ASSOCIATED PRIME IDEALS

Let $R$ be a commutative Noetherian ring and $M$ a non-zero finitely generated $R$-module. Let $I = \text{Ann}(M)$. The case $M = R/I$ is of particular interest. We sketch the theory of associated prime ideals of $M$, following Matsumura (pp 39-40) and, in particular, of the height of ideals. The height of a prime ideal $P$ is the Krull dimension of the localization $R_P$, that is the maximal length of a chain of prime ideals contained in $P$. The height of $I$ is the minimum of the heights of $P$, where $P$ ranges over the prime ideals that contain $I$ (or, equivalently, are minimal among the prime ideals that contain $I$).

**Definition 0.1.** The support of $M$, $\text{Supp}(M)$, is the set of prime ideals such that the localization $M_P$ is non-zero.

A prime containing a prime in $\text{Supp}(M)$ is also in $\text{Supp}(M)$. For example, the support of $R/I$ is $V(I)$.

**Definition 0.2.** The associated primes of $M$ are those primes $P$ that coincide with the annihilator of some non-zero element $x \in M$. Observe that $I \subseteq P$ for any such $P$ since $I = \text{Ann}(M) \subseteq \text{Ann}(x)$. The set of associated prime ideals of $M$ is denoted $\text{Ass}(M)$ or, when necessary for clarity, $\text{Ass}_R(M)$.

The definition of localization implies the following observation.

**Lemma 0.3.** $M_P$ is non-zero if and only if there is an element $x \neq 0$ in $M$ such that $\text{Ann}(x) \subset P$. Therefore $\text{Ass}(M)$ is contained in $\text{Supp}(M)$.

**Proposition 0.4.** Let $\mathcal{I}$ be the set of ideals of the form $\text{Ann}(x)$ for some $x \neq 0$ in $M$. An ideal $P$ that is maximal in the set $\mathcal{I}$ is prime. In particular, $\text{Ass}(M)$ is non-empty.

**Proof.** Let $rs \in P$, where $P = \text{Ann}(x)$ and $s \notin P$. Then $r \in \text{Ann}(sx) \supset \text{Ann}(x)$. By the maximality of $P$, $\text{Ann}(sx) = \text{Ann}(x)$ and therefore $r \in P$. □

**Corollary 0.5.** The set of zero-divisors for $M$ is the union of the primes in $\text{Ass}(M)$.

**Proof.** Clearly, any element of an associated prime is a zero-divisor for $M$. If $rx = 0$, then $(r) \subset \text{Ann}(x)$, and $\text{Ann}(x) \subset P$ for some $P \in \text{Ass}(M)$ by the proposition. □

**Proposition 0.6.** If $S$ is a multiplicative subset of $R$ and $N$ is a finitely generated $R_S$-module, then $\text{Ass}_R(N) = \text{Ass}_{R_S}(N)$, where we view $\text{Spec}(R_S)$ as contained in $\text{spec}(R)$. Therefore $\text{Ass}_R(M_S) = \text{Ass}_{R_S}(M_S)$.

**Proof.** Inspection of the definitions. See Matsumura, p. 39. □

**Corollary 0.7.** $P$ is in $\text{Ass}_R(M)$ if and only if $PR_P$ is in $\text{Ass}_{R_P}(M_P)$.

**Proposition 0.8.** If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact, then $\text{Ass}(M) \subset \text{Ass}(M') \cup \text{Ass}(M'')$.
Proof. If $P = \text{Ann}(x)$, then $Rx$ is a copy $N$ of $R/P$ contained in $M$. Since $P$ is prime $\text{Ann}(y) = P$ for any non-zero $y \in N$. If $N \cap M' \neq 0$, this implies $P \in \text{Ann}(M')$. If $N \cap M' = 0$, then $N$ is isomorphic to its image in $M''$ and $P \in \text{Ann}(M'')$. □

**Proposition 0.9.** There is a chain of submodules 

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_i/M_{i−1} \cong R/P_i$ for some prime ideal $P_i$.

Proof. Proceed inductively, starting with $M_1 = Rx$ where $\text{Ann}(x)$ is prime. If $M_i \neq M$, choose $P_i \in \text{Ass}(M/M_{i−1})$ to obtain a copy of $R/P_i$ in $M/M_{i−1}$. □

**Theorem 0.10.** $\text{Ass}(M)$ is a finite subset of $\text{Supp}(M)$, and the minimal elements of $\text{Ass}(M)$ and $\text{Supp}(M)$ coincide.

Proof. The set of prime ideals containing any non-zero proper ideal in a commutative Noetherian ring is finite, by consideration of the topology on $\text{Spec}(R)$. But we have a different proof here: the finiteness of $\text{Ass}(M)$ follows inductively from the previous two propositions. Let $P$ be a minimal element of $\text{Supp}(M)$. Then $M_P \neq 0$ and, by minimality and results above,

$$\emptyset \neq \text{Ass}_R(M_P) = \text{Ass}_R(M) \cap \text{Spec}(R_P) \subset \text{Supp}(M) \cap \text{Spec}(R_P) = \{P\}.$$ 

Therefore $P \in \text{Ass}(M)$. □