DERIVED CATEGORIES FROM A TOPOLOGICAL POINT OF VIEW

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Contents

This is an edited version of Part III of [8]. It is an elementary introduction to the theory of derived categories that is based on slavish immitation of the theory of CW complexes, or rather cell complexes, in algebraic topology. The basic theory is in Sections 1, 2, 3, and 5. The other sections, although important, are a little more advanced.

Let k be a commutative ring. We consider $\mathbb{Z}$-graded chain complexes of k-modules, which we abbreviate to k-complexes. Such an $X = \{X_q\}$ has a differential $d : X_q \to X_{q-1}$. The cohomologically minded reader can reindex by setting $X^q = X_{-q}$ and $d^q = d_{-q}$, so that the differential raises degree. A k-chain map $f : X \to Y$ is a sequence of maps $f\colon X_q \longrightarrow Y_q$ that commute with the differentials, $d\circ f = f\circ d$; f is a quasi-isomorphism if it induces an isomorphism on homology. The tensor product (over k) $X \otimes Y$ of k-complexes X and Y is given by

$$(X \otimes Y)_n = \oplus_{p+q=n} X_p \otimes Y_q,$$

with differential

$$d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y)$$

if $\deg(x) = p$.

Let A be a differential graded associative and unital k-algebra (= DGA). This means that A is a k-complex with a unit element $1 \in A_0$ and an associative and unital product map $A\otimes A \longrightarrow A$ of chain complexes. Writing products by juxtaposition, $d(xy) = d(x)y+(-1)^p xd(y)$. The novice in homological algebra may think of the simple case $A = k$, concentrated in degree zero and given zero differential, but the general case is no more difficult and is very important in modern mathematics. By an A-module M, we mean a left A-module such that the action $A \otimes M \longrightarrow M$ is given by a k-chain map.

As many topologists recognize, there is a close analogy between the derived category $\mathscr{D}_A$ of differential graded A-modules and the homotopy category of spaces in algebraic topology. The analogy becomes much closer if one considers the stable homotopy category of spectra, but the novice is not expected to know about that. We here give a topologically motivated, but purely algebraic, exposition of the classical derived categories associated to DGA's. These categories admit remarkably simple and explicit descriptions in terms of "cell A-complexes". These are the precise algebraic analogs of "cell complexes" in topology, which are defined in the same way as CW complexes except that cells need not be attached only to cells of lower dimension.

Such familiar topological results as Whitehead's theorem and Brown's representability theorem transcribe directly into algebra. There is also a theory of CW modules, but these are less useful, due to the limitations of the algebraic cellular approximation theorem. Derived tensor products and Hom functors, together with differential Tor and Ext functors and Eilenberg-Moore (or hyperhomology) spectral sequences for their computation, drop out quite easily. Since [8] assumed familiarity with spectral sequences, which the novice will not have seen, we put the relevant material into a small section of its own. In the lectures, I intend to interpolate a brief discussion of the classical Tor and Ext functors from the present point of view by specialization of Sections 1–3.

Our methods can be abstracted and applied more generally, and some of what we do can be formalized in Quillen's context of closed model categories. We prefer to be more concrete and less formal. We repeat that many topologists have long known some of this material. For the expert, we emphasize that k is an arbitrary commutative ring and we nowhere impose boundedness or flatness hypotheses. The novice will wonder why such hypotheses were ever thought to be needed.

1. Cell A-modules

We begin with some trivial notions, expressed so as to show the analogy with topology. Let I denote the "unit interval k-complex". It is free on generators $[0]$ and $[1]$ of degree 0 and $[I]$ of degree 1, with $d[I] = [0] - [1]$. A homotopy is a map $X \otimes I \to Y$. The cone CX is the quotient module $X \otimes (I/k[1])$ and the suspension ΣX is $X \otimes (I/\partial I)$, where ∂I has basis $[0]$ and $[1]$. Additively, CX is the sum of copies of X and ΣX, but with differential arranged so that $H_*(CX) = 0$. The usual algebraic notation for the suspension is $\Sigma X = X[1]$, and $(\Sigma X)_q \cong X_{q-1}$. Since we have tensored the interval coordinate on the right, the differential on ΣX is the same as the differential on X, without the introduction of a sign.

The cofiber of a map $f : X \to Y$ is the pushout of f along the inclusion $X = X \otimes [0] \to CX$. There results a short exact sequence

$$0 \to Y \to Cf \to \Sigma X \to 0.$$

Up to sign, the connecting homomorphism of the resulting long exact homology sequence is f_*. Explicitly, $(Cf)_q = Y_q \oplus X_{q-1}$, with differential

$$d(y, x) = (dy + (-1)^q fx, dx).$$

The sequence

$$X \to Y \to Cf \to \Sigma X$$

is called a cofiber sequence, or an exact triangle.

Now assume given a DGA A over k. If X is a k-complex and M is an A-module, then $M \otimes X$ is an A-module, and the notion of a homotopy between maps of A-modules is defined by taking $X = I$. Since we defined cofiber sequences in terms of tensoring with k-complexes, the cofiber sequence generated by a map of A-modules is clearly a sequence of A-modules. Let $\mathscr{M}_A$ denote the category of A-modules and $h\mathscr{M}_A$ be its homotopy category. Its objects are the A-modules, and its maps are the homotopy classes of maps of A-modules. Then the derived category $\mathscr{D}_A$ is obtained from $h\mathscr{M}_A$ by adjoining formal inverses to the quasi-isomorphisms of A-modules. In Construction 2.7 below, we shall give an explicit description that makes it clear that there are no set theoretic difficulties. (This point is typically ignored in algebraic geometry and obviated by concrete construction in algebraic topology.)

The sequences isomorphic to cofiber sequences in the respective categories give $h\mathscr{M}_A$ and $\mathscr{D}_A$ classes of "exact triangles" with respect to which they become triangulated categories in the sense of Verdier [14] (see also [13]), but we do not go seriously into that here.

It is also convenient to think of the suspension functors in a slightly different way. Let S^q be the free k-complex generated by a cycle i_q of degree q, where $q \in \mathbb{Z}$. Then our suspension functors are just

$$\Sigma^q M = M \otimes S^q.$$

We think of the S^q as sphere k-complexes. We let $S^q_A = A \otimes S^q$ and think of the S^q_A as sphere A-modules; they are free on the generating cycles i_q. The reader puzzled by the analogy with spaces might prefer to take all k-complexes X to satisfy $X_q = 0$ for $q < 0$, as holds for the chains of a space.

Digressively, as noted before, the closer analogy is with stable homotopy theory, since that is a place in which negative dimensional spheres live topologically. In fact, one description of the stable homotopy category ([9]) translates directly into our new description of the derived category. (The preamble of [9] explains the relationship with earlier treatments of the stable homotopy category, which did not have the same flavor.) In brief, one sets up a category of spectra. In that category, one defines a theory of cell and CW spectra that allows negative dimensional spheres. One shows that a weak homotopy equivalence between cell spectra is a homotopy equivalence and that every spectrum is weakly homotopy equivalent to a cell spectrum. The stable homotopy category is obtained from the homotopy category of spectra by formally inverting the weak homotopy equivalences, and it is described more concretely as the homotopy category of cell spectra. With spectra and weak homotopy equivalences replaced by A-modules and quasi-isomorphisms, precisely the same pattern works algebraically, but far more simply.

Definitions 1.1. (i) A cell A-module M is the union of an expanding sequence of sub A-modules M_n such that $M_0 = 0$ and M_{n+1} is the cofiber of a map $\phi_n : F_n \to M_n$, where F_n is a direct sum of sphere modules S^q_A (of varying degrees q). The restriction of ϕ_n to a summand S^q_A is called an attaching map and is determined by the "attaching cycle" $\phi_n(i_q)$. An attaching map $S^q_A \to M_n$ induces a map

$$CS^q_A = A \otimes CS^q \to M_{n+1} \subset M,$$

and such a map is called a $(q+1)$-cell. Thus M_{n+1} is obtained from M_n by adding a copy of S^{q+1}_A for each attaching map with domain S^q_A, but giving the new generators $j_{q+1} = i_q \otimes [I]$ the differentials

$$d(j_{q+1}) = (-1)^q \phi_n(i_q).$$

We call such a copy of S_A^{q+1} in M an open cell; if we ignore the differential, then M is the direct sum of its open cells.
(ii) A map $f : M \to N$ between cell A-modules is cellular if $f(M_n) \subset N_n$ for all n.
(iii) A submodule L of a cell A-module M is a cell submodule if L is a cell A-module such that $L_n \subset M_n$ and the composite of each attaching map $S_A^q \to L_n$ of L with the inclusion $L_n \to M_n$ is an attaching map of M. Every cell of L is a cell of M.

We call $\{M_n\}$ the sequential filtration of M. It is essential for inductive arguments, but it should be regarded as flexible and subject to change whenever convenient. It merely records the order in which cells are attached and, as long as the cycles to which attachment are made are already present, it doesn't matter when we attach cells.

Lemma 1.2. *Let $f : M \to N$ be an A-map between cell A-modules. Then M admits a new sequential filtration with respect to which f is cellular.*

Proof. Assume inductively that M_n has been filtered as a cell A-module $M_n = \cup M'_r$ such that $f(M'_r) \subset N_r$ for all r. Let $x \in M_n$ be an attaching cycle for the construction of M_{n+1} from M_n and let $\chi : CS_A^q \to M_{n+1}$ be the corresponding cell. Let r be minimal such that both $x \in M'_r$ and $f \circ \chi$ has image in N_{r+1}. Extend the filtration of M_n to M_{n+1} by taking x to be a typical attaching cycle of a cell $CS_A^q \to M'_{r+1}$. □

Definition 1.3. The dimension of a cell $CS_A^q \to M_{n+1}$ is $q+1$. A cell A-module M is said to be a CW A-module if each cell is attached only to cells of lower dimension, in the sense that the defining cycles $\phi_n(i_q)$ are elements in the sum of the images of cells of dimension at most q. The n-skeleton M^n of a CW A-module is the sum of the images of its cells of dimension at most n, so that $M^n \subset M^{n+1}$. We require of cellular maps $f : M \to N$ between CW A-modules that they be "bicellular", in the sense that both $f(M^n) \subset N^n$ and $f(M_n) \subset N_n$ for all n. By Lemma 1.2, the latter condition can be arranged by changing the order in which the cells of M are attached.

Definition 1.4. A cell A-module is finite dimensional if it has cells in finitely many dimensions. It is finite if it has finitely many cells.

Just as finite cell spectra are central to the topological theory, so finite cell A-modules are central here, especially when we restrict to commutative DGA's and discuss duality. The collection of cell A-modules enjoys the following closure properties, which imply many others.

Proposition 1.5. *(i) A direct sum of cell A-modules is a cell A-module.*
(ii) If L is a cell submodule of a cell A-module M, N is a cell A-module, and $f : L \to N$ is a cellular map, then the pushout $N \cup_f M$ is a cell A-module with sequential filtration $\{N_n \cup_f M_n\}$. It contains N as a cell submodule and has one cell for each cell of M not in L.
(iii) If L is a cell submodule of a cell A-module M and X is a cell submodule of a cell k-complex Y, then $M \otimes Y$ is a cell A-module with sequential filtration $\{\sum_p (M_p \otimes Y_{n-p})\}$. It contains $L \otimes Y + M \otimes X$ as a cell submodule and has a $(q+r)$-cell for each pair consisting of a q-cell of M_p and an r-cell of Y_{n-p}, $0 \le p \le n$.
(iv) The mapping cylinder $Mf = N \cup_f (L \otimes I)$ of $f : L \to N$ is the pushout defined by taking $L = L \otimes k[0] \subset L \otimes I$. If f is a cellular map between cell A-modules, then

Mf is a cell A-module, $L = L \otimes k[1]$ is a cell submodule, the inclusion $N \to Mf$ is a homotopy equivalence, and $Cf = Mf/L$.

Proof. Parts (i) and (ii) are easy and (iv) follows from (ii) and (iii). For (iii), observe that there are evident canonical isomorphisms

$$S^q \otimes S^r \cong S^{q+r} \text{ and } S^q_A \otimes S^r \cong S^{q+r}_A.$$

$M \otimes Y$ has an open cell S^{q+r}_A for each open cell S^q_A of M and S^r of Y; the differential on its canonical basis element is the cycle

$$d(j_q) \otimes j_r + (-1)^q j_q \otimes d(j_r). \quad \square$$

2. Whitehead's theorem and the derived category

A quick space level version of some of the results of this section may be found in [11], and the spectrum level model is given in [9, I §5]. We construct the derived category explicitly in terms of cell modules. As in topology, the "homotopy extension and lifting property" is pivotal. It is a direct consequence of the following trivial observation. Let i_0 and i_1 be the evident inclusions of M in $M \otimes I$.

Lemma 2.1. *Let $e : N \to P$ be a map such that $e_* : H_*(N) \to H_*(P)$ is a monomorphism in degree q and an epimorphism in degree $q+1$. Then, given maps f, g, and h such that $f|S^q_A = hi_0$ and $eg = hi_1$ in the following diagram, there are maps $\tilde{g}$ and $\tilde{h}$ that make the entire diagram commute.*

$$\begin{tikzcd}
S^q_A \arrow[rr, "i_0"] \arrow[dd] & & S^q_A \otimes I \arrow[dl, "h"'] \arrow[dd] & & S^q_A \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& P & & N \arrow[ll, "e"'] & \\
CS^q_A \arrow[ur, "f"] \arrow[rr, "i_0"'] & & CS^q_A \otimes I \arrow[ul, "\tilde{h}"] & & CS^q_A \arrow[ll, "i_0"] \arrow[ul, "\tilde{g}"']
\end{tikzcd}$$

Proof. Let $i = i_q \otimes [0]$ and $j = i_q \otimes [I]$ be the basis elements of CS^q_A, so that $d(j) = (-1)^q i$. Then $eg(i) = h(i \otimes [1])$ and $f(i) = h(i \otimes [0])$, hence

$$d(h(i \otimes [I]) - f(j)) = (-1)^{q+1} eg(i).$$

Since $eg(i)$ bounds in P, $g(i)$ must bound in N, say $d(n') = g(i)$. Then

$$p \equiv e(n') + (-1)^q (h(i \otimes [I]) - f(j))$$

is a cycle. There must be a cycle $n \in N$ and a chain $c \in P$ such that

$$d(c) = p - e(n).$$

Define $\tilde{g}(j) = (-1)^q (n' - n)$ and $\tilde{h}(j \otimes [I]) = c$. $\square$

Theorem 2.2 (HELP). *Let L be a cell submodule of a cell A-module M and let $e : N \to P$ be a quasi-isomorphism of A-modules. Then, given maps $f : M \to P$,*

$g : L \to N$, and $h : L \otimes I \to P$ such that $f|L = hi_0$ and $eg = hi_1$ in the following diagram, there are maps $\tilde{g}$ and $\tilde{h}$ that make the entire diagram commute.

$$\begin{tikzcd}
L \arrow[rr, "i_0"] \arrow[dd] & & L \otimes I \arrow[dl, "h"'] \arrow[dd] & & L \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& P & & N \arrow[ll, "e"'] & \\
M \arrow[ur, "f"] \arrow[rr, "i_0"'] & & M \otimes I \arrow[ul, "\tilde{h}"] & & M \arrow[ll, "i_1"] \arrow[ul, "\tilde{g}"']
\end{tikzcd}$$

Proof. By induction up the filtration $\{M_n\}$ and pullback along cells not in L, this quickly reduces to the case $(M, L) = (CS^q_A, S^q_A)$ of the lemma. □

For objects M and N of any category Cat, let Cat(M, N) denote the set of morphisms in Cat from M to N.

Theorem 2.3 (Whitehead)**.** *If M is a cell A-module and $e : N \to P$ is a quasi-isomorphism of A-modules, then $e_* : h\mathscr{M}_A(M, N) \to h\mathscr{M}_A(M, P)$ is an isomorphism. Therefore a quasi-isomorphism between cell A-modules is a homotopy equivalence.*

Proof. Take $L = 0$ in HELP to see the surjectivity. Replace (M, L) by the pair $(M \otimes I, M \otimes (\partial I))$ to see the injectivity. When N and P are cell A-modules, we may take $M = P$ and obtain a homotopy inverse $f : P \to N$. □

Theorem 2.4 (Cellular approximation)**.** *Assume that $H_q(N/N^q) = 0$ for all q and all CW A-modules N. Let L be a cell submodule of a CW A-module M, let N be a CW A-module , and let $f : M \to N$ be a map whose restriction to L is cellular. Then f is homotopic relative to L to a cellular map. Therefore any map $M \to N$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.*

Proof. By Lemma 1.2, we may change the sequential filtration of M to one for which f is sequentially cellular. Proceeding by induction up the filtration $\{M_n\}$, we construct compatible cellular maps $g_n : M_n \to N_n$ and a homotopy $h_n : M_n \otimes I \to N_n$ from $f|M_n$ to g_n. The result quickly reduces to the case of a single cell of M that is not in L and thus to the case when $(M, L) = (CS^q_A, S^q_A)$. The conclusion follows by application of Lemma 2.1 to the inclusions $e : (N_n)^{q+1} \to N_n$. □

Remark 2.5. If $H_q(A) = 0$ for all $q < 0$, then the homological hypothesis holds and we can work throughout with CW A-modules and cellular maps rather than with cell A-modules. This matches the intuition: CW theory works topologically because the homotopy groups of a sphere S^q are zero in degrees less than q. In many of the motivating algebraic examples, the natural grading is cohomological, with the cohomology groups of the spheres S^q_A zero in degrees less than q. In homological grading, this reverses the inequality, and the homological hypothesis of the cellular approximation theorem then fails in general. That is why we focus on cellular rather than CW theory here.

Theorem 2.6 (Approximation by cell modules)**.** *For any A-module M, there is a cell A-module N and a quasi-isomorphism $e : N \to M$.*

Proof. We construct an expanding sequence N_n and compatible maps $e_n : N_n \to M$ inductively. Choose a cycle ν in each homology class of M, let N_1 be the direct sum of A-modules S^q_A, one for each ν of degree q, and let $e_1 : N_1 \to M$ send the νth canonical basis element to the cycle ν. Inductively, suppose that $e_n : N_n \to M$ has been constructed. Choose a pair of cycles (ν, ν') in each pair of unequal homology classes on N_n that map under $(e_n)_*$ to the same element of $H_*(M)$. Let N_{n+1} be the "homotopy coequalizer" obtained by adjoining a copy of $S^q_A \otimes I$ to N_n along the evident map $S^q_A \otimes \partial I \to N_n$ determined by each such pair (ν, ν') of degree q. Proposition 1.5 implies that N_{n+1} is a cell A-module such that N_n is a cell submodule. Any choice of chains $\mu \in M$ such that $d(\mu) = \nu - \nu'$ determines an extension of $e_n : N_n \to M$ to $e_{n+1} : N_{n+1} \to M$. Let N be the colimit of the N_n and $e : N \to M$ be the resulting map. Clearly, N is a cell module, e induces an epimorphism on homology since e_1 does, and e induces a monomorphism on homology by construction. □

Construction 2.7. For each A-module M, choose a cell A-module ΓM and a quasi-isomorphism $\gamma : \Gamma M \to M$. By the Whitehead theorem, for a map $f : M \to N$, there is a map $\Gamma f : \Gamma M \to \Gamma N$, unique up to homotopy, such that the following diagram is homotopy commutative.

$$\begin{tikzcd} \Gamma M \arrow[r, "\Gamma f"] \arrow[d, "\gamma"'] & \Gamma N \arrow[d, "\gamma"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Thus Γ is a functor $h\mathscr{M}_A \to h\mathscr{M}_A$, and γ is natural. The derived category $\mathscr{D}_A$ can be described as the category whose objects are the A-modules and whose morphisms are specified by

$$\mathscr{D}_A(M,N) = h\mathscr{M}_A(\Gamma M, \Gamma N),$$

with the evident composition. When M is a cell A-module,

$$\mathscr{D}_A(M,N) \cong h\mathscr{M}_A(M,N).$$

Using the identity function on objects and Γ on morphisms, we obtain a functor $i : h\mathscr{M}_A \to \mathscr{D}_A$ that sends quasi-isomorphisms to isomorphisms and is universal with this property. Let $\mathscr{C}_A$ be the full subcategory of $\mathscr{M}_A$ whose objects are the cell A-modules. Then the functor Γ induces an equivalence of categories $\mathscr{D}_A \to h\mathscr{C}_A$ with inverse the composite of i and the inclusion of $h\mathscr{C}_A$ in $h\mathscr{M}_A$.

Therefore the derived category and the homotopy category of cell modules can be used interchangeably. Homotopy-preserving functors on A-modules that do not preserve quasi-isomorphisms are transported to the derived category by first applying Γ, then the given functor. Much more emphasis is placed on this simple procedure in the algebraic than in the topological literature, as is reflected in the respective notational conventions.

Digressively, we note that topologists routinely transport constructions to the stable homotopy category by passing to CW spectra, without change of notation. In fact, while a great deal of modern work in stable homotopy theory depends heavily on having a good underlying category of spectra, earlier constructions of the stable homotopy category did not even allow spectra that were more general than CW spectra. For this and other reasons, topologists are accustomed to work with CW spectra and their cells in a concrete calculational way, not as something esoteric but rather as something much more basic and down to earth than general spectra. An analogous view of differential graded A-modules is rather intriguing.

3. Derived tensor product and Hom functors: Tor and Ext

We first record some elementary facts about tensor products with cell A-modules.

Lemma 3.1. *Let N be a cell A-module. Then the functor $M \otimes_A N$ preserves exact sequences and quasi-isomorphisms in the variable M.*

Proof. With differential ignored, N is a free A-module, and preservation of exact sequences follows. The sequential filtration of N gives short exact sequences of free A-modules

$$0 \longrightarrow N_n \longrightarrow N_{n+1} \longrightarrow N_{n+1}/N_n \longrightarrow 0,$$

where the subquotients N_{n+1}/N_n are direct sums of sphere A-modules. The preservation of quasi-isomorphisms holds trivially if N is a sphere A-module, and the general case follows by passage to direct sums, induction up the filtration, and passage to colimits. □

It is usual to define the derived tensor product, denoted $M \otimes_A^L N$, by replacing the left A-module N (or the right A-module M) by a suitable resolution P and taking the ordinary tensor product $M \otimes_A P$, in line with the standard rubric of derived functors (see e.g. Verdier [14], who restricts to bounded below modules). Our procedure is the same, except that we take approximation by quasi-isomorphic cell A-modules as our version of a resolution and, following the pedantically imprecise tradition in algebraic topology, we prefer not to change notation. That is, in $\mathscr{D}_k$, $M \otimes_A N$ *means* $M \otimes_A \Gamma N$ (or $\Gamma M \otimes_A \Gamma N$ or $\Gamma M \otimes_A N$: the three are canonically isomorphic in $\mathscr{D}_k$). The lemma shows that the definition makes sense. We can also use the lemma to show that the derived category $\mathscr{D}_A$ depends only on the quasi-isomorphism type of A.

Proposition 3.2. *Let $\phi : A \to A'$ be a quasi-isomorphism of DGA's. Then the pullback functor $\phi^* : \mathscr{D}_{A'} \to \mathscr{D}_A$ is an equivalence of categories with inverse given by the extension of scalars functor $A' \otimes_A (?)$.*

Proof. For $M \in \mathscr{M}_A$ and $M' \in \mathscr{M}_{A'}$, we have

$$\mathscr{M}_{A'}(A' \otimes_A M, M') \cong \mathscr{M}_A(M, \phi^* M').$$

The functor $A' \otimes_A (?)$ preserves sphere modules and therefore cell modules. This implies formally that the adjuction passes to derived categories, giving

$$\mathscr{D}_{A'}(A' \otimes_A M, M') \cong \mathscr{D}_A(M, \phi^* M').$$

If M is a cell A-module, then

$$\phi \otimes \mathrm{id} : M \cong A \otimes_A M \longrightarrow \phi^*(A' \otimes_A M)$$

is a quasi-isomorphism of A-modules. These maps give the unit of the adjunction. Its counit is given by the maps of A'-modules

$$\mathrm{id} \otimes_\phi \gamma : A' \otimes_A \Gamma M' \longrightarrow A' \otimes_{A'} M' \cong M',$$

where $\Gamma M'$ is a cell A-module and $\gamma : \Gamma M' \longrightarrow M'$ is a quasi-isomorphism of A-modules. Since the composite of this map with the quasi-isomorphism $\phi \otimes \mathrm{id}$ for the A-module $\Gamma M'$ coincides with γ, this map too is a quasi-isomorphism. □

For left A-modules M and N, let $\mathrm{Hom}_A(M,N)_q$ be the k-complex of homomorphisms of A-modules of degree q (components $f: M_n \longrightarrow N_{n+q}$ for all n) with the differential $(df)(m) = d(f(m)) - (-1)^q f(d(m))$. It is usual to regrade this cohomologically, but it is nicer to view it as an object in our original category of $\mathbb{Z}$-graded k-complexes. For k-complexes L,

$$\mathscr{M}_A(L \otimes M, N) \cong \mathscr{M}_k(L, \mathrm{Hom}_A(M,N)), \tag{3.3}$$

where A acts on $L \otimes M$ through its action on M (with the usual sign convention: $a(\ell \otimes m) = (-1)^{\deg a \deg \ell} \ell \otimes am$). This isomorphism clearly passes to homotopy categories. Letting L run through the sphere k-complexes and using (3.1) and the Whitehead theorem, we see that if M is a cell A-module then the functor $\mathrm{Hom}_A(M,N)$ preserves quasi-isomorphisms in N.

This allows us to define $\mathrm{Hom}_A(M,N)$ in $\mathscr{D}_A$ for arbitrary modules M and N by first replacing M by a cell approximation ΓM and then taking $\mathrm{Hom}_A(\Gamma M, N)$ on the level of modules. Thus, in $\mathscr{D}_k$, $\mathrm{Hom}_A(M,N)$ *means* $\mathrm{Hom}_A(\Gamma M, N)$. This gives a well-defined functor such that

$$\mathscr{D}_A(L \otimes M, N) \cong \mathscr{D}_k(L, \mathrm{Hom}_A(M,N)). \tag{3.4}$$

Remark 3.5. The argument we have just run through is a special case of a general one. If S and T are left and right adjoint functors between two categories of the sort that we are considering, then S preserves objects of the homotopy type of cell modules if and only T preserves quasi-isomorphisms, and in that case the resulting induced functors on derived categories are still adjoint. See [9, I.5.13] for a precise categorical statement.

We can now define differential Tor and Ext (or hyperhomology and hypercohomology) groups as follows. We ignore questions of justification in terms of standard homological terminology, some of which we believe to be antiquated.

Definition 3.6. Working in derived categories, define

$$\mathrm{Tor}^A_*(M,N) = H_*(M \otimes_A N) \text{ and } \mathrm{Ext}^A_*(M,N) = H_*(\mathrm{Hom}_A(M,N)).$$

It is usual to regrade Ext cohomologically, along with Hom. If we specialize by setting $k = \mathbb{Z}$ and letting A be a ring, thought of as a DGA concentrated in degree zero and with zero differential, then these groups Tor and Ext are the Tor and Ext groups of classical homological algebra. We can check this by comparing definitions, but it is more satisfactory to make this treatment self-contained by checking that our functors satisfy the axioms that characterize the classical functors.

4. Some spectral sequences

This section is addressed to those who have seen spectral sequences. It will not be used in later sections. No matter how Tor and Ext are defined in the generality of modules over DGA's, the essential point is to have Eilenberg-Moore, or hyperhomology, spectral sequences for their calculation.

Theorem 4.1. *There are natural spectral sequences of the form*

$$E^2_{p,q} = Tor^{H_*A}_{p,q}(H_*M, H_*N) \Longrightarrow Tor^A_{p+q}(M,N) \tag{4.2}$$

and

$$E^2_{p,q} = Ext^{H_*A}_{p,q}(H_*M, H_*N) \Longrightarrow Ext^A_{p+q}(M,N). \tag{4.3}$$

These are both spectral sequences of homological type, with

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}. \tag{4.4}$$

In (4.2), p is the usual homological degree, the spectral sequence is non-zero only in the right half-plane, and it converges strongly. In (4.3), p is the negative of the usual cohomological degree, the spectral sequence is non-zero only in the left half plane, and it converges strongly if, for each fixed (p,q), only finitely many of the differentials (4.4) are non-zero. (The best study of the convergence of spectral sequences, is given in [1].)

Our construction of the spectral sequences follows [6], which is a precursor of the present approach to derived categories. Let $\epsilon : P \to N$ be a quasi-isomorphism of left A-modules, where P is a cell A-module. Rewrite the cellular filtration of P by setting $F_nP = P_{n+1}$. Thus

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \cdots \subset F_nP \subset \cdots .$$

The filtration gives rise to a spectral sequence that starts from

$$E^0_{p,q}P = (F_pP/F_{p-1}P)_{p+q} \cong A \otimes (\bar{P}_{p,*})_{p+q},$$

where $\bar{P}_{p,*}$ is k-free on the canonical basis elements of the open cells of P_p. The definition of a cell module implies that $d_0 = d \otimes 1$. Therefore

$$E^1_{p,*}P \cong H_*(A) \otimes \bar{P}_{p,*}.$$

Thinking of N as filtered with $F_{-1}N = 0$ and $F_pN = N$ for $p \geq 0$, we see that $E^1_{*,*}P$ gives a complex of left $H_*(A)$-modules

$$\cdots \to E^1_{p+1,*}P \to E^1_{p,*}P \to \cdots \to E^1_{0,*}P \to H_*(N) \to 0. \tag{4.5}$$

Definition 4.6. Let P be a cell A-module. A quasi-isomorphism $\epsilon : P \to N$ is said to be a distinguished resolution of N if the sequence (4.5) is exact, so that $\{E^1_{p,*}P\}$ is a free $H_*(A)$-resolution of $H_*(N)$.

Observe that $\epsilon : P \to N$ is necessarily a homotopy equivalence if N is a cell A-module, by Whitehead's theorem. The following result, which is due to Gugenheim and myself [6, 2.1] and will not be reproven here, should be viewed as a greatly sharpened version of Theorem 2.6: it gives cell approximations with precisely prescribed algebraic properties.

Theorem 4.7 (Gugenheim-May)**.** *For any A-module N, every free $H_*(A)$-resolution of $H_*(N)$ can be realized as $\{E^1_{p,*}P\}$ for a distinguished resolution $\epsilon : P \to N$.*

A distinguished resolution $\epsilon : P \to N$ of a cell A-module N induces a homotopy equivalence $M \otimes_A P \to M \otimes_A N$ for any (right) A-module M. Filtering $M \otimes_A P$ by

$$F_p(M \otimes_A P) = M \otimes_A (F_pP), \quad p \geq 0,$$

we obtain the spectral sequence (4.2).

Similarly, a distinguished resolution $\epsilon : P \to M$ of a cell A-module A-module M induces a homotopy equivalence $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_A(P,N)$ for any (left) A-module N, and the filtration

$$F_p\mathrm{Hom}_A(P,N) = \mathrm{Hom}_A(P/F_{-p-1}P, N), \quad p \leq 0,$$

gives rise to the spectral sequence (4.3).

5. Commutative DGA's and duality

Let A be commutative throughout this section. We give $\mathscr{D}_A$ a structure of a closed symmetric monoidal category in the sense of [10, 7]. This means that we have a tensor product, $M \otimes_A N$, which is associative, commutative, and unital (with unit A) up to coherent natural isomorphism and that we also have internal hom objects, $\mathrm{Hom}_A(M, N)$, with the property that

$$\mathscr{D}_A(L \otimes_A M, N) \cong \mathscr{D}_A(L, \mathrm{Hom}_A(M, N)). \tag{5.1}$$

We also discuss duality, characterizing the (strongly) dualizable objects or, in another language, identifying the largest rigid tensored subcategory of $\mathscr{D}_A$. Again, in $\mathscr{D}_A$, $M \otimes_A N$ *means* $M \otimes_A \Gamma N$. Since A is commutative, this is an A-module. From our present point of view, it makes good sense to resolve both variables since we now have canonical isomorphisms of A-modules

$$S^q_A \otimes_A S^r_A \cong S^{q+r}_A$$

As in Proposition 1.5(iii), this directly implies that tensor products of cell A-modules are cell A-modules.

Proposition 5.2. *If M and M' are cell A-modules, then $M \otimes_A M'$ is a cell A-module with sequential filtration $\{\sum_p (M_p \otimes_A N_{n-p})\}$. It has a $(q+r)$-cell for each pair consisting of a q-cell of M_p and an r-cell of M'_{n-p}, $0 \le p \le n$.*

For A-modules M and N, $\mathrm{Hom}_A(M, N)$ is an A-module such that

$$\mathscr{M}_A(L \otimes_A M, N) \cong \mathscr{M}_A(L, \mathrm{Hom}_A(M, N)). \tag{5.3}$$

In $\mathscr{D}_A$, $\mathrm{Hom}_A(M, N)$ *means* $\mathrm{Hom}_A(\Gamma M, N)$, and we have the isomorphism (5.1).

There are general accounts of duality theory in the context of symmetric monoidal categories in the literature of both algebraic geometry [4, §1], [3] and algebraic topology [5], [9, III§1]. I have recently given what I hope is an easily readable exposition [12]. I will recall some of the ideas. Observe first that, by an easy direct inspection of definitions, the functor $\mathrm{Hom}_A(M, N)$ preserves cofiber sequences in both variables. (Actually, in the variable M, the functor $\mathrm{Hom}_A(M, N)$ converts an exact triangle into the negative of an exact triangle.)

The dual of an A-module M, denoted $M^\vee$ (in algebraic geometry) or DM, is defined to be $\mathrm{Hom}_A(M, A)$. The adjunction (5.1) specializes to give an evaluation map $\epsilon : DM \otimes_A M \to A$ and a map $\eta : A \to \mathrm{Hom}_A(M, M)$. It also leads to a natural map

$$\nu : \mathrm{Hom}_A(L, M) \otimes_A N \to \mathrm{Hom}_A(L, M \otimes_A N), \tag{5.4}$$

which specializes to

$$\nu : DM \otimes_A M \to \mathrm{Hom}_A(M, M). \tag{5.5}$$

M is said to be "dualizable" or "finite" or "rigid" if, in $\mathscr{D}_A$, there is a coevaluation map $\eta : A \to M \otimes_A DM$ such that the following diagram commutes, where τ is the commutativity isomorphism.

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, "\eta"'] & M \otimes_A DM \arrow[d, "\tau"] \\ \mathrm{Hom}_A(M, M) & DM \otimes_A M \arrow[l, "\nu"] \end{tikzcd} \tag{5.6}$$

The definition has many purely formal implications. The map ν of (5.4) is an isomorphism (in $\mathscr{D}_A$) if either L or N is dualizable. The map ν of (5.5) is an isomorphism if and only if M is dualizable, and the coevaluation map η is then the composite $\gamma\nu^{-1}\eta$ in (5.6). The natural map

$$\rho : M \to DDM$$

is an isomorphism if M is dualizable. The natural map

$$\otimes : \mathrm{Hom}_A(M, N) \otimes_A \mathrm{Hom}_A(M', N') \to \mathrm{Hom}_A(M \otimes_A M', N \otimes_A N')$$

is an isomorphism if M and M' are dualizable or if M is dualizable and $N = A$.

Say that a cell A-module N is a direct summand up to homotopy of a cell A-module M if there is a homotopy equivalence of A-modules between M and $N \oplus N'$ for some cell A-module N'.

Theorem 5.7. *A cell A-module is dualizable if and only if it is a direct summand up to homotopy of a finite cell A-module.*

Proof. Observe first that S^q_A is dualizable with dual S^{-q}_A, hence any finite direct sum of A-modules S^q_A is dualizable. Observe next that the cofiber of a map between dualizable A-modules is dualizable. In fact, the evaluation map ϵ induces a natural map

$$\epsilon_\# : \mathscr{D}_A(L, N \otimes_A DM) \to \mathscr{D}_A(L \otimes_A M, N),$$

and M is dualizable if and only if $\epsilon_\#$ is an isomorphism for all L and N [9, III.3.6]. Since both sides turn cofiber sequences in the variable M into long exact sequences, the five lemma gives the observation. We conclude by induction on the number of cells that a finite cell A-module is dualizable. It is formal that a direct summand in $\mathscr{D}_A$ of a dualizable A-module is finite. For the converse, let M be a dualizable cell A-modulewith coevaluation map $\eta\colon A \to M \otimes_A DM$. Clearly η factors through $N \otimes_A DM$ for some finite cell subcomplex N of M. By a diagram chase ([9, III.1.2]), the bottom composite in the following commutative diagram is the identity (in $\mathscr{D}_A$):

$$\begin{tikzcd}
 & N \otimes_A DM \otimes_A M \arrow[r, "1\wedge\epsilon"] \arrow[d] & N \otimes_A A \arrow[r, "\cong"] \arrow[d] & N \arrow[d] \\
M \cong A \otimes_A M \arrow[ur] \arrow[r, "\eta\wedge 1"'] & M \otimes_A DM \otimes_A M \arrow[r, "1\wedge\epsilon"'] & M \otimes_A A \arrow[r, "\cong"] & M
\end{tikzcd}$$

Therefore M is a retract up to homotopy and thus, by a comparison of exact triangles, a direct summand up to homotopy of N. (Retractions split in triangulated categories.) $\square$

Let $\mathscr{F}_A$ be the full subcategory of $\mathscr{M}_A$ whose objects are the direct summands up to homotopy of finite cell A-modules. In the language of [4, 1.7], the theorem states that the homotopy category $h\mathscr{F}_A$ is the largest rigid tensored subcategory of the derived category $\mathscr{D}_A$. Note that the sequential filtration of a finite cell A-module can be arranged so that a single cell is attached at each stage. That is, such a module is just a finite sequence of extensions by free modules on a single generator, and each quotient module M_n/M_{n-1} has the form S^q_A for some q. A direct summand up to homotopy of a finite cell A-module, which is the appropriate analog in $\mathscr{D}_A$ of a finitely generated projective A-module, need not be an actual direct summand and need not be isomorphic in $\mathscr{D}_A$ to a finite cell A-module. The situation demands the introduction and study of the K-theory group $K_0(\mathscr{F}_A)$, but we shall desist.

6. Brown's representability theorem

We revert to a general DGA A, not necessarily commutative. Functors of cohomological type on $\mathscr{D}_A$ are of considerable interest, and we here recall a categorical result that characterizes when they can be represented in the form $\mathscr{D}_A(?, N)$. The topological analogue has long played an important role.

We have said that we think of the S^q_A as analogs of sphere spectra. Just as maps out of spheres calculate homotopy groups and therefore detect weak equivalences, so maps out of the S^q_A calculate homology groups and therefore detect quasi-isomorphisms. We display several versions of this fact for later use: for all A-modules N,

$$H_q(N) \cong h\mathscr{M}_k(k, N \otimes S^{-q}) \cong h\mathscr{M}_k(S^q, N) \cong h\mathscr{M}_A(S^q_A, N) \cong \mathscr{D}_A(S^q_A, N). \tag{6.1}$$

The category $\mathscr{D}_A$ has "homotopy limits and colimits". These are weak limits and colimits in the sense that they satisfy the existence but not the uniqueness property of categorical limits and colimits. For example, the homotopy pushout of maps $f : L \to M$ and $g : L \to N$ is obtained from $M \oplus (L \otimes I) \oplus N$ by identifying $l \otimes [0]$ with $f(l)$ and $l \otimes [1]$ with $g(l)$. More precisely, we first apply cell approximation and then apply the cited construction. We used a similar homotopy coequalizer in the proof of Theorem 2.6. The homotopy colimit, or telescope $\mathrm{Tel} M_i$, of a sequence of maps $f_i : M_i \to M_{i+1}$ is the homotopy coequalizer of Id: $\oplus M_i \to \oplus M_i$ and $\oplus f_i : \oplus M_i \to \oplus M_i$; equivalently, it is the cofiber of $g : \oplus M_i \to \oplus M_i$, where $g(m) = m - f_i(m)$ for $m \in M_i$. We now have enough information to quote the categorical form of Brown's representability theorem given in [2], but we prefer to run through a quick concrete version of the proof.

Theorem 6.2 (Brown)**.** *A contravariant functor $J : \mathscr{D}_A \to$ Sets is representable in the form $J(M) \cong \mathscr{D}_A(M, N)$ for some A-module N if and only if J converts direct sums to direct products and converts homotopy pushouts to weak pullbacks.*

Proof. Necessity is obvious. Thus assume given a functor J that satisfies the specified direct sum and Mayer-Vietoris axioms. Since homotopy coequalizers and telescopes can be constructed from sums and homotopy pushouts, J converts homotopy coequalizers to weak equalizers and telescopes to weak limits. Write $f^* = J(f)$ for a map f. Consider pairs (M, μ) where M is an A-module and $\mu \in J(M)$.

Starting with an arbitrary pair (N_0, ν_0), we construct a sequence of pairs (N_i, ν_i) and maps $f_i : N_i \to N_{i+1}$ such that $f_i^*(\nu_{i+1}) = \nu_i$. Let $N_1 = N_0 \oplus (\oplus S^q_A)$, where there is a copy of S^q_A for each element ϕ of each set $J(S^q_A)$. Let ν_1 have coordinates ν and the elements ϕ, and let $f_0 : N_0 \to N_1$ be the inclusion. Inductively, given (N_i, ν_i), let L_i be the sum of a copy of S^q_A for each q and each unequal pair (x, y) of elements of $H_q(N_i)$ such that, when thought of as maps $S^q_A \to N_i$ in $\mathscr{D}_A$, $x^*(\nu_i) = y^*(\nu_i)$. Let $f_i : N_i \to N_{i+1}$ be the coequalizer of the pair of maps $L_i \to N_i$ given by the x's and the y's. By the weak equalizer property, there is an element $\nu_{i+1} \in J(N_{i+1})$ such that $f_i^*(\nu_{i+1}) = \nu_i$.

Let $N = \mathrm{Tel}\, N_i$. By the weak limit property, there is an element $\nu \in J(N)$ that pulls back to ν_i for each i. For an A-module M, define $\theta_\nu : \mathscr{D}_A(M, N) \to J(M)$ by

$\theta_\nu(f) = f^*(\nu)$. Then, by construction, θ_ν is a bijection for all S^q_A. We claim that θ_ν is a bijection for all M.

Suppose given elements $x, y \in \mathscr{D}_A(M, N)$ such that $\theta_\nu(x) = \theta_\nu(y)$. Replacing M by a cell approximation if necessary, we can assume that x and y are given by maps $M \to N$. Let $c : N \to N'_0$ be the homotopy coequalizer of x and y and choose an element $\nu'_0 \in J(N'_0)$ such that $c^*(\nu'_0) = \nu$. Construct a pair (N', ν') by repeating the construction above, but starting with the pair (N'_0, ν'_0). Let $j : N'_0 \to N'$ be the evident map such that $j^*(\nu') = \nu'_0$. Then, since $(jc)^*(\nu') = \nu$ and both θ_ν and $\theta_{\nu'}$ are bijections for all S^q_A, $jc : N \to N'$ is an isomorphism in $\mathscr{D}_A$. Since $cx = cy$ by construction, it follows that $x = y$. Therefore θ_ν is an injection for all A-modules M.

Finally, let $\omega \in J(M)$ for any module M. Repeat the construction, starting with the zeroth pair $(M \oplus N, (\omega, \nu))$. We obtain a new pair (N', ν') together with a map $i : M \to N'$ such that $i^*(\nu') = \omega$ and a map $j : N \to N'$ such that $j^*(\nu') = \nu$. Again, j is an isomorphism in $\mathscr{D}_A$ since both θ_ν and $\theta_{\nu'}$ are bijections for all S^q_A. Therefore $\omega = (ij^{-1})^*(\nu)$ and θ_ν is a surjection for all A-modules M. □

Observe that we can start with $N_0 = 0$, in which case N can be given the structure of a cell A-module. It is formal that the module N that represents J is unique up to isomorphism in $\mathscr{D}_A$ and that natural transformations between representable functors are represented by maps in $\mathscr{D}_A$.

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