CHAPTER 1

Classical groups and bundle theory

We introduce the spaces we shall study and review the fundamentals of bundle theory in this chapter. Aside from a few arguments included for didactic purposes, proofs are generally sketched or omitted. However, §3 and §6 contain some material that is hard to find or missing from the literature, and full proofs of such statements have been supplied.

All of the spaces that we encounter will be of the homotopy types of CW complexes, so we agree once and for all to restrict attention to such spaces. This ensures that a weak homotopy equivalence, namely a map that induces isomorphisms of homotopy groups for all choices of basepoints, is a homotopy equivalence. By the basic results of Milnor [**9**] (see also Schon [**13**]), this is not a very serious restriction. As usual in algebraic topology, we assume that all spaces are compactly generated [**11**, Ch 5]. We also assume that all spaces are paracompact. This ensures that all bundles are numerable in the sense specified in §2. Since all subspaces of CW complexes [**7**, §I.3], all metric spaces, and all countable unions of compact spaces are paracompact, this assumption is also not unduly restrictive.

1. The classical groups

All of our work will deal with the classical Lie groups and related spaces defined in this chapter. Some general references for this section are [**1, 2, 4**].

Let $\mathbb{K}$ denote any one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the real numbers, complex numbers, or quaternions. For $\alpha \in \mathbb{K}$, let $\bar{\alpha}$ denote the conjugate of α. A right inner product space over $\mathbb{K}$ is a right $\mathbb{K}$-module W together with a function $(-,-) : W \times W \longrightarrow \mathbb{K}$ that satisfies the following properties.

(i) $(x, y + y') = (x, y) + (x, y')$
(ii) $(x, y\alpha) = (x, y)\alpha$
(iii) $(x, y) = \overline{(y, x)}$
(iv) $(x, x) \in \mathbb{R}$, $(x, x) \geq 0$, and $(x, x) = 0$ if and only if $x = 0$.

The unmodified term inner product space will mean right inner product space. All inner product spaces will be finite or countably infinite dimensional; we write $\dim W = \infty$ in the latter case.

We say that a $\mathbb{K}$-linear transformation $T : W \longrightarrow W$ is of finite type if W contains a finite dimensional subspace V invariant under T such that T restricts to the identity on $V^{\perp}$. The *classical groups* are

$$GL(W) = \{T : W \longrightarrow W \mid T \text{ is invertible and of finite type}\}$$
$$U(W) = \{T : W \longrightarrow W \mid T \in GL(W) \text{ and } T \text{ is an isometry}\}$$

and, if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$,

$$SL(W) = \{T : W \longrightarrow W \,|\, T \in GL(W) \text{ and } \det T = 1\}$$
$$SU(W) = \{T : W \longrightarrow W \,|\, T \in U(W) \text{ and } \det T = 1\}.$$

The finite type requirement ensures that the determinant is well-defined. By choosing a fixed orthonormal basis for W, we can identify $GL(W)$ with the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix},$$

where A is an invertible $n \times n$ matrix for some $n < \infty$. Such a matrix is in $U(W)$ if and only if $A^{-1} = \bar{A}^t$, where $\bar{A}$ is obtained from A by conjugating each entry and $(-)^t$ denotes transpose.

Topologize inner product spaces as the union (or colimit) of their finite dimensional subspaces. By choosing a fixed orthonormal basis and using matrices, the classical groups of W can be topologized as subspaces of $\mathbb{K}^{n^2}$ when $n = \dim W < \infty$. The same topology can also be specified either in terms of the metric given by norms of linear transformations or as the compact open topology obtained by regarding these groups as subsets of the space of maps $W \longrightarrow W$. With this topology, $G(W)$ is a Lie group ($G = GL$, U, SL, or SU), and $U(W)$ and $SU(W)$ are compact. When $\dim W = \infty$, $G(W)$ is topologized as the union of its subgroups $G(V)$, where V runs through all finite dimensional subspaces of W or, equivalently, through those V in any expanding sequence with union W.

A matrix A is hermitian if $A^t = \bar{A}$. The characteristic values of A are then real, and A is said to be positive definite if its characteristic values are positive. A standard theorem of linear algebra states that any element of $GL(W)$ can be written uniquely as the product of a positive definite hermitian transformation and an element of $U(W)$, and similarly with GL and U replaced by SL and SU. Since the space of positive definite hermitian matrices is contractible, it follows that the inclusions $U(W) \longrightarrow GL(W)$ and $SU(W) \longrightarrow SL(W)$ are homotopy equivalences. For our purposes, it suffices to restrict attention to $U(W)$ and $SU(W)$, and we do so from now on.

A convenient framework in which to view the classical groups is as follows. Let $\mathscr{I}_{\mathbb{K}}$ denote the category of finite or countably infinite dimensional inner product spaces over $\mathbb{K}$ with linear isometries as morphisms (and note that isometries need not be surjective). Then U and SU are functors from $\mathscr{I}_{\mathbb{K}}$ to the category of topological groups. For a linear isometry $f : V \longrightarrow W$ and a $T \in U(V)$ that is the identity on the orthogonal complement of a finite dimensional $X \subset V$, $T(f) = f \circ T \circ f^{-1}$ on $f(X)$ and $T(f) = \mathrm{id}$ on $f(X)^{\perp}$. Obviously, if V and W are objects in $\mathscr{I}_{\mathbb{K}}$ of the same dimension, then there is an isomorphism $V \cong W$ in $\mathscr{I}_{\mathbb{K}}$, and this induces isomorphisms $U(V) \cong U(W)$ and $SU(V) \cong SU(W)$.

This formulation has the conceptual clarity common to basis free presentations and will be useful in our proof of Bott periodicity. However, for calculational purposes, it is more convenient to deal with particular representations of the classical groups. We define canonical examples as follows, where $\mathbb{K}^n$ has its standard inner product.

$$
\begin{array}{rll}
O(n) = U(\mathbb{R}^n), & O = U(\mathbb{R}^\infty) & \text{the } \textit{orthogonal groups} \\
SO(n) = SU(\mathbb{R}^n), & SO = SU(\mathbb{R}^\infty) & \text{the } \textit{special orthogonal groups} \\
U(n) = U(\mathbb{C}^n), & U = U(\mathbb{C}^\infty) & \text{the } \textit{unitary groups} \\
SU(n) = SU(\mathbb{C}^n), & SU = SU(\mathbb{C}^\infty) & \text{the } \textit{special unitary groups} \\
Sp(n) = U(\mathbb{H}^n), & Sp = U(\mathbb{H}^\infty) & \text{the } \textit{symplectic groups}
\end{array}
$$

There is another family of classical groups not included in this scheme, namely the *spinor groups* $Spin(n)$ for $n > 2$ and $Spin = Spin(\infty)$. We define $Spin(n)$ to be the universal covering group of $SO(n)$. Each $Spin(n)$ for $n < \infty$ is a compact Lie group, and $Spin$ is the union of the $Spin(n)$. Since $\pi_1(SO(n)) = \mathbb{Z}_2$, $Spin(n)$ is a 2-fold cover of $SO(n)$. An alternative and in many ways preferable description of the spinor groups in terms of Clifford algebras is given, for example, in [**4**, p. 65].

There are forgetful functors

$$(-)^{\mathbb{R}} : \mathscr{I}_{\mathbb{C}} \longrightarrow \mathscr{I}_{\mathbb{R}} \quad \text{and} \quad (-)^{\mathbb{C}} : \mathscr{I}_{\mathbb{H}} \longrightarrow \mathscr{I}_{\mathbb{C}}.$$

If W is in $\mathscr{I}_{\mathbb{C}}$, then $W^{\mathbb{R}}$ is the underlying real vector space with inner product the real part of the inner product of W. This induces an inclusion $U(W) \subset SU(W^{\mathbb{R}})$. Thus

$$U(n) \subset SO(2n) \quad \text{and} \quad U \subset SO.$$

Similarly, for W in $\mathscr{I}_{\mathbb{H}}$, $U(W) \subset SU(W^{\mathbb{C}})$ and thus

$$Sp(n) \subset SU(2n) \quad \text{and} \quad Sp \subset SU.$$

There are also extension of scalars functors

$$(-)_{\mathbb{C}} : \mathscr{I}_{\mathbb{R}} \longrightarrow \mathscr{I}_{\mathbb{C}} \quad \text{and} \quad (-)_{\mathbb{H}} : \mathscr{I}_{\mathbb{C}} \longrightarrow \mathscr{I}_{\mathbb{H}}.$$

If W is in $\mathscr{I}_{\mathbb{R}}$, then $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ with inner product

$$(v \otimes \alpha, w \otimes \beta) = \bar{\alpha}(v, w)\beta.$$

This induces inclusions $U(W) \longrightarrow U(W_{\mathbb{C}})$ and $SU(W) \longrightarrow SU(W_{\mathbb{C}})$ that send T to $T \otimes \mathrm{id}$. Thus

$$O(n) \subset U(n), \quad O \subset U, \quad SO(n) \subset SU(n), \quad \text{and} \quad SO \subset SU.$$

Similarly, for W in $\mathscr{I}_{\mathbb{C}}$, $W_{\mathbb{H}} = W \otimes_{\mathbb{C}} \mathbb{H}$ as a right $\mathbb{H}$-space. In this case, the noncommutativity of $\mathbb{H}$ requires careful attention and we are forced to the (formally identical) formula

$$(v \otimes \alpha, w \otimes \beta) = \bar{\alpha}(v, w)\beta$$

for the inner product. This gives $U(W) \subset U(W_{\mathbb{H}})$ and thus

$$U(n) \subset Sp(n) \quad \text{and} \quad U \subset Sp.$$

These inclusions are summarized in the following diagram, the vertical arrows of which are extensions by scalars.

$$
\begin{array}{ccccc}
SO(n) & \subset & O(n) & & \\
\downarrow & & \downarrow & & \\
SU(n) & \subset & U(n) & \subset & SO(2n) \\
 & & \downarrow & & \downarrow \\
 & & Sp(n) & \subset & SU(2n).
\end{array}
$$

In low dimensions we have the following identifications:

$$SO(1) = SU(1) = \{e\} \quad \text{and} \quad O(1) = \mathbb{Z}_2$$

$$SO(2) \cong U(1) = T^1 = S^1 \ \ \text{(the circle group of unit complex numbers)}$$

$$Spin(3) \cong SU(2) \cong Sp(1) = S^3 \ \ \text{(the group of unit quaternions)}$$

$$Spin(4) \cong Sp(1) \times Sp(1)$$

$$Spin(5) \cong Sp(2)$$

$$Spin(6) \cong SU(4).$$

Together with the 2-fold covers $Spin(n) \longrightarrow SO(n)$, this list gives all local isomorphisms among the classical Lie groups.

The following theorem will be essential to our work. Recall that a torus is a Lie group isomorphic to $T^n = (T^1)^n$ for some n.

THEOREM 1.1. *A compact connected Lie group G contains maximal tori. Any two such are conjugate, and G is the union of its maximal tori.*

Actually, we shall only use particular maximal tori in our canonical examples of compact connected classical Lie groups. In $U(n)$, the subgroup of diagonal matrices is a maximal torus T^n. In $SU(n)$, the subgroup of diagonal matrices of determinant 1 is a maximal torus T^{n-1}. In $Sp(n)$, the subgroup of diagonal matrices with complex entries is a maximal torus T^n. In $SO(2n)$ or $SO(2n+1)$, the subgroup of matrices of the form $diag(A_1, \ldots, A_n)$ or $diag(A_1, \ldots, A_n, 1)$ with each $A_i \in SO(2) \cong T^1$ is a maximal torus T^n.

The quotient N/T, where T is a maximal torus in a compact Lie group G and N is the normalizer of T in G, is a finite group called the Weyl group of G and denoted $W(G)$. We shall say more about these groups where they are used.

2. Fiber bundles

Although our main interest will be in vector bundles, we prefer to view them in their proper general setting as examples of fiber bundles. This section and the next give an exposition of the more general theory. We essentially follow Steenrod [**14**] (see also Husemoller [**8**]), but with a number of modifications and additions reflecting more recent changes in point of view.

Recall that a cover $\{V_j\}$ of a space B is said to be *numerable* if it is locally finite[1] and if each V_j is $\lambda_j^{-1}[0,1)$ for some map $\lambda_j : B \longrightarrow I$. Since every open cover of a paracompact space has a numerable refinement, we agree to restrict attention to numerable covers throughout. One motivation for doing so is the following standard result [**11**, §7.4].

THEOREM 2.1. *A map $p : E \longrightarrow B$ is a fibration if and only if it restricts to a fibration $p^{-1}U \longrightarrow U$ for all U in a numerable open cover of B.*

Here, by a *fibration* we understand a map $p : E \longrightarrow B$ that satisfies the covering homotopy property: for any map $f : X \longrightarrow E$ and homotopy $h : X \times I \longrightarrow B$ of $p \circ f$, there is a homotopy $H : X \times I \longrightarrow E$ of f such that $p \circ H = h$. It follows

[1]Locally finite means that each point has a neighborhood that intersects only finitely many sets in the cover; this notion was misdefined in [**11**, p. 49].

that, for any basepoint in any fiber $F = p^{-1}(b)$, there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n F \longrightarrow \pi_n E \longrightarrow \pi_n B \longrightarrow \pi_{n-1} F \longrightarrow \cdots .$$

A fiber bundle is a locally trivial fibration with coordinate patches glued together continuously by means of some specified group. To be precise, recall that a (left) action by a topological group G on a space F is a map $G \times F \longrightarrow F$ such that $g(g'f) = (gg')f$ and $ef = f$ for all $g, g' \in G$ and $f \in F$, where e denotes the identity element of G. An action is said to be *effective* if $gf = g'f$ for all $f \in F$ implies $g = g'$; equivalently, the only element of G that acts trivially on F is e. The reader may want to think in terms of $G = U(W)$ and $F = W$ for some inner product space W.

DEFINITION 2.2. A *coordinate bundle* $\xi = (E, p, B, F, G, \{V_j, \phi_j\})$ is a surjective map $p : E \longrightarrow B$, a space F with an effective action of a topological group G, a numerable cover $\{V_j\}$ of B, and homeomorphisms $\phi_j : V_j \times F \longrightarrow p^{-1}(V_j)$, called *coordinate charts*, such that the following properties hold. For $x \in V_j$, define $\phi_{j,x} : F \longrightarrow p^{-1}(x)$ by $\phi_{j,x}(f) = \phi_j(x, f)$.

(i) $p \circ \phi_j : V_j \times F \longrightarrow V_j$ is projection on the first variable.
(ii) For $x \in V_i \cap V_j$, $\phi_{j,x}^{-1} \circ \phi_{i,x} : F \longrightarrow F$ coincides with action by a (necessarily unique) element $g_{ji}(x) \in G$.
(iii) The function $g_{ji} : V_i \cap V_j \longrightarrow G$ is continuous.

Two coordinate bundles are *strictly equivalent* if they have the same *base space* B, *total space* E, *projection* p, *fiber* F, and *group* G and if the union of their *atlases* $\{V_j, \phi_j\}$ is again the atlas of a coordinate bundle. A *fiber bundle*, or *G-bundle with fiber F*, is a strict equivalence class of coordinate bundles.

DEFINITION 2.3. A map of coordinate bundles is a pair of maps $f : B \longrightarrow B'$ and $\tilde{f} : E \longrightarrow E'$ such that the following properties hold.

(i) $f \circ p = p' \circ \tilde{f}$ and, for $x \in B$, $\tilde{f} : p^{-1}(x) \longrightarrow {p'}^{-1}(f(x))$ is a homeomorphism, denoted $\tilde{f}_x$.
(ii) For $x \in V_j \cap f^{-1}(V'_k)$, ${\phi'_{k,f(x)}}^{-1} \circ \tilde{f}_x \circ \phi_{j,x} : F \longrightarrow F$ coincides with action by a (necessarily unique) element $\bar{g}_{k,j}(x) \in G$.
(iii) The function $\bar{g}_{k,j} : V_j \cap f^{-1}(V'_k) \longrightarrow G$ is continuous.

Note that $\tilde{f}$ is determined by f and the $\bar{g}_{kj}$ via the formula

$$\tilde{f}(y) = \phi'_k(f(x), \bar{g}_{kj}(x)\phi_{j,x}^{-1}(y))$$

for $x \in V_j \cap f^{-1}(V'_k)$ and $y \in p^{-1}(x)$. If f is a homeomorphism, then $\tilde{f}$ is also a homeomorphism and $(\tilde{f}^{-1}, f^{-1})$ is again a bundle map. Two coordinate bundles with the same base space, fiber, and group are said to be *equivalent* if there is a bundle map between them which is the identity on the base space. Two fiber bundles are said to be equivalent if they have equivalent representative coordinate bundles.

These notions can be described directly in terms of *systems of transition functions* $\{V_j, g_{ji}\}$, namely numerable covers $\{V_j\}$ of B together with maps $g_{ji} : V_i \cap V_j \longrightarrow G$ that satisfy the "cocycle condition"

$$g_{kj}(x)g_{ji}(x) = g_{ki}(x) \quad \text{for} \quad x \in V_i \cap V_j \cap V_k$$

(from which $g_{ii}(x) = e$ and $g_{ij}(x) = g_{ji}(x)^{-1}$ follow). The maps g_{ji} of Definition 2.2 certainly satisfy this condition.

THEOREM 2.4. *If G acts effectively on F, then there exists one and, up to equivalence, only one G-bundle with fiber F, base space B, and a given system $\{V_j, g_{ji}\}$ of transition functions. If ξ and ξ' are G-bundles with fiber F over B and B' determined by $\{V_j, g_{ji}\}$ and $\{V'_j, g'_{ji}\}$ and if $f : B \longrightarrow B'$ is any map, then a bundle map $(\tilde{f}, f) : \xi \longrightarrow \xi'$ determines and is determined by maps $\bar{g}_{kj} : V_j \cap f^{-1}(V'_k) \longrightarrow G$ such that*

$$\bar{g}_{kj}(x)g_{ji}(x) = \bar{g}_{ki}(x) \quad \textit{for} \quad x \in V_i \cap V_j \cap f^{-1}(V'_k)$$

and

$$g'_{hk}(f(x))\bar{g}_{kj}(x) = \bar{g}_{hj}(x) \quad \textit{for} \quad x \in V_j \cap f^{-1}(V'_k \cap V'_h).$$

When $B = B'$ and f is the identity map, these conditions on $\{\bar{g}_{kj}\}$ prescribe equivalence. When, further, ξ and ξ' have the same coordinate neighborhoods (as can always be arranged up to strict equivalence by use of intersections), ξ and ξ' are equivalent if and only if there exist maps $\psi_j : V_j \longrightarrow G$ such that

$$g'_{ji}(x) = \psi_j^{-1}(x)g_{ji}(x)\psi_i(x) \quad \textit{for} \quad x \in V_i \cap V_j.$$

For the first statement, the total space E can be constructed from $\coprod V_j \times F$ by identifying $(x, f) \in V_i \times F$ with $(x, g_{ji}(x)f) \in V_j \times F$ whenever $x \in V_i \cap V_j$. For the second statement $\tilde{f}$ can and must be specified by the formula in Definition 2.3. For the last statement, set

$$\psi_j(x) = \bar{g}_{jj}(x)^{-1} \quad \text{and} \quad \bar{g}_{kj}(x) = \psi_k(x)^{-1}g_{kj}(x)$$

to construct $\{\psi_j\}$ from $\{\bar{g}_{kj}\}$ and conversely. The requisite verifications are straightforward; see Steenrod [**14**, §§2,3].

Fiber bundles are often just called G-bundles since the theorem makes clear that the fiber plays an ancillary role. In particular, we have described equivalence independently of F. Therefore, the set of equivalence classes of G-bundles with fiber F is the same for all choices of F. We shall return to this point in the next section, where we consider the canonical choice $F = G$. Note too that the effectiveness of the action of G on F is not essential to the construction. In other words, if in Definitions 2.2 and 2.3, we assume given maps g_{ji} and $\bar{g}_{kj}$ with the prescribed properties, then we may drop the effectiveness since we no longer make use of the clauses "(necessarily unique)" in parts (ii) of these definitions.

The basic operations on fiber bundles can be described conveniently directly in terms of transition functions. The product $\xi_1 \times \cdots \times \xi_n$ of G-bundles ξ_q with fibers F_q and systems of transition functions $\{(V_j)_q, (g_{ji})_q\}$ is the $(G_1 \times \cdots \times G_n)$-bundle with fiber $F_1 \times \cdots \times F_n$ and system of transition functions given by the evident n-fold products of neighborhoods and maps. Its total space, base space, and projection are also the evident products.

For a G-bundle ξ with fiber F, base space B, and system of transition functions $\{V_j, g_{ji}\}$ and for a map $f : A \longrightarrow B$, $\{f^{-1}(V_j), g_{ji} \circ f\}$ is a system of transition functions for the *induced G-bundle* $f^*\xi$ with fiber F over A. The total space of $f^*\xi$ is the pullback of f and the projection $p : E \longrightarrow B$. If $(\tilde{f}, f) : \xi' \longrightarrow \xi$ is any bundle map, then ξ' is equivalent to $f^*\xi$. That is, up to equivalence, any bundle map displays an induced bundle. A crucially important fact is that homotopic maps induce equivalent G-bundles; see [**14**, p. 53] or [**8**, p. 51]. This is the second place where numerability plays a key role.

For our last construction, let $\gamma : G \longrightarrow G'$ be a continuous homomorphism, F a G-space, and F' a G'-space. If ξ is a G-bundle with fiber F, base space B, and system of transition functions $\{V_j, g_{ji}\}$, then $\{V_j, \gamma(g_{ji})\}$ is a system of transition functions for the *coinduced* G'-bundle $\gamma_*\xi$ with fiber F' over B. In the most important case, $F = F'$ and G acts on F through γ, $gf = \gamma(g)f$. We then say that $\gamma_*\xi$ is obtained from ξ by *extending its group* to G'. We say that the group of a G'-bundle ξ' with fiber F is *reducible* to G if ξ' is equivalent as a G'-bundle to some extended bundle $\gamma_*\xi$. Such an equivalence is called a *reduction* of the structural group. This language is generally used only when γ is the inclusion of a closed subgroup, in which case the last statement of Theorem 2.4 has the following immediate consequence.

COROLLARY 2.5. *Let H be a closed subgroup of G. A G-bundle ξ specified by a system of transition functions $\{V_j, g_{ji}\}$ has a reduction to H if and only if there exist maps $\psi_j : V_j \longrightarrow G$ such that*

$$\psi_j(x)^{-1} g_{ji}(x)\psi_i(x) \in H \quad \textit{for all} \quad x \in V_i \cap V_j.$$

A G-bundle is said to be trivial if it is equivalent to the G-bundle given by the projection $B \times F \longrightarrow B$ or, what amounts to the same thing, if its group can be reduced to the trivial group.

3. Principal bundles and homogeneous space

The main reason for viewing vector bundles in the context of fiber bundles is that the general theory allows the clearer understanding of the structure of vector bundles that comes from the comparison of general fiber bundles to principal bundles.

DEFINITION 3.1. A *principal G-bundle* is a G-bundle with fiber G. The principal G-bundle specified by the same system of transition functions as a given G-bundle ξ is called its associated principal G-bundle and denoted $\mathrm{Prin}(\xi)$. It is immediate from Theorem 2.4 that two G-bundles with the same fiber are equivalent if and only if their associated principal G-bundles are equivalent. Two G-bundles with possibly different fibers are said to be *associated* if their associated principal bundles are equivalent.

If $\pi : Y \longrightarrow B$ is a principal G-bundle, then G acts from the right on Y in such a way that the coordinate functions $\pi_j : V_j \times G \longrightarrow Y$ are G-maps, where G acts on $V_j \times G$ by right translation on the second factor. Moreover, B may be identified with the orbit space of Y with respect to this action. The following description of the construction of general fiber bundles from principal bundles is immediate from the proof of Theorem 2.4.

PROPOSITION 3.2. *Let $\pi : Y \longrightarrow B$ be a principal G-bundle. The associated G-bundle $p : E \longrightarrow B$ with fiber the G-space F has total space*

$$E = Y \times_G F \equiv Y \times F/(\sim), \quad \textit{where} \quad (yg, f) \sim (y, gf).$$

The map p is induced by passage to orbits from the projection $Y \times F \longrightarrow Y$.

The construction of $\mathrm{Prin}(\xi)$ from ξ is less transparent (and will not be used in our work). If ξ is given by $p : E \longrightarrow B$ and has fiber F, we say that a map $\psi : F \longrightarrow p^{-1}(x) \subset E$ is admissible if $\phi_{j,x}^{-1} \circ \psi : F \longrightarrow F$ coincides with action by an element of G, where $x \in V_j$; admissibility is independent of the choice of coordinate neighborhood V_j. The total space Y of $\mathrm{Prin}(\xi)$ is the set of admissible maps $F \longrightarrow E$. Its projection to B is induced by p, and its right action by G is given by composition of maps. Provided that the topology on G coincides with that obtained by regarding it as a subspace of the function space of maps $F \longrightarrow F$, Y is topologized as a subspace of the function space of maps $F \longrightarrow E$. The proviso is satisfied in all of our examples.

The following consequence of Corollary 2.5 is often useful.

PROPOSITION 3.3. *A principal G-bundle $\pi : Y \longrightarrow B$ is trivial if and only if it admits a cross section $\sigma : B \longrightarrow Y$.*

PROOF. Necessity is obvious. Given σ and an atlas $\{V_j, \phi_j\}$, the maps $\psi_j : V_j \longrightarrow G$ specified by $\psi_j(x) = \phi_{j,x}^{-1}\sigma(x)$ satisfy

$$\psi_j(x)^{-1} g_{ji}(x) \psi_i(x) = e \quad \text{for} \quad x \in V_i \cap V_j. \quad \square$$

Another useful fact is that the local continuity conditions (iii) in Definitions 2.2 and 2.3 can be replaced by a single global continuity condition in the case of principal bundles.

DEFINITION 3.4. Let Y be a right G-space and let $Orb(Y)$ denote the subspace of $Y \times Y$ consisting of all pairs of points in the same orbit under the action of G. Say that Y is a *principal G-space* if G acts freely on Y, $gy = y$ implies $g = e$, and if the function $\tau : Orb(Y) \longrightarrow G$ specified by $\tau(y, yg) = g$ is continuous. Let $B = Y/G$ with projection $\pi : Y \longrightarrow B$. Say that Y is *locally trivial* if B has a numerable cover $\{V_j\}$ together with homeomorphisms $\phi_j : V_j \times G \longrightarrow \pi^{-1}(V_j)$ such that $\pi \circ \phi_j$ is the projection $V_j \times G \longrightarrow V_j$ and ϕ_j is a right G-map.

PROPOSITION 3.5. *A map $\pi : Y \longrightarrow B$ is a principal G-bundle if and only if Y is a locally trivial principal G-space and, up to homeomorphism under Y, $B = Y/G$ and π is the projection onto orbits. If $\pi : Y \longrightarrow B$ and $\pi' : Y \longrightarrow B'$ are principal G-bundles, then maps $\tilde{f} : Y \longrightarrow Y'$ and $f : B \longrightarrow B'$ specify a bundle map $\pi \longrightarrow \pi'$ if and only if $\tilde{f}$ is a right G-map such that $f \circ \pi = \pi' \circ \tilde{f}$ and, up to homeomorphism, f is obtained from $\tilde{f}$ by passage to orbits.*

PROOF. Since any right G-map $G \longrightarrow G$ is left multiplication by an element of G, conditions (i) and (ii) of Definitions 2.2 and 2.3 certainly hold for $\{V_j, \phi_j\}$ as in Definition 3.4. It is only necessary to relate the continuity conditions (iii) to the continuity of τ. Since

$$\phi_i(x, e) = \phi_{i,x}(e) = \phi_{j,x}(g_{ji}(x)) = \phi_j(x, e) g_{ji}(x) \quad \text{for} \quad x \in V_i \cap V_j,$$

the following diagram commutes, where $\omega(h, g) = g^{-1}hg$:

$$\begin{array}{ccc} (V_i \cap V_j) \times G & \xrightarrow{(\phi_i, \phi_j)} & Orb(\pi^{-1}(V_i \cap V_j)) \\ {\scriptstyle g_{ji} \times \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle \tau} \\ G \times G & \xrightarrow[\omega]{} & G. \end{array}$$

Moreover, (ϕ_i, ϕ_j) is a homeomorphism. It follows that τ is continuous if and only if all g_{ji} are so. Similarly, with the notations of Definition 2.3, the following diagram commutes:

$$\begin{array}{ccc} (V_j \cap f^{-1}(V'_k)) \times G & \xrightarrow{(\tilde{f}\phi_j, \phi'_k f)} & Orb({\pi'}^{-1}(f(V_j) \cap V'_k)) \\ {\scriptstyle \tilde{g}_{kj}\times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \tau} \\ G \times G & \xrightarrow[\omega]{} & G. \end{array}$$

Therefore the $\tilde{g}_{kj}$ are continuous if τ is so. □

If H is a closed subgroup of a topological group G, we denote by G/H the space of left cosets gH in G with the quotient topology. Such a coset space is called a *homogeneous space*. It is a G-space with G acting by left translation. The basic method in our study of the cohomology of classical groups will be the inductive analysis of various bundles relating such spaces. We need some preliminary observations in order to state the results which provide the requisite bundles. Subgroups are understood to be closed throughout.

Let $H_0 \subset G$ be the subset of those elements g that act trivially on G/H. Explicitly, H_0 is the intersection of the conjugates of H in G. Then H_0 is a closed normal subgroup of G contained in H, and it is the largest subgroup of H which is normal in G. The quotient group G/H_0 acts effectively on G/H.

Note that G, and thus also G/H_0, acts transitively on G/H. That is, for every pair of cosets x, x', there exists g such that $gx = x'$. Conversely, if G acts transitively on a space X and if H is the *isotropy group* of a given point $x \in X$, namely the subgroup of elements that fix x, then H is a closed subgroup of G and the map $p : G \longrightarrow X$ that sends g to gx induces a continuous bijection $q : G/H \longrightarrow X$. Of course, q^{-1} is continuous if and only if q is an open map, and this certainly holds when G is compact Hausdorff. We shall make frequent use of such homeomorphisms q and shall generally regard them as identifications.

In many cases of interest to us, the group H_0 is trivial by virtue of the following observation.

LEMMA 3.6. *For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the largest subgroup of $U(\mathbb{K}^{n-1})$ that is normal in $U(\mathbb{K}^n)$ is the trivial group.*

PROOF. The space $U(\mathbb{K}^n)/U(\mathbb{K}^{n-1})$ is homeomorphic to the unit sphere S^{dn-1}, $d = \dim_{\mathbb{R}} \mathbb{K}$, on which $U(\mathbb{K}^n)$ itself acts effectively. □

We need one other concept. Let $p : G \longrightarrow G/H$ be the quotient map. A *local cross section* for H in G is a neighborhood U of the basepoint eH in G/H together with a map $s : U \longrightarrow G$ such that $p \circ s = \mathrm{id}$ on U. When G is a Lie group, a local cross section always exists [**4**, p. 110]. By [**3**, p. 5-10], the infinite classical groups are enough like Lie groups that essentially the same argument works for such G and reasonable H. The idea is that if G has Lie algebra $\mathscr{G}$ and H has Lie algebra $\mathscr{H} \subset \mathscr{G}$ with $\mathscr{G} = \mathscr{H} \oplus \mathscr{H}^{\perp}$, then the exponential local homeomorphism of pairs $\exp : (\mathscr{G}, \mathscr{H}) \longrightarrow (G, H)$ can be used to show that there is a homeomorphism $\phi : V \times H \longrightarrow W$ specified by $\phi(v, h) = \exp(v)h$, where V is a suitably small open neighborhood of 0 in $\mathscr{H}^{\perp}$ and W is an open neighborhood of e in G. Then $U = p(W)$ and $s(u) = \exp(v)$ if $p\phi(v, h) = u$ specify the required local cross section.

PROPOSITION 3.7. *If H has a local cross section in G, then $p : G \longrightarrow G/H$ is a principal H-bundle.*

PROOF. If $s : U \longrightarrow G$ is a local cross section, then $\{gU \mid g \in G\}$ is an open cover of G/H and the right H-maps $\phi_g : gU \times H \longrightarrow p^{-1}(gU)$ specified by $\phi_g(gu, h) = gs(u)h$ are homeomorphisms. The continuity of $\tau : Orb(G) \longrightarrow H$ is clear, hence the conclusion is immediate from Proposition 3.5. □

The proposition admits the following important generalization.

PROPOSITION 3.8. *If H has a local cross section in G and $\pi : Y \longrightarrow B$ is a principal G-bundle, then the projection $q : Y \longrightarrow Y/H$ is a principal H-bundle.*

PROOF. Let $\{V_j, \phi_j\}$ be an atlas for π. With the notations of the previous proof, let

$$W_{j,g} = q(\phi_j(V_j \times p^{-1}(gU))) \subset Y/H.$$

Define right H-maps $\omega_{j,g} : W_{j,g} \times H \longrightarrow q^{-1}(W_{j,g})$ by

$$\omega_{j,g}(q(\phi_j(x, \phi_g(gu, e))), h) = \phi_j(x, gs(u))h.$$

Then the $\omega_{j,g}$ are homeomorphisms, and the conclusion is again immediate from Proposition 3.5. □

These principal bundles appear in conjunction with associated bundles with homogeneous spaces as fibers.

LEMMA 3.9. *If $\pi : Y \longrightarrow B$ is a principal G-bundle, then passage to orbits gives a G/H_0-bundle $Y/H \longrightarrow B$ with fiber G/H.*

PROOF. This is immediate by passage to orbits on the level of coordinate functions. □

This leads to the following generalization of Proposition 3.3.

PROPOSITION 3.10. *If H has a local cross section in G and $\pi : Y \longrightarrow B$ is a principal G-bundle, then π admits a reduction of its structural group to H if and only if the orbit bundle $Y/H \longrightarrow B$ admits a cross section.*

PROOF. We use Corollary 2.5 and the notations of the previous two proofs. Given $\psi_j : V_j \longrightarrow G$ such that

$$\psi_j(x)^{-1} g_{ji}(x) \psi_i(x) \in H \quad \text{for all} \quad x \in V_i \cap V_j,$$

the formula $\sigma(x) = \phi_j(x, \psi_j(x))H$ for $x \in V_j$ specifies a well-defined global cross section $\sigma : B \longrightarrow Y/H$. Conversely, given σ, the maps $\psi_j : V_j \longrightarrow G$ specified by $\psi_j(x) = gs(u)$ if $\sigma(x) = q(\phi_j(x, gs(u)))$ satisfy the cited condition. □

Finally, we note that Lemma 3.9 and Proposition 3.7 together imply most of the following generalization of the latter.

PROPOSITION 3.11. *Let $J \subset H \subset G$ and let H admit a local cross section in G. Then the inclusion of cosets map $G/J \longrightarrow G/H$ is an H/J_0-bundle with fiber H/J, where J_0 is the largest subgroup of J that is normal in H. Moreover, left translation by elements of G specifies self-maps of this bundle, and its associated principal bundle is $G/J_0 \longrightarrow G/H$.*

4. Vector bundles; Stiefel and Grassmann manifolds

By a *vector bundle*, we understand a $U(W)$-bundle ξ with fiber W, where W is any finite dimensional inner product space over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. When $W = \mathbb{K}^n$, we refer to ξ as a (real, complex, or quaternionic) n-plane bundle. Taking the group to be $U(W)$ rather than $GL(W)$ ensures that we can give ξ an inner product metric. That is, if the projection of ξ is $p : E \longrightarrow B$, we can transport the inner product of W onto fibers $p^{-1}(x)$ by means of the coordinate functions, and the positive definite quadratic forms given by the real numbers (y, y) for $y \in p^{-1}(x)$ then specify a continuous function $\mu : E \longrightarrow \mathbb{R}$. The map μ is generally called a *Euclidean metric* in the real case and a *Hermitian metric* in the complex case.

Vector bundles are our basic objects of study, and we need various operations on them. We give a generic construction. Let $\mathscr{I}^*_{\mathbb{K}} \subset \mathscr{I}_{\mathbb{K}}$ denote the category of finite dimensional inner product spaces over $\mathbb{K}$ and their linear isometric isomorphisms. This is a topological category. That is, its hom sets are spaces and composition is continuous. A functor between topological categories is said to be continuous if it induces continuous maps on hom sets. Suppose given such a functor

$$T : \mathscr{I}^*_{\mathbb{K}_1} \times \cdots \times \mathscr{I}^*_{\mathbb{K}_n} \longrightarrow \mathscr{I}^*_{\mathbb{K}}$$

(where $\mathbb{K}$ and each $\mathbb{K}_q$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$). We allow contravariance. That is, some of the $\mathscr{I}^*_{\mathbb{K}_q}$ may be replaced by their opposite categories, and then $U(W_q)$ is to be replaced by its opposite group below. The functor T gives continuous homomorphisms

$$T : U(W_1) \times \cdots \times U(W_n) \longrightarrow U(T(W_1, \cdots, W_n)), \quad W_q \in \mathscr{I}^*_{\mathbb{K}_q}.$$

Suppose given vector bundles ξ_q with fibers W_q over base spaces B_q and suppose that ξ_q is specified by a system of transition functions $\{(V_j)_q, (g_{ji})_q\}$. We obtain a vector bundle with fiber $T(W_1, \cdots, W_n)$ over $B_1 \times \cdots \times B_n$ by virtue of the system of transition functions obtained by composing products of maps $(g_{ji})_q : (V_i)_q \cap (V_j)_q \longrightarrow U(W_q)$, or $(g_{ji})_q^{-1}$ if T is contravariant in the qth variable, with the above map T. This defines the external operation on bundles determined by T. With $B_1 = \cdots = B_n = B$, the pullback of this external operation along the diagonal $\Delta : B \longrightarrow B^n$ gives the corresponding internal operation. The notation $T(\xi_1, \cdots, \xi_n)$ will be reserved for the internal operation.

The most important example is the *Whitney sum* $\xi \oplus \xi'$ obtained via $\oplus : \mathscr{I}^*_{\mathbb{K}} \times \mathscr{I}^*_{\mathbb{K}} \longrightarrow \mathscr{I}^*_{\mathbb{K}}$. Observe that if η is a subbundle of a vector bundle ξ, so that each fiber of η is a sub inner product space of the corresponding fiber of ξ, then η has a complement $\eta^\perp$ such that $\eta \oplus \eta^\perp = \xi$.

Clearly an n-plane bundle over B is trivial if and only if it is equivalent to the Whitney sum of n copies of the trivial line bundle over B. This holds if and only if ξ admits n orthonormal cross sections and, by fiberwise Gram-Schmidt orthogonalization, this holds if and only if ξ admits n linearly independent cross sections. Indeed, a nowhere zero cross section prescribes a $\mathbb{K}$-line subbundle, and the conclusion follows by induction.

Other examples are given by such functors as Hom, tensor product, and exterior powers. Note in particular that our general context includes such operations as

$$\otimes_{\mathbb{H}} : \mathscr{I}^*_{\mathbb{H}} \times \mathscr{I}^*_{\mathbb{H}} \longrightarrow \mathscr{I}^*_{\mathbb{R}} \quad \text{and} \quad \otimes_{\mathbb{R}} : \mathscr{I}^*_{\mathbb{H}} \times \mathscr{I}^*_{\mathbb{R}} \longrightarrow \mathscr{I}^*_{\mathbb{H}}.$$

These will be useful in the study of Bott periodicity.

We next recall some of the classical examples of homogeneous spaces and vector bundles.

Definition 4.1. Let W be an inner product space over $\mathbb{K}$. A q-frame in W is an ordered q-tuple of orthonormal vectors. A q-plane in W is a sub inner product space of dimension q over $\mathbb{K}$. Let $V_q(W)$ be the set of q-frames in W topologized as a subspace of W^q. Let $G_q(W)$ be the set of q-planes in W, let $\pi : V_q(W) \longrightarrow G_q(W)$ be the map that sends a q-frame to the q-plane it spans, and give $G_q(W)$ the resulting quotient topology. The spaces $V_q(W)$ and $G_q(W)$ are called the *Stiefel manifolds* and *Grassmann manifolds* of W. Clearly $U(W)$ acts transitively on these spaces. If $x \in V_q(W)$ spans $X \in G_q(W)$, then $U(X^\perp)$ fixes x and $U(X) \times U(X^\perp)$ fixes X. There result homeomorphisms

$$V_q(W) \cong U(W)/U(X^\perp) \quad \text{and} \quad G_q(W) \cong U(W)/U(X) \times U(X^\perp),$$

and $\pi : V_q(W) \longrightarrow G_q(W)$ is a principal $U(X)$-bundle. The associated bundle with fiber X has total space

$$\{(Y, y) \mid Y \text{ is a } q\text{-plane in } W \text{ and } y \text{ is a vector in } Y\},$$

topologized as a subspace of $G_q(W) \times W$ and given the evident projection to $G_q(W)$. These are the *classical universal vector bundles*.

Remark 4.2. For $n > 0$ or $n = \infty$, the space $G_1(\mathbb{K}^n)$ is the projective space $\mathbb{K}P^{n-1}$ of lines through the origin in $\mathbb{K}^n$ and $V_1(\mathbb{K}^n)$ is the unit sphere S^{dn-1} in $\mathbb{K}^n$, where $d = \dim_{\mathbb{R}} \mathbb{K}$. The 1-plane bundles associated to the principal S^{d-1}-bundles $S^{dn-1} \longrightarrow \mathbb{K}P^{n-1}$ are called the *canonical line bundles* over projective spaces. However, the reader should be warned that, in the complex case, some authors take the conjugates of these line bundle to be "canonical".

We also need the oriented variants of these spaces.

Definition 4.3. An orientation of an inner product space Y of dimension q over $\mathbb{R}$ or $\mathbb{C}$ is an equivalence class of q-frames, where q-frames y and z are equivalent if the element $g \in U(Y)$ such that $z = gy$ has determinant one. Let $\tilde{G}_q(W)$ be the set of oriented q-planes in W, let $\tilde{\pi} : V_q(W) \longrightarrow \tilde{G}_q(W)$ send a q-frame to the oriented q-plane it determines, and give $\tilde{G}_q(W)$ the resulting quotient topology. Let $p : \tilde{G}_q(W) \longrightarrow G_q(W)$ be given by neglect of orientation, so that $p \circ \tilde{\pi} = \pi : V_q(W) \longrightarrow G_q(W)$. The space $\tilde{G}_q(W)$ is called the *oriented Grassmann manifold* of W. For any chosen $X \in \tilde{G}_q(W)$, there is a homeomorphism

$$\tilde{G}_q(W) \cong U(W)/SU(X) \times U(X^\perp);$$

$\tilde{\pi} : V_q(W) \longrightarrow \tilde{G}_q(W)$ is a principal $SU(X)$-bundle and $p : \tilde{G}_q(W) \longrightarrow G_q(W)$ is a principal S^{d-1}-bundle, where S^{d-1} is $\mathbb{Z}_2$ in the real case and T^1 in the complex case. The associated bundle with fiber X has total space

$$\{(Y, y) \mid Y \text{ is an oriented } q\text{-plane in } W \text{ and } y \text{ is a vector in } Y\}.$$

The following lemma will imply that the bundles $V_q(W) \longrightarrow G_q(W)$ and $V_q(W) \longrightarrow \tilde{G}_q(W)$ are in fact "universal", in a sense to be made precise in the next section, when $\dim W = \infty$.

Lemma 4.4. *If $\dim W = \infty$, then $V_q(W)$ is contractible for all q.*

PROOF. The space $\mathscr{I}(V,W)$ of linear isometries $V \longrightarrow W$ is contractible (e.g. [**12**, I.1.3]). Let $X \subset W$ have dimension q. Since $\dim X^{\perp} = \infty$, the inclusion $X^{\perp} \longrightarrow W$ is homotopic through isometries to an isomorphism. Thus the inclusion $U(X^{\perp}) \longrightarrow U(W)$ is homotopic to a homeomorphism and is thus a homotopy equivalence. Therefore $\pi_*(V_q(W)) = 0$ by the long exact sequence of homotopy groups of the fibration sequence $U(X^{\perp}) \longrightarrow U(W) \longrightarrow V_q(W)$. The conclusion follows. □

For calculational purposes, we record the canonical examples of Stiefel manifolds and the various bundles relating them. For $0 < q \leq n$, the Stiefel manifold $V_q(\mathbb{K}^n)$ is the homogeneous space $U(\mathbb{K}^n)/U(\mathbb{K}^{n-q})$. That is,

$$V_q(\mathbb{R}^n) \cong O(n)/O(n-q), \;\; V_q(\mathbb{C}^n) \cong U(n)/U(n-q), \;\; V_q(\mathbb{H}^n) \cong Sp(n)/Sp(n-q).$$

LEMMA 4.5. *For $q < n$, the natural maps*

$$SO(n)/SO(n-q) \longrightarrow V_q(\mathbb{R}^n) \quad \textit{and} \quad SU(n)/SU(n-q) \longrightarrow V_q(\mathbb{C}^n)$$

are homeomorphisms. There are canonical homeomorphisms

$$\begin{array}{lll} V_n(\mathbb{R}^n) = O(n) & V_n(\mathbb{C}^n) = U(n) & V_n(\mathbb{H}^n) = Sp(n) \\ V_{n-1}(\mathbb{R}^n) = SO(n) & V_{n-1}(\mathbb{C}^n) = SU(n) & \\ V_1(\mathbb{R}^n) = S^{n-1} & V_1(\mathbb{C}^n) = S^{2n-1} & V_1(\mathbb{H}^n) = S^{4n-1}. \end{array}$$

The results of the previous section have the following immediate consequence.

PROPOSITION 4.6. *For $0 < p < q \leq n$, canonical maps give a commutative diagram*

$$\begin{array}{ccccc} U(\mathbb{K}^{n-q}) & = & U(\mathbb{K}^{n-q}) & & \\ \downarrow & & \downarrow & & \\ U(\mathbb{K}^{n-p}) & \longrightarrow & U(\mathbb{K}^n) & \longrightarrow & V_p(\mathbb{K}^n) \\ \downarrow & & \downarrow & & \| \\ V_{q-p}(\mathbb{K}^{n-p}) & \longrightarrow & V_q(\mathbb{K}^n) & \longrightarrow & V_p(\mathbb{K}^n) \end{array}$$

in which the left two columns are principal $U(\mathbb{K}^{n-q})$-bundles, the map between these columns is a bundle map, the middle row is a principal $U(\mathbb{K}^{n-p})$-bundle and the bottom row is its associated bundle with fiber $V_{q-p}(\mathbb{K}^{n-p})$.

The case $q = p + 1$ is of particular interest since it allows inductive study of these spaces for fixed n.

5. The classification of bundles and characteristic classes

While vector bundles are our ultimate objects of study (although they themselves are of greatest interest as a tool for the study of manifolds), our calculational focus will be on classifying spaces. From the point of view of algebraic topology, this focus yields the most efficient proofs and the greatest insight, especially as there is still no satisfactory intrinsic calculational theory of characteristic classes for less structured classes of bundles and fibrations than vector bundles, such as PL and topological sphere bundles and spherical fibrations.

Definition 5.1. A *universal bundle* for a topological group G is a principal G-bundle $\pi : EG \longrightarrow BG$ such that EG is contractible. Any such base space BG is called a *classifying space* for G.

By Proposition 3.11 and Lemma 4.4, if G is a closed subgroup of $U(\mathbb{K}^q)$, then the principal G-bundle

$$V_q(\mathbb{K}^q \oplus \mathbb{K}^\infty) \cong U(\mathbb{K}^q \oplus \mathbb{K}^\infty)/U(\mathbb{K}^\infty) \longrightarrow U(\mathbb{K}^q \oplus \mathbb{K}^\infty)/G \times U(\mathbb{K}^\infty)$$

is universal. Recall that any compact Lie group embeds in some $U(q)$.

Theorem 5.2 (The classification theorem). *If G acts effectively on a space F, then equivalence classes of G-bundles over X with fiber F are in natural bijective correspondence with homotopy classes of maps $X \longrightarrow BG$.*

The correspondence assigns to a map $f : X \longrightarrow BG$ the G-bundle with fiber F associated to the induced principal G-bundle $f^*\pi$. Every topological group G has a universal bundle π, and any two universal bundles are canonically equivalent, by the theorem. In particular, any two classifying spaces for G are canonically homotopy equivalent.

The classification theorem admits various proofs, such as those of Steenrod [**14**], Dold [**6**], and tom Dieck [**5**], the last being particularly elegant. The proof I gave in [**10**] has the advantage that it applies equally well to the classification of fibrations and of both bundles and fibrations with various kinds of additional structure. I won't repeat any of the proofs here.

Let $[X, Y]$ denote the set of homotopy classes of maps $X \longrightarrow Y$. If $\mathscr{B}_G(X)$ denotes the set of equivalence classes of G-bundles with fiber F over X, then the classification theorem states that there is a natural isomorphism

$$\mathscr{B}_G(X) \cong [X, BG].$$

When X and Y have basepoints (denoted $*$), let $[X, Y]_*$ denote the set of based homotopy classes of based maps $X \longrightarrow Y$. If the inclusion $* \longrightarrow X$ is a cofibration, as always holds if X is a CW complex, then $\pi_1(Y)$ acts on $[X, Y]_*$ since evaluation at $*$ is a fibration $Y^X \longrightarrow Y$ [**11**, §7.6]. The action is trivial if Y is an H-space (has a product such that the basepoint is a two-sided homotopy unit), by [**15**, p. 119]. The map $[X, Y]_* \longrightarrow [X, Y]$ obtained by forgetting basepoints induces a bijection from the orbit set $[X, Y]/\pi_1(Y)$ to $[X, Y]$. We shall see in the next section that BG is simply connected if G is connected. Thus $[X, BG]_* = [X, BG]$ if G is connected or if BG is an H-space. The latter condition holds when G is Abelian since the product $G \times G \longrightarrow G$ is then a homomorphism and so induces a product $BG \times BG \longrightarrow BG$, by Proposition 5.3 below.

Two familiar examples are

$$B\mathbb{Z}_2 = BO(1) = \mathbb{R}P^\infty = K(\mathbb{Z}_2, 1)$$

and

$$BT^1 = BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2),$$

where $K(\pi, n)$ denotes a space with nth homotopy group π and remaining homotopy groups zero. Such Eilenberg-Mac Lane spaces represent cohomology,

$$\tilde{H}^n(X; \pi) = [X, K(\pi, n)]_*,$$

and we conclude from the classification theorem that $O(1)$-bundles and $U(1)$-bundles over X are in natural bijective correspondence with elements of $H^1(X;\mathbb{Z}_2)$ and $H^2(X;\mathbb{Z})$, respectively.

Many important properties of classifying spaces can be deduced directly from the classification theorem. Some details of proof may help clarify this fundamentally important transition back and forth between bundle theory and homotopy theory.

PROPOSITION 5.3. *Up to homotopy, passage to classifying spaces specifies a product-preserving functor from topological groups to topological spaces.*

PROOF. If $\pi_i : EG_i \longrightarrow BG_i$ is a universal bundle for groups G_i, $i = 1$ and $i = 2$, then

$$\pi_1 \times \pi_2 : EG_1 \times EG_2 \longrightarrow BG_1 \times BG_2$$

is clearly a universal bundle for $G_1 \times G_2$. Therefore $BG_1 \times BG_2$ is a classifying space for $G_1 \times G_2$. For the functoriality, if $\gamma : G \longrightarrow G'$ is a continuous homomorphism, coinduction assigns a principal G'-bundle $\gamma_*\xi$ to a principal G-bundle ξ (see §2). This assignment is natural with respect to maps of base spaces. That is, γ_* is a natural transformation

$$\gamma_* : \mathscr{B}_G(X) \longrightarrow \mathscr{B}_{G'}(X).$$

Any map $f : BG \longrightarrow BG'$ induces a natural transformation

$$f_* : [X, BG] \longrightarrow [X, BG'],$$

by composition of maps $X \longrightarrow BG$ with f, and we may use the classification theorem to interpret f_* as a natural transformation $\mathscr{B}_G \longrightarrow \mathscr{B}_{G'}$. We specify $B\gamma : BG \longrightarrow BG'$ to be the classifying map (well-defined up to homotopy) of the principal G'-bundle over BG coinduced from the universal G-bundle over BG. With this choice, $(B\gamma)_*$ coincides with γ_*. □

While the proposition suffices for our purposes and will lead to the identification of particular maps $B\gamma$ in the next section, the precise construction of classifying spaces and universal bundles by use of the "geometric bar construction" yields much sharper results. Indeed, with this construction, E and B are functors before passage to homotopy, $EG_1 \times EG_2$ and $BG_1 \times BG_2$ are naturally homeomorphic to $E(G_1 \times G_2)$ and $B(G_1 \times G_2)$, and B is homotopy preserving in the sense that if γ and γ' are homotopic through homomorphisms, then $B\gamma$ is homotopic to $B\gamma'$. The reader is referred to [**10**] for an exposition of these and related results.

It is immediate from the proposition and the generic construction of operations on vector bundles in the previous section that if T is a continuous functor of the sort considered there, then

$$BT : BU(W_1)\times\cdots\times BU(W_n) \simeq B(U(W_1)\times\cdots\times U(W_n)) \longrightarrow BU(T(W_1\oplus\cdots\oplus W_n))$$

induces the corresponding external operation on bundles via composition with products of classifying maps. If $f_q : X \longrightarrow BU(W_q)$ classifies ξ_q, then the following composite classifies the internal operation $T(\xi_1, \cdots, \xi_n)$:

$$X \xrightarrow{\Delta} X^n \xrightarrow{f_1\times\cdots\times f_n} BU(W_1) \times \cdots \times BU(W_n) \xrightarrow{BT} BU(T(W_1 \oplus \cdots \oplus W_n)).$$

If T is contravariant in the qth variable, then, to arrange that $BU(W_q)^{\mathrm{op}}$ rather than $BU(W_q)$ appears as the qth space in the domain of BT, we must interpolate $B\chi : BU(W_q) \longrightarrow BU(W_q)^{\mathrm{op}}$, where $\chi : G \longrightarrow G^{\mathrm{op}}$ is the isomorphism $\chi(g) = g^{-1}$.

We shall study such operations calculationally via canonical examples. In particular, for any classical groups G ($G = O$, U, etc), Whitney sums are induced by the maps

$$p_{m,n} : BG(m) \times BG(n) \longrightarrow BG(m+n) \tag{5.4}$$

obtained from the block sum of matrices homomorphisms

$$G(m) \times G(n) \longrightarrow G(m+n).$$

The standard inclusion $G(n) \longrightarrow G(n+1)$ is block sum with $I \in G(1)$, hence the corresponding map

$$i_n : BG(n) \longrightarrow BG(n+1) \tag{5.5}$$

induces addition of a trivial line bundle.

Say that two vector bundles ξ and ξ' over the same base space X are *stably equivalent* if $\xi \oplus \varepsilon$ is equivalent to $\xi' \oplus \varepsilon'$ for some trivial bundles ε and ε'. With our explicit Grassmann manifold construction of classifying spaces, we have that i_n is an inclusion and the union of the $BG(n)$ is a classifying space for the infinite classical group G. For a finite dimensional CW complex X, a map $X \longrightarrow BG$ necessarily factors through some $BG(n)$. For such spaces, $[X, BG]$ is in natural bijective correspondence with the set of stable equivalence classes of vector bundles over X. The natural map $BG(n) \longrightarrow BG$ induces the natural transformation that sends an n-plane bundle to its stable equivalence class.

The map $BO(n) \longrightarrow BU(n)$ induced by the inclusion $O(n) \longrightarrow U(n)$ represents complexification of real vector bundles, and similarly for our other forgetful and extension of scalars maps between classical groups. In sum, we may think of the classification theorem as providing an equivalence between the theory of vector bundles and the study of classifying spaces of classical groups.

We shall exploit this equivalence for the study of characteristic classes.

DEFINITION 5.6. Let G be a topological group. A *characteristic class* c for G-bundles associates to each G-bundle ξ over X a cohomology class $c(\xi) \in H^*(X)$ (for any cohomology theory H^*), naturally with respect to bundle maps; that is, if $(\tilde{f}, f) : \xi \longrightarrow \xi'$ is a map of G-bundles, then $f^*c(\xi') = c(\xi)$.

We have not mentioned a fiber. While the definition implicitly assumes a fixed choice, any choice will do. We adopt the convention that associated bundles have the same characteristic classes since they have the same classifying maps.

LEMMA 5.7. *Characteristic classes for G-bundles are in canonical bijective correspondence with elements of $H^*(BG)$.*

PROOF. We may as well restrict attention to principal G-bundles. If $\pi : EG \longrightarrow BG$ is universal, c is a characteristic class, and $f : X \longrightarrow BG$ classifies ξ, then $c(\xi) = f^*c(\pi)$. Thus c is determined by $c(\pi) \in H^*(BG)$. Conversely, $\gamma \in H^*(BG)$ determines a characteristic class c by $c(\pi) = \gamma$ and naturality. □

Categorically, this is a special case of an observation about representable functors known as the Yoneda lemma: natural transformations $[-, Y] \longrightarrow F(-)$ are in bijective correspondence with elements of $F(Y)$ for any set-valued contravariant homotopy functor F.

The lemma is the philosophical basis for our calculations. Observe that we may study the effect on characteristic classes of operations on vector bundles by calculating the induced map on the cohomology of the relevant classifying spaces. For example, the effect of Whitney sum on characteristic classes can be deduced from the map $p^*_{m,n}$.

6. Some homotopical properties of classifying spaces

We collect a few miscellaneous facts about classifying spaces for later use.

Let $\pi : EG \longrightarrow BG$ be a universal G-bundle. If $h : EG \times I \longrightarrow EG$ is a contracting homotopy, $h(y, 0) = *$ and $h(y, 1) = y$ for some choice of basepoint $*$, let $\tilde{h} : EG \longrightarrow PBG$ be the map specified by $\tilde{h}(y)(t) = \pi h(y, t)$, where PBG is the space of paths in BG that start at $\pi(*)$. Let $p : PBG \longrightarrow BG$ be the end-point projection, so that $p^{-1}(\pi(*))$ is the loop space ΩBG. Then $p \circ \tilde{h} = \pi$ and $\tilde{h}$ restricts to a map $\zeta : G \longrightarrow \Omega BG$, where G is identified with the fiber $\pi^{-1}(\pi(*))$. Thus the following diagram commutes:

$$\begin{array}{ccccc} G & \longrightarrow & EG & \xrightarrow{\pi} & BG \\ \Big\downarrow{\scriptstyle \zeta} & & \Big\downarrow{\scriptstyle \tilde{h}} & & \Big\| \\ \Omega BG & \longrightarrow & PBG & \xrightarrow[p]{} & BG. \end{array}$$

By comparison of long exact sequences of homotopy groups, this yields the following result.

Proposition 6.1. *The space BG is connected and $\pi_{n+1}(BG)$ is naturally isomorphic to $\pi_n(G)$ for $n > 0$. The map $\zeta : G \longrightarrow \Omega BG$ is a homotopy equivalence.*

In particular, BG is simply connected if G is connected.

We have observed that BG is a functor of G. We need several results about the behavior of this functor on particular kinds of maps. These will all be consequences of the following criterion for recognizing when a map of classifying spaces is $B\gamma$ for a given homomorphism γ.

Lemma 6.2. *Let $\gamma : G \longrightarrow G'$ be a continuous homomorphism and let $\pi : EG \longrightarrow BG$ and $\pi' : EG' \longrightarrow BG'$ be universal bundles. If $f : EG \longrightarrow EG'$ is any map such that $f(yg) = f(y)\gamma(g)$ for all $y \in EG$ and $g \in G$, then the map $BG \longrightarrow BG'$ obtained from f by passage to orbits is in the homotopy class $B\gamma$.*

Proof. Regarding G' as a left G-space, we see by inspection of definitions that $\gamma_*(\pi)$ is the principal G'-bundle $EG \times_G G' \longrightarrow BG$. By Proposition 3.5, the composite of

$$f \times_G \mathrm{id} : EG \times_G G' \longrightarrow EG' \times_G G'$$

with the map $EG' \times_G G' \longrightarrow EG'$ induced by the right action of G' on EG' is a right G'-map that gives a bundle map $\gamma_*(\pi) \longrightarrow \pi'$. The conclusion follows from the specification of $B\gamma$ in the proof of Proposition 5.3. □

As a first example, we have the following important observation.

Lemma 6.3. *If $g \in G$ and $\gamma_G : G \longrightarrow G$ is given by conjugation, $\gamma_g(h) = g^{-1}hg$, then $B\gamma_g$ is the identity map of BG.*

Proof. The map $EG \longrightarrow EG$ given by right multiplication by g satisfies the prescribed equivariance property and induces the identity map of BG on passage to orbits. □

Henceforward in this section, let H be a closed subgroup with a local cross section in G and let $i : H \longrightarrow G$ denote the inclusion. By Proposition 3.8, $EG \longrightarrow EG/H$ is a universal H-bundle. Applying Lemma 6.2 to the identity map of EG, we obtain the following observation.

LEMMA 6.4. *The map $Bi : BH \longrightarrow BG$ is the bundle*

$$BH = EG/H \longrightarrow EG/G$$

with fiber G/H.

Now assume further that H is normal in G with quotient group $K = G/H$, so that we have an extension

$$1 \longrightarrow H \xrightarrow{\ i\ } G \xrightarrow{\ j\ } K \longrightarrow 1 .$$

Let $EG \longrightarrow BG$ and $EK \longrightarrow BK$ be universal bundles for G and K and take $EG \longrightarrow EG/H = BH$ to be the universal bundle for H. Choose a map $Ej : EG \longrightarrow EK$ such that $(Ej)(yg) = (Ej)(y)j(g)$, so that Ej induces Bj on passage to orbits. Since $j(h) = e$ for $h \in H$, Ej factors through BH, and we obtain a bundle map

$$\begin{array}{ccc} BH & \longrightarrow & EK \\ {\scriptstyle Bi}\downarrow & & \downarrow \\ BG & \xrightarrow[Bj]{} & BK. \end{array}$$

In particular, this square is a pullback.

Recall that the homotopy fiber $F(f)$ of a based map $f : X \longrightarrow Y$ is defined by the pullback diagram

$$\begin{array}{ccc} Ff & \longrightarrow & PY \\ \downarrow & & \downarrow {\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Equivalently, $F(f)$ is the actual fiber over the basepoint of the map $NX \longrightarrow Y$ obtained by turning f into a fibration via the standard mapping path fibration construction [**11**, §§8.5, 8.6]. There is thus a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_{n+1}Y \longrightarrow \pi_n F(f) \longrightarrow \pi_n X \longrightarrow \pi_n Y \longrightarrow \cdots .$$

Using the diagram above Proposition 6.1, we see by the universal property of pushouts that there is a map $\theta : BH \longrightarrow F(Bj)$ such that the following diagram commutes:

$$\begin{array}{ccccc} K & \longrightarrow & BH & \xrightarrow{Bi} & BG \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \theta} & & \| \\ \Omega BK & \longrightarrow & F(Bj) & \longrightarrow & BG \end{array}$$

By the five lemma, θ induces an isomorphism on homotopy groups and is thus a homotopy equivalence. This conclusion may be restated as follows.

PROPOSITION 6.5. *Up to homotopy, the sequence*

$$BH \xrightarrow{Bi} BG \xrightarrow{Bj} BK$$

is a fiber sequence.

Finally, retaining the hypotheses above, assume further that K is discrete. For $g \in G$, we have the conjugation homomorphism $\gamma_g : H \longrightarrow H$. On the other hand, $Bi : BH \longrightarrow BG$ is a principal K-bundle, hence we have a right action of K on BH. In fact, Bi is a regular cover and K is its group of covering transformations. We have the following generalization of Lemma 6.3.

LEMMA 6.6. *If $k = gH \in K = G/H$, then the covering transformation $k : BH \longrightarrow BH$ is homotopic to $B\gamma_g : BH \longrightarrow BH$.*

PROOF. The action of K on $BH = EG/H$ is induced from the action of G on EG, and right translation by g gives a map $EG \longrightarrow EG$ with the equivariance property prescribed for γ_g in Lemma 6.2. $\square$

We shall later apply this to the extension

$$1 \longrightarrow SO(n) \longrightarrow O(n) \longrightarrow \mathbb{Z}_2 \longrightarrow 1 .$$

It also applies to the extension

$$1 \longrightarrow T \longrightarrow N \longrightarrow W(G) \longrightarrow 1 ,$$

where T is a maximal torus in a compact Lie group G, N is the normalizer of T in G, and $W(G)$ is the Weyl group. The covering transformations of BT given by the elements of $W(G)$ are of fundamental importance because of the following result.

LEMMA 6.7. *For $\sigma \in W(G)$, the following diagram is homotopy commutative:*

$$\begin{array}{ccc} BT & \xrightarrow{Bi} & BG \\ {\scriptstyle\sigma}\downarrow & & \| \\ BT & \xrightarrow[Bi]{} & BG. \end{array}$$

PROOF. If $g \in N$ has image σ, then $\sigma = B\gamma_g$, by the previous lemma. Since $\gamma_g \circ i = i \circ \gamma_g$ and $B\gamma_g = \text{id}$ on BG by Lemma 6.3, the conclusion follows. $\square$

Bibliography

[1] J.F. Adams. Lectures on Lie groups. W.A. Benjamin. 1969. (Reprinted by the University of Chicago Press, 1982).

[2] T. Bröcker and T. tom Dieck. Representations of compact Lie groups. Springer-Verlag. 1985.

[3] H. Cartan and J.C. Moore. Périodicité des groupes d'homotopie stables des groupes classiques, d'après Bott. Séminaire Henri Cartan 1959/60. École Normale Supérieure. 1961.

[4] C. Chevalley Theory of Lie groups I. Princeton University Press. 1946.

[5] T. tom Dieck. Klassifikation numerierbarer Bündel. Arch. Math.(Basel) 17(1966), 395-399.

[6] A. Dold. Partitions of unity in the theory of fibrations. Annals of Math. 78(1963), 223-255.

[7] R. Fritsch and R.A. Piccinini. Cellular structures in topology. Cambridge University Press. 1990.

[8] D. Husemoller. Fiber bundles. Springer-Verlag. 1966. (Second edition, 1975).

[9] J. Milnor. On spaces having the homotopy type of a CW-complex. Trans. Amer. Math. Soc. 90(1959), 272-280.

[10] J.P. May. Classifying spaces and fibrations. Memoirs Amer. Math. Soc. No 155. Amer. Math. Soc. 1975.

[11] J.P. May. A concise course in algebraic topology. University of Chicago Press. 1999.

[12] J. P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞-ring spaces and E_∞-ring spectra. Springer Lecture Notes in Mathematics Vol. 577. 1977.

[13] R. Schon. Fibrations over a CWh-space. Proc. Amer. Math. Soc. 62(1976), 165-166.

[14] N.E. Steenrod. The topology of fiber bundles. Princeton University Press. 1951.

[15] G.W. Whitehead. Elements of homotopy theory. Springer-Verlag. 1978.