1. Self-dual Hopf algebras

The homology Hopf algebras \( H_*(BU; \mathbb{Z}) \) and \( H_*(BO; \mathbb{F}_2) \) enjoy a very special property: they are self-dual, so that they are isomorphic to the cohomology Hopf algebras \( H^*(BU; \mathbb{Z}) \) and \( H^*(BO; \mathbb{F}_2) \). The proof of this basic result is purely algebraic and explicitly determines the homology Hopf algebras from the cohomology Hopf algebras (or vice versa if one calculates in the opposite order). We assume that the reader knows that the cohomology Hopf algebras are given by

\[
H^*(BU; \mathbb{Z}) = P\{c_i \mid i \geq 1\} \quad \text{with} \quad \psi(c_n) = \sum_{i+j=n} c_i \otimes c_j
\]

and

\[
H^*(BO; \mathbb{F}_2) = P\{w_i \mid i \geq 1\} \quad \text{with} \quad \psi(w_n) = \sum_{i+j=n} w_i \otimes w_j.
\]

The calculations of \( H^*(BU(n); \mathbb{Z}) \) and \( H^*(BO(n); \mathbb{F}_2) \) are summarized in [?, pp 187, 195], and passage to colimits over \( n \) gives the stated conclusions. Thus determination of the homology algebras is a purely algebraic problem in dualization.\(^1\)

Recall that the dual coalgebra of a polynomial algebra \( P[x] \) over \( R \) is written \( \Gamma[x] \); when \( P[x] \) is regarded as a Hopf algebra with \( x \) primitive, \( \Gamma[x] \) is called a divided polynomial Hopf algebra.

Clearly \( H^*(BU(1); \mathbb{Z}) = P[c_1] \) and \( H^*(BO(1); \mathbb{F}_2) = P[w_1] \) are quotient algebras of \( H^*(BU; \mathbb{Z}) \) and \( H^*(BO; \mathbb{F}_2) \). Write \( H_*(BU(1); \mathbb{Z}) = \Gamma[\gamma_1] \); it has basis \( \{\gamma_i \mid i \geq 0\} \) and coproduct \( \psi(\gamma_n) = \sum_{i+j=n} \gamma_i \otimes \gamma_j \), where \( \gamma_0 = 1 \) and \( \gamma_i \) is dual to \( c_i^1 \). Write \( H_*(BO(1); \mathbb{F}_2) = \Gamma[\gamma_1] \) similarly. The inclusions \( BU(1) \rightarrow BU \) and \( BO(1) \rightarrow BO \) induce identifications of these homologies with sub coalgebras of \( H_*(BU; \mathbb{Z}) \) and \( H_*(BO; \mathbb{F}_2) \), and we shall prove that these sub coalgebras freely generate the respective homology algebras.

**Theorem 1.3.** \( H_*(BU; \mathbb{Z}) = P\{\gamma_i \mid i \geq 1\} \), where \( \gamma_i \in H_*(BU(1); \mathbb{Z}) \) is dual to \( c_i^1 \). The basis \( \{p_i\} \) for the primitive elements of \( H_*(BU; \mathbb{Z}) \) such that \( \langle c_1, p_i \rangle = 1 \) is specified inductively by

\[
p_1 = \gamma_1 \quad \text{and} \quad p_i = (-1)^{i+1} i \gamma_i + \sum_{j=1}^{i-1} (-1)^{j+1} \gamma_j p_{i-j} \quad \text{for} \ i > 0.
\]

This recursion formula is generally ascribed to Newton, of course in a different but related context, although the following explicit evaluation was known even earlier (to Girard, in a 1629 paper).

**Remark 1.4.** An explicit formula for \( p_i \) is given by

\[
p_i = \sum_E (-1)^{|E|+i} \frac{(|E|-1)!i!}{e_1! \cdots e_r!} \gamma^E.
\]

Here the sum is taken over all sequences \( E = (e_1, \ldots, e_r) \) with \( e_q \geq 0 \) and \( \sum q e_q = i \); \( |E| = \sum e_q \) and \( \gamma^E = \gamma_1^{e_1} \cdots \gamma_r^{e_r} \).

\(^1\text{We thank John Rognes, who texed this section from the first author’s notes in 1996.}\)
Theorem 1.5. $H^*(BO; \mathbb{F}_2) = P\{\gamma_i \mid i \geq 1\}$, where $\gamma_i \in H^*(BO(1); \mathbb{F}_2)$ is dual to $w^1_i$. The nonzero primitive elements of $H^*(BO; \mathbb{F}_2)$ are specified inductively by

$$p_1 = \gamma_1 \quad \text{and} \quad p_i = i\gamma_i + \sum_{j=1}^{i-1} \gamma_j p_{i-j} \quad \text{for } i > 0.$$

Comparison of these theorems to (1.1) and (1.2) shows that $H^*(BU; \mathbb{Z})$ and $H^*(BO; \mathbb{F}_2)$ are self-dual; that is, they are isomorphic as Hopf algebras to their own duals. Following Moore [?], we shall carry out the proofs by considering self-duality for certain general types of Hopf algebras.

We work in the category of connected free $R$-modules $X$ of finite type, so that $X_i = 0$ for $i < 0$ and $X_0 = R$. Throughout the discussion, all algebras are to be commutative and all coalgebras are to be cocommutative. Thus all Hopf algebras are to be commutative and cocommutative.

Definition 1.6. We define some universal Hopf algebras.

(i) A universal enveloping Hopf algebra of a coalgebra $C$ is a Hopf algebra $LC$ together with a morphism $i: C \rightarrow LC$ of coalgebras which is universal with respect to maps of coalgebras $f: C \rightarrow B$, where $B$ is a Hopf algebra. That is, any such $f$ factors uniquely as $\tilde{f} \circ i$ for a morphism $\tilde{f}: LC \rightarrow B$ of Hopf algebras.

(ii) A universal covering Hopf algebra of an algebra $A$ is a Hopf algebra $MA$ together with a morphism $p: MA \rightarrow A$ of algebras which is universal with respect to maps of algebras $f: B \rightarrow A$, where $B$ is a Hopf algebra. That is, any such $f$ factors uniquely as $p \circ \tilde{f}$ for a morphism $\tilde{f}: B \rightarrow MA$ of Hopf algebras.

Lemma 1.7. Universal Hopf algebras exist and are unique. That is,

(i) any coalgebra $C$ admits a universal enveloping Hopf algebra $i: C \rightarrow LC$;
(ii) any algebra $A$ admits a universal covering Hopf algebra $p: MA \rightarrow A$.

Proof. Of course, uniqueness up to isomorphism follows from universality. For (i), we have $C = R \oplus JC$, where $JC$ is the module of positive degree elements of $C$. As an algebra, we take $LC = A(JC)$, the free (graded) commutative algebra generated by $JC$. Let $i: C \rightarrow LC$ be the natural inclusion $JC \rightarrow LC$ in positive degrees and the identity map id of $R$ in degree zero. If $\psi$ is the coproduct of $C$, the coproduct of $LC$ is defined to be the unique map of algebras $\psi: LC \rightarrow LC \otimes LC$ that makes the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\psi} & C \otimes C \\
\downarrow i & & \downarrow i \otimes i \\
LC & \xrightarrow{\psi} & LC \otimes LC.
\end{array}
$$

That $\psi$ defines a coalgebra and thus a Hopf algebra structure on $LC$ and that $i: C \rightarrow LC$ is universal follow directly from the universal property of $LC$ as an algebra. For (ii), since all modules are taken to be free of finite type, $p: MA \rightarrow A$ can be specified as $i^*: (L(A^*))^* \rightarrow A^{**} = A$. \qed

Remark 1.8. Similar constructions may be obtained when we omit some or all of the commutativity hypotheses. We can define universal enveloping commutative
Hopf algebras for arbitrary coalgebras and universal covering cocommutative Hopf algebras for arbitrary algebras. These will coincide with our present constructions under our hypotheses. The universal enveloping non–commutative Hopf algebra is of course a quite different construction.

We shall shortly require a pair of dual lemmas, for which we need some notations. For an $R$-module $X$, let $X^n$ denote the $n$-fold tensor product of $X$ with itself. With the usual sign $(-1)^{\text{deg } x \cdot \text{deg } y}$ inserted when $x$ is permuted past $y$, the symmetric group $\Sigma_n$ acts on $X^n$. If $X$ is an algebra or coalgebra, then so is $X^n$, and $\Sigma_n$ acts as a group of automorphisms. Let $\Sigma_n$ act trivially on $LC$ and $MA$.

**Lemma 1.9.** Let $C$ be a coalgebra. For $n > 0$, define $\iota_n : C^n \to LC$ to be the composite of $i^n : C^n \to (LC)^n$ and the iterated product $\psi : (LC)^n \to LC$. Then $\iota_n$ is a morphism of both $\Sigma_n$-modules and coalgebras. If $C_q = 0$ for $0 < q < m$, then $\iota_n$ is an epimorphism in degrees $q \leq mn$.

**Proof.** The first statement is immediate from the definitions and the second statement follows from the fact that the image of $\iota_n$ is the span of the monomials in $C$ of length at most $n$. □

**Lemma 1.10.** Let $A$ be an algebra. For $n > 0$, define $\pi_n : MA \to A^n$ to be the composite of the iterated coproduct $\psi : MA \to (MA)^n$ and $p^n : (MA)^n \to A^n$. Then $\pi_n$ is a morphism of both $\Sigma_n$-modules and algebras. If $A_q = 0$ for $0 < q < m$, then $\iota_n$ is a monomorphism in degrees $q \leq mn$.

**Proof.** This follows by dualizing the previous lemma. □

**Definition 1.11.** Let $X$ be positively graded $R$-module, so that $X_i = 0$ for $i \leq 0$. Define $LX = L(R \oplus X)$, where $R \oplus X$ is $R$ in degree zero and has the trivial coalgebra structure, in which every element of $X$ is primitive. Define $MX = M(R \oplus X)$, where $R \oplus X$ has the trivial algebra structure, in which the product of any two elements of $X$ is zero. There is a natural morphism of Hopf algebras $\lambda : LMX \to MLX$, which is defined in two equivalent ways. Indeed, consider the following diagram:

\[
\begin{array}{ccc}
LMX & \lambda & MLX \\
\downarrow i & & \downarrow \mu \\
MX & \nu & RX \\
\downarrow p & & \downarrow \iota \\
& LX.
\end{array}
\]

Define $\mu$ to be $A(p) : A(JMX) \to A(X)$, which is the unique morphism of algebras that extends $i \circ p$, and then obtain $\lambda$ by the universal property of $p : MLX \to LX$. Define $\nu$ to be the dual of $A(i^*) : A(JLX)^* \to A(X^*)$, so that $\nu$ is the unique morphism of coalgebras that covers $i \circ p$, and obtain $\lambda$ by the universal property of $i : MX \to LMX$. To see that the two definitions coincide, note that if $\lambda$ is defined by the first property, then $\lambda \circ i = \nu$ by uniqueness and so $\lambda$ also satisfies the second property.

Observe that $(R \oplus X)^*$ may be identified with $R \oplus X^*$. Since $MA = (L(A^*))^*$, it follows that

\[
MX \equiv M(R \oplus X) = (L(R \oplus X^*))^* \equiv (L(X^*))^*.
\]

In turn, with $A = L(X^*)$, this implies

\[
ML(X^*) = MA = (L(A^*))^* = (L(L(X^*))^*)^* = (LMX)^*.
\]
If \( X \) is \( R \)-free on a given basis, then the isomorphism \( X \cong X^* \) determined by use of the dual basis induces an isomorphism of Hopf algebras

\[ \beta: MLX \cong ML(X^*) = (LMX)^*. \]

When \( \lambda: LMX \rightarrow MLX \) is an isomorphism, it follows that \( LMX \) is self-dual. While \( \lambda \) is not always an isomorphism, it is so in the cases of greatest topological interest. We now regard \( i: C \rightarrow LC \) as an inclusion, omitting \( i \) from the notation. Write \( \langle -, - \rangle \) for the usual pairing between a free \( R \)-module and its dual.

**Theorem 1.12.** Let \( X \) be free on one generator \( x \) of degree \( m \), where either \( m \) is even or \( R \) has characteristic two. Then \( \lambda: LMX \rightarrow MLX \) is an isomorphism. Moreover if

\[ c_i = \gamma_i(x) \in \Gamma[x] = MX \quad \text{and} \quad \gamma_i = (\beta \circ \lambda)(c_i) \in (LMX)^*, \]

then \( \gamma_i \) is the basis element dual to \( c_i^* \) and the basis \( \{p_i\} \) for the primitive elements of \( (LMX)^* \) such that \((c_i, p_i) = 1\) is specified inductively by

\[ p_1 = \gamma_1 \quad \text{and} \quad p_i = (-1)^{i+1}i\gamma_i + \sum_{j=1}^{i-1}(-1)^{i+j}p_{i-j} \quad \text{for } i > 0. \]

Here \( LMX = P\{c_i \mid i \geq 1\} \) with \( \psi(c_n) = \sum_{i+j=n} c_i \otimes c_j \), where \( c_0 = 1 \). When \( R = \mathbb{Z} \) and \( m = 2 \), \( LMX \) may be identified with \( H^*(BU_{\mathbb{Z}}) \) and \( (LMX)^* \) may be identified with \( H_*(BU_{\mathbb{Z}}) \). Thus this result immediately implies Theorem 1.3. Similarly, with \( R = \mathbb{F}_2 \) and \( m = 1 \), it implies Theorem 1.5. The rest of the section will be devoted to the proof.

**Proof.** Note that \( LX = P[x] \) and write \( P[x]^n = P[x_1, \ldots, x_n] \), where \( x_i = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \) with \( x \) in the \( i \)th position. Let \( \sigma_1, \ldots, \sigma_n \) be the elementary symmetric functions in the \( x_i \). Consider \( \pi_n \lambda: LMX \rightarrow P[x]^n \), where \( \pi_n = p^n \psi: MP[x] \rightarrow P[x]^n \) is as specified in Lemma 1.10. From the diagram which defines \( \lambda \), we see that \( p\lambda: LMX \rightarrow P[x] \) is given on generators by

\[ p\lambda c_j = ipc_j = \begin{cases} x & \text{if } j = 1 \\ 0 & \text{if } j > 1. \end{cases} \]

Since \( \lambda \) is a morphism of Hopf algebras, it follows that

\[ \pi_n p\lambda c_j = p^n \psi \lambda c_j = p^n \lambda^n \psi c_j = (p\lambda)^n \left( \sum_{i_1 + \cdots + i_n = j} c_{i_1} \otimes \cdots \otimes c_{i_n} \right) = \begin{cases} \sigma_j & \text{if } j \leq n \\ 0 & \text{if } j > n. \end{cases} \]

Since \( LMX = P[c_i] \), the map \( \pi_n \lambda: P[c_i] \rightarrow P[\sigma_1, \ldots, \sigma_n] \) is an isomorphism in degrees \( q \leq mn \). By Lemma 1.10, \( \pi_n \) also takes values in \( P[\sigma_1, \ldots, \sigma_n] \) and is a monomorphism in degrees \( q \leq mn \). Therefore \( \pi_n \) and \( \lambda \) are both isomorphisms in degrees \( q \leq mn \). Since \( n \) is arbitrary, this proves that \( \lambda \) is an isomorphism.

To see the duality properties of the \( \gamma_i \), consider the map \( \nu: \Gamma[x] \rightarrow MP[x] \) in the diagram defining \( \lambda \). Here \( \nu \) is dual to \( A(i^*): A(JT[x]) \rightarrow P[x^*] \), where \( x^* \) is the basis element of \( X^* \) dual to \( x \), and \( i^* \) maps \( \gamma_1(x^*) \) to \( x^* \) and annihilates \( \gamma_i(x^*) \) for \( i > 1 \). Since \( c_i = \gamma_i(x) \) is dual to \( (x^*)^i \), \( \nu(c_i) \) is dual to \( \gamma_1(x^*)^i \) and thus \( i\nu(c_i) = \gamma_i \) is dual to \( c_i^* \).

Since the primitive elements of \( (LMX)^* \) are dual to the indecomposable elements of \( LMX \), they are free on one generator dual to \( c_i \) in each degree \( mi \). We shall prove inductively that this generator is \( p_i \), the case \( i = 1 \) having been handled.
above. Consider the term \( \gamma_j p_{i-j}, \) \( 1 \leq j \leq i - 1, \) in the iterative expression for \( p_i. \) Let \( c^E \) be a monomial in the \( c_k, \) so that \( E = (e_1, \ldots, e_r) \) and \( c^E = c_1^i \cdots c_r^e. \) Then

\[
\langle c^E, \gamma_j p_{i-j} \rangle = \langle \psi c^E, \gamma_j \otimes p_{i-j} \rangle = \langle \psi c^E, (c_1^i)^* \otimes c_r^e \rangle
\]

by the induction hypothesis and the calculation above. Consideration of the form of \( \psi c^E \) shows that this is zero unless \( c^E \) is either \( c_1^i c_{i-j} \) or \( c_1^{i-1} c_{r-j+1}. \) When it is one in all cases except the case \( \langle c_1, \gamma_{i-1} p_1 \rangle = 1. \) It follows that \( \langle c^E, p_i \rangle = 0 \) except for the case \( \langle c_i, p_i \rangle = 1. \) An alternative argument is to verify inductively that each \( p_i \) is in fact primitive and then check that \( \langle c_i, p_i \rangle = 1. \)

2. The homotopy groups of \( MO \) and other Thom spectra

In [?, Ch. 25], we explained Thom’s classical computation of the real cobordism of smooth manifolds. In fact, the exposition was something of a cheat. Knowing the splitting theorems of §2? and the self-duality theorem of §1, the senior author simply transcribed the first and quoted the second to give the main points of the calculation. That obscures the conceptual simplicity of the idea and its implementation. We explain in this section how the general theory applies. A punch line, explained at the end of the section, is that the conceptual argument applies to much more sophisticated cobordism theories, where the actual calculations are far more difficult. We take all homology and cohomology with coefficients in \( F_2 \) in this section.

Recall the description of the Hopf algebra \( H^*(BO) \) from (1.2). The structure of the dual Hopf algebra \( H_*(BO) \) is given in Theorem 1.5. To conform to the notation of [?, Ch. 25], write \( \gamma_i = b_i. \) It is the image in \( H_*(BO) \) of the non-zero class \( x_i \in H_*(\mathbb{R}P^\infty). \) Thus \( H_*(BO) \) is the polynomial algebra on the \( b_i, \) and \( \psi(b_k) = \sum_{i+j=k} b_i \otimes b_j. \)

The Thom prespectrum \( TO \) and its associated Thom spectrum \( MO \) are described in [?, pp. 216, 229], but we are not much concerned with the foundations of stable homotopy theory here. The ring structure on \( TO \) gives its homology an algebra structure, and the Thom isomorphism \( \Phi: H_*(TO) \rightarrow H_*(BO) \) is an isomorphism of algebras [?, p. 221]. Write \( a_i = \Phi^{-1}(b_i). \) The Thom space \( TO(1) \) of the universal line bundle is equivalent to \( \mathbb{R}P^\infty \) and, with \( a_0 = 1, \) \( a_i \) is the image of \( x_{i+1} \) in \( H_*(TO). \)

Let \( A \) be the mod 2 Steenrod algebra and \( A_\ast \) be its dual. Then \( A \) acts on the cohomology of spaces, prespectra, and spectra, and the action of \( A \) on the cohomology of a ring prespectrum \( T \) dualizes to give \( H_*(T) \) a structure of left \( A \)-comodule algebra, as in ???. The composite

\[
\pi = (id \otimes \varepsilon) \nu: H_*(TO) \rightarrow A_\ast \otimes H_*(TO) \rightarrow A_\ast
\]

is computed on [?, p. 224]. The computation just translates the easy computation of the action of \( A \) on \( H^*(\mathbb{R}P^\infty) \) to a formula for the coaction of \( A_\ast. \) As an algebra, \( A_\ast \) is a polynomial algebra on certain generators \( \xi_r \) of degree \( 2^r - 1, \) and \( \pi(a_{2^r-1}) = \xi_r. \) Thus \( \pi \) is an epimorphism.

By ??, this implies that there is an isomorphism

\[
A_\ast \otimes P_{A_\ast}(H_*(TO)) \cong H_*(TO)
\]

of left \( A_\ast \)-comodules and right \( P_{A_\ast}(H_*(TO))-\)modules. Since we know that \( A_\ast \) and \( H_*(TO) \) are polynomial algebras such that the generators of \( A_\ast \) map to some of
the generators of $H_\ast(\text{TO})$, it is clear that $P_{A_\ast}(H_\ast(\text{TO})) \equiv N_\ast$ must be a polynomial algebra on (abstract) generators $u_i$ of degree $i$, where $i > 1$ and $i \neq 2r - 1$. Dually $H^\ast(\text{TO}) = H^\ast(\text{MO})$ is isomorphic as an $A$-module to $A \otimes N^\ast$. As explained informally in [?, §25.7], this implies that $\text{MO}$ is a product of suspensions of Eilenberg-Mac Lane spectrum $HF_2$ and that $\pi_\ast(\text{MO}) \cong N_\ast$ as an algebra. This gives the now standard way of obtaining Thom’s calculation [?] of $\pi_\ast(\text{MO})$.

The theorem applies to unoriented smooth manifolds, but one might consider less structured manifolds, such as piecewise linear or topological manifolds. Focusing on PL manifolds for definiteness, which makes sense since the theory of PL-manifolds was designed to get around the lack of obvious transversality in the theory of topological manifolds, one can adapt Thom’s theorem to prove geometrically that the $PL$-cobordism groups are isomorphic to the homotopy groups of a Thom prespectrum $TPL$. By neglect of structure, we obtain a map of Thom prespectra $\text{TO} \longrightarrow TPL$. We have the same formal structure on $TPL$ as we have on $\text{TO}$, and we have a commutative diagram

$$
\begin{array}{ccc}
H_\ast(\text{TO}) & \longrightarrow & H_\ast(TPL) \\
\pi \downarrow & & \uparrow \pi \\
& A_\ast & \\
\end{array}
$$

Even without any calculational knowledge of $H_\ast(\text{BPL})$ and $H_\ast(TPL)$, we conclude that $\pi$ on the right must also be an epimorphism.

Therefore, as a matter of algebra, ?? gives us an isomorphism

$$A_\ast \otimes P_{A_\ast}(H_\ast(TPL)) \cong H_\ast(TPL)$$

of left $A_\ast$-comodules and right $P_{A_\ast}(H_\ast(TPL))$-algebras. Here again, the Thom isomorphism $\Phi: H_\ast(TPL) \longrightarrow H_\ast(\text{BPL})$ is an isomorphism of algebras. Therefore, if we can compute $H_\ast(\text{BPL})$ as an algebra, then we can read off what $P_{A_\ast}(H_\ast(TPL))$ must be as an algebra. The same formal argument as for $\text{MO}$ shows that $\text{MP}$ is a product of suspensions of $HF_2$ and that $\pi_\ast(\text{MP}) \cong P_{A_\ast}(H_\ast(TPL))$ as algebras. In fact, this argument was understood and explained in [?] well before $H_\ast(\text{BPL})$ was determined. The calculation of $H_\ast(\text{BPL}; \mathbb{F}_p)$ at all primes $p$ is described in [?, ?], but that is another story. In any case, this sketch should give some idea of the algebraic power of the splitting theorems in §??.

3. A PROOF OF THE BOTT PERIODICITY THEOREM

The self duality of $H^\ast(\text{BU})$ described in (1.1) and Theorem 1.3 also plays a central role in a quick proof of (complex) Bott periodicity. We describe how that works in this section. As discussed briefly in [?, §24.2], the essential point is to prove the following result. Homology and cohomology are to be taken with coefficients in $\mathbb{Z}$ in this section.

**Theorem 3.1.** There is a map $\beta: \text{BU} \longrightarrow \Omega SU$ of $H$-spaces which induces an isomorphism on homology.

It follows from the dual Whitehead theorem that $\beta$ must be an equivalence.

---

2 It is part of the 1970’s story of infinite loop space theory and $E_\infty$ ring spectra; see [?] for a 1970’s overview and [?] for a modernized perspective.
We begin by defining the Bott map $\beta$, following Bott [?]. Write $U(V)$ for the compact Lie group of unitary transformations $V \to V$ on a complex vector space $V$ with a given Hermitian product. If $V$ is of countable dimension, let $U(V)$ denote the colimit of the $U(W)$ where $W$ runs through the finite dimensional subspaces of $V$ with their induced Hermitian products. Fixing the standard inclusions $\mathbb{C}^n \to \mathbb{C}^\infty$, we specify $BU = U/U \times U$ to be the colimit of the Grassmannians $U(2n)/U(n) \times U(n)$. We let $U$ be the colimit of the $U(2n)$ and $SU$ be its subgroup colim $SU(2n)$ of unitary transformations with determinant one. For convenience, we write $V = \mathbb{C}^\infty$ and let $V^n$ denote the direct sum of $n$ copies of $V$.

It is also convenient to use paths and loops of length $\pi$. Taking $0 \leq \theta \leq \pi$, define $\nu(\theta) \in U(V^2)$ by

$$\nu(\theta)(z', z'') = (e^{i\theta}z', e^{-i\theta}z'').$$

Note that $\nu(0)$ is multiplication by 1, $\nu(\pi)$ is multiplication by $-1$, and $\nu(\theta)^{-1} = \nu(-\theta)$. Define

$$\beta: U(\mathbb{C}^\infty \oplus \mathbb{C}^\infty) \to \Omega SU(\mathbb{C}^\infty \oplus \mathbb{C}^\infty)$$

by letting

$$\beta(T)(\theta) = [T, \nu(\theta)] = T\nu(\theta)T^{-1}\nu(-\theta)$$

where $T \in U(V^2)$. Clearly $[T, \nu(\theta)]$ has determinant one and $\beta(T)$ is a loop at the identity element $e$ of the group $SU(V^2)$. Moreover, since $\nu(\theta)$ is just a scalar multiplication on each summand $V$, if $T = T' \times T'' \in U(V) \times U(V)$, then $\beta(T)(\theta) = e$. Therefore $\beta$ passes to orbits to give a well-defined map

$$\beta: BU = U/U \times U \to \Omega SU.$$

To define the $H$-space structure on $BU$, choose a linear isometric isomorphism $\xi: V^2 \to V$ and let the product $T_1T_2$ be the composite

$$V^2 \xrightarrow{(\xi^{-1})^2} V^4 \xrightarrow{T_1 \oplus T_2} V^4 \xrightarrow{id \oplus \gamma \oplus id} V^4 \xrightarrow{\xi^2} V^2,$$

where $\gamma: V^2 \to V^2$ interchanges the two summands. Up to homotopy, the product is independent of the choice of $\xi$. The $H$-space structure we use on $\Omega SU$ is the pointwise product, $(\omega_1\omega_2)(\theta) = \omega_1(\theta)\omega_2\theta$. We leave it as an exercise to verify that $\beta$ is an $H$-map.\footnote{This is also part of the 1970’s infinite loop space story; details generalizing these $H$-space structures and maps to the context of actions by an $E_\infty$ operad may be found in [?, pp. 9-17].}

Let $\{e'_1\}$ and $\{e''_1\}$ denote the standard bases of two copies of $V$ and let $\mathbb{C}_1^n$ and $\mathbb{C}_2^n$ be spanned by the first $n$ vectors in each of these bases. Let

$$j: U(\mathbb{C}_1^n \oplus \mathbb{C}_2^n) \to U(\mathbb{C}_1^n \oplus \mathbb{C}_2^n)$$

be the inclusion. Restrictions of $\beta$ give a commutative diagram

$$\begin{array}{ccc}
CP^n = U(\mathbb{C}_1^n \oplus \mathbb{C}_2^n)/U(\mathbb{C}_1^n) \times U(\mathbb{C}_2^n) & \xrightarrow{\alpha} & \Omega SU(\mathbb{C}_1^n \oplus \mathbb{C}_2^n) = \Omega SU(n+1) \\
\downarrow j & & \downarrow \beta \\
U(2n)/U(n) \times U(n) = U(\mathbb{C}_1^n \oplus \mathbb{C}_2^n)/U(\mathbb{C}_1^n) \times U(\mathbb{C}_2^n) & \xrightarrow{\beta} & \Omega SU(\mathbb{C}_1^n \oplus \mathbb{C}_2^n) = \Omega SU(2n).
\end{array}$$
Passing to colimits over $n$, we obtain the commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}P^\infty & \overset{\alpha}{\longrightarrow} & \Omega SU \\
\downarrow j & & \downarrow \sim \\
BU & \overset{\beta}{\longrightarrow} & \Omega SU.
\end{array}
$$

The right arrow is an equivalence, as we see from a quick check of homology or homotopy groups.

We claim that $H_*(\Omega SU)$ is a polynomial algebra on generators $\delta_i$ of degree $2i$, $i \geq 1$, and that $\alpha_* : H_*(\mathbb{C}P^\infty) \to H_*(\Omega SU)$ is a monomorphism onto the free abelian group spanned by suitably chosen polynomial generators $\delta_i$. The algebra in §1 implies the topological statement that $j_* : H_*(\mathbb{C}P^\infty) \rightarrow H_*(BU)$ is a monomorphism onto the free abelian group generated by a set $\{\gamma_i\}$ of polynomial generators for $H_*(BU)$, hence the claim will complete the proof of Theorem 3.1.

Think of $S^1$ as the quotient of $[0, \pi]$ obtained by setting $0 = \pi$. Let

$$i : U(\mathbb{C}^{n-1} \oplus \mathbb{C}_1^1) \longrightarrow U(\mathbb{C}_1^n \oplus \mathbb{C}_2^1)$$

be the inclusion. It induces a map $i : \mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^n$ that leads to the left diagram below, and the right diagram is its adjoint.

\begin{align*}
\begin{array}{ccc}
\mathbb{C}P^{n-1} & \overset{\alpha}{\longrightarrow} & \Omega SU(n) \\
\downarrow i & & \downarrow \Omega i \\
\mathbb{C}P^n & \overset{\alpha}{\longrightarrow} & \Omega SU(n + 1) \\
\downarrow \rho & & \downarrow \Omega \pi \\
S^{2n} & \overset{\alpha}{\longrightarrow} & \Omega S^{2n+1} \\
\end{array} \quad \quad \begin{array}{ccc}
\Sigma \mathbb{C}P^{n-1} & \overset{\alpha}{\longrightarrow} & \Sigma SU(n) \\
\downarrow \Sigma i & & \downarrow i \\
\Sigma \mathbb{C}P^n & \overset{\alpha}{\longrightarrow} & \Sigma SU(n + 1) \\
\downarrow \Sigma \rho & & \downarrow \Sigma \pi \\
\Sigma S^{2n} & \overset{\alpha}{\longrightarrow} & \Sigma S^{2n+1} \\
\end{array}
\end{align*}

Here $\rho : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n / \mathbb{C}P^{n-1} \cong S^{2n}$ is the quotient map and $\pi(T) = T(e'_n)$.

**Lemma 3.3.** The composite $\Omega \pi \circ \alpha \circ \Sigma i$ is trivial, so that $\Omega \pi \circ \alpha$ factors as the composite $h \rho$ for a map $h$. Moreover, the adjoint $h$ of $h$ is a homeomorphism.

**Proof.** Let $T \in U(\mathbb{C}_1^n \oplus \mathbb{C}_2^1)$ represent $\tilde{T} \in \mathbb{C}P^n$ and let $T_1^{-1}$ and $T_2^{-1}$ denote the projections of $T^{-1}$ on $\mathbb{C}_1^n$ and on $\mathbb{C}_2^n$. We have

\begin{align*}
(\Omega \pi) \alpha(T)(\theta) &= T \nu(\theta)T^{-1} \nu(-\theta)(e'_n) \\
&= T \nu(\theta)T^{-1}(e^{-i\theta} e'_n) \\
&= T(T_1^{-1}(e'_n), e^{-2i\theta} T_2^{-1}(e'_n)) \\
&= e'_n + (e^{-2i\theta} - 1) TT_2^{-1}(e'_n)
\end{align*}

as we see by adding and subtracting $TT_2^{-1}(e'_n)$. If $T(e'_n) = e'_n$, so that $T$ is in the image of $U(\mathbb{C}_1^n \oplus \mathbb{C}_2^n)$ and $\tilde{T}$ is in the image of $\mathbb{C}P^n$, then $T_2^{-1}(e'_n) = 0$ and thus $(\Omega \pi) \alpha(T)(\theta) = e'_n$ for all $\theta$. To prove that $h$ is a homeomorphism, it suffices to check that it is injective. Its image will then be open by invariance of domain and closed by the compactness of $\Sigma S^{2n}$, hence will be all of $S^{2n+1}$ since $S^{2n+1}$ is connected. Denote points of $\Sigma X$ as $[x, \theta]$ for $x \in X$ and $\theta \in S^1$. We have

$$h(\Sigma \rho)[T, \theta] = \pi \hat{\alpha}[T, \theta] = (\Omega \pi) \alpha(T)(\theta) = e'_n + (e^{-2i\theta} - 1) TT_2^{-1}(e'_n).$$
Since \( T^{-1} \) is the conjugate transpose of \( T \), \( T_{2}^{-1}(e_{n}') = \bar{c}e_{n}' \), where \( c \) is the coefficient of \( e_{n}' \) in \( T(e_{n}') \). Here \( T \notin \mathbb{C}P^{n-1} \) if and only if \( c \neq 0 \) and then \( TT_{2}^{-1}(e_{n}') = e_{n}' + T'(e_{n}') \), where \( T' \) denotes the projection of \( T \) on \( \mathbb{C}^{n-1} \oplus \mathbb{C}_{2}^{1} \). Therefore

\[
\hat{h}[\rho(T), \theta] = e^{-2i\theta}e_{n}' + T'(e_{n}')
\]

when \( \bar{T} \notin \mathbb{C}P^{n-1} \). The injectivity is clear from this. \( \square \)

Armed with this elementary geometry, we return to homology. The rightmost column in the second diagram of (3.2) is a fibration, and we use it to compute \( H_{*}(\Omega SU(n+1)) \) by induction on \( n \). We have \( SU(2) \cong S^{3} \), and we claim inductively that the cohomology Serre spectral sequence of this fibration satisfies \( E_{2} = E_{\infty} \). This leads to a quick proof that

\[
H_{*}(SU(n+1)) = E\{y_{2i+1} | 1 \leq i \leq n\}
\]

as a Hopf algebra, where \( y_{2i+1} \) has degree \( 2i + 1 \) and \( \pi_{*}(y_{2n+1}) \) is a generator of \( H_{2n+1}(S^{2n+1}) \). Indeed, assume that we know this for \( SU(n) \). Then, since the cohomology spectral sequence is multiplicative and the exterior algebra generators of \( H^{*}(SU(n)) = E_{2}^{0,*} \) have degrees less than \( 2n \), they must be permanent cycles. Therefore \( E_{2} = E_{\infty} \). This implies that \( H^{*}(SU(n+1)) \) is an exterior algebra. Moreover, by the edge homomorphisms, \( i^{*} \) is an isomorphism in degrees less than \( 2n + 1 \) and the last exterior algebra generator is \( \pi^{*}(i_{2n+1}) \). Inductively, the exterior generators in degrees less than \( 2n \) are primitive. Since \( i \) is a map of topological groups, \( i^{*} \) is a map of Hopf algebras. Since \( i^{*}\pi^{*} = 0 \), inspection of the coproduct shows that the generator in degree \( 2n + 1 \) must also be primitive.

Using the Serre spectral sequence of the path space fibration over \( SU(n+1) \), we conclude that

\[
H_{*}(\Omega SU(n+1)) \cong P\{\delta_{i} | 1 \leq i \leq n\},
\]

where \( \delta_{i} \) has degree \( 2i \). The classical way to see this is to construct a test multiplicative spectral sequence with

\[
E_{2, i}^{*} = P\{\delta_{i} | 1 \leq i \leq n\} \otimes E\{y_{2i+1} | 1 \leq i \leq n\}
\]

and with differentials specified by requiring \( y_{2i+1} \) to transgress to \( \delta_{i} \). This ensures that \( E_{\infty} \) is zero except for \( E_{0,0} = E_{0,0}^{\infty} \). We can map the test spectral sequence to the homology Serre spectral sequence of the path space fibration by a map that is the identity on \( E_{0,0}^{\infty} \) and commutes with the transgression. The conclusion follows by the comparison theorem, ???. The argument shows that the polynomial generators transgress to the exterior algebra generators and thus that the exterior algebra generators suspend to the polynomial algebra generators. At the risk of belaboring the obvious, we spell things out explicitly via the following commutative diagram, in which the unlabelled isomorphisms are suspension isomorphisms.
Here $\varepsilon$ denotes the evaluation map of the $(\Sigma, \Omega)$ adjunction, and the suspension $\sigma$ is defined to be the composite of $\varepsilon_*$ and the suspension isomorphism. The algebra generator $\delta_{2n}$ maps to a fundamental class under $\pi_* \sigma$. By the diagram, so does the basis element $x_{2n} \in H_{2n}(\mathbb{C}P^n)$. Therefore, modulo decomposable elements which are annihilated by $\sigma$, $\alpha_*(x_{2i}) = \delta_i$ as claimed.