory, we have

- $\mathscr{B}^{op}$ reverse 1-cells
- $\mathscr{B}^{co}$ reverse 2-cells
- $\mathscr{B}^{coop}$ reverse both

In the case of a monoidal category (a 1-object bicategory) $\mathscr{V}$, then $\mathscr{V}^{op}$ reverses 1-cells (which is $\mathscr{V}^{co}$ as a bicategory) and $\mathscr{V}^{rev}$ reverses the tensor product ($\mathscr{V}^{op}$ as a bicategory).

Steve believes there is no perfect way to name things 'op's and 'co's, although people have expressed other opinions.

0.5. **Formal category theory.** One point of view is that a 2-category is a generalized category (add 2-cells). Another important one is that an *object of* a 2-category is a generalized category (since **Cat** is the primordial 2-category). This is 'formal category theory': think of a 2-category as a collection of category-like-things.

We can't always think of a $\mathscr{V}$-category as an element of a 2-category, just as we can't always think of a group as just being an object of **Grp**, but many things work out well when we take the 'element-free' approach. You tend to avoid talking about objects of a category, instead talking about morphisms into the category.

The fundamental paper of formal category theory, and where its name comes from, is Ross Street's "formal theory of monads". Its origin is related to this question: what is the enriched version of universal algebra (monads)? It uses all four dualities to incredible effect. There was also a book by Gray called "formal category theory".

Let $\mathcal{K}$ be a 2-category. If we started with a bicategory, we could replace it by a biequivalent 2-category, and in this context you probably should. Can translate everything to bicategory language, but I don't think you gain anything, so let's keep things simple.

Can define an *adjunction* $f\colon A \rightleftarrows B : u$ in $\mathcal{K}$ with 2-cells $\eta\colon 1 \to uf$, $\varepsilon\colon fu \to 1$, and triangle equations. In a lot of 2-categories, this is a good thing to study. We mentioned **MonCat**. Also good to study it in **Mod** (Morita theory). If η and ε are invertible, we have an *adjoint equivalence*.

Another way to define adjunctions is with bijections on hom-sets; this also works fine here: to give a 2-cell $fa \to b$ is the same as to give a 2-cell $f \to ub$. The proof is identical to the usual one; the usual case is when the source of a and b is 1. The reason for this is that *2-functors preserve adjunctions*, and we have representable 2-functors $\mathcal{K}(X, -)$, giving adjunctions

$$\mathcal{K}(X, f)\colon \mathcal{K}(X, A) \rightleftarrows \mathcal{K}(X, B) : \mathcal{K}(X, u)$$

(A 2-functor preserves 2-category structure strictly. Pseudofunctors also preserve adjunctions.)

There are also the contravariant representable functors

$$\mathcal{K}(-, X)\colon \mathcal{K}^{op} \to \mathbf{Cat}$$

which also preserve adjunctions.

Exercise 0.1. f is a left adjoint in $\mathcal{K}$ iff it is a right adjoint in $\mathcal{K}^{co}$ iff it is a right adjoint in $\mathcal{K}^{op}$.

Thus to give a 2-cell $s \to tf$ is the same as one $su \to t$. This is true even in **Cat**, and very useful and perhaps less well-known than it should be there.

Can also combine these two. Given a pair of adjunctions, then squares

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$$

correspond to squares

$$\begin{tikzcd} A \arrow[d, "a"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[l, "u"'] \arrow[d, "b"] \\ A' & B' \arrow[l, "u'"] \end{tikzcd}$$

These pairs of 2-cells are called *mates*. This is easy once you know it's true, but you might not think of it.

0.6. **Things to fix up.** Didn't exactly explain what slashing means. Slashing doesn't just mean how you write an arrow in a bicategory, it's how you write certain *sorts* of arrows, the ones that are more like relations/modules instead of functions/functors. In an abstract bicategory, wouldn't slash anything, but if they have the flavor of relations, modules, or parametrized spectra, write them as $\nrightarrow$.

Also the dichotomy between a straight arrow $\to$ and a squiggly arrow $\rightsquigarrow$ for a weaker type of morphism. This is sometimes useful. It doesn't usually get taken as far as to have a notation for lax things.

Insofar as comparing categories to $\mathscr{V}$-categories, would be much more likely to say a category is a special sort of $\mathscr{V}$-category, rather than a $\mathscr{V}$-category being a category.

1. 2-categorical things, continued

1.1. **Extensions.** Let $\mathcal{K}$ be a 2-category. What is the universal solution to extending f along j?

$$\begin{tikzcd} & B \arrow[dr, dashed] & \\ A \arrow[ur, "j"] \arrow[rr, "f"'] & & C \end{tikzcd}$$

Call this $\mathrm{Lan}_j\, f$, then by universal we mean a bijection

$$\frac{f \longrightarrow gj}{\mathrm{Lan}_j\, f \longrightarrow g}.$$

When such a $\mathrm{Lan}_j\, f$ exists in $\mathcal{K}$, it is called a *left extension* of f along j.

It looks like a Kan extension and we'd like it to be one. For Kan extensions we have a coend formula

$$(\mathrm{Lan}_j\, f)b = \int^a B(ja, b) \cdot fa$$

but we don't yet know quite what this means in $\mathcal{K}$. (Kan extensions exist which are not 'pointwise', i.e. don't satisfy this formula, but they're in general the 'wrong' ones.)

Consider an object $b \in B$ as a morphism $b\colon 1 \to B$, then this formula is telling us what the composite

$$\begin{tikzcd} j/b \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\swArrow"] & A \arrow[d] \arrow[dr, "f"] & \\ 1 \arrow[r, "b"'] & B \arrow[r, "\mathrm{Lan}_j f"] & C \end{tikzcd}$$

Here j/b is the comma category. The really good notion of extension is when in addition $(\mathrm{Lan}_j\, f)(b) = \mathrm{Lan}_c(fd)$ for all morphisms b with codomain B (in **Cat**, it's enough to have b with domain 1). In this case we say $\mathrm{Lan}_j\, f$ is the *pointwise extension.* This is a really good notion for lots of 2-categories, such as internal ones, but not quite enough for the enriched case.

Let's leave the pointwise aspect aside and go back to extensions.

- A left extension in $\mathcal{K}^{co}$ (reverse the 2-cells) is a *right extension.*
- A left extension in $\mathcal{K}^{op}$ (reverse the 1-cells) is a *left lifting.*
- A left extension in $\mathcal{K}^{coop}$ (reverse both) is a *right lifting.*

The right lifting has the property that

$$\frac{pg \longrightarrow f}{g \longrightarrow r = f \lhd p}$$

which is a sort of internal-hom.

Look at the special case of adjunctions

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u"] \end{tikzcd}$$

Then

$$\begin{tikzcd} & A \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow"] & \\ X \arrow[ur, "a"] \arrow[rr, "b"'] & {} & B \end{tikzcd}$$

and the correspondence $B(fa, b) = A(a, ub)$ shows that $ub = b \lhd f$. That lies behind a lot of the stuff Niles was talking about.

In particular, $u = 1 \lhd f$. Thus every adjunction gives a right lifting, and a right lifting $u = 1 \lhd f$ is an adjunction iff it is 'respected' by any b, i.e. $ub = b \lhd f$. Similarly for all the dual versions. A *closed* structure for the bicategory is about right extensions and right liftings only.

Let me point out a little lemma which everyone knows for **Cat**, but which is true for 2-categories basically because everything is representable. If we have an adjunction $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u"] \end{tikzcd}$ with unit $1 \to uf$ invertible, then f is representably fully faithful, i.e. for all X, the functor $\mathcal{K}(X, A) \xrightarrow{\mathcal{K}(X,f)} \mathcal{K}(X, B)$ is fully faithful. Also u is 'co-fully-faithful', meaning that all functors $\mathcal{K}(f, X)$ are fully faithful (i.e. u is fully faithful in $\mathcal{K}^{op}$).

In places like **Mod**, extensions tend to exist. In places like **Cat**, it's a condition.

1.2. **Monads.** If $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u"] \end{tikzcd}$ is an adjunction, we get a 1-cell $t = uf \colon A \to A$, with unit $1 \to [\eta]t$ and multiplication $\mu \colon ufuf \xrightarrow{u\varepsilon f} uf$. Exactly as for categories. We have the notion of a *monad* (t, μ, η) in a 2-category $\mathcal{K}$ (on a 0-cell A). Sometimes write (A, t) for a monad t on A, although this is obviously stupid since we haven't specified the most important things μ and η.

Never talk about a monad *in* a category for the classical situation; call it a monad *on* the category, since the monad lives *in* the 2-category (in this case, **Cat**).

Example 1.1. Monads in **Cat**. Know that.

Example 1.2. Monads in $\mathbf{Span}(\mathbb{E})$. We have an object E_0, a 1-cell $t \colon E_0 \nrightarrow E_0$, i.e. a span

$$\begin{tikzcd} & E_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\ E_0 & & E_0 \end{tikzcd}$$

(a 'graph') with a multiplication

$$\mu\colon E_1 \times_{E_0} E_1 \to E_1$$

from the 'object of composable pairs' to the arrows, giving a composite, have associativity = associativity, and a unit $1 \xrightarrow{\eta} t$ gives $E_0 \to E_1$ since the identity span is

$$\begin{tikzcd} & E_0 \arrow[dl] \arrow[dr] & \\ E_0 & & E_0. \end{tikzcd}$$

Thus *a monad in* $\mathbf{Span}(\mathbb{E})$ *is the same as an internal category in* $\mathbb{E}$.

This is one of the most important reasons for having any interest in the span construction.

Example 1.3. Monads in $\mathscr{V}$-**Mat**. We have an object X, which is just a set, a 1-cell $X \nrightarrow X$, i.e. $X \times X \to \mathscr{V}$, which we think about as sending $(x, y) \mapsto \mathscr{C}(x, y)$, a 'hom-object'. Another Australian thing is to never write the word 'hom', only the name of the category: $\mathscr{C}(x, y)$ instead of $\hom_{\mathscr{C}}(x, y)$. The multiplication map goes from the matrix product

$$\sum_y \mathscr{C}(y, z) \otimes \mathscr{C}(x, y) \longrightarrow \mathscr{C}(x, z)$$

and gives a composition map. Again associativity = associativity, unit = unit, and we get that *a monad in* $\mathscr{V}$-**Mat** *is the same as a category enriched over* $\mathscr{V}$.

A morphism of monads (which we haven't defined yet) in $\mathbf{Span}(\mathbb{E})$ is, however, *not* an internal functor, since it will involve a 1-cell $E_0 \nrightarrow F_0$; but rather it is an internal *profunctor* or *module*. Similarly, morphisms of monads in $\mathscr{V}$-**Mat** give modules between $\mathscr{V}$-categories.

Of course, you can get the internal and enriched functors if you use double categories.

To get 2-cells, you don't use the (obvious) notion of 2-cells between monads in 'The Formal Theory of Monads', but rather in 'The Formal Theory of Monads II' (by Street and SL). May talk about that later.

1.3. **Lax morphisms.** I switch seemingly at random between 2-categories and bicategories. All that was 2-categories, and I feel like using the word bicategory for now.

A 'morphism' or *lax functor* from $\mathscr{A}$ to $\mathscr{B}$ sends objects A to objects FA, has functors $F\colon \mathscr{A}(A, B) \to \mathscr{B}(FA, FB)$ (thus preserving 2-cell composition in a strict way), and has some comparison maps $\varphi\colon Fg \cdot Ff \to F(gf)$ and $\varphi_0\colon 1_{FA} \to F(1_A)$ and some coherence conditions, which are the same as the coherence conditions for monoidal functors, since it's the same thing.

There are several systems of naming:

Strict	strong	– (unadjectived)	op–
2- or strict	pseudo	lax	oplax

but sometimes the unadjectived one is the pseudo version (such as for 2-algebras).

All the good things that happen for (lax) monoidal functors happen for lax functors, such as taking monoids to monoids, only here we call them monads instead. A monad in $\mathscr{B}$ on X is the same as a monoid in the monoidal category $\mathscr{B}(X, X)$.

For example, consider the identity monad 1 in the 2-category 1. Then for any lax functor $1 \to \mathscr{B}$, the object $*$ gets set to $F* = A$, 1 is sent to $F1 = t$, the comparison maps become $\mu \colon tt \to t$ and $\eta \colon 1 \to t$, and the coherence conditions make this precisely a monad. In fact, *monads in* $\mathscr{B}$= *lax functors* $1 \to \mathscr{B}$. For Benabou, this was really the reason to consider 'morphisms' of bicategories, rather than the stronger version. This is also true for transformations, perhaps with an op. Perhaps due to some historical error, the 'right' sort of transformations frequently come with an op.

In particular, $\mathscr{V}$-categories are the same as lax functors $1 \to \mathscr{V}$-**Mat**. This is the same as a set X together with a lax functor

$$X_{ch} \xrightarrow{lax} \Sigma\mathscr{V}$$

where X_{ch} is X made into a chaotic/indiscrete category (every object is uniquely isomorphic to every other), then made a bicategory with only identity 2-cells, and $\Sigma\mathscr{V}$ means pretend a monoidal category is a 1-object bicategory. Why? We send each x to $*$, we have a functor

$$X_{ch}(x, y) \to \Sigma\mathscr{V}(*, *)$$

which is just $1 \to \mathscr{V}$ picking out the hom $\mathscr{C}(x, y)$, and the lax comparison maps φ become the composition and identity maps.

If we replace $\Sigma\mathscr{V}$ by an arbitrary bicategory $\mathscr{W}$, we get the notion of a $\mathscr{W}$-enriched category: a set X with a lax functor

$$X_{ch} \xrightarrow{lax} \mathscr{W}$$

Another way to think about X_{ch}, as a bicategory, is to say that the unique map $X \to 1$ is fully faithful. But we can also consider, more generally, a pair of bicategories with a partial map

$$\begin{tikzcd} & \mathscr{D} \arrow[dl, "{strict, f+f}"'] \arrow[dr, "lax"] & \\ \mathscr{A} & & \mathscr{B} \end{tikzcd}$$

This partial map is called a *2-sided enrichment* or a *category enriched from* $\mathscr{A}$ *to* $\mathscr{B}$. If $\mathscr{A}$ is 1, it's just a category enriched over $\mathscr{B}$. Using the notion of composition for these things is very helpful in analyzing the change of base between different bicategories and enrichment.

1.4. **Pseudofunctors and 2-functors.** These are the lax functors for which φ and φ_0 are invertible.

Example 1.4. $\mathscr{B}$ a bicategory, then the representable things

$$\mathscr{B} \xrightarrow{\mathscr{B}(B,-)} \mathbf{Cat}$$

are (not strict) pseudofunctors.

Example 1.5. Indexed categories: a pseudofunctor $\mathscr{B}^{op} \to \mathbf{Cat}$, where magically and confusingly, here $\mathscr{B}$ stands more for 'base' than for 'bicategory'. Often $\mathscr{B}$ is just a category here. Then such a pseudofunctor corresponds to a fibration $\mathscr{E} \to \mathscr{B}$ in the 'Grothendieck picture'.

Similarly, '2-functors' (or 'strict homomorphisms' between bicategories) are where φ and φ_0 are identities. This makes it preserve the associativity and so on, in addition to the composition strictly.

2-functors are much nicer to work with, but often we only have a pseudofunctor. One reason you might prefer them is not to have to worry about coherence. Moreover, 2-functors have better properties than pseudofunctors. For example, in the category of 2-categories and 2-functors, you have limits and colimits, but this is not true for 2-categories and pseudofunctors.

The (1-)category $\mathbf{2Cat}_{ps}$ of 2-categories and pseudofunctors is neither complete nor cocomplete. For example,

$$\begin{tikzcd} & (0 \to 1) \\ (1) \arrow[ur] \arrow[dr] & \\ & (1 \to 2) \end{tikzcd}$$

has no pushout. Why? Such a pushout would have to have $(0 \to 1 \to 2)$ and a composite, but in some other $\mathcal{K}$ there is nowhere we know how to send the composite. Of course, this is being a little perverse; in a higher-dimensional world, we have an appropriate notion. But it can be nice to restrict to this situation, if we can.

On the other hand, even if you start in the world of 2-categories and 2-functors, you may be forced out of it. A 2-functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ is a *biequivalence* if $\mathscr{A}(A, B) \to \mathscr{B}(FA, FB)$ are equivalences and it is 'bi-essentially surjective', i.e. for all $X \in \mathscr{B}$, there exists an $A \in \mathscr{A}$ and an equivalence $FA \simeq X$. This is the 'right notion' of equivalence for 2-functors.

The point is that you'd like something going back the other way, and you have it, but it's just not a 2-functor in general. Given $X \in \mathscr{B}$, pick $FA \simeq X$ and let $GX = A$. Given $X \xrightarrow{x} Y$, we can bring it across the equivalences $FA \simeq X$ and $FB \simeq Y$ to get $\overline{x} \colon FA \to FB$, and since F is locally an equivalence, $\overline{x} \cong Fa$ some $a \colon A \to B$, let $Gx = a$. This all works, but since everything is only defined up to isomorphism, there's no way you can possibly hope for G to preserve things strictly.

There is a model structure on **2-Cat** for which the weak equivalences are the biequivalences, and clearly getting something going the other way has something to do with $\mathscr{A}$ being cofibrant.

2. 2-dimensional universal algebra

'Universal 2-algebra'? Maybe no reason not to call it that.

Lots of categorical approaches to universal algebra. Theories, operads, sketches (a bit wilder), but I'll mostly talk about monads. Certainly today, 'universal algebra' will mean monads. Although you may see parallels, certainly with operads and also with theories.

The ordinary universal algebra picture you should have in mind is monoids (or groups, rings, etc.) living over sets. Doesn't have to be single-sorted; could be living over some power of sets. Abstractly, of course, we could be living over almost everything. For a small category $\mathscr{C}$, the functor category $[\mathscr{C}, \mathbf{Set}]$ living over $[ob\,\mathscr{C}, \mathbf{Set}]$ is a good one to have in mind. If $\mathscr{C}$ has one object, we get M-sets for a monoid M.

When we come to 2-categories, we might generalize monoids over sets to

$$\begin{tikzcd} \text{monoidal categories} \arrow[d] \\ \mathbf{Cat} \end{tikzcd}$$

(won't prejudge yet what the morphisms in monoidal categories are, strict, strong, lax). Also categories with finite products, or coproducts, or both, maybe with a distributive law.

Corresponding to diagrams we have

$$\begin{tikzcd} \mathrm{Hom}(\mathscr{B}, \mathbf{Cat}) \arrow[d] \\ {[ob\,\mathscr{B}, \mathbf{Cat}]} \end{tikzcd}$$

(Homomorphisms from some bicategory $\mathscr{B}$ to $\mathbf{Cat}$. Haven't talked about transformations between pseudofunctors yet, will do that at some point, but it will actually pop out of the general theory.)

Today particularly, will see a lot of interplay between 2-category theory and $\mathbf{Cat}$-category theory. Trying not to really assume that you know any $\mathscr{V}$-category theory, so tend to do things like go over the ordinary case and then tell you it works when you do it it for $\mathscr{V}$, and concentrate more on how to modify it for the 2-category situation.

2.1. **2-monads.** When I say '2-' that means strict, here and elsewhere. Still strict, but just when we do stuff, we'll do it in a different way, maybe. A *2-monad* consists of a 'good' 2-category $\mathcal{K}$ (like $\mathbf{Cat}$, maybe, complete, cocomplete, etc.), with $T = (T, m, i)$ where

$$T\colon \mathcal{K} \to \mathcal{K}$$
$$m\colon T^2 \to T$$
$$i\colon 1 \to T$$

With the usual associativity and unit rules. Here m, i are 2-natural transformations and T is a strict 2-functor. This is precisely a monad in the 2-category of 2-categories, 2-functors, and 2-natural transformations. Perhaps a bit funny to think of that as a 2-category, you can think of it as a 3-category if you like. So far, it's just about $\mathscr{V}$. A $\mathscr{V}$-monad is just a monad in $\mathscr{V}$-**Cat**, so far we haven't departed from that.

A (strict) T-*algebra* is the usual thing, an $A \in \mathcal{K}$ with $TA \xrightarrow{a} A$ and the usual equations. This point is where something happens, can't just keep making things strict. A *lax T-morphism* $(A, a) \to (B, b)$ is a morphism $f\colon A \to B$ in $\mathcal{K}$, but

$$\begin{tikzcd}
TA \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow \bar{f}"] & TB \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

with some coherence conditions, which I will write down, since perhaps not as familiar. If I forced you to go away and guess, you'd probably get it right. First we have

$$\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\swarrow T\bar{f}"] & T^2B \arrow[d, "Tb"] \\
TA \arrow[r, "Tf"'] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow \bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[r, "T^2f"] \arrow[d, "mA"'] & T^2B \arrow[d, "mB"] \\
TA \arrow[r, "Tf"'] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow \bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

(note that the outer 1-cells are the same; I wouldn't write this down if they weren't) and

$$\begin{tikzcd}
A \arrow[d, "i"'] & \\
TA \arrow[r] \arrow[d] & TB \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

should be the identity.

Let's do a little baby example: $\mathcal{K} = \mathbf{Cat}$ and $TA = \sum_n A^n$ the usual free monoid construction. The T-algebras are strict monoidal categories, and a lax morphism is a square

$$\begin{tikzcd}
\sum_n A^n \arrow[r] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\swarrow"] & \sum_n B^n \arrow[d, "\otimes"] \\
A \arrow[r] & B
\end{tikzcd}$$

so we have transformations

$$f(a_1) \otimes \ldots \otimes f(a_n) \longrightarrow f(a_1 \otimes \ldots \otimes a_n)$$

when I defined monoidal functor, I just did the case $n = 0$ and $n = 2$, but you can build it up in the usual way, and the coherence conditions just say that you did it in the only sensible way, not some stupid way. This should be some sort of evidence for the reasonableness of the definition.

As more motivation, here's an abstract reason why this definition pops out. There's a 2-category $\mathbf{Lax}(\nvDash, \mathcal{K})$ where $\nvDash$ is the arrow category. In detail:

- An object is an arrow in $\mathcal{K}$

- A 1-cell is a square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ B \arrow[r] & B' \end{tikzcd}$$

- A 2-cell. . . you can decide for yourself.

Since this is functorial, we get a 2-monad $\mathbf{Lax}(\nvDash, T)$ on $\mathbf{Lax}(\nvDash, \mathcal{K})$. Then a $\mathbf{Lax}(\nvDash, T)$-algebra, in the strict sense, is the same as a lax T-morphism. Our coherence is the usual axioms for something to be an algebra.

We also have, for lax T-morphsims $(f, \overline{f}), (g, \overline{g}) \colon (A, a) \to (B, b)$, a *$T$-transformation* is a 2-cell $\rho \colon f \to g$ in $\mathcal{K}$ such that

$$\begin{tikzcd} TA \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow T\rho"] \arrow[d] & TB \arrow[d] \arrow[dl, phantom, "\Swarrow \overline{g}"] \\ A \arrow[r] & B \end{tikzcd} = \begin{tikzcd} TA \arrow[r] \arrow[d] & TB \arrow[d] \arrow[dl, phantom, "\Swarrow \overline{f}"] \\ A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\rho"] & B \end{tikzcd}$$

In the baby example, for $n = 2$ this says that

$$\begin{tikzcd} fa_1 \otimes fa_2 \arrow[r] \arrow[d, "\rho a_1 \otimes \rho a_2"'] & f(a_1 \otimes a_2) \arrow[d, "\rho"] \\ ga_1 \otimes ga_2 \arrow[r] & g(a_1 \otimes a_2) \end{tikzcd}$$

so you get a monoidal natural tranformation, just as you'd like.

You can play the $\mathbf{Lax}(\nvDash, \mathcal{K})$-sort of game with the transformations as well, but I won't.

We get a 2-category $T\text{-}\mathbf{Alg}_\ell$ of T-algebras, lax Tmorphisms, and T-trasnformations, and a forgetful 2-functor

$$T\text{-}\mathbf{Alg}_\ell \xrightarrow{U_\ell} \mathcal{K}$$

Sometimes this is what we really want to study, sometimes we want things to go pseudo. If $\overline{f}$ is invertible, we say that $(f, \overline{f})$ is a *pseudo T-morphism* or just a *T-morphism* (privileging these over the strict or the lax, since in a lot of cases we care most about those). We get

$$T\text{-}\mathbf{Alg} \xrightarrow{U} \mathcal{K}$$

with strict algebras and pseudo morphisms.

Some people think that using strict algebras is some sort of cop-out, because it turns out that in reasonable cases you can always replace T by some other 2-monad T' (a 'cofibrant replacement') such that pseudo T-algebras are the same as strict T'-algebras. But there's also a really practical reason. It's true that the pseudo-algebras for the strict-monoidal-categories 2-monad gives the monoidal categories, but that's a relatively hard fact. Some cofibrant replacements are bigger than others, and here there's another, easier one, that gives you exactly monoidal categories without having to deal with the huge T' thing. Later, not very much today, about 'presentations' for monads, which is how we could write down a 2-monad whose strict algebras are monoidal categories.

We also have the *strict* T-morphisms when $\overline{f}$ is an identity (normally you say that the square just commutes). Still want the condition on 2-cells where we put the $\overline{f} = \mathrm{id}$ in. These give

$$T\text{-}\mathbf{Alg}_s \xrightarrow{U_s} \mathcal{K}$$

Each of these 2-categories have the same objects, and we have

$$\begin{tikzcd}
 & & & T\text{-}\mathbf{Alg}_c \\
T\text{-}\mathbf{Alg}_s \arrow[r, "J"'] \arrow[rr, bend left, "J_\ell"] \arrow[dr] & T\text{-}\mathbf{Alg} \arrow[r] \arrow[urr] \arrow[d] & T\text{-}\mathbf{Alg}_\ell \arrow[dl] & \\
 & \mathcal{K} & &
\end{tikzcd}$$

(we've added the one with co-lax morphisms at the upper right, which you can have fun formulating as lax things on $\mathcal{K}^{co}$.) We also have the category $\mathbf{Ps}{-}T\text{-}\mathbf{Alg}$ of pseudo T-algebras, which do come up sometimes, but in practice are usually less important.

Remark 2.1. 2-monads are also algebras. There is a $\mathcal{K}$ and a T for which 2-monads on a fixed 2-category are the T-algebras. The point for doing this is a notion of pseudo-morphism of 2-monad, which will be really important. Also the $'$ thing. Will come back to this. Also we have presentations (later on), which are colimits of monads.

For now let T be reasonable, by which we mean that $\mathcal{K}$ is cocomplete and T has *rank*, which in turn means that $T\colon \mathcal{K} \to \mathcal{K}$ preserves α-filtered colimits for some α, which means that it describes some sensible structure that you could actually write down. For ordinary monads on categories, it says that we can describe the structure in terms of not necessarily finitary operations, but α-ary for some very large cardinal α. Much of the time, finitary is good enough. The famous example of a monad on **Set** which is not α-filtered is the covariant power set monad.

[*MS: what about, say, 'complete categories'?*]

[*EC: you can't do that, size issues.*]

Actually, you can fix that, do things with universes, but once you do that, then the same problems are fixed

Then

$$T\text{-}\mathbf{Alg}_s \xrightarrow{J} T\text{-}\mathbf{Alg}$$

$$T\text{-}\mathbf{Alg}_s \xrightarrow{J_\ell} T\text{-}\mathbf{Alg}_\ell$$

have left adjoints. What does this mean? That we have a bijection

$$\frac{A \rightsquigarrow B}{A' \to B}$$

We see that the left adjoint to J is called $(-)'$ (which is sort of an embarrasing name for a functor). These are *2-adjoints*, which means **Cat**-adjoint, so the above bijection underlies an isomorphism of categories

$$T\text{-}\mathbf{Alg}_s(A', B) \cong T\text{-}\mathbf{Alg}(A, JB)$$

which is 2-natural in A and B. We usually omit writing the J, since it is the identity on objects. From this we get a unit

$$p\colon A \rightsquigarrow A'$$

and counit

$$q\colon A' \to A$$

and one of the triangle equations tells you that $qp = 1$. The other one says maybe $qp' = 1$, meaningfully interpreted.

This is actually the same as the prime we saw earlier, when we remember that 2-monads are also algebras.

2.2. **Sketch proof of the existence of $(-)'$.** There are two proofs that I know; will sketch both, one in much more detail.

Step 1. **T-$\mathbf{Alg}_s$ is cocomplete.**

This is essentially where you use the reasonableness assumption (the only place, in this proof). Colimits of algebras, as we know, are generally hard. The problem is essentially that it's a 'quadratic' thing, have $TA \to A$ with two As. We 'linearize' and it becomes easy. What does that mean?

Take the T-algebra and forget any conditions, and also forget that the two As are the same, so consider it only as a map $TA \to B$. This defines the objects of a new category; morphisms are

$$\begin{tikzcd} TA \arrow[r] \arrow[d, "Tf"'] & TB \arrow[d, "g"] \\ TC \arrow[r] & D \end{tikzcd}$$

Recall that T-$\mathbf{Alg}_s$ is the ordinary $\mathbf{Cat}$-enriched thing. We just did this for ordinary categories, you can enrich it yourself. You might call this the comma category $T/\mathcal{K}$. The point is that we have a full embedding

$$T\text{-}\mathbf{Alg}_s \longrightarrow T/\mathcal{K}$$

(check that in a morphism in $T/\mathcal{K}$ between two T-algebras, the two maps f and g must be the same, using the unit condition).

The point is that colimits in $T/\mathcal{K}$ are easy. Say we have a diagram of things $TA_i \to B_i$. Take the colimits in $\mathcal{K}$ and take the pushout

$$\begin{tikzcd} \operatorname{colim} TA_i \arrow[r] \arrow[d] & \operatorname{colim} B_i \arrow[d] \\ T \operatorname{colim} A_i \arrow[r] & {} \end{tikzcd}$$

to get the colimits in $T/\mathcal{K}$. The hard bit is that we have a left adjoint $T/\mathcal{K} \to T$-$\mathbf{Alg}_s$, which is where we use the reasonableness assumption. Big and nasty transfiniteness, as you expect once we write down something involving α-filtered colimits.

[*JPM: Classically, there is an easy proof when T preserves reflexive coequalizers. Does this generalize?*]

I guess so. If so, then the same construction should work. Does this depend on any nice property of $\mathscr{V}$? Disagreement.

In general, if T preserves α-filtered colimits, it's enough. If T preserves more colimits, then it's easier, and should it preserve *all* colimits, it becomes trivial: you get them all constructed pointwise. In particular, this is true for diagram categories.

Also true that for part 2, we don't need *all* colimits, only rather special ones, which have the flavor of reflexive coequalizers.

Part 2. Let (A, a) an algebra, want to construct A'. A T-morphism $(A, a) \to (B, b)$ consists of various stuff. Want to translate all the stuff into $T\text{-}\mathbf{Alg}_s$. It's some sort of bar construction, you've probably realized this already.

A (pseudo) T-morphism $A \to B$ consists of

- A morphism $f\colon A \to B$ in $\mathcal{K}$, which we can model in $T\text{-}\mathbf{Alg}_s$ by a morphism $g\colon TA \to B$ where $g = b \cdot Tf$.
- A square
$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow \overline{f}"] & TB \arrow[d, "b"] \\ A \arrow[r] & B \end{tikzcd}$$
which will become a square
$$\begin{tikzcd} T^2A \arrow[r, "mA"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\swarrow \zeta"] & TA \arrow[d, "g"] \\ TA \arrow[r, "g"'] & B \end{tikzcd}$$
in $T\text{-}\mathbf{Alg}_s$
$$b.T(fa) = b.Tf.Ta = g.Ta$$
$$b.T(b.Tf) = b.Tb.T^2f = \cdots = g.mA$$
- The condition $\overline{f}.iA = \mathrm{id}$ corresponds to saying that $\zeta.TiA = \mathrm{id}$
- The other one becomes
$$\begin{tikzcd} & T^2A \arrow[r, "mA"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\swarrow \zeta"] & TA \arrow[d, "g"] \\ T^3A \arrow[ur, "mTA"] \arrow[dr, "T^2a"'] & TA \arrow[r, "g"'] \arrow[dr, phantom, "\searrow \zeta"] & B \\ & T^2A \arrow[u, "mA"] \arrow[r, "Ta"'] & TA \arrow[u, "g"'] \end{tikzcd} = \text{the other way}$$

Had A not been an algebra but a pseudo-algebra, then some of these commuting squares would have some stuff in there, but everything would still work, to get a left adjoint defined on pseudo-algebras.

We have a truncated simplicial gizmo:
$$\begin{tikzcd} T^3A \arrow[r, shift left=2, "T^2a"] \arrow[r, "TmA" description] \arrow[r, shift right=2, "mTA"'] & T^2A \arrow[r, shift left=2, "Ta"] \arrow[r, shift right=2, "mA"'] & A \arrow[l, "TiA" description] \end{tikzcd}$$

and we take the *codescent object* of this construction. Alternately, we can do it working our way up step by step. We first universally insert a 2-cell in between (a *coinserter*)

$$T^2A \underset{mA}{\overset{Ta}{\rightrightarrows}} A$$

and then do a *coequifier* forcing the next-level equation to hold. Will talk about 2-categorical limits later.

Another fact is that the thing TiA going back the other way makes it look like a reflexive coequalizer, which makes it have a better chance of being presered by T than a random one.

Alternative proof. This was actually the original proof. It involves embedding T-**Alg** $\hookrightarrow T/_{ps}\mathcal{K}$ in some pseudo-comma category and constructing a reflection there.

2.3. **More.** Power has just proven a corresponding theorem for the 3-dimensional version, using a **Gray**-monad on a **Gray**-category. The equations are only invertible 3-cells satisfying some coherence conditions (a 4-cube). In his way, the cubes are not quite so evident. Haven't really understood the suitable notion of codescent object yet.

Recall we have

$$\begin{tikzcd} & A' \arrow[dr] & \\ A \arrow[ur, rightsquigarrow] & & A \end{tikzcd}$$

with $qp = 1$. It's also true that $pq \cong 1$, so that this is an equivalence, and thus $A \simeq A'$ in T-**Alg** (but not in T-$\mathbf{Alg}_s$).

If q has a section in T-$\mathbf{Alg}_s$ (as opposed to T-**Alg**), then A is said to be *flexible* (which is another word for *cofibrant*). Think of A' as being a cofibrant replacement for A; could make this precise, but won't today. The q is a strict morphism which is an equivalence in T-**Alg**, which should be thought of as the weak equivalences. There is some model structure on T-$\mathbf{Alg}_s$ in which flexible becomes the same as cofibrant.

Exercise 2.2. If A is flexible, then any wobbly $A \rightsquigarrow$ is isomorphic to a strict $A \to$.

The fact of $A \simeq A'$ gives us a 'coherence result for morphisms'. Could also be thinking about algebras; recall there was

$$T\text{-}\mathbf{Alg}_s \to T\text{-}\mathbf{Alg} \to \mathbf{Ps}T\text{-}\mathbf{Alg}$$

and the composite still has a left adjoint, which you could almost still call $(-)'$ (as we remarked earlier the same thing works if you stick an extra isomorphism in here and there). If A is a pseudo T-algebra, we get a strict T-algebra A' and a pseudo map $A \rightsquigarrow A'$ which is universal in the sense that any $A \rightsquigarrow B$ factors through as $A' \rightsquigarrow B$.

Here the counit is a little different; have to start with a *strict* algebra. If B is strict, we get $q\colon B' \to B$, but just starting with a pseudo-algebra A there is no reason to have such. In some cases, $p\colon A \rightsquigarrow A'$ is an equivalence. For example, it's an equivalence if T preserves these codescent objects, since can then construct it downstairs and use the universal property there to get a map going back the other way. There are various sufficient conditions for this to work.

Tend to think of the existence of $(-)'$ and then the fact of $A \rightsquigarrow A'$ being an equivalence as the 'full coherence result'. But in practice, not that important, since usually the strict algebras are what you want; you wanted to use some smaller cofibrant replacement for T that T'.

Know an example of a 2-monad whose pseudo-algebras don't strictify, but it's horrible on some horrible $\mathcal{K}$; don't know one with these nice conditions but for which $A \rightsquigarrow A'$ fails to be an equivalence.

3. Presentations for 2-monads

'Presentations' means free things and colimits of things. First of all, ought to justify why colimits of monads are good things; this isn't really 2-categorical at all.

3.1. Endomorphism Monads. T 2-monad on $\mathcal{K}$ (really a 2-category, might just as well be a category), $\mathcal{K}$ complete. Then given objects

$$\begin{tikzcd} & \mathcal{K} \arrow[dr, "\langle A,B\rangle"] & \\ I \arrow[ur, "A"] \arrow[rr, "B"'] & & \mathcal{K} \end{tikzcd}$$

The right Kan extension is a functor defined as

$$\langle A, B\rangle C = \mathcal{K}(C, A) \pitchfork B$$

where $\pitchfork$ means the cotensor, defined as

$$\mathcal{K}(D, X \pitchfork B) \cong \mathbf{Cat}(X, \mathcal{K}(D, B))$$

Thus in our case, this becomes the universal property of the right Kan extension. In particular, we have bijections of 2-natural transformations.

$$\frac{T \longrightarrow \langle A, B\rangle}{TA \longrightarrow B}$$

This is starting to look like something you might want to do if T is a monad.

We have a natural 'composition' map

$$\langle B, C\rangle\langle A, B\rangle \longrightarrow \langle A, C\rangle$$

which makes $\mathcal{K}$ actually enriched over $[\mathcal{K}, \mathcal{K}]$. Writing down where this comes from is a good exercise. We also have a unit

$$1 \to \langle A, A\rangle$$

Thus, as always, $\langle A, A\rangle$ becomes a monoid in $[\mathcal{K}, \mathcal{K}]$, i.e. a monad. And monad maps

$$\frac{T \xrightarrow{\text{monad}} \langle A, A\rangle}{TA \xrightarrow{alg.str} A}$$

correspond to T-algebra structures on A.

This tells us that colimits of monads are interesting. For example, algebras for $S + T$ (coproduct as monads) are objects with an algebra structure for S and T with no particular relationship between them.

What are the algebras for $\langle A, A\rangle$? A has an obvious one given by evaluation, but maybe there aren't any other interesting ones? Certainly if $B \cong A$ then B has an algebra structure.

This is exactly like the endomorphism operad of an object, except that instead of the n-ary operations, we are looking at the collection of 'C-ary operations' for *all* objects $C \in \mathcal{K}$:

$$\langle A, A\rangle C = \mathcal{K}(C, A) \pitchfork A$$

So far, everything is $\mathscr{V}$-categorical for any $\mathscr{V}$. Can do the same thing for morphisms. For $A \xrightarrow{f} B$ we get the lower right corner,

$$\begin{tikzcd} T \arrow[r] \arrow[d] & \langle A, A\rangle \arrow[d] \\ \langle B, B\rangle \arrow[r] & \langle A, B\rangle \end{tikzcd}$$

and so if we have algebras $T \to \langle A, A\rangle$ and $T \to \langle B, B\rangle$, this square commuting means that f is a strict map of algebras. Don't really want this to commute strictly, only pseudo or laxly. To give a 2-cell

$$\begin{tikzcd} TA \arrow[r] \arrow[d] \arrow[dr, phantom, "\overline{f}\Nearrow"] & A \arrow[d] \\ TB \arrow[r] & B \end{tikzcd}$$

is the same as

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \langle A, A\rangle \arrow[d] \\ \langle B, B\rangle \arrow[r] & \langle A, B\rangle \end{tikzcd}$$

Thus if we form the *comma object*

$$\begin{tikzcd} \{f, f\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \langle A, A\rangle \arrow[d] \\ \langle B, B\rangle \arrow[r] & \langle A, B\rangle \end{tikzcd}$$

to give a 2-cell as above is the same as to give a 1-cell $T \to \{f, f\}$.

Now $\{f, f\}$ becomes a monad. This is an exercise involving pasting together squares. And in such a way that

$$\begin{tikzcd} \{f, f\} \arrow[r] \arrow[d] & \langle A, A\rangle \\ \langle B, B\rangle & \end{tikzcd}$$

are monad maps (although $\langle A, B\rangle$ is not a monad). Finally, $T \to \{f, f\}$ is a monad map iff $(f, \overline{f})$ is a T-morphism. And of course there is a pseudo version of this. Thus we can work out the algebras and morphisms for a monad just by looking at monad morphisms out of T. This is supposed to justify the importance of free monads and and colimits of monads. Exercise: describe the T-transformations in this way.

3.2. **Pseudomorphisms of monads.** In addition to strict monad maps, where the good colimits live, there are also pseudo maps of monads. A *pseudomorphism* of 2-monads on $\mathcal{K}$ is $T \to S$ which is 2-natural (might also want to consider it being only pseudonatural) and 'looks like a strong monoidal functor'

$$\begin{tikzcd}
T^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & T \arrow[d] \\
T \arrow[r] & S \\
1 \arrow[u] \arrow[ur] \arrow[u, phantom, shift left=3, "\cong"] &
\end{tikzcd}$$

satisfying the same usual coherence conditions. These will be wobbly, as usual.

Note that maps in the 'underlying' place, not preserving anything at all, somehow become straight arrows again.

Now to give

$$T \rightsquigarrow \langle A, A\rangle$$

becomes $TA \xrightarrow{a} A$ in the underlying world, and the 2-cells and their coherence conditions unravel to make A precisely a pseudo-T-algebra (which we never wrote down the full definition of before).

Now, build up to a presentation of the 2-monad for monoidal categories. Put it together step by step, using free monads for operations, and colimits of monads, for more stuff.

3.3. **Locally finitely presentable 2-categories.** The problem is that free monads don't exist and colimits of monads don't exist, for stupid size reasons, so we have to do something to tame them. Assume that $\mathcal{K}$ is a locally finitely presentable 2-category. Various ways I could tell you what that means:

- Some definition, which you don't need to know because I'm not going to prove anything. A cocomplete 2-category with a small full subcategory which is a strong generator and consists of finitely presentable objects.
- Complete and cocomplete and transfinite arguments work more than usual. They almost just work.
- Things (2-equivalent to things) of the form $\mathrm{Lex}(\mathscr{C}, \mathbf{Cat})$ for $\mathscr{C}$ a small 2-category with finite limits, and 'Lex' meaning finite-limit-preserving 2-functors. Can take $\mathscr{C}$ to be $\mathcal{K}_f^{op}$ where $\mathcal{K}_f$ is the full subcategory of finitely presentable objects.
- Examples: can also take $[\mathscr{A}, \mathbf{Cat}]$ for any small 2-category $\mathscr{A}$, or $\mathbf{Cat}^X$ for any set X. The 2-category of groupoids is another example.
- Full reflective subcategories of presheaf categories which are closed under filtered colimits.
- Complete and cocomplete, and is the free cocompletion under filtered colimits of some small thing (an Ind-completion). The latter, plus cocomplete, implies complete as well.

Again, this is all $\mathscr{V}$, nothing particularly special about $\mathbf{Cat}$, although need $\mathscr{V}$ to be somewhat nicer than usual.

Then to give a finitary 2-functor $\mathcal{K} \to \mathcal{K}$ (meaning filtered-colimit-preserving) is to give an arbitrary one $\mathcal{K}_f \to \mathcal{K}$. Write $\mathrm{End}_f(\mathcal{K}) \simeq [\mathcal{K}_f, \mathcal{K}]$ for the monoidal category of finitary endo(-2-)functors on $\mathcal{K}$. Unlike $[\mathcal{K}, \mathcal{K}]$ this is locally small, since $\mathcal{K}_f$ is small.

We also have $\mathrm{Mnd}_f(\mathcal{K})$, the finitary 2-monads on $\mathcal{K}$, meaning that the functor part is finitary. This is just the monoids in $\mathrm{End}_f(\mathcal{K})$. In this world, we do have a left adjoint

$$\begin{tikzcd}
\mathrm{Mnd}_f(\mathcal{K}) \arrow[d, shift left] \\
\mathrm{End}_f(\mathcal{K}) \arrow[u, bend left]
\end{tikzcd}$$

Can also regard this as a partial adjunction in the unrestricted case, defined on the finitary objects. That sort of is important because the $\langle A, A\rangle$ things are not finitary, although we can sort of coreflect them into the finitary things.

Everything is still true if you replace 'finite' by some regular cardinal α, if you like that sort of thing.

Moreover, $\mathrm{End}_f(\mathcal{K})$ and $\mathrm{Mnd}_f(\mathcal{K})$ are themselves lfp 2-categories, so they are complete and cocomplete. Moreover, colimits in the good 2-category of finitary 2-monads are still colimits upstairs in the category of all 2-monads.

Moreover, the adjunction is monadic; there is a 2-monad on $\mathrm{End}_f(\mathcal{K})$ for which $\mathrm{Mnd}_f(\mathcal{K})$ is the strict algebras and strict morphisms. We can drop down even further to get

$$\begin{tikzcd}
\mathrm{Mnd}_f(\mathcal{K}) \arrow[d, shift left, "W"] \arrow[dd, bend left=60, "U"] \\
\mathrm{End}_f(\mathcal{K}) \arrow[u, bend left, "H"] \arrow[d, shift left, "V"] \\
{[ob\, \mathcal{K}_f, \mathcal{K}]} \arrow[u, bend left, "H"] \arrow[uu, bend left=60, "F"]
\end{tikzcd}$$

and go back up by left Kan extension. The lower adjunction is also monadic, and actually all of these are monadic.

Depending on which monad you think of the top as algebras over, it affects what the pseudomorphisms and pseudoalgebras will be. Dropping down one level, the transformations are 2-natural ones, while if we drop down the whole way, they will be only pseudonatural.

3.4. **Presentations.** What I regard as the most basic generator for a 2-monad is an object of the bottom, i.e. an $X\colon ob\,\mathcal{K}_f \to K$, $c \mapsto Xc$. Can then construct the free 2-monad on such a thing, FX.

What is an FX-algebra? A monad map

$$FX \to \langle A, A\rangle$$

which is the same as

$$X \to U\langle A, A\rangle.$$

(Here we're being a little innacurate, since $\langle A, A\rangle$ is not finitary, but you can either coreflect (given by restriction) or do stuff we said before). This just means that for each c, we have

$$Xc \to \langle A, A\rangle c$$

which unravels to a functor

$$\mathcal{K}(c, A) \to \mathcal{K}(Xc, A)$$

between hom-categories. Since $\mathcal{K}$ is cocomplete (including tensored), this is the same as a map

$$\sum_c \mathcal{K}(c, A) \cdot Xc \longrightarrow A$$

Thus we can think of Xc as the 'object of all c-ary operations'.

Example 3.1. Let $\mathcal{K} = \mathbf{Cat}$, so $\mathcal{K}_f$ is the finitely presentable categories, and X assigns to every such c a category Xc of c-ary operations. We take

$$c \mapsto \begin{cases} 1 & c = 0, 2 = 1 + 1 \\ 0 & o/w \end{cases}$$

(2 is the discrete category 2). Thus we have one binary operation and one nullary operation. An FX-algebra is then a category A with maps as above. If Xc is empty, then $\mathcal{K}(Xc, A)$ is terminal, so there's nothing to do. In the other cases, we get maps

$$A^2 \to A$$

when $c = 2$ and

$$A^0 = 1 \to A$$

when $c = 0$. This is the first step along our path of building up the 2-monad for monoidal categories. The morphisms, which we can work out, will preserve both of these $\otimes$ and I, up to isomorphism, but nothing else to do, since FX is free.

Example 3.2. Again let $\mathcal{K} = \mathbf{Cat}$, and let

$$X^*c = \begin{cases} \nvDash & c = 1 \\ 0 & o/w \end{cases}$$

Then an FX^* algebra is a category with a map

$$A \to A^{\nvDash}$$

in other words, a pair of maps with a natural transformation

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd} .$$

This is an example when Xc is not discrete.

In the case of monoidal categories, there are operations of 'type arrow' but I'm not going to construct them that way, although you could. It's not a nice way.

3.5. **Monoidal categories.** Actually,, let's forget about the units, just worry about the binary operation. Then Xc is 1 if $c = 2$ and 0 otherwise, so an FX-algebra is a category with a single binary operation. Then we have

$$\begin{tikzcd} & FX\text{-}\mathbf{Alg}_s \arrow[dl, "U_s"'] \arrow[dr, "U_s"] & \\ \mathbf{Cat} \arrow[dr, "\mathbf{Cat}(3,-)"'] & & \mathbf{Cat} \arrow[dl, "\mathbf{Cat}(1,-)"] \\ & \mathbf{Cat} & \end{tikzcd}$$

which sends

$$\begin{tikzcd} & (A,\otimes) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ A \arrow[dr, mapsto] & & A \arrow[dl, mapsto] \\ & A^3 \rightrightarrows A & \end{tikzcd}$$

and we have the two maps $\otimes(\otimes 1)$ and $\otimes(1\otimes)$ which are natural, so we get two transformations in the above square. Want to make them isomorphic.

Take the mate along the adjoint of U_s to get

$$\begin{tikzcd} & FX\text{-}\mathbf{Alg}_s \arrow[dr, "U_s"] & \\ \mathbf{Cat} \arrow[ur, "F_s"'] \arrow[dr, "\mathbf{Cat}(3,-)"'] & & \mathbf{Cat} \arrow[dl, "\mathbf{Cat}(1,-)"] \\ & \mathbf{Cat} & \end{tikzcd}$$

with two 2-cells in the middle. Note that $U_sF_s = FX$ is the monad. Now we can drop the identity functor $\mathbf{Cat}(1,-)$ (which we put in to clarify more general versions) and we have have these two transformations

$$\mathbf{Cat}(3,-) \rightrightarrows FX,$$

which are morphisms of endofunctors. We can now soup them up to get monad morphisms

$$H\mathbf{Cat}(3,-) \rightrightarrows FX.$$

Then if we also have

$$H\mathbf{Cat}(3,-) \rightrightarrows FX \longrightarrow \langle A, A\rangle,$$

restriction along the two arrows is picking out the two structure maps $\otimes(\otimes 1)$ and $\otimes(1\otimes)$. Now in the 2-category $\mathrm{Mnd}_f(\mathcal{K})$ we construct the universal thing $FX \to S$ with an isomorphism between the two restrictions. This is called a *co-iso-inserter*, another 2-categorical colimit (completely strict) which we will meet later on.

Now, an S-algebra is a category A with a functor $\otimes\colon A^2 \to A$ and a natural isomorphism $\alpha\colon \otimes(1\otimes) \cong \otimes(\otimes 1)$. Can also write down what it means to be a pseudo or lax morphism of such algebras, and it's what you want it to be; the tensor-preserving isomorphisms are forced to commute with this associator.

To do the coherence condition, we have a pair of 2-cells

$$\begin{tikzcd} H\mathbf{Cat}(4,-) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & S \end{tikzcd}$$

which encode the two isomorphisms that one wants to make equal. Now we form the *coequifier* $q\colon S \to T$, in the category of monads, of these two 2-cells, the universal thing with the property that $q\gamma = q\beta$ of making them equal.

Then the 2-category T-$\mathbf{Alg}$ is the 2-category of 'semigroupoidal categories' and strong morphisms (we can get the strict and lax morphisms in the obvious way too). All this follows from the universal property of the monad T.

3.6. **Some More Examples.**

3.6.1. *Terminal Objects.* Consider the structure of *category with terminal object*. How do you say algebraically that a category has a terminal object? Give an object

$$1 \xrightarrow{t} A$$

with a natural transformation

$$\begin{tikzcd}
A \arrow[rr, bend left, "A"] \arrow[dr] \arrow[rr, phantom, "\Downarrow\tau"] & & A \\
& 1 \arrow[ur] &
\end{tikzcd}$$

such that the component of t

$$\begin{tikzcd}
1 \arrow[r, "t"] & A \arrow[rr, bend left, "A"] \arrow[dr] \arrow[rr, phantom, "\Downarrow\tau"] & & A \\
& & 1 \arrow[ur] &
\end{tikzcd}$$

is the identity. This is a baby example, but you can do any limist you like once you understand this example.

Let's give a presentation for it. Two ways we can think of doing it. The way from last time was starting with basic operations and build up more stuff using colimits. Never said this, but there was this mystical thing: all the object-type stuff at the beginning, then all the colimits were never imposing equations between objects, only making objects isomorphic or 2-cells equal.

Could do that in this case. t is a nullary operation, so that would involving giving something

$$\mathbf{Cat}(0, A) \to \mathbf{Cat}(1, A)$$

which corresponds to an X which takes value 1 on $c = 0$ and 0 everywhere else. Could now form the corresponding free monad and then do some coinserters and coequifiers.

Here's a slightly different way. Start the same, with a nullary operation

$$\mathbf{Cat}(0, A) \xrightarrow{t} \mathbf{Cat}(1, A)$$

and then put in a unary operation of type arrow:

$$\mathbf{Cat}(1, A) \xrightarrow{\tau} \mathbf{Cat}(\not\models, A)$$

which specifies two endormorphisms of A and a natural transformation between them.

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\tau"] & A
\end{tikzcd}$$

Now have to say what f and g are. Can now start building up 'derived operations' which are produced by composites and functoriality from the original ones; we see them in the free monad. We have

$$\begin{tikzcd} \mathbf{Cat}(1,A) \arrow[r, "\tau"'] \arrow[rr, bend left, "1"] & \mathbf{Cat}(\nvDash,A) \arrow[r, "\mathbf{Cat}(d_0,A)"'] & \mathbf{Cat}(1,A) \end{tikzcd}$$

and the equality of these says that $f = 1_A$. Then we have

$$\begin{tikzcd} \mathbf{Cat}(1,A) \arrow[r, "\tau"] \arrow[d, "\mathbf{Cat}(!,A)"'] & \mathbf{Cat}(\nvDash,A) \arrow[d, "\mathbf{Cat}(d_1,A)"] \\ \mathbf{Cat}(0,A) \arrow[r, "t"'] & \mathbf{Cat}(1,A) \end{tikzcd}$$

whose commutativity makes $g = t\circ !$. Finally, we have

$$\begin{tikzcd} \mathbf{Cat}(0,A) \arrow[r, "t"] \arrow[d, "t"'] & \mathbf{Cat}(1,A) \arrow[d, "\mathbf{Cat}(1,A)"] \\ \mathbf{Cat}(1,A) \arrow[r, "\tau"'] & \mathbf{Cat}(\nvDash,A) \end{tikzcd}$$

whose commutativity makes the last axiom true.

Note that here the algebras are categories with a *chosen* terminal object. This seems strange, but not really a problem. The strict morphisms preserve the chosen terminal object strictly, which is stupid, but the pseudo morphisms just preserve it in the usual way.

3.6.2. *Bicategories.* Two reasons for doing this one: an example when $\mathcal{K} \neq \mathbf{Cat}$, and also important for 2-nerves.

Let $\mathcal{K} = \mathbf{Cat}$-Grph, the 2-category of category-enriched graphs. A $\mathbf{Cat}$-graph consists of a set of things $G, H, \dots$ and hom-categories $\mathscr{G}(G,H) \in \mathbf{Cat}$. (Can do this for any $\mathscr{V}$, of course.) A morphism is a function on sets and functors on hom-categories. Can't express naturality for the 2-cells, and we could take them to be just components that have no naturality, but here what we do is require that for a 2-cell

$$\begin{tikzcd} \mathscr{G} \arrow[r, bend left, "F"] \arrow[r, bend right, "F1"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{H} \end{tikzcd}$$

to exist, we must have $F = F'$ on objects, and then we ask for natural transformations

$$\begin{tikzcd} \mathscr{G}(G,H) \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{H}(FG,FH) \end{tikzcd}$$

on all hom-categories.

Now, given a $\mathbf{Cat}$-graph, what do you need to do to turn it into a bicategory? Have to give compositions

$$\mathscr{G}(H,K) \times \mathscr{G}(G,H) \longrightarrow \mathscr{G}(G,K).$$

Let c be the (locally discrete) **Cat**-graph $(\cdot \to \cdot \to \cdot)$. Then it turns out that

$$\mathcal{K}(c, scrG) = \sum_{G,H,K} \mathscr{G}(H,K) \times \mathscr{G}(G,H).$$

If we define $Xc = (\cdot \to \cdot)$, then

$$\mathcal{K}(Xc, scrG) = \sum_{G,K} \mathscr{G}(G,K).$$

To get the individual maps, not just the sums, we need to put in an equation. Have two maps $1 \to Xc$, compose it with both of those and say that it's the right thing.

That's all I want to say about the construction. There's a lot more stuff of course. What we get is:

- An algebra is a bicategory.
- A lax morphism is a lax functor.
- A pseudo morphism is a pseudo functor.
- A strict morphism is a strict functor.
- A 2-cell is an *icon*. This is an oplax natural transformation (which we haven't officially met yet) for which the 1-cell components are identities. ICON stands for 'Identity Component Oplax Natural-transformation'.

$$\begin{tikzcd} FA \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\swarrow"] & GA \arrow[d] \\ FB \arrow[r, equal] & GB \end{tikzcd}$$

These are what turn out to be the monoidal natural transformations when we restrict to one-object ones.

These guys are also just nice enough to give us a 2-category of bicategories. In general, lax natural transformations between lax functors can't even be whiskered by lax functors.

$$\begin{tikzcd} {} \arrow[r] & {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {} \arrow[r] & {} \end{tikzcd}$$

For pseudo things, we can define it, but composition fails to be associative, so we only get a tricategory. But with just icons, we do get a 2-category, which is moreover a category of algebras for (this) 2-monad. Still not perfect: this restriction is quite severe and for some things you just don't want to have to do that, need to go to the full tricategory, but it's good enough for lots of things.

For example, in this 2-category, it's true that every bicategory is equivalent (in the 2-category) to a 2-category, since we don't have to do anything to the objects to strictify. This 2-category also turns out to be a full sub-2-category of simplicial objects in **Cat**, via nerve constructions, once we restrict to normal homomorphisms by working with *reflexive* **Cat**-graphs to start out with.

The reason oplax transformations are called 'oplax' is due to Benabou. He got the functors right: lax functors are much more important than oplax ones. But it definitely seems that he got it wrong for the transformations: oplax ones are clearly the good ones.

3.6.3. *Cartesian closed categories.* Could be monoidal closed, symmetric monoidal closed, doesn't make much difference for the point I want to make. Here we have this internal-hom thing

$$A^{op} \times A \to A$$

and we're just not allowed to talk about A^{op} the way we're doing things. Our operations are supposed to be of the sort $A^c \to A$. How can we deal with this?

In fact, it's a theorem that cccs *don't* have the form T-**Alg** over **Cat**. Have to change our base 2-category again, take $\mathcal{K} = \mathbf{Cat}_g$, the 2-category of categories, functors, and natural *isomorphisms*. Recall that $\mathbf{Cat}(2, A)$ is just $A \times A$, but in $\mathbf{Cat}_g(2, A)$ we have only $A_{iso} \times A_{iso}$. The internal-hom *does* give us a map

$$A_{iso} \times A_{iso} \longrightarrow A_{iso}$$

which is

$$\mathbf{Cat}_g(2, A) \longrightarrow \mathbf{Cat}_g(1, A)$$

since we can turn around an isomorphism in the first variable to make contravariant into covariant. This gives a new problem; we have to put in the rest of the functoriality separately by hand, which is a real pain if you try to do it in detail.

Put in another operation

$$\mathbf{Cat}_g(\nvDash + \nvDash, A) \longrightarrow \mathbf{Cat}_g(\nvDash, A)$$

which encodes the effect on morphisms of $[-,-]$. Then have to specify domains and codomains, encode that this is a functor, that it has the right universal property. We also have to make the product functorial by hand.

Observe that anything monadic over **Cat** gives induced monads on $\mathbf{Cat}_g$ and on the 1-category $\mathbf{Cat}_0$ (since things are stable under change of base enriching category, categories to groupoids to sets). But at each stage, to present the same structure becomes harder. We want to stop at the groupoid-enriched stage so that we can talk about pseudo morphisms, which we wouldn't be able to do in $\mathbf{Cat}_0$. We don't have an independent notion of lax morphism any more, though, since they all reduce to pseudo ones. For some things this is quite adequate, but for others it's not so good; that's life.

3.6.4. *Diagram 2-categories.* The first version of this is not really an example of a presentation at all, since the 2-monad pops out for free. Let $\mathscr{C}$ be a small 2-category, and consider the 2-category $[\mathscr{C}, \mathbf{Cat}]$ of (strict) 2-functors, 2-natural transformations, and modifications. This is the **Cat**-enriched functor category. Can now forget down:

$$\begin{tikzcd}
{[\mathscr{C}, \mathbf{Cat}]} \arrow[d, "U_s"] \\
{[ob\,\mathscr{C}, \mathbf{Cat}]} \arrow[u, bend left] \arrow[u, bend right]
\end{tikzcd}$$

which has both adjoints given by left and right Kan extension. The existence of the right adjoint tells us that the forgetful functor preserves all colimits. In this case U_s is strictly monadic (purely enriched notion); easy to prove with the enriched Beck's theorem. The induced monad T then preserves *all* colimits, and we can write, using the Kan extension formula,

$$(TX)c = \sum_d \mathscr{C}(d, c) \cdot Xd.$$

It's now fairly easy to check that

- pseudo T-algebras are pseudo-functors,
- lax algebras are lax functors,
- pseudo morphisms are pseudo-natural transformations,
- etc.

Everything works (perhaps the lax and the oplax get switched?). Note that when you write down the coherence conditions for a lax morphism it will tell you more than is in the *definition* of a lax functor, which is because the latter is the 'minimum' necessary to make 'everything' true.

Now let $\mathscr{C}$ be a bicategory. Could try to do the same game, but wouldn't get a 2-monad, since the associativity of the multiplication for the monad corresponds to the associativity of composition in $\mathscr{C}$, so we'd just get a pseudo-monad. We could do this, but we've been avoiding pseudo-monads, so let's continue. We can instead give a presentation for a 2-monad T on $[ob\,\mathscr{C}, \mathbf{Cat}]$ whose

- (strict) algebras are pseudofunctors $\mathscr{C} \to \mathbf{Cat}$,
- pseudomorphisms of algebras are pseudonatural transformations,
- etc.

The target doesn't really need to be **Cat**. Need to have coproducts at least for the first case. In the second case, need all colimits. But then that's enough.

4. Limits

Let's start with some concrete examples. Talk about limits in T-**Alg**, as a stepping-stone to the general notion.

4.1. Limits in T-Alg.

As always, T-**Alg** is the strict algebras and pseudo morphisms for a nice 2-monad T on a nice 2-category $\mathcal{K}$.

4.1.1. *Terminal objects.* Let's start with something really easy: terminal objects. Let 1 be terminal in $\mathcal{K}$; we have a unique map $T1 \to 1$, making 1 a T-algebra, and then for any other A we have a unique $! : A \to 1$, and

$$\begin{tikzcd} TA \arrow[r, "T!"] \arrow[d] & T1 \arrow[d] \\ A \arrow[r, "!"'] & 1 \end{tikzcd}$$

commutes strictly, so there's a unique *strict* algebra morphism $A \to 1$. Moreover, by the 2-universal property of 1, there's a unique isomorphism in the above square, which happens to be an identity; thus there is only one pseudo morphism as well (which happens to be strict). A similar argument works for endomorphisms of this morphism; thus

$$T\text{-}\mathbf{Alg}((A, a), (1, !)) \cong 1$$

so $(1, !)$ is a terminal object in T-**Alg**.

4.1.2. *Products.* Similarly for products, we have an obvious map

$$T(A \times B) \to TA \to TB \to A \times B$$

and so on. This is just like ordinary monads, nothing 2-categorical going on here. The point is that if we have some *pseudo* morphisms

$$\begin{tikzcd} TC \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & TA \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

$$\begin{tikzcd} TC \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & TB \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

we get

$$\begin{tikzcd} TC \arrow[r] \arrow[d] & T(A \times B) \arrow[d] \\ C \arrow[r, "(f,g)"'] & A \times B \end{tikzcd}$$

And to give a 2-cell in here is, by the universal property of $A \times B$, just to give two 2-cells, one into A and one into B, so we get one. Can check the universal property, so we get

$$T\text{-}\mathbf{Alg}(C, A \times B) \cong T\text{-}\mathbf{Alg}(C, A) \times T\text{-}\mathbf{Alg}(C, B)$$

These are strict 2-limits (enriched **Cat**-limits). Might consider weakening these things, which can be a good thing to do, but it's even better here.

Note that the projections $A \times B \to A$ and $A \times B \to B$ are actually strict maps, by construction. Moreover, they jointly 'detect strictness': a map into $A \times B$ is strict if and only if its composites into A and B are strict. This is a useful technical property.

Actually, we didn't really need to check anything, since we've already seen that $T\text{-}\mathbf{Alg}_s \hookrightarrow T\text{-}\mathbf{Alg}$ has a left adjoint, hence preserves all limits, and in the case of terminal objects and products, the thing we're taking a limit of is just objects, so it already exists in the strict world. Although having the adjoint involves some transfinite constructions, which might not exist in some places, but anyway you can just check that the limits are preserved.

4.1.3. *Equalizers.* Now let's look at equalizers. Here it's different, because the morphisms might not be strict morphisms. If they are, then the equalizer exists in $T\text{-}\mathbf{Alg}_s$ and is preserved, but if they aren't, the adjunction doesn't help. In fact, in general equalizers of pseudo morphisms need *not* exist.

For example, let T be the 2-monad on **Cat** for categories with a terminal object. Let 1 be the terminal category and let $\mathscr{I}$ be the free-living isomorphism. Clearly both have a terminal object, and both inclusions are pseudo morphisms. But any functor which equalizes them has to have empty domain, and no category with an empty domain has a terminal object.

4.1.4. *Equifiers.* Thus T-**Alg** is not complete, and that's that. But we can look at some of the limits that it does have. Consider *equifiers.* Here we have a parallel pair of 1-cells with a parallel pair of 2-cells between them:

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow\ \Downarrow\beta"] & B
\end{tikzcd}$$

and we want the universal 1-cell $K \xrightarrow{k} A$ such that $\alpha k = \beta k$. In other words, $\mathcal{K}(D, C)$ is *isomorphic* (not equivalent) to the category of morphisms $D \xrightarrow{h} A$ with $\alpha h = \beta h$. We do have equifiers. This is as we expect; we can make 2-cells equal, but not 1-cells, because of the pseudo morphisms.

In T-**Alg**, we have

$$\begin{tikzcd}
CT \arrow[r, "Tk"] \arrow[d, dotted, "c"'] & TA \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "T\alpha\Downarrow\ \Downarrow T\beta"] \arrow[d] \arrow[dr, phantom, "\Swarrow\overline{f}\quad f\quad \Swarrow\overline{g}"] & TB \arrow[d] \\
C \arrow[r, "k"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow\ \Downarrow\beta"] & B
\end{tikzcd}$$

with isomorphisms $\overline{f}$ on the back and $\overline{g}$ on the front, making the cylinders commute. Let $C \xrightarrow{k} A$ be the equifier in $\mathcal{K}$; we want to make C an algebra such that the left square commutes. To factor through C (in fact strictly) we just need to show that $\alpha a.Tk = \beta a.Tk$. We have the composite

$$\begin{tikzcd}
TC \arrow[r, "Tk"] & TA \arrow[r, "a"'] \arrow[rr, bend left=50, "Tf"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[u, phantom, "\Downarrow\overline{f}"] & B
\end{tikzcd}$$

Since α is an algebra 2-cell, we can replace $\alpha\overline{f}$ with $\overline{g}.T\alpha$:

$$\begin{tikzcd}
 & & TB \arrow[dr, "b"] & \\
TC \arrow[r, "Tk"] & TA \arrow[ur, bend left, "Tf"] \arrow[ur, bend right, "Tg"'] \arrow[ur, phantom, "\Swarrow T\alpha"] \arrow[dr, "a"'] & \overline{g} & B \\
 & & A \arrow[ur, "g"'] &
\end{tikzcd}$$

But $\alpha k = \beta k$, so by functoriality $T\alpha.Tk = T\beta.Tk$, getting

$$\begin{tikzcd}
 & & TB \arrow[dr, "b"] & \\
TC \arrow[r, "Tk"] & TA \arrow[ur, bend left, "Tf"] \arrow[ur, bend right, "Tg"'] \arrow[ur, phantom, "\Swarrow T\beta"] \arrow[dr, "a"'] & \overline{g} & B \\
 & & A \arrow[ur, "g"'] &
\end{tikzcd}$$

and then we use the fact that β is an algebra 2-cell backwards, to get that the first composite is equal to

$$\begin{tikzcd} TC \arrow[r, "Tk"] & TA \arrow[r, "a"'] \arrow[rr, bend left=50, "Tf"] & A \arrow[u, phantom, "\Downarrow\overline{f}"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd} .$$

Finally, since $\overline{f}$ is invertible, we can cancel them off the pasting diagrams to get $\alpha a.Tk = \beta a.Tk$ as desired. Thus we get the unique induced c as desired. Have to check that $TC \xrightarrow{c} C$ is an algebra ($c.iC = 1$ and $c.Tc = c.mC$), which is just like the usual case, since C is a limit and both of these are things going into C. We now have a strict algebra morphism $C \to A$; then also check the universal property for this in T-**Alg**.

Observe that once again, the projection map k of the limit is actually a strict map, and detects strictness of incoming maps.

What fails for the equalizer? Could do the same thing, but to get a c we'd need to know that $f.a.Ti = g.a.Ti$, but we'd only get these isomorphic to $b.Tf.Ti$ and $b.Tg.Ti$, respectively, which are equal, but this only tells us that $f.a.Ti \cong g.a.Ti$, rather than equal, so we don't get a factorization. This is stupid thing you shouldn't expect, so we shouldn't really expect to have equalizers.

4.1.5. *Inserters.* What about *inserters*? These play a bit like the role of equalizers; you can't make 1-cells equal, but you can make them isomorphic, or (as here) just put a 2-cell in between them. Given $A \rightrightarrows B$, this is the universal $C \to [k]A$ together with $fk \to [\kappa]gk$. In detail, $\mathcal{K}(D, C)$ should be isomorphic to the category whose objects are morphisms $D \to [\ell]A$ equipped with $f\ell \to [\lambda]f\ell$, and whose morphisms are 2-cells

$$\begin{tikzcd} D \arrow[r, bend left, "\ell"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd}$$

such that

$$\begin{tikzcd} f\ell \arrow[r, "\lambda"] \arrow[d, "f\alpha"'] & g\ell \arrow[d, "g\alpha"] \\ fm \arrow[r, "\kappa"'] & gm \end{tikzcd}$$

commutes.

Once again, inserters in $\mathcal{K}$ lift to T-**Alg**, where they have strict projections and detect strictness. Given a pair

$$\begin{tikzcd} (A, a) \arrow[r, bend left, "(f, \overline{f})"] \arrow[r, bend right, "(g, \overline{g})"'] & (B, b) \end{tikzcd}$$

of pseudo morphisms, we construct the inserter (k, κ) of f and g, and want to make it an algebra. We need to get from $f.a.Tk$ to $g.a.Tk$ to induce c, so we follow our nose:

$$f.a.Tk \cong b.Tf.Tk \xrightarrow{b.T\kappa} b.Tg.Tk \cong g.a.Tk$$

This thing then must be κc for a unique c, by the universal property of the inserter in $\mathcal{K}$. Check that c makes C into an algebra, and so on; everything goes through just as before.

Observe that inserters in a category with no nonidentity 2-cells is the same as an equalizer.

4.1.6. *PIE-limits.* Thus T-**Alg** has products, inserters, and equifiers, so it has *PIE-limits*, defined to be precisely the limits we can build up out of products, inserters, and equifiers. This is quite a good class, for various reasons. Equalizers aren't in here, but we do get the following:

- *iso-inserters*, which are inserters where we ask the 2-cell to be invertible. We insert something going the other way, then equify their composites with identities. But iso-inserters don't suffice to construct inserters.
- *cotensors* by categories. Given a category C, we first use products to do discrete C, then use inserters to put the morphisms in, and then do some equifiers to make it functorial.
- *inverters*, where we start with a 2-cell α and make it invertible, i.e. k is universal such that αk is invertible. Do this as before: insert something going back the other way, then equify composites with the identity. Observe that *coinverters* in **Cat** are also known as 'categories of fractions' $\mathscr{C}[\Sigma^{-1}]$.

4.2. **A Meta-Comment.** I haven't been very careful about describing results, and probably won't. I have started preparing a references; maybe wait until finished and then give it to you. Not intended to be complete, but contains all my favorite stuff.

4.3. **Weighted Limits.** Let $\mathscr{C} \xrightarrow{S} \mathcal{K}$ be a functor between, say, ordinary categories. The limit is supposed to be defined by the fact that maps

$$\begin{aligned}\mathcal{K}(A, \lim S) &\cong \mathrm{Cone}(A, S)\\ &= [\mathscr{C}, \mathcal{K}](\Delta A, S)\\ &= [\mathscr{C}, \mathbf{Set}](\Delta 1, \mathcal{K}(A, S)).\end{aligned}$$

In other words, for each $C \in \mathscr{C}$ we have a map $1 \to \mathcal{K}(A, SC)$ in **Set**, hence a map $A \to SC$, and naturality in C is precisely what gives you a cone. Don't assign any importance to the difference between $\cong$ and $=$. This last is the definition of limit we're going to use today, and ask what happens if we change $\Delta 1$ to something else.

Example 4.1. No one really uses this in practice, but it's useful to think about. Let $\mathscr{C} = 2$ have two objects, so a functor $S\colon \mathscr{C} \to \mathcal{K}$ is a pair of objects B and C, and a weight is a functor $J\colon \mathscr{C} \to \mathbf{Set}$, say it sends one to 2 and the other to 3. Then

$$[\mathscr{C}, \mathbf{Set}](J, \mathcal{K}(A, S))$$

consists of $2 \to \mathcal{K}(A, B)$ and $3 \to \mathcal{K}(A, C)$, thus 2 arrows $A \to B$ and 3 arrows $A \to C$. Thus this becomes an arrow $A \to B^2 \times C^3$, so we get a 'weighted' product. This is quite contrived, but you see where the word 'weight' comes from.

For general $\mathscr{V}$, have $S\colon \mathscr{C} \to \mathcal{K}$ and $J\colon \mathscr{C} \to \mathscr{V}$ both $\mathscr{V}$-functors, and we consider

$$[\mathscr{C}, \mathscr{V}](J, \mathcal{K}(A, S)).$$

If this is representable as a functor of A, the representing object is called the *J-weighted limit* of S and written $\{J, S\}$. Thus we have a natural isomorphism

$$\mathcal{K}(A, \{J, S\}) \cong [\mathscr{C}, \mathscr{V}](J, \mathcal{K}(A, S)).$$

which defines the limit.

Exercise 4.2. If $\mathcal{K} = \mathscr{V}$, then $\{J, S\}$ is the internal hom $[\mathscr{C}, \mathscr{V}](J, S)$.

That's all I want to say about general $\mathscr{V}$.

When $\mathscr{V} = \mathbf{Set}$, this is a good thing to think about, but it doesn't give you any *new* limits that didn't exist already. I.e. if $\mathcal{K}$ is an ordinary category which is complete in the usual sense of having all conical limits ($J = \Delta 1$), then it also has all of these weighted limits. But the weighted ones are more expressive, so it's still useful to think about them. Think of the weight J as determining the type of limit, not just the diagram shape $\mathscr{C}$.

But for $\mathscr{V} \neq \mathbf{Set}$ it's not in general true. But if you have all conical limits *and* cotensors, you can construct all weighted limits. A little bit subtle because the constant functor at 1 is usually not what you want to look at, nor need it even exist. Instead of maps out of 1, you want to look at maps out of I, the unit object of $\mathscr{V}$. And ΔI may not exist either, unless $\mathscr{C}$ is the free $\mathscr{V}$-category on some ordinary category $\mathscr{B}$. But we can talk about those sorts, and together with cotensors you can construct all weighted limits from them.

4.4. **Cat-weighted limits.**

Example 4.3 (Inserters). Let $\mathscr{C} = (\rightrightarrows)$, so S is determined by a parallel pair of arrows $A \rightrightarrows B$. The weight $J\colon S \to \mathbf{Cat}$ sends it to $(1 \rightrightarrows \nvDash)$. Then to give a natural transformation $J \to \mathcal{K}(C, S)$ gives us $1 \to \mathcal{K}(C, A)$, hence an arrow $h\colon C \to A$, and $\nvDash \to \mathcal{K}(C, B)$, hence

$$\begin{tikzcd} C \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} .$$

But by naturality the two 1-cells must be fh and gh, so the data consists of $h\colon C \to A$ and a 2-cell $\beta\colon fh \to gh$.

Need then to check that it gets the arrows right, since we have not just a 1-dimensional property but a 2-dimensional one. But in the presence of tensors, we get that for free.

Example 4.4 (Equifiers). Here, our category $\mathscr{C}$ is

$$\begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Downarrow"] & {} \end{tikzcd}$$

and our weight is

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Downarrow"] & \nvDash \end{tikzcd}$$

in which the two 2-cells get mapped to the same 2-cell in **Cat**.

Example 4.5 (Comma objects). $\mathscr{C}$ is the same shape for pullbacks

$$\begin{tikzcd} & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

and J is

$$\begin{tikzcd} & 1 \arrow[d, "1"] \\ 1 \arrow[r, "0"'] & \nvDash \end{tikzcd} .$$

There is no 2-cell in $\mathscr{C}$, since we don't *start* with a 2-cell, we only add one universally (hence it is in J).

Example 4.6 (Inverters). Recall, this is where we start with a 2-cell and make it universally invertible. Then $\mathscr{C}$ is

$$\begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {} \end{tikzcd}$$

and J is

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathscr{I} \end{tikzcd}$$

where $\mathscr{I}$ is the 'free-living isomorphism' $\cdot \rightleftarrows \cdot$.

Suppose in general we have some 'limit-notion' which we know in advance is a weighted limit, but we don't know what the weight is, here's what you can do. (This is the sort of trick you should never give away. I'm going to give it away, but I'm not going to give away the reason, so maybe it'll still be slightly mystical.) Consider the version of the Yoneda embedding $\mathscr{C} \to [\mathscr{C}, \mathscr{V}]^{op}$ and take its 'limit', for the notion of limit we're interested in. Never actually seen that written down anywhere.

4.5. **Pseudolimits.** Now we are interested in

$$\mathcal{K}(A, \operatorname{pslim} S) \cong \operatorname{Ps}(\mathscr{C}, \mathbf{Cat})(\Delta 1, \mathcal{K}(A, S)).$$

where $\operatorname{Ps}(\mathscr{A}, \mathscr{B})$ is the 2-category of 2-functors, pseudonaturals, and modifications from $\mathscr{A}$ to $\mathscr{B}$. The right side is what we mean by a *pseudo-cone*. Note that this is still an *isomorphism* of categories, not an equivalence.

Example 4.7 (Pseudopullbacks). Again we take $\mathscr{C}$ to be

$$\begin{tikzcd} & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

A pseudo-cone then consists of

$$\begin{tikzcd} {} \arrow[drr, dotted] \arrow[ddr, dotted] \arrow[ddrr, dotted] \arrow[ddrr, phantom, bend left, "\cong"] \arrow[ddrr, phantom, bend right, "\cong"] \\ && {} \arrow[d] \\ & {} \arrow[r] & {} \end{tikzcd}$$

with isomorphisms in each triangle. We have made the cones commute only up to isomorphisms, but the universal property and factorizations are still strict. Note that the pseudopullback is *equivalent* (not isomorphic) to the *isocomma object* (assuming both exist). In the latter, we specify $fa \cong gb$ without specifying the middle diagonal arrow. Of course, we can take it to be fa, or gb, so we get ways of going back and forth.

But they are not in general equivalent to the pullback ('not all pullbacks are homotopy pullbacks'). But if either is a fibration in the suitable sense (the categorical model structure on **Cat**), then they are equivalent to the pullback.

Again, given a weight $J \colon \mathscr{C} \to \mathbf{Cat}$, the *weighted pseudolimit* is defined by

$$\mathcal{K}(C, \{J, S\}_{ps}) \cong \operatorname{Ps}(\mathscr{C}, \mathbf{Cat})(J, \mathcal{K}(C, S)).$$

I don't really want to do any examples of this one. I want to do some general nonsense instead.

Remember than $\operatorname{Ps}(\mathscr{C}, \mathbf{Cat}) = T\text{-}\mathbf{Alg}$ for some T on $ob\,\mathscr{C}$-indexed families of **Cat**. And we have $T\text{-}\mathbf{Alg}_s = [\mathscr{C}, \mathbf{Cat}]$, so

$$[\mathscr{C}, \mathbf{Cat}] \hookrightarrow \operatorname{Ps}(\mathscr{C}, \mathbf{Cat})$$

has a left adjoint which rejoices under the name of $(-)'$. Thus

$$\operatorname{Ps}(\mathscr{C}, \mathbf{Cat})(J, \mathcal{K}(C, S)) \cong [\mathscr{C}, \mathbf{Cat}](J', \mathcal{K}(C, S))$$

which just defines the universal property for the J'-weighted limit. In other words, *pseudolimits are not some more general thing, but a special case of ordinary (weighted) limits*. Thus we say that a weight 'is' a pseudolimit if it is J' for some J.

This is typical. Recall, for example, that pseudo-algebras for monads are strict algebras over a cofibrant replacement monad. Thus talking about things of the form PsT-**Alg** is actually *less* general than things of the form T-**Alg**, since everything of the former form has the latter form, but not conversely.

Modulo details, it is also true that homotopy limits are a special case of weighted limits.

4.6. **PIE-limits.** Recall that these are the limits constructible from products, inserters, and equifiers.

Fact 4.8. *Pseudolimits are PIE-limits.*

But not conversely. For example, inserters are not pseudolimits (proof: why should it be?). Neither are iso-comma objects, although they're pretty close (as we saw above).

This is because of the construction of $(-)'$ that I gave. By Yoneda, applying colimits to the weights corresponds to limits of the diagrams; more precisely, to say that pseudolimits are PIE-limits is to say that all pseudolimit weights J' can be constructed by PIE-colimits from representables (which, as weights, correspond to evaluation at one spot, by Yoneda again). But recall that we constructed J' using coinserters and coequifiers applied to free algebras (for which we need coproducts as well). So we use precisely the PIE-limits.

Remember that T-**Alg** had all PIE-limits. It therefore has all pseudolimits as well. But consider the class of all limits (weights) which are equivalent (in $[\mathscr{C}, \mathbf{Cat}]$, so that the equivalences are 2-natural) to pseudolimits. It is not the case that T-**Alg** has all of those limits. So equivalence of limits is not always totally trivial.

For example, consider splitting of idempotent equivalences, which seems like a very benign thing to do.

$$\begin{tikzcd} TA \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & TA \arrow[d] \\ T \arrow[r, "e"] & T \end{tikzcd}$$

If we split this idempotent equivalence, we won't necessarilly get a T-algebra back, only a pseudo-algebra. For, say, strict monoidal categories, it isn't.

4.7. **Bilimits.** "I'm going to write down all the same symbols, but they'll just mean different things."

$C \xrightarrow{S} \mathcal{K}$ and $\mathscr{C} \xrightarrow{J} \mathbf{Cat}$ are now homomorphisms (pseudofunctors) between bicategories. The *weighted bilimit* is defined by an *equivalence*

$$\mathcal{K}(C, \{J, S\}_b) \simeq \mathrm{Hom}(\mathscr{C}, \mathbf{Cat})(J, \mathcal{K}(C, S)).$$

Now our limits are determined only up to equivalence, instead of up to isomorphism.

In the case when $\mathscr{C}$ and $\mathcal{K}$ are 2-categories and J and S are 2-functors, then the RHS is equal to the RHS for pseudolimits, just by definition (since $\mathrm{Ps}(\mathscr{C}, \mathbf{Cat}) \hookrightarrow \mathrm{Hom}(\mathscr{C}, \mathbf{Cat})$ is locally an isomorphism). Thus *every pseudolimit is a bilimit*.

On the other hand, if just $\mathcal{K}$ is a 2-category, then you can always change $\mathscr{C}$ so that $\mathscr{C}$ is a 2-category and J and S are 2-functors, so that the RHS stays the same up to equivalence. So for limits in a 2-category (which we might as well talk about), you may as well suppose $\mathscr{C}$ is a 2-category and J and S are 2-functors, without any essential loss of generality. And for any type of bilimit, there is a corresponding type of pseudolimit which, if it exists in a 2-category, is the given type of bilimit (but the bilimit might exist without the pseudolimit existing). So this is a difference you should ignore. The *real* difference is that there is an equivalence in the definition instead of an isomorphism.

"But if you do have the isomorphism, then you'd be a bloody fool not to use it." Sometimes, as for T-**Alg**, it's easier to check the strict universal property. Similarly, the 2-category of (bounded) toposes has PIE-colimits, but limits are more interesting and harder to construct, and much easier to use the PIE-colimits than the bicolimits.

4.8. **Colimits.** Colimits in $\mathcal{K}$ are limits in $\mathcal{K}^{op}$. That's really all you have to say, but I should show you the notation. Usually we switch back to $\mathcal{K}$ and replace $\mathscr{C}$ by $\mathscr{C}^{op}$, so we consider

$$S\colon \mathscr{B} \to \mathcal{K}$$
$$J\colon \mathscr{B}^{op} \to \mathbf{Cat}$$

with the weighted colimit notated $J \star S$.

A form of the **Yoneda lemma** says that

$$J \cong J \star Y$$

where $Y\colon \mathscr{B} \to [\mathscr{B}^{op}, \mathbf{Cat}]$. This is the explanation of why you can construct the weighting from a 'limit-notion' by $J \cong \{J, \widetilde{Y}\}$.

"Possibly the shortest ever treatment of Yoneda in the history of the world."

4.9. T**-Alg again.** When we say T-**Alg** we always mean for a reasonable $\mathcal{K}$, complete and cocomplete, and a reasonable T, having a rank. In particular, if $\mathcal{K}$ is locally finitely presentable and T is finitary. In this case T-**Alg** has PIE-limits, as we saw, and so has pseudo-limits, and so has bilimits. So from a bicategorical perspective, we have everything we might want.

Also, T-**Alg** has *bicolimits* (but not, in general, PIE-colimits). But unlike the PIE-limits, they are not constructed as downstairs. I'm not going to prove this, but state a more general theorem from which it follows.

Fact 4.9. *Suppose we have some other 2-functor G*

$$T\text{-}\mathbf{Alg}_s \xrightarrow{I} T\text{-}\mathbf{Alg} \xrightarrow{G} \mathscr{L}$$

such that the composite GI has a left adjoint F. Then IF is left biadjoint to G.

In other words, we have a pseudonatural equivalence

$$T\text{-}\mathbf{Alg}(IFL, A) \simeq \mathscr{L}(L, GA).$$

Note that we assume a strict left 2-adjoint F, but conclude a left biadjoint. Probably F being only a left biadjoint isn't good enough.

This implies that T-**Alg** has bicolimits, and also that given a monad morphism $S \xrightarrow{f} T$, the induced forgetful map $f^*\colon T\text{-}\mathbf{Alg} \to S\text{-}\mathbf{Alg}$ has a left biadjoint. (For which we use the general enriched-category-theory fact that $T\text{-}\mathbf{Alg}_s \to S\text{-}\mathbf{Alg}_s$ has a strict left adjoint.)

Why is this? To give $L \to GIA$, corresponds by adjunction to $FL \to A$, and thus from $IFL \to IA$; but there could be more pseudo-maps arising. What we need is that FL is 'cofibrant' enough. It's the fact that the right adjoint GI factors through T-**Alg** that forces the left adjoint to land in 'cofibrant' things. The unit for the original adjunction $1 \to GIF$ will also be the unit for the biadjunction (so it's a bit special as a biadjunction, since both functors are 2-functors and the unit is 2-natural).

We have a wobbly $FL \rightsquigarrow (FL)'$, and applying G we get $GFL \to G((FL)') = GI((FL)')$. Composing with the unit $L \to GFL$, we get

$$\frac{L \to GI((FL)')}{FL \to (FL)'}$$

which is then a strict map, and you check it's isomorphic to the wobbly you started with. Hence FL is equivalent to $(FL)'$ in T-$\mathbf{Alg}_s$, which is what we want.

Now, we want a biadjoint to f^*:

$$\begin{tikzcd} T\text{-}\mathbf{Alg}_s \arrow[r] \arrow[d, "I"'] & S\text{-}\mathbf{Alg}_s \arrow[l, bend right] \arrow[d] \\ T\text{-}\mathbf{Alg} \arrow[r] & S\text{-}\mathbf{Alg} \arrow[u, bend right, "(-)'"'] \end{tikzcd}$$

We then apply the above result to the two known 2-adjunctions; one is prime, and the other is the strict version that we know exists by general enriched category theory..

To see that T-**Alg** has bicolimits, take $S\colon \mathscr{C} \to T\text{-}\mathbf{Alg}$ and $J\colon \mathscr{C}^{op} \to \mathbf{Cat}$ 2-functors on a 2-category $\mathscr{C}$, and we have

$$T\text{-}\mathbf{Alg}(S,1)\colon T\text{-}\mathbf{Alg} \longrightarrow \mathrm{Hom}(\mathscr{C}^{op}, \mathbf{Cat})$$

which sends $A \mapsto T\text{-}\mathbf{Alg}(S,A)$, and we want a left biadjoint to this (which would send J to $J \star S$). Then the composite

$$T\text{-}\mathbf{Alg}_s \longrightarrow T\text{-}\mathbf{Alg} \longrightarrow \mathrm{Hom}(\mathscr{C}^{op}, \mathbf{Cat})$$

is still $T\text{-}\mathbf{Alg}(S,I)$, which is the same as $T\text{-}\mathbf{Alg}_s(S',1)$ (apply prime pointwise to S). But (under assumptions of rank) T-$\mathbf{Alg}_s$ is cocomplete, so it has pseudocolimits, and hence this functor has a left 2-adjoint.

How would you ever hope to prove that T-**Alg** has bicolimits directly? That would be a disaster. But using the *strict* notion of pseudolimits we can do it.

5. Model categories, 2-categories, and 2-monads

Lots of things this could mean, and four that it will.

(1) Model structures *on* 2-categories: If $\mathcal{K}$ is a (nice) 2-category, we get a model structure on its underlying ordinary category $\mathcal{K}_0$, whose weak equivalences are the 2-categorical equivalences. This is totally uninteresting in its own right, but we can use it to build up to other things.
(2) Model categories *for* 2-categories: There's a model structure on the category of 2-categories and 2-functors, and one for bicategories too.
(3) For T a 2-monad on $\mathcal{K}$, the ordinary category $(T\text{-}\mathbf{Alg}_s)_0$ of strict algebras and strict morphisms has a model structure induced in a standard way by the one mentioned above on $\mathcal{K}$, but no longer of that form; now the weak equivalences are those which become an equivalence downstairs in $\mathcal{K}$.
(4) The category $\mathrm{Mnd}_f(\mathcal{K})$ of finitary 2-monads *also* has a model structure. In fact, it has two, one induced by endofunctors and one by object-indexed things. This is related to the above.

Other things that it might mean but won't:

(1) What is a 'model 2-category'? Some 1-morphism weak equivalences which would become equivalences in a 'homotopy 2-category', but also some 2-morphism weak equivalences? Think I know what this might mean, but not for today.

5.1. **Model structures on 2-categories.** Let $\mathcal{K}$ be a 2-category with finite limits and colimits. In particular, $\mathcal{K}_0$ has finite limits and colimits, and also $\mathcal{K}$ has tensors and cotensors with $\not\models$. (In fact, this is equivalent.)

In particular, we have the *pseudolimit of an arrow* $E \to B$. Ordinarily we don't talk about limits of an arrow, since in an ordinary category the limit is just E, but the pseudolimit is equivalent to E but not equal to it. It's the universal picture that looks like

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "u"'] & \arrow[d, phantom, "\Downarrow\cong"] & B \\ & A \arrow[ur, "v"'] & \end{tikzcd}$$

such that given $fa \cong b$ there is a unique $c\colon X \to L$ with $uc = a$ and $vc = b$. In this case, u is an equivalence, because $(1, f)$ factors through by a $D\colon A \to L$, and $ud = 1$ while $du \cong 1$.

The model structure is:

- The weak equivalences are the equivalences;
- The fibrations are the *isofibrations*, the maps such that

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\swarrow\cong"] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

lifts to a

$$\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "a'"'] \arrow[r, phantom, "\Downarrow\cong"] \arrow[dr, bend right, "b"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

- The cofibrations have the left lifting property.

The trivial fibrations are then the *surjective equivalences*, i.e. p such that there is an s with $ps = 1$ and $sp \cong 1$.

In particular, for $\mathcal{K} = \mathbf{Cat}$, you get the 'categorical model structure' or 'folklore model structure'. (There are other model structures on $\mathbf{Cat}$, in particular the famous one due to Thomason that gives you a homotopy theory equivalent to simplicial sets.)

The pseudo limit of f gives us, for any f, a factorization $f = vd$ where v is a fibration (which follows from the universal property of the pseudolimit) and d is an equivalence. In the case of $\mathbf{Cat}$, you could stop there and d would already be a trivial cofibration, but in general need to keep going. We've reduced the problem to factorizing an equivalence, however.

The way you do that is also the way you get the other factorization: use the dual thing. Form the pseudo**co**limit of the arrow f, and use $(f, 1)$ to induce an e with

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend left, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\cong"] & C \arrow[r, "e"] & B \\ B \arrow[ur, "j"] \arrow[urr, bend right, "1"'] & & \end{tikzcd}$$

This time i is a cofibration and e is a trivial fibration, and if f itself is an equivalence, then i has the left lifting property with respect to the fibrations (so it's what's going to become a trivial cofibration).

That's all I'll say about the proof. There is, of course, a dual structure in which the cofibrations are characterized and the fibrations are defined by a right lifting property. For $\mathbf{Cat}$, these coincide. In general, I know less about the dual one; lifting it to T-$\mathbf{Alg}$ later on would probably be harder.

When $\mathcal{K}$ is arbitrary, no reason this should be cofibrantly generated. Certainly for $\mathbf{Cat}$ it is, and have lots of other examples, but I can't tell you exactly when it is.

As evidence for the triviality of this:

- All objects are cofibrant and fibrant.
- The morphisms in the homotopy category $\mathrm{Ho}(\mathcal{K}_0)$ are the isomorphism classes of 1-cells in $\mathcal{K}$.

Could also take categories internal to some other category. If your starting category is reasonable, internal categories will be good too. Could do the above for this 2-category, but that's not what people tend to do with internal categories. Given $F\colon \mathbb{A} \to \mathbb{B}$ you tend to say it's an equivalence if it's full and faithful and essentially surjective in an internal sense. For $\mathbf{Cat}$ this is equivalent to the usual notion by the axiom of choice, but in general it won't be. It's the weak equivalences in this sense that people tend to use as their weak equivalences for $\mathbf{Cat}(\mathscr{E})$. When

$\mathscr{E}$ is a topos, this was studied by Joyal and Tierney, and there's been recent work on other cases, when $\mathscr{E}$ is groups (get crossed modules) or abelian groups.

5.2. **Model structures for T-algebras.** Now let T be a (finitary) 2-monad on $\mathcal{K}$ (locally finitely presentable), and $T\text{-}\mathbf{Alg}_s$ the strict algebras and strict morphisms. Then the underlying ordinary category $(T\text{-}\mathbf{Alg}_s)_0$ is the algebras for the underlying ordinary monad T_0 on $\mathcal{K}_0$, so we have adjunctions

$$\begin{tikzcd} T\text{-}\mathbf{Alg}_s \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathcal{K} \arrow[u, bend left] \end{tikzcd} \qquad \begin{tikzcd} (T\text{-}\mathbf{Alg}_s)_0 \arrow[d, bend left] \arrow[d, phantom, "\dashv"] \\ \mathcal{K}_0 \arrow[u, bend left] \end{tikzcd}$$

There's this usual recipe for lifting model structures. Gives a model structure on $(T\text{-}\mathbf{Alg}_s)_0$ where f is a weak equivalence or fibration iff Uf is one in $\mathcal{K}_0$. The cofibrations are defined by the llp. Play the usual silly games. Not completely automatic in general, there are a few things you have to check, but they are easy to check here, especially because all objects in $\mathcal{K}$ are fibrant.

Starts to become a little more interesting, since it doesn't have this trivial form any more. In general, if $(A, a) \xrightarrow{(f,\overline{f})} (B, b)$ is a pseudomorphism of T-algebras and $A \xrightarrow{f} B$ is an equivalence, then choosing an inverse $B \xrightarrow{g} A$ naturally becomes an equivalence upstairs. This is a 2-categorical analogue of the fact that if an algebra morphism is a bijection, its inverse also preserves the algebra structure.

But, if f is strict ($\overline{f}$ is an identity), there is no reason why its inverse equivalence should also be strict. For example, a strict monoidal functor which is an equivalence of categories has an inverse which is strong monoidal, but not in general strict.

Recall the inclusion

$$\begin{tikzcd} T\text{-}\mathbf{Alg}_s \arrow[r, bend left, "(-)'"] \arrow[r, phantom, "\perp"] & T\text{-}\mathbf{Alg} \arrow[l, bend left] \end{tikzcd}$$

has a left adjoint $(-)'$, so that

$$\frac{A \rightsquigarrow B}{A' \to B}.$$

This fits into the model category theory very nicely. The counit of this adjunction

$$A' \xrightarrow{q} A$$

is a cofibrant replacement: a trivial fibration with A' being cofibrant. So we see that T-$\mathbf{Alg}$, which is the thing we're more interested in, is starting to come out of the picture. This is a much tighter thing than the general philosophy that 'we should think of maps in the homotopy category as maps out of a cofibrant replacement.'

An algebra turns out to be cofibrant iff $A' \xrightarrow{q} A$ has a section *in* $T\text{-}\mathbf{Alg}_s$. (There's always a wobbly one.) One direction is obvious as q is a trivial fibration, and otherwise A is a retract of A' which is cofibrant. Recall that such an object is said to be *flexible*. These notions somehow developed in parallel; probably cofibrant is a bit older. Again, it's a tighter sort of situation than you might have in general.

5.3. **Model structures for 2-monads.** Recall now that we have adjunctions

$$\begin{tikzcd}
\mathrm{Mnd}_f(\mathcal{K}) \arrow[d, bend left, "W"] \arrow[d, phantom, "\dashv"] \\
\mathrm{End}_f(\mathcal{K}) \arrow[u, bend left, "H"] \arrow[d, bend left, "V"] \arrow[d, phantom, "\dashv"] \\
{[ob\,\mathcal{K}_f, \mathcal{K}]} \arrow[u, bend left, "H"]
\end{tikzcd}$$

both of which are monadic, as is the composite. Thus $\mathrm{Mnd}_f(\mathcal{K})$ is both M-$\mathbf{Alg}_s$ and N-$\mathbf{Alg}_s$ where M is the induced monad on $\mathrm{End}_f(\mathcal{K})$ and N is the induced monad on $[ob\,\mathcal{K}_f, \mathcal{K}]$.

Thus $\mathrm{Mnd}_f(\mathcal{K})$ has *two* lifted model structures. They're not the same, since something can be an equivalence all the way downstairs without being one in $\mathrm{End}_f(\mathcal{K})$ (which is itself algebras for another induced monad on $[ob\,\mathcal{K}_f, \mathcal{K}]$).

A monad map $S \xrightarrow{f} T$ is a 2-natural transformation compatible with the unit and multiplication. If the 2-natural transformation is an equivalence in $\mathrm{End}_f(\mathcal{K})$, these are the weak equivalences for the M-model structure. If the *components* of the 2-natural transformation are equivalences, these are the weak equivalences for the N-model structure.

First consider the M-model structure, induced by

$$\begin{tikzcd}
\mathrm{Mnd}_f(\mathcal{K}) \arrow[r, equal] \arrow[d, bend left, "W"] \arrow[d, phantom, "\dashv"] & M\text{-}\mathbf{Alg}_s \\
\mathrm{End}_f(\mathcal{K}) \arrow[u, bend left, "H"]
\end{tikzcd}$$

Here we have the prime construction, which will classify pseudomorphisms of monads. These are precisely the things that arise when talking about pseudoalgebras: recall that a pseudo-T-algebra was an object A with a pseudo-morphism

$$T \rightsquigarrow \langle A, A\rangle$$

into the 'endomorphism 2-monad' of A, corresponding to maps $TA \xrightarrow{a} A$ which are associative and unital up to coherent isomorphism.

This corresponds to a strict map $T' \to \langle A, A\rangle$, so that T'-$\mathbf{Alg} = \mathrm{Ps}T$-$\mathbf{Alg}$. (This is the part of the justification for working with strict algebras that people tend to understand first.)

Now we have $T' \xrightarrow{q} T$. If q has a section in M-$\mathbf{Alg}_s = \mathrm{Mnd}_f(\mathcal{K})$, then T is said to be *flexible* (= *cofibrant*). This was the context in which the notion of 'flexible' was first introduced. **Any monad that you can give a presentation for without having to use equations between objects is always flexible.** For example, the monad for monoidal categories is flexible, but the monad for strict monoidal categories is not. These are the cases in which it's true that every pseudo-algebra is equivalent to a strict one. Remember that the importance of pseudo-algebras is *not* for describing concrete things, but for the theoretical side, since various constructions don't preserve strictness of algebras. For particular structures like monoidal categories, better off choosing the right monad to start with.

5.4. **Model structure on 2-Cat.** **2-Cat** is the category of 2-categories and 2-functors. It underlies a 3-category, and a 2-category, and perhaps more importantly a Gray-category. Want to describe a model structure on this, analogous to the one above for **Cat**.

The weak equivalences will be the *biequivalences*, which somehow we haven't talked about yet. $F\colon \mathscr{A} \to \mathscr{B}$ is a biequivalence if

- $\mathscr{A}(A,B) \xrightarrow{F} \mathscr{B}(FA,FB)$ is an equivalence of categories, and
- F is 'biessentially surjective' on objects, i.e. if $C \in \mathscr{B}$, there exists an $A \in \mathscr{A}$ with $FA \simeq C$ in $\mathscr{B}$.

General remarks on notation:

- $=$ is equality
- $\cong$ is isomorphism
- $\simeq$ is equivalence (these are quite widespread)
- $\sim$ is sometimes used for biequivalence, although it's a bit unsatisfactory, and it's not at all clear what to do next.

Of course, for equivalences you have the corresponding thing of having a functor going back the other way. Here too: you can build a thing $G\colon \mathscr{B} \to \mathscr{A}$ with $GF \simeq 1$ and $FG \simeq 1$. You can make G a pseudofunctor, but generally not a 2-functor, even when F is (just like in the last section). That's sort of the whole point of the model structure. Also the equivalences will be pseudonatural, and that's generally as good as it gets.

Thus the analogy with equivalences 'breaks down' here, if you regard it as breaking down; maybe you should regard pseudofunctors as the 'natural' thing. But if you allow arbitrary pseudofunctors, you don't have limits and colimits. Of course, if you pass up to the full tricategory structure, you do in a suitable sense, but the suitable sense is kind of messy and if you can avoid it, that's good.

Clearly biequivalence is the right notion of 'sameness' for bicategories, or 2-categories, but there is this 'stability' problem.

The fibrations are similar to the case of 2-categories. There we lifted invertible 2-cells; here we lift equivalences. $F\colon \mathscr{A} \to \mathscr{B}$ is a fibration if

(1) given an equivalence downstairs, we have a lift

$$\begin{tikzcd}
A' \arrow[r, dotted, "\simeq"] \arrow[d, dotted, mapsto] & A \arrow[d, mapsto] & \mathscr{A} \arrow[d, "F"] \\
B \arrow[r, "\simeq"] & FA & \mathscr{B}
\end{tikzcd}$$

(2) given an isomorphism 2-cell downstairs, we have a lift

$$\begin{tikzcd}
A' \arrow[r, bend left] \arrow[r, dotted] \arrow[r, phantom, "\cong"] \arrow[d, mapsto] & A \arrow[d, mapsto] & \mathscr{A} \arrow[d, "F"] \\
B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\cong"] & FA & \mathscr{B}
\end{tikzcd}$$

That is to say, all the functors $\mathscr{A}(A_1,A_2) \to \mathscr{B}(FA_1,FA_2)$ are an (iso)fibration in **Cat**.

Note that the notion of biequivalence is *not* internal to the 3-category or Gray-category of 2-categories, which maybe speaks against the existence of an general construction on an arbitrary 3-category or Gray-category, at one which would reduce to this one.

There's an equivalent way of expressing these which is useful. Keep the second as is, but modify the first to lifting of full adjoint equivalences, which is equivalent in the presence of the second:

$$\begin{tikzcd} A' \arrow[r, bend left, "\simeq"] \arrow[r, phantom, "\dashv"] \arrow[d, dotted] & A \arrow[l, bend left] \arrow[d] & & \mathscr{A} \arrow[d, "F"] \\ B \arrow[r, bend left, "\simeq"] \arrow[r, phantom, "\dashv"] & FA \arrow[l, bend left] & & \mathscr{B} \end{tikzcd}$$

Clearly this implies lifting of equivalences, since we can complete any equivalence to an adjoint equivalence, but the converse is only true when we can lift 2-cell isomorphisms.

This is related to a mistake I made in my first paper on this topic, where I used a condition like this on lifting equivalences that aren't necessarily adjoint equivalences. Regard 'being an equivalence' as a property, and 'an adjoint equivalence' as a structure, but be wary of regarding 'a not-necessarily-adjoint equivalence' as a structure. Adjoint equivalences are now completely algebraic, classified by maps out of 'the free-living adjoint equivalence', which is biequivalent to the terminal 2-category 1. A 'free-living not-adjoint equivalence' would *not* be biequivalent to 1.

The trivial fibrations, which are the things which are both, can be characterized as the 2-functors which are

- Surjective on objects, and
- $\mathscr{A}(A_1, A_2) \to \mathscr{B}(FA_1, FA_2)$ are surjective equivalences (the trivial fibrations in **Cat**).

Note that the trivial fibrations don't use the 2-category structure; you don't need anything about the composition to know what these things are, only the '2-graph' structure. So they're much simpler to work with.

There's an obvious ω-categorical analogue to these things, which is throughout Makkai's work on ω-categories. You don't need the ω-category structure, only a globular set, to say what this means. The corresponding notion of 'cofibrant object' is then what he calls a 'computad'.

It's a bit less trivial than the other cases to prove that this all works, but it's not really hard. Everything is directly a lifting property (once you use the version with adjoint equivalences), so finding generating sets is easy.

Here all objects are fibrant. But *not* all objects are cofibrant. We have a special cofibrant replacement $\mathscr{A}' \xrightarrow{q} \mathscr{A}$ with the property that

$$\frac{\mathscr{A} \rightsquigarrow \mathscr{B}}{\mathscr{A}' \to \mathscr{B}}$$

with the same sorts of properties. And $\mathscr{A}$ is cofibrant (flexible) iff the trivial fibration q has a section in 2-**Cat**. This happens exactly when the underlying category $\mathscr{A}_0$ is free on some graph (i.e. you haven't imposed any equations on 1-cells, only isomorphisms). In principle, it could be a retract of something free, but it turns out that in this case that actually makes you free.

Two things of interest to me in this paper. This (cofibrant = flexible) was the first one. The second was monoidal structures. The model structure is *not* compatible with the cartesian product $\times$. The thing you should have in mind is that the locally discrete 2-category $\nvDash$ is cofibrant, but $\nvDash \times \nvDash$ is not, since a commutative square

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

is not free. There are various tensor products you can put on 2-**Cat**. The cartesian product is also called the *ordinary product* (since it is also a special case of the tensor product of $\mathscr{V}$-categories), but I like to call it the *black product* since the square is 'filled in'.

There's also the *white* or *funny* product, in which the square has nothing in it at all. It's a theorem that on **Cat** there are exactly 2 symmetric monoidal closed structures: the ordinary one and the funny one. The closed structure corresponding to the funny product is the not-necessarily-natural transformations (just components). Enriching over this structure gives you a 'sesquicategory' (perhaps an unfortunate name, but you can see how it came about), which has hom-categories and whiskering, but no middle-four interchange, hence no well-defined horizontal composition of 2-cells.

Finally, there's the *Gray* or *grey* tensor product in which you put an isomorphism in the square, so it's 'partially filled in'.

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

The first two make sense for any $\mathscr{V}$ at all, but this one doesn't. The reason I started thinking about this at all was to think about Gray tensor product from various different points of view. For general $\mathscr{V}$ there's a canonical comparison from the funny product to the ordinary one, and you'd like to put a 'cofibrant replacement' in between. I fought with some general theory using factorization systems for a while, never got anywhere, then tried model categories. Model categories were never really going to work, since going to weak factorizations lose some control, but here's what happens.

The Gray tensor product $\nvDash \otimes \nvDash$ is cofibrant. More generally, the model structure is compatible with the Gray tensor product.

The Gray tensor product is not what Gray defined: he defined the lax version. And I'm told that for some time he refused to believe that there was such a tensor product, while Joyal and Street insisted there should be, since the closed structure of pseudonatural transformations is obviously there.

Now consider **Bicat**, the category of bicategories and strict morphisms. Everything before looks exactly the same, i.e. the full inclusion 2-**Cat** $\hookrightarrow$ **Bicat** reflects weak equivalences and fibrations. This inclusion has a left adjoint 'strictification'

$$\begin{tikzcd} 2\text{-}\mathbf{Cat} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{Bicat} \arrow[l, bend left] \end{tikzcd} .$$

It's not a prime, since all morphisms are strict, and it's not *st* either. In general the unit is *not* a biequivalence. This fits well into the Quillen picture: in fact this (obviously Quillen, by definition) adjunction is a Quillen equivalence, for which we need the components of the unit at cofibrant objects only to be weak equivalences. There exist bicategories (even monoidal categories) for which there does not exist a strict map into a 2-category which is a biequivalence, although we know that any $\mathscr{B}$ has a wobbly $\mathscr{B} \rightsquigarrow \mathscr{B}_{st}$. But it is true for cofibrant ones, since this says 'there aren't any equations between 1-cells'. (The classification of cofibrant objects in **Bicat** is a little less tight than for 2-**Cat**.)

As for 2-**Cat**, we have a prime construction which models weak homomorphisms. And also the cartesian product is not compatible. Don't know if there exists a Gray tensor product for bicategories; once you pass to bicategories, generally tend to just make everything weak.

The model structure on 2-**Cat** is proper; showing that it's left proper (biequivalences are stable under pushout along cofibrations) was the hardest part of that paper. Right proper is trivial since everything is fibrant; didn't check left proper for **Bicat**. Much less pleasant to deal with free bicategories.

5.5. **Back to 2-monads.** Recall we had also an adjunction

$$\begin{tikzcd} \mathrm{Mnd}_f(\mathcal{K}) \arrow[r, equal] \arrow[d, bend left, "\dashv"'] & N\text{-}\mathbf{Alg}_s \\ {[ob\,\mathcal{K}_f, \mathcal{K}]} \arrow[u, bend left] \end{tikzcd}$$

giving an N-model structure on $\mathrm{Mnd}_f(\mathcal{K})$. Why is this interesting?

There are some annoying details here I haven't completely thought about. Consider 2-**CAT**, the category of possibly-large 2-categories and functors between them. Everything I said before you can do in this case too. Need to move to a higher universe for small object arguments to work. I want instead to pass to the slice 2-**CAT**/**Cat**, which inherits a model structure in the usual way.

The point is that we have a 2-functor

$$\mathrm{Mnd}_f(\mathbf{Cat})^{op} \xrightarrow{\text{sem}} 2\text{-}\mathbf{CAT}/\mathbf{Cat}$$

which you might call *semantics*, defined by:

$$T \mapsto (T\text{-}\mathbf{Alg} \xrightarrow{U} \mathbf{Cat}).$$

and

$$(S \xrightarrow{f} T \mapsto (T\text{-}\mathbf{Alg} \xrightarrow{f^*} S\text{-}\mathbf{Alg}).$$

since if $TA \to A$ is a T-algebra, then $SA \to TA \to A$ is an S-algebra, since f is a morphism of monads.

In the ordinary unenriched, or $\mathscr{V}$-enriched, case (or here, if we took T-$\mathbf{Alg}_s$), this functor would be fully faithful. But for T-**Alg**, it isn't true any more. In fact, to give a map in 2-**CAT**/**Cat** corresponds to giving a 'pseudo-morphism' of 2-monads, not in the old (and usual) sense but now in the sense given by the N-model structure. Instead of a 2-natural transformation which is compatible with multiplication and unit up to isomorphism, it is a *pseudo*natural transformation which is so compatible.

This thing sem preserves limits, fibrations, and trivial fibrations (all proven using these things $\langle A, A\rangle$, $\{f, f\}$, and so on). All these are interpreted in $\mathrm{Mnd}_f(\mathbf{Cat})^{op}$, hence correspond to colimits, cofibrations, and trivial cofibrations. Thus, it should in principle be the right adjoint part of a Quillen adjunction. It's not, of course, because of size problems. I'm moderately confident that at some point I'm going to sit down and get some decent full subcategory of 2-**CAT**/**Cat** so that both those problems are resolved.

I suspect that sem doesn't preserve weak equivalences in general, in particular the weak equivalence $T' \to T$. But it does preserve weak equivalences between cofibrant objects, i.e. flexible monads. The problem with preserving weak equivalences is that you'll get out of the strict algebras into pseudo ones, but with a flexible monad you can rectify that at the end.

6. The formal theory of monads

6.1. **Generalized Algebras.** The formal theory of monads is the jewel in the crown of formal category theory; it's a remarkable thing. Let's start by thinking about ordinary monads.

Let A be a category, $t = (t, \mu, \eta)$ a monad on A. Can from the Eilenberg-Moore category A^t, the category of algebras. The starting point is to think about the universal property of this construction. What is it to give a functor $C \xrightarrow{a} A^t$? We give for each $c \in C$, an algebra ac, which we also use for the name of the underlying object, with structure map $tac \xrightarrow{\alpha c} ac$. And for every $\gamma\colon c \to d$, we have an $a\gamma\colon ac \to ad$ with a square

$$\begin{tikzcd} tac \arrow[r, "\alpha c"] \arrow[d, "ta\gamma"'] & ac \arrow[d, "a\gamma"] \\ tad \arrow[r, "\alpha d"'] & ad. \end{tikzcd}$$

This square looks an awful lot like a naturality square; it wants to say that α is natural with respect to γ.

What we're actually doing is giving a functor $C \xrightarrow{a} A$ and a 2-cell

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[dr, bend right, "a"'] \arrow[dr, phantom, "\Swarrow\alpha"] & A \arrow[d, "t"] \\ & A \end{tikzcd}$$

with equations of natural transformations

$$\begin{tikzcd} t^2a \arrow[r, "\mu a"] \arrow[d, "t\alpha"'] & ta \arrow[d, "\alpha"] & a \arrow[l, "\eta a"'] \arrow[dl, "1"] \\ ta \arrow[r, "\alpha"'] & a & \end{tikzcd}$$

which just says that on components, it makes each ac into a t-algebra.

You might call this a *generalized algebra*, or a *t-algebra with domain C*. Think of a usual algebra as one with domain 1.

Similarly, you can look at natural transformations. To give a natural transformation

$$\begin{tikzcd} C \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow"] & A^t \end{tikzcd}$$

amounts to giving

$$\begin{tikzcd} C \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\varphi"] & A \end{tikzcd}$$

which is suitably compatible, i.e.

$$\begin{tikzcd} ta \arrow[r, "t\varphi"] \arrow[d, "\alpha"'] & tb \arrow[d, "\beta"] \\ a \arrow[r, "\varphi"'] & b. \end{tikzcd}$$

So the algebras construction has a universal property. This is the starting point of the whole thing.

I've been talking all along about categories, but once we've decided that algebras with domain 1 don't have such a special role, there's no reason to restrict that way, so we can instead talk about a monad on an object A in any 2-category $\mathcal{K}$. We can't just 'build it up' as we did before, but we can *ask* whether there exists an object A^t with this universal property, i.e. there exists a 1-cell $A^t \xrightarrow{u} A$ such that maps into A^t correspond to maps into A with an action.

(Remember that the notion of monad has not been weakened or made higher-dimensional in any way (yet). The 2-category $\mathcal{K}$ might be **Cat**, or 2-**Cat**, or $\mathscr{V}$-**Cat**, but we use the same definition.)

You can make this precise, but a slick way to say it is that the hom-category $\mathcal{K}(C, A)$ has a monad $\mathcal{K}(C, t)$ on it (since 2-functors take monads to monads), and this is the ordinary type of monad in **Cat**. The endofunctor part of this monad sends $a\colon C \to A$ to $ta\colon C \to A$. This generalized notion of algebra is then nothing but the usual sort of algebra for the ordinary monad $\mathcal{K}(C, t)$. So what we want is

$$\mathcal{K}(C, A^t) \cong \mathcal{K}(C, A)^{\mathcal{K}(C,t)}$$

naturally in C (where the RHS means the ordinary Eilenberg-Moore category of algebras for the ordinary monad $\mathcal{K}(C, t)$).

It turns out that in some places, such as **Cat**, it's enough to check that this is true for $C = 1$, but in an abstract 2-category there may not be a 1, and if there is, it may not be enough to get the full universal property.

Now we can start reworking and reformulating these things in various 2-categorical ways to get interesting things. This maybe looks a bit like a limit, and we'll see that in a bit, but first let's do a different point of view.

6.2. **Monads in $\mathcal{K}$.** For a 2-category $\mathcal{K}$, $\mathrm{Mnd}(\mathcal{K})$ is a 2-category of monads in $\mathcal{K}$. Note thath previously we'd been talking about $\mathrm{Mnd}_f(\mathcal{K})$ which meant the finitary *2*-monads *on* $\mathcal{K}$ (as a fixed base object), while now we mean all the *internal* monads *in* $\mathcal{K}$, with base objects varying.

- Its objects are monads in $\mathcal{K}$.
- Its 1-cells are supposed to correspond to morphisms which lift to the level of algebras:

$$\begin{tikzcd} A^t \arrow[r, "\overline{m}"] \arrow[d, "u^t"'] & B^t \arrow[d, "u^s"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

  and we can think of this as an identity 2-cell and take its mate, since the the us are right adjoints:

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "f^t"'] \arrow[dr, phantom, "\swarrow"] & B \arrow[d, "f^s"] \\ A^t \arrow[r, "\overline{m}"] \arrow[d, "u^t"'] & B^t \arrow[d, "u^s"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

which we then paste together to get a 2-cell, and the forgetful-free composite gives us the monads. Thus we should define a morphism of monads to be $A \xrightarrow{m} B$ with a 2-cell $sm \xrightarrow{\varphi} mt$ such that

$$\begin{tikzcd} s^m \arrow[r, "s\varphi"] \arrow[d, "\mu m"'] & smt \arrow[r, "\varphi t"] & mt^t \arrow[d, "m\mu"] \\ sm \arrow[rr, "\varphi"'] & & mt \end{tikzcd}$$

plus another one for the identities. We could take the first square as a definition, provided the Eilenberg-Moore objects exist, which they don't need to in an abstract 2-category; in general giving a φ with this condition will be the same as giving a lifting of m to the algebra objects.

- The 2-cells are

$$\begin{tikzcd} A \arrow[r, bend left, "m"{name=U}] \arrow[r, bend right, "n"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\rho"] \end{tikzcd}$$

with a compatibility condition, which you could express as saying that ρ lifts to a $\overline{\rho}$ betwen algebra objects, or saying that

$$\begin{tikzcd} sm \arrow[r, "\varphi"] \arrow[d, "s\rho"'] & mt \arrow[d, "\rho t"] \\ sn \arrow[r, "\psi"] & nt \end{tikzcd}$$

There's a full embedding $\mathrm{Id}\colon \mathcal{K} \hookrightarrow \mathrm{Mnd}(\mathcal{K})$ sending A to the identity monad $(A, 1)$ on A, and the obvious thing for 1-cells and 2-cells. This is particularly clear in the algebra-objects picture (which doesn't always make sense), since if $t = 1$ then $A^t = A$, so obviously m will lift uniquely to an $\overline{m}$ (which is what fully faithfulness of Id says).

A trivial observation is that for any monad we can always choose to forget the monad and be left with the object, and this is left adjoint to Id. The interesting thing is a right adjoint. To give a right adjoint to Id is exactly to give a choice of an Eilenberg-Moore object for each monad in $\mathcal{K}$. Why? Look at the universal property. If $A \mapsto A^t$ is the right adjoint, this says that

$$\mathcal{K}(C, A^t) \cong \mathrm{Mnd}(\mathcal{K})((C, 1), (A, t))$$

The key point is that the RHS is equal to $\mathcal{K}(C, A)^{\mathcal{K}(C,t)}$, since we get

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, "1"'] \arrow[dr, phantom, "\swarrow\alpha"] & A \arrow[d, "t"] \\ C \arrow[r, "a"'] & A \end{tikzcd}$$

which is the same as for a generalized algebra, and so on. If you write them down, they really are identical except for the extra 1-cells sitting around.

Now the really beautiful thing happens: we can start looking at duals of $\mathcal{K}$ and see what happens. Consider first $\mathcal{K}^{co}$, where we reverse the 2-cells but not the 1-cells. A monad in $\mathcal{K}^{co}$ is then a *comonad* in $\mathcal{K}$. And an EM-object in $\mathcal{K}^{co}$ is the obvious analogue for comonads. If $\mathcal{K} = \mathbf{Cat}$, we get ordinary comonads, and the EM-object is the usual category of coalgebras for the comonad.

That's nice, but not incredibly surprising. What's more interesting is what happens in $\mathcal{K}^{op}$. A monad in $\mathcal{K}^{op}$ consists of an object A, a morphism not from A to A, but rather from A to A:

$$A \leftarrow A$$

and for the multiplication, you have to make sure when you compose t with itself, you do it in the reverse order, and so on. Thus we get just a monad in $\mathcal{K}$.

But what about the EM-object? The arrows are reversed, so we get a different universal property. An algebra for this monad consists of

$$\begin{tikzcd} A \arrow[d, "a"'] \arrow[dr, phantom, "\Searrow\alpha"] & A \arrow[l, "t"'] \arrow[dl, bend left, "a"] \\ C & \end{tikzcd}$$

The amazing thing is that this is the same thing as a map $A_t \to C$ where A_t is the familiar Kleisli object!

By the way, it's true (in any 2-category) that, as classically stated, the EM-object is the terminal adjunction giving rise to the monad, and the Kleisli object is the initial one, but this universal property is 'richer' in that it refers to maps with arbitrary domains.

Using $\mathcal{K}^{coop}$, of course, gives you Kleisli objects for comonads.

6.3. Mnd **as a monad.** Now, where does the construction $\mathrm{Mnd}(\mathcal{K})$ really live? Consider 2-**Cat** as a *2-category* of 2-categories, 2-functors, and 2-natural transformations. Completely banish from your mind all concerns about size, which doesn't have any role here. What I've done is shown that if you start with a 2-category $\mathcal{K}$, you get a 2-category $\mathrm{Mnd}(\mathcal{K})$, and this is clearly completely 2-functorial, so we get

$$\mathrm{Mnd}\colon 2\text{-}\mathbf{Cat} \to 2\text{-}\mathbf{Cat}$$

and the inclusion Id is clearly natural in $\mathcal{K}$, so we get

$$\begin{tikzcd} 2\text{-}\mathbf{Cat} \arrow[r, bend left, "1"] \arrow[r, bend right, "\mathrm{Mnd}"'] \arrow[r, phantom, "\Downarrow\mathrm{Id}"] & 2\text{-}\mathbf{Cat} \end{tikzcd}$$

A certain sort of person is tempted to pose the question of whether this is part of the structure of a monad on 2-Cat! We do have a composition

$$\begin{tikzcd}[column sep=large] 2\text{-}\mathbf{Cat} \arrow[r, bend left, "\mathrm{Mnd}^2"] \arrow[r, bend right, "\mathrm{Mnd}"'] \arrow[r, phantom, "\Downarrow\mathrm{comp}"] & 2\text{-}\mathbf{Cat} \end{tikzcd}$$

This is the other beautiful thing about the paper.

What is a monad in $\mathrm{Mnd}(\mathcal{K})$? It's

- a monad (A, t) in $\mathcal{K}$,
- an endo-1-cell, which consists of a morphism $A \xrightarrow{s} A$ in $\mathcal{K}$ with a 2-cell $ts \xrightarrow{\lambda} st$ (with conditions)
- A multiplication $(s, \lambda)(s, \lambda) \to (s, \lambda)$, corresponding to $s^2 \xrightarrow{\nu} s$ (with conditions)
- a unit $1 \to (s, \lambda)$ corresponding to $1 \to s$ (with conditions)

As well as the conditions for these to be 1-cells and 2-cells in $\mathrm{Mnd}(\mathcal{K})$, we need the conditions for this to be a monad there. These make s itself into a monad on A in $\mathcal{K}$. The 2-cell λ is now what's called a *distributive law* between these two monads, which is exactly what you need to compose these two monads and get another monad.

Think about this as like the tensor product of rings. $R \otimes S$ is the tensor product of the underlying abelian groups, with multiplication

$$R \otimes S \otimes R \otimes S \xrightarrow{\mathrm{tw}} R \otimes R \otimes S \otimes S \xrightarrow{m_R \otimes m_S} R \otimes S$$

The point is we're trying to do something very similar, but here we're in a world where the tensor product is not commutative, so we don't have the twist. So λ plays the role of the twist; it's a 'local' commutativity or 'braiding' that only applies to these two objects. The conditions put on it are exactly what we need to make the composite st into a monad.

For example, the multiplication on st is then

$$stst \xrightarrow{s\lambda t} sstt \xrightarrow{ss\mu} sst \xrightarrow{\mu t} st$$

The notion of distributive law, in the ordinary case of categories, is due to Jon Beck, and he proved that we have a bijection between distributive laws $ts \to st$ and 'compatible' monad structures on st, and also to liftings of s to A^t (whenever A^t exists). It's not as well-known as it should be and is frequently rediscovered.

Can do this also for $\mathcal{K}^{co}$ or $\mathcal{K}^{op}$ or $\mathcal{K}^{coop}$, of course. If you do it in $\mathcal{K}^{op}$, then a distributive law becomes the other way round, hence an extension to the Kleisli object rather than a lifting to the EM-object. It's easy to get confused, and sometimes it doesn't go quite the way you expected.

Note that operads are monoids in a monoidal category, and passing to the associated monad preserves the composition (via a functor from the monoidal category of collections to the monoidal category of endofunctors), so the conditions for one operad to distribute over another are formally exactly the same. But it's not clear that every distributive law for monads comes from a distributive law for the associated operads.

Peter: the notion of action of one operad on another, as formulated, involves diagonals, so it only makes sense cartesian? Algebraic distributivity $x(y + z) = xy + xz$ involves the diagonal on x.

Semidirect products: if the underlying category is cartesian, we have an obvious twist $S \times R \to R \times S$, but we can use a different one. If we have an action $S \times R \to R$, we can then put it together with the diagonal of S to get a *different* distributive law $S \times R \to R \times S$. That does depend on cartesian setting.

6.4. **Making it into Limits.** There are two ways to do this. Remember that way back in the first week, we saw that monads t in $\mathcal{K}$ correspond to lax functors $\widetilde{t} \colon 1 \to \mathcal{K}$. Then the *lax limit* of $\widetilde{t}$ is exactly the EM-object A^t.

I didn't quite explicitly talk about lax limits of lax functors, but it's the same thing as pseudo limits of pseudo functors. This is incredibly nice, but sometimes it's nice not to have to deal with lax functors, so I'll also tell you another way to do it using 2-functors instead.

Let's recall how $\widetilde{t}$ works. We send $*$ to A, and 1_* to an endo-cell $t \colon A \to A$, the unit is the lax unit comparison, and the multiplication is the lax composition comparison. To understand the lax limit of these sorts of things, we should think about lax cones. A lax cone would involve a vertex C of $\mathcal{K}$, with just one component $C \xrightarrow{a} A$, and a lax naturality 2-cell for every 1-cell in 1:

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "\swArrow_\alpha"] & A \arrow[d, "t"] \\ C \arrow[r, "a"'] & A \end{tikzcd}$$

and some conditions.

There's old paper of Street called 'two constructions on lax functors', and this was the first one. This was the motivating example. The other was the lax colimit, which gives the Kleisli construction.

We have the corresponding construction to prime (which replaces pseudo maps by strict maps) which replaces *lax* maps by strict maps. Thus there's a 2-category **mnd** which is the universal 2-category with a monad in it, i.e.

$$\frac{\text{2-functors } \mathbf{mnd} \longrightarrow \mathcal{K}}{\text{lax functors } 1 \longrightarrow \mathcal{K}}$$

Remember that a monad in $\mathcal{K}$ is the same as a monoid in a hom-category, and we know the universal monoidal category containing a monoid is the 'algebraic $\mathbf{\Delta}$', the category of (possibly empty) finite ordinals. Thus **mnd** has one object $*$ and $\mathbf{mnd}(*, *) = \mathbf{ord}_f = \mathbf{\Delta}$ (not the simplicial sets delta, since it has an extra object).

Now it's an exercise for Eugenia: give a limit notion and ask for the weight. There exists a 2-functor $\mathbf{mnd} \xrightarrow{\text{alg}} \mathbf{Cat}$ such that given $\mathbb{T}\colon \mathbf{mnd} \to \mathcal{K}$, the weighted limit $\{\text{alg}, \mathbb{T}\}$ is exactly the EM-object of the corresponding monad. It follows from what we've done so far that there has to be such an *alg*, since we expressed it in a representable way, and I gave you a recipe for calculating what it must be: apply the corresponding colimit-notion to the yoneda-functor $\mathbf{mnd}^{op} \to [\mathbf{mnd}, \mathbf{Cat}]$. In our case, the Yoneda lemma determines a particular monad T in $[\mathbf{mnd}, \mathbf{Cat}]$, and we take its the Kleisli object; we do this pointwise, and what we get is the weight for EM-objects.

Of course, such limits may or may not exist. But we can also build them up using limits that we know already, if those exist.

- First form the inserter of $\begin{tikzcd} A \arrow[r, bend left, "t"] \arrow[r, bend right, "1"'] & A \end{tikzcd}$. This is an $A_1 \xrightarrow{k} A$ equipped with a 2-cell $tk \xrightarrow{\kappa} k$.
- Then take the equifier of $k(\eta k)$ and 1 to get an $A_2 \xrightarrow{k'} A_1$ such that the identity law holds.
- Finally take the equifier of something else to get the associativity.

In particular, this shows that EM-objects are PIE-limits, in fact *finite* PIE-limits.

6.5. **Limits in T-$\mathbf{Alg}_c$ and T-$\mathbf{Alg}_\ell$.** These are, recall, strict T-algebras with lax and colax morphisms. Recall that we had nice pseudo-limits in T-**Alg**. Here it's much harder.

In T-$\mathbf{Alg}_\ell$, you have oplax limits, and in T-$\mathbf{Alg}_c$ you have lax limits (an unfortunate switch happens; you could by definition make them match up better, but there'd be a kink somewhere else). These are not nearly as good: no inserters, no equifiers in general, so life is a bit harder.

I described how to construct inserters and equifiers in T-**Alg**: form the limit downstairs and show that the thing you get canonically becomes an algebra. You'll see if you look carefully at that that you use invertibility for one of f and g, but not the other one. So you can form inverters and equifiers in $T\text{-}\mathbf{Alg}_\ell$ if one of the cells is pseudo and the other is lax, if you get it the right way round.

Now suppose we're trying to make EM-objects in $T\text{-}\mathbf{Alg}_c$. Then it turns out to be the right way round, we can get the inserter, since 1 is pseudo. And so on: it turns out that $T\text{-}\mathbf{Alg}_c$ does have EM-objects. The most important case of this is where T-**Alg** is monoidal categories, so that $T\text{-}\mathbf{Alg}_c$ has opmonoidal functors. That comes up in Hopf monads and things like that.

6.6. **FTM 2.** We can now see EM-objects as weighted limits in the strict sense, and there's a well-developed theory of free completions under classes of weighted limits. So we can form the free completion $EM(\mathcal{K})$ of a 2-category $\mathcal{K}$ under EM-objects. We also have $KL(\mathcal{K})$ which freely adds Kleisli objects.

The colimit side is more familiar to construct. To freely add all colimits to an ordinary category, we take the presheaf category; to add a restricted class, we take the closure in the presheaf category under the limits we want to add. So here, to get $KL(\mathcal{K})$, we take the closure of the representables in $[\mathcal{K}^{op}, \mathbf{Cat}]$ under Kleisli objects. It's part of a general theorem that this works, at least when $\mathcal{K}$ is small. Of course, $EM(\mathcal{K})$ is dual.

Sometimes it can be not so easy to calculate exactly which things appear in this completion process. You start with the representables and chuck in the Kleisli object for any monad. Usually this is an iterative process, since there will be new diagrams appearing at each step and you have to go on possibly transfinitely. The nice thing about this particular case is that it stops after one step. Colimits in the functor category are pointwise, so we construct Kleisli objects as in **Cat**. The key facts are:

(1) The Kleisli adjunctions in **Cat** are precisely the bijective-on-objects left adjoints.
(2) These are closed under composition.

Now, given a monad t on A we throw in the Kleisli object A_t in $[\mathcal{K}^{op}, \mathbf{Cat}]$, which may have a new monad s on it. We then throw in its Kleisli object for s to get $(A_t)_s$, but then the composite

$$A \longrightarrow A_t \longrightarrow (A_t)_s$$

is also a bijective-on-objects left adjoint, hence $(A_t)_s$ is also a Kleisli object for a monad on A. Thus this is a 1-step process.

Therefore, we can identify (up to equivalence) the objects of $KL(\mathcal{K})$ with monads in $\mathcal{K}$, and then explicitly describe morphisms and 2-cells between them in terms of $\mathcal{K}$ itself.

In the dual case, where $EM(\mathcal{K}) = (KL(\mathcal{K}^{op}))^{op}$, we get

- The objects are the monads in $\mathcal{K}$,
- The morphisms are the monad morphisms (same as in $\mathrm{Mnd}(\mathcal{K})$), and

- The 2-cells $(A,t) \underset{(n,\psi)}{\overset{(m,\varphi)}{\Downarrow}} (B,s)$ are

$$\begin{tikzcd} A \arrow[rr, bend left, "m"] \arrow[dr, "n"'] & \Downarrow & {} \\ & B \arrow[ur, "s"'] & \end{tikzcd}$$

(which should look sort of Kleisli) with some compatibility with t.

Composition is a also Kleisli sort of thing. Think of sn as the 'free s-algebra on n', so using the universal property of free algebras, can express this as something $sm \to sn$, and express compatibility that way.

Why is this a good thing to do?

(1) We still have $\mathcal{K} \xrightarrow{\mathrm{Id}} EM(\mathcal{K})$, and a right adjoint to this is just, by general nonsense for limit-completions, to give a choice of EM-objects in $\mathcal{K}$.
(2) This comes up in examples. If we start with **Span**, we've seen that monads in **Span** are categories and the morphisms are (pro)functors, but the 2-cells didn't match up. But here they do: we have
$$\mathbf{Cat} \hookrightarrow KL(\mathbf{Span})$$
which is bijective-on-objects and locally fully faithful (restrict to the functors). Can also do this for $\mathbf{Cat}(\mathbb{E})$, or $\mathscr{V}$-**Cat**, or generalized multicategories
(3) Remember that before a distributive law was a monad in $\mathrm{Mnd}(\mathcal{K})$. The multiplication and unit are 2-cells, so if we change the 2-cells, the notion of monad changes. A monad in $EM(\mathcal{K})$ is more general: we call it a *wreath*, since the composition operation is a wreath product.

For example, think about group extensions. Let G be a group acting on an abelian group A, and we have some normalized 2-cocycle $G \times G \xrightarrow{\rho} A$. We consider A and G as monoids, hence monoids in the monoidal category **Set**. (Everything here can be done for bicategories, such as monoidal categories, of course. Better to do for 2-categories since the theory of free completions works better in the enriched setting, but all the definitions make sense for bicategories.)

A wreath still lives on a monad (A,t) in $\mathcal{K}$. We have an endo-thing $A \xrightarrow{s} A$ as before, along with a $\lambda\colon ts \to st$ with some conditions as before. (But s is not any more a monad.) The multiplication is now something $ss \xrightarrow{\nu} st$, and the unit is $1 \to st$. You can still make sense of associativity and unit using λ, but everything ends up in st. Ultimately this gives a monad structure on st, which is called the *wreath product* or composite of s and t.

Let's come back to the example. A is our monoid. G happens to also be a group, but the group structure isn't encoded anywhere. We have the action

$$\lambda\colon G \times A \to A \times G$$
$$(g,a) \mapsto ({}^{g}a, g)$$

and our

$$\nu\colon G \times G \longrightarrow A \times G$$

$$(g, h) \mapsto (\rho(g, h), gh)$$

this is a wreath, so it induces a monoid structure $A \rtimes G$ (which is actually a group). The multiplication is the usual one coming from the cocycle.

There's a corresponding thing for Hopf algebras. Not the most general type of 'twisted smash products', but they are known.

We have two right adjoints to Id defined on different categories $\mathrm{Mnd}(\mathcal{K})$ and $EM(\mathcal{K})$, but the adjunction involves things mapping out of monads whose multiplications are identities, in which case the change in 2-cells doesn't matter. In the new setting, the 2-cells are *just* 2-cells upstairs, that don't need to be lifting anything downstairs:

$$\begin{tikzcd} A^t \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] \arrow[d] & B^s \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

6.7. **Another point of view on $EM(\mathcal{K})$.** This is due to me, but I'm going to follow something due to Aaron Lauda, which makes the terminology more sensible. Of course, that's only a bicategory, but you shouldn't pay any attention at all to that fact. I'm going to confuse the 2-category and rings-and-modules setting, however I feel like it; could be a bumpy ride.

Recall that a morphism $(f,\varphi)\colon (A,t) \to (B,s)$ consists of $f\colon A \to B$ and $sf \xrightarrow{\varphi} ft$. If $\mathcal{K} = \mathbf{Ab}$, the monoidal category (1-object bicategory) of abelian groups, then a monad (monoid) is a ring R. These are the objects of $EM(\mathbf{Ab})$. A 1-cell $R \nrightarrow S$ then consists of an abelian group M and a map $S \otimes M \to M \otimes R$. You should think of this as being a bimodule structure on $M \otimes R$; the left action is

$$S \otimes M \otimes R \longrightarrow M \otimes R \otimes R \longrightarrow M \otimes R$$

and the right action is the free one, and the conditions on φ make it work. Thus the 1-cells are the *right-free bimodules*. The 2-cells are then just module maps.

Composition of 1-cells is the ordinary module composition, but because of the freeness condition, don't need to use any coequalizers. If I were to look at $KL(\mathcal{K})$, we'd get the *left-free* modules. We might also consider the construction $\mathbf{Mod}(\mathcal{K})$, but for that we need some condition on local coequalizers.

In the other construction $\mathbf{Mnd}(\mathcal{K})$, we get only the maps $M \otimes R \to N \otimes R$ of modules which are induced by a map $M \to N$.

7. Pseudomonads

Relationship with monoidal categories. A pseudomonad involves a thing T, which plays the role of a category, a multiplication $T^2 \xrightarrow{m} T$, a unit $1 \xrightarrow{i} T$, an associativity isomorphism

$$\begin{tikzcd} T^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & T^2 \arrow[d] \\ T^2 \arrow[r] & T \end{tikzcd}$$

unit isomorphims λ, ρ, and so on, looking very like a monoidal category.

If we want this to be internal to a **Gray**-category $\mathbb{A}$, this is a different from monoidal categories. Monoidal categories are pseudo-monoids in the monoidal 2-category **Cat**, which is not strict, but doesn't have the Gray property. Here, the associativity pentagon becomes a cube, relating ways to go from $T^4 \to T$, involving a bunch of μs and a pseudonaturality isomorphism. In monoidal categories, one side of the cube corresponds to

$$\begin{tikzcd} ((A \otimes B) \otimes C) \otimes D \arrow[d] & & \\ (A \otimes (B \otimes C)) \otimes D \arrow[dr] & & A \otimes (B \otimes (C \otimes D)) \\ & A \otimes ((B \otimes C) \otimes D) \arrow[ur] & \end{tikzcd}$$

while the other side corresponds to

$$\begin{tikzcd}[column sep=tiny] & (A \otimes B) \otimes (C \otimes D) \arrow[r, equal] & (A \otimes B) \otimes (C \otimes D) \arrow[dr] & \\ ((A \otimes B) \otimes C) \otimes D \arrow[ur] & & & A \otimes (B \otimes (C \otimes D)) \end{tikzcd}$$

while in general, the equality will be replaced by an isomorphism, saying that it doesn't matter whether we tensor A and B first, then C and D, or vice versa.

Our unit isomorphisms will be

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr] & T^2 \arrow[d] \arrow[dl, phantom, near start, "\Swarrow"] \\ T^2 \arrow[r] \arrow[ur, phantom, near start, "\Nearrow"] & T \end{tikzcd}$$

we could take them going the same way, to avoid using inverses (say, if we cared about lax things), but this way will be convenient for the coherence result.

7.1. **Coherence.** The coherence result describes the fact that **there's a universal Gray-category with a pseudo-monad in it.** I.e. there's a **Gray**-category **Psm** such that to give a Gray-functor **Psm** $\to \mathbb{A}$ corresponds to a pseudomonoid in $\mathbb{A}$. Want to make everything enriched-categorical since we have a lot of machinery there to help.

Psm is sort of a cofibrant resolution of $\mathbf{\Delta}$. Could prime $\mathbf{\Delta}$ to get a 2-category, but that's 'too big'; there are some composites that should 'come for free'. The way it works is that **Psm** has a single object $*$ and $\mathbf{Psm}(*,*)$ is a cofibrant replacement of $\mathbf{\Delta}$. The underlying category of $\mathbf{Psm}(*,*)$ (which is a 2-category, since **Psm** is a Gray-category) is freely generated by the face and degeneracy maps in $\mathbf{\Delta}$ (forget the relations we expect to hold)

$$\longrightarrow \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \cdots$$

Since this graph G generates $\mathbf{\Delta}$, we have a map $FG \to \mathbf{\Delta}$ which is bijective on objects and surjective on objects, so we can factor it as a b.o.b.a. 2-functor followed by a l.f.f. one (throw in isomorphisms between the things that would become equal in $\mathbf{\Delta}$), to get

$$\begin{tikzcd} FG \arrow[rr] \arrow[dr] & & \mathbf{\Delta} \\ & \mathbf{Psm}(*,*) \arrow[ur] & \end{tikzcd}$$

To construct prime, we forgot all the way down to the underlying graph of $\mathbf{\Delta}$, rather than a *generating* graph for it, and that would produce all sorts of stuff that we don't really need. That would, maybe, correspond to an 'unbiased' notion of pseudo-monad, where we included a generating operation $T^n \to T$ for all n

We also saw something like this for the Gray tensor product, factoring the map from the funny tensor to the ordinary one.

You now have to define the composition in **Psm**

$$\mathbf{Psm}(*,*) \otimes \mathbf{Psm}(*,*) \longrightarrow \mathbf{Psm}(*,*)$$

to make it a Gray-category. You basically take the composition in $\mathbf{\Delta}$, use that to define it on the generators, then build it up to deal with arbitrary 1-cells, but since the relations only hold up to isomorphism, that's why the Gray-tensor appears.

Now prove that this has the universal property that I said it does, so it really does classify pseudo-monads in a Gray-category. I'm certainly not going to do that. Roughly, how does it go? Given a pseudo-monad, we have

$$\begin{tikzcd} 1 \arrow[r, "i"] & T \arrow[r, shift left=6, "iT"] & T^2 \arrow[l, "m"'] \arrow[from=l, shift right=6, "Ti"'] & \cdots \end{tikzcd}$$

and so on, which defines it on objects, 1-cells, and 2-cells, then have to do it on 3-cells. That's where the fun starts; have μ, λ, and ρ, and need to build up all the 3-cells we need.

Idea: show that for any 2-cell f in **Psm** (i.e. a 1-cell in the above picture, generated by ms and is), there is a normal form $\overline{f}$ and an (unique) isomorphism $f \cong \overline{f}$ built up out of μ, λ, ρ. (This is imprecise, of course; there are things in **Psm** which we should call μ, λ, ρ). And if $f \cong g$, then $\overline{f} = \overline{g}$. Then to define $\mathbf{Psm} \to \mathbb{A}$ on 3-cells (the rest is easy), if $f \cong g$, then compose these isomorphisms through $\overline{f} = \overline{g}$. This is fairly technical; define this rewrite system, show that it has to terminate, etc. You don't want to know any more right now.

7.2. **Algebras.** Can now define a particular weight $\mathbf{Psm} \longrightarrow \mathrm{Psa}\mathbf{Gray}$ such that for any Gray-functor $\mathbb{T} \to \mathbf{Psm} \to \mathbb{A}$, the weighted limit $\{\mathrm{Psa}, \mathbb{T}\}$ 'is' the object of pseudo-algebras, pseudo-morphisms, and algebra 2-cells (all suitably defined) for the pseudo-monad corresponding to $\mathbb{T}$. Again, this is sort of a 'cofibrant replacement' for the corresponding one for 2-categories, although the domain has changed.

Want to explain why the fact that we have pseudo-morphisms here comes from the fact that we're working over Gray. Recall that for ordinary monads, we talked about the fact that to give something $C \to A^t$ is the same as $m\colon C \to A$ with an action $tm \xrightarrow{\alpha} m$, where $c \mapsto (tac \xrightarrow{\alpha c} ac)$, and $\gamma\colon c \to d$ is sent to

$$\begin{tikzcd} tac \arrow[r, "\alpha c"] \arrow[d, "ta\gamma"'] & ac \arrow[d, "a\gamma"] \\ tad \arrow[r, "\alpha d"'] & ad \end{tikzcd}$$

and the fact that $a\gamma$ is a homomorphism can be seen as the naturality of α. One sees something corresponding for operads and Lawvere theories: the actions are natural with respect to homomorphisms.

Then when we come up to the Gray situation, we are thinking of pseudonatural transformations, hence the square commutes up to isomorphism, so we get pseudo-morphisms, not strict ones. That's the 'reason' for making the formal theory of pseudo-monads live in the Gray context. Even if you wanted only to consider 3-categories $\mathbb{A}$, the fact of working over **Gray** gives you the pseudo-morphisms.

This work came before FTM2. In the final section, I talked a bit about distributive laws, with some things called 'theorems'. But trying to make everything live in Gray, only some of the ways of looking at distributive laws go through nicely in the enriched way. After FTM2, thought this would solve everything, but still no, don't know how to deal well with distributive laws nicely in the enriched view.

Since then, Hyland, Power, Cheng have done it ‘tricategorically’, which is a good thing to do, but I kept holding out against it. It also turns out to be stricter than it might be, and you don’t try to make it maximally weak, you are happy not to have to deal with trilimits.

8. 2-Nerves

For ordinary nerves, given a category $\mathscr{C}$, you associate a simplicial set $N\mathscr{C}$ called its *nerve* in which

- a 0-simplex is an object
- a 1-simplex is a morphism
- a 2-simplex is a composable pair and its composite
- and so on.

This process gives a fully faithful embedding

$$\mathbf{Cat} \hookrightarrow [\mathbf{\Delta}^{op}, \mathbf{Set}]$$

(where now $\mathbf{\Delta}$ has gone back to being the topologists’ delta).

If you were to start with a bicategory $\mathscr{B}$, we can still do

- a 0-simplex is an object
- a 1-simplex is a morphism
- a 2-simplex is a 2-cell living in a triangle

$$\begin{tikzcd} {} \arrow[r] \arrow[dr] & {} \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ & {} \end{tikzcd}$$

These 2-simplices are being overworked; they have to express at the same time composition of 1-cells, at least in some weak way, and what the 2-cells are. The problem is that they don’t really ever say what the composite of a 1-cell is, only what maps out of it are. For some purposes, that’s not so great, although it has its advantages.

Thus, we can’t expect to get the strict homomorphisms coming out, since we don’t have any control over what the composites actually are. What you actually get is

$$\mathbf{Bicat}_{\text{normal,lax}} \hookrightarrow [\mathbf{\Delta}^{op}, \mathbf{Set}]$$

where lax means noninvertible comparison, and normal means that identities actually are preserved strictly. There are some subtleties as to what that actually means, which I’ll ignore.

A lot of the time you want to talk about homomorphisms rather than lax ones, so if you want to get your hands on those there are various possibilites. One is to have a bit more structure than a simplicial set: specify as extra data which 2-simplices actually have an equality, or an isomorphism (and similar for higher simplices). This gives a *stratified simplicial set.* This gives rise to the notion of a *complicial set*, if you go up to ω-categories. All sorts of amazing stuff there, see Dominic Verity.

But a *different* way of getting more stuff in there is to consider something which I’m calling 2-nerves, perhaps not the best name. Here we send a bicategory $\mathscr{B}$ to $N_2\mathscr{B}$ (just $N\mathscr{B}$, from now on) which is a functor $\mathbf{\Delta}^{op} \to \mathbf{Cat}$, a simplicial object in $\mathbf{Cat}$.

- $(N\mathscr{B})_0$ is the discrete category of the objects.

- $(N\mathscr{B})_1$ is category whose objects are morphisms and whose morphisms are 2-cells $\begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {} \end{tikzcd}$. Think of this as being an 'enriched' nerve, although it wouldn't work quite as well for a general $\mathscr{V}$.
- $(N\mathscr{B})_2$ doesn't need to include the 2-cells, since we already have them, so we can have the objects be isomorphisms

$$\begin{tikzcd} {} \arrow[r] \arrow[dr] & {} \arrow[d] \arrow[dl, phantom, "\swarrow\cong"] \\ & {} \end{tikzcd}$$

(could use equalities too, but we won't) and the morphisms consist of three 2-cells which commute in the obvious way (in particular, the objects are all fixed).
- $(N\mathscr{B})_3$ has objects being tetrahedra whose faces are all isomorphisms, and so on.

Want a nice definition which is functorial. Consider $N\,\mathrm{Hom}$, the 2-category of bicategories, normal homomorphisms, and icons (which, recall, are oplax natural transformations all of whose 1-cell components are identities). Now $\mathbf{Cat} \hookrightarrow N\,\mathrm{Hom}$, where $\mathbf{Cat}$ is the locally discrete 2-category consisting of categories, functors, and only identity natural transformations, embedding as a full sub-2-category consisting of the locally discrete bicategories. Note that an icon between functors can only be an identity. And of course we have $\mathbf{\Delta} \hookrightarrow \mathbf{Cat}$, so the composite fully fathful $J \colon \mathbf{\Delta} \hookrightarrow N\,\mathrm{hom}$ induces

$$N\,\mathrm{Hom}(J, 1) \colon N\,\mathrm{Hom} \longrightarrow [\mathbf{\Delta}^{op}, \mathbf{Cat}]$$

sending $\mathscr{B}$ to $N\,\mathrm{Hom}(J-, \mathscr{B})$.

For instance, $0 \in \mathbf{\Delta}$ goes to the terminal bicategory, and a normal homomorphism from that into $\mathscr{B}$ is just an object of $\mathscr{B}$, with no room for icons. $1 \in \mathbf{\Delta}$ goes to $(\rightarrow)$, so a homomorphism from this into $\mathscr{B}$ is an arrow, and the icons are exactly what we want. And so on.

Theorem 8.1. $N\,\mathrm{Hom}(J, 1) = N$ *is a fully faithful 2-functor (in a completely strict sense) and has a left biadjoint (following from 2-categorical nonsense).*

How can we characterize the image? $X \in [\mathbf{\Delta}^{op}, \mathbf{Cat}]$ is isomorphic to some $N\mathscr{B}$ iff

1. X_0 is discrete,
2. X is *3-coskeletal*, i.e. isomorphic to the right Kan extension of its 3-truncation. The idea is that 4-simplices and higher are uniquely determined by their boundary.
3. $X_2 \to \mathrm{cosk}_1(X)_2$ is a *discrete isofibration.* A functor $p \colon A \to B$ is a discrete isofibration if given $e \in E$ and $\beta \colon b \cong pe$, there exists a *unique* $\varepsilon \colon e' \cong e$ with $p\varepsilon = \beta$. This implies that if

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\varepsilon"] & E \end{tikzcd}$$

and $p\varepsilon = \mathrm{id}$, then $\varepsilon = \mathrm{id}$.
4. $X_3 \to \mathrm{cosk}_1(X)_3$ (could also use the 2-coskeleton) is also a d.i.f.

(5) The Segal maps are equivalences. (Don't need to say it for all of them, since being 3-coskeletal and so on, but still true.)

A Tamsamani weak 2-category would be one of these such that just the first and the last holding, so this shows that the 2-nerve of a bicategory is a Tamsamani weak 2-category.

This is also a little stricter than some notion of 'internal quasi-category in **Cat**'.

Tamsamani also mentions some sort of '2-nerve' of a bicategory, but what he gets isn't this. There seems to be something missing there. Something to do with algebraic-ness?

The inclusion of $N\,\mathrm{Hom}$ into Tamsamani 2-categories looks like it should be a biequivalence, but it's not quite. If so, then you have to change his morphisms and take the ('normal'?) *pseudo*-natural transformations. The 2-nerves of bicategories here have a 'fibrancy' property, which Simpson talks about in different terms, so here you can get away with just using the 2-natural transformations. Could also restrict to the Tamsamani things which are fibrant; still a bit bigger than 2-nerves?

What you might guess for the nerve of a bicategory is to have

- $N\mathscr{B}_0$ the objects
- $N\mathscr{B}_1 = \sum_{x,y} \mathscr{B}(x,y)$
- $N\mathscr{B}_2 = \sum_{x,y,z} \mathscr{B}(y,z) \times \mathscr{B}(x,y)$

which is what we do for the **Cat** case. If you try to do this, you don't get a simplicial object at the next level, due to failure of associativity. Actually, what we do is to take the pseudo-limit of the composition functor

$$\sum_{x,y,z} \mathscr{B}(y,z) \times \mathscr{B}(x,y) \longrightarrow \sum_{x,y} \mathscr{B}(x,y).$$

And this goes on; for composable triples, we have

$$\begin{tikzcd} \mathscr{B}^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathscr{B}^2 \arrow[d] \\ \mathscr{B}^2 \arrow[r] & \mathscr{B} \end{tikzcd}$$

and $N\mathscr{B}_3$ is the pseudo-limit of this whole diagram. Going on, we can take the pseudo-limit of all sorts of various higher cubes. It's actually even true at $N\mathscr{B}_1$, if you say what you mean, but not very helpful.