

ASSOCIATED PRIME IDEALS

Let R be a commutative Noetherian ring and M a non-zero finitely generated R -module. Let $I = \text{Ann}(M)$. The case $M = R/I$ is of particular interest. We sketch the theory of associated prime ideals of M , following Matsumura (pp 39-40) and, in particular, of the height of ideals. The height of a prime ideal P is the Krull dimension of the localization R_P , that is the maximal length of a chain of prime ideals contained in P . The height of I is the minimum of the heights of P , where P ranges over the prime ideals that contain I (or, equivalently, are minimal among the prime ideals that contain I).

Definition 0.1. The support of M , $\text{Supp}(M)$, is the set of prime ideals such that the localization M_P is non-zero.

A prime containing a prime in $\text{Supp}(M)$ is also in $\text{Supp}(M)$. For example, the support of R/I is $V(I)$.

Definition 0.2. The associated primes of M are those primes P that coincide with the annihilator of some non-zero element $x \in M$. Observe that $I \subset P$ for any such P since $I = \text{Ann}(M) \subset \text{Ann}(x)$. The set of associated prime ideals of M is denoted $\text{Ass}(M)$ or, when necessary for clarity, $\text{Ass}_R(M)$.

The definition of localization implies the following observation.

Lemma 0.3. M_P is non-zero if and only if there is an element $x \neq 0$ in M such that $\text{Ann}(x) \subset P$. Therefore $\text{Ass}(M)$ is contained in $\text{Supp}(M)$.

Proposition 0.4. Let \mathcal{S} be the set of ideals of the form $\text{Ann}(x)$ for some $x \neq 0$ in M . An ideal P that is maximal in the set \mathcal{S} is prime. In particular, $\text{Ass}(M)$ is non-empty.

Proof. Let $rs \in P$, where $P = \text{Ann}(x)$ and $s \notin P$. Then $r \in \text{Ann}(sx) \subset \text{Ann}(x)$. By the maximality of P , $\text{Ann}(sx) = \text{Ann}(x)$ and therefore $r \in P$. \square

Corollary 0.5. The set of zero-divisors for M is the union of the primes in $\text{Ass}(M)$.

Proof. Clearly, any element of an associated prime is a zero-divisor for M . If $rx = 0$, then $(r) \subset \text{Ann}(x)$, and $\text{Ann}(x) \subset P$ for some $P \in \text{Ass}(M)$ by the proposition. \square

Proposition 0.6. If S is a multiplicative subset of R and N is a finitely generated R_S -module, then $\text{Ass}_R(N) = \text{Ass}_{R_S}(N)$, where we view $\text{Spec}(R_S)$ as contained in $\text{spec}(R)$. Therefore $\text{Ass}_R(M_S) = \text{Ass}_{R_S}(M_S)$.

Proof. Inspection of the definitions. See Matsumura, p. 39. \square

Corollary 0.7. P is in $\text{Ass}_R(M)$ if and only if PR_P is in $\text{Ass}_{R_P}(M_P)$.

Proposition 0.8. If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact, then

$$\text{Ass}(M) \subset \text{Ass}(M') \cup \text{Ass}(M'')$$

Proof. If $P = \text{Ann}(x)$, then Rx is a copy N of R/P contained in M . Since P is prime $\text{Ann}(y) = P$ for any non-zero $y \in N$. If $N \cap M' \neq 0$, this implies $P \in \text{Ann}(M')$. If $N \cap M' = 0$, then N is isomorphic to its image in M'' and $P \in \text{Ann}(M'')$. \square

Proposition 0.9. *There is a chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i-1} \cong R/P_i$ for some prime ideal P_i .

Proof. Proceed inductively, starting with $M_1 = Rx$ where $\text{Ann}(x)$ is prime. If $M_i \neq M$, choose $P_i \in \text{Ass}(M/M_{i-1})$ to obtain a copy of R/P_i in M/M_{i-1} . \square

Theorem 0.10. *$\text{Ass}(M)$ is a finite subset of $\text{Supp}(M)$, and the minimal elements of $\text{Ass}(M)$ and $\text{Supp}(M)$ coincide.*

Proof. The set of prime ideals containing any non-zero proper ideal in a commutative Noetherian ring is finite, by consideration of the topology on $\text{Spec}(R)$. But we have a different proof here: the finiteness of $\text{Ass}(M)$ follows inductively from the previous two propositions. Let P be a minimal element of $\text{Supp}(M)$. Then $M_P \neq 0$ and, by minimality and results above,

$$\emptyset \neq \text{Ass}_R(M_P) = \text{Ass}_R(M) \cap \text{Spec}(R_P) \subset \text{Supp}(M) \cap \text{Spec}(R_P) = \{P\}.$$

Therefore $P \in \text{Ass}(M)$. \square