## 1. Symmetric monoidal categories and enriched categories

In practice, categories come in nature with more structure than just sets of morphisms. This extra structure is central to all of category theory, homotopical or not. While every mathematician who makes use of categories should understand enrichment, this is not the place for a full exposition. The most thorough source is Kelly's book [**3**], and an introduction can be found in Borceux [**2**, Ch. 6]. We outline what is most relevant to model categories in this section.

A monoidal structure on a category $\mathscr{V}$ is a product, $\otimes$ say, and a unit object $\mathbf{I}$ such that the product is associative and unital up to coherent natural isomorphisms; $\mathscr{V}$ is symmetric if $\otimes$ is also commutative up to coherent natural isomorphism. Informally, coherence means that diagrams that intuitively should commute do in fact commute. (The symmetry coherence admits a weakening that gives braided monoidal categories, but those will not concern us.) A symmetric monoidal category $\mathscr{V}$ is closed if it has internal hom objects $\underline{\mathscr{V}}(X, Y)$ in $\mathscr{V}$ together with adjunction isomorphisms

$$\mathscr{V}(X \otimes Y, Z) \cong \mathscr{V}(X, \underline{\mathscr{V}}(Y, Z)).$$

These isomorphisms of hom sets imply isomorphisms of internal hom objects in $\mathscr{V}$

$$\underline{\mathscr{V}}(X \otimes Y, Z) \cong \underline{\mathscr{V}}(X, \underline{\mathscr{V}}(Y, Z)).$$

The proof is an exercise in the use of the Yoneda lemma: these two objects represent isomorphic functors of three variables.

From now on, we let $\mathscr{V}$ be a bicomplete closed symmetric monoidal category. Such categories appear so often in nature that category theorists have invented a name for them: such a category is often called a "cosmos". We will require our cosmos $\mathscr{V}$ to be a model category in the next section, but we ignore model category theory for the moment. When $\otimes$ is the cartesian product, we say that $\mathscr{V}$ is cartesian closed, but the same category $\mathscr{V}$ can admit other symmetric monoidal structures.

Examples 1.1. We give examples of cosmoi $\mathscr{V}$.

(i) The category $\mathscr{S}et$ of sets is closed cartesian monoidal.
(ii) The category $\mathscr{U}$ of (compactly generated) spaces is cartesian closed. The space $\underline{\mathscr{U}}(X, Y)$ is the function space of maps $X \longrightarrow Y$ with the $k$-ification of the compact open topology.
(iii) The category $\mathscr{U}_*$ of based spaces is closed symmetric monoidal under the smash product. The smash product would not be associative if we used just spaces, rather than compactly generated spaces [**6**, §1.7]. The based space $\underline{\mathscr{U}_*}(X, Y)$ is the function space $F(X, Y)$ of based maps $X \longrightarrow Y$.
(iv) The category $s\mathscr{S}et$ of simplicial sets is cartesian closed. A simplicial set is a contravariant functor $\Delta \longrightarrow \mathscr{S}et$, where $\Delta$ is the category of sets $n = \{0, 1, \cdots, n\}$ and monotonic maps. There are $n$-simplices $\Delta[n]$ in $s\mathscr{S}et$, and $n \mapsto \Delta[n]$ gives a covariant functor $\Delta \longrightarrow s\mathscr{S}et$. The internal hom [**5**, §I.6] in $s\mathscr{S}et$ is specified by

$$\underline{s\mathscr{S}et}(X, Y)_n = s\mathscr{S}et(X \times \Delta[n], Y).$$

(v) For a commutative ring $R$, the category $\mathscr{M}_R$ of $R$-modules is closed symmetric monoidal under the functors $\otimes_R$ and $\mathrm{Hom}_R$; in particular, the category $\mathscr{A}b$ of abelian groups is closed symmetric monoidal.

(vi) For a commutative ring $R$, the category $Ch_R$ of $\mathbb{Z}$-graded chain complexes of $R$-modules (with differential lowering degree) is closed symmetric monoidal under the graded tensor product and hom functors

$$(X \otimes_R Y)_n = \Sigma_{p+q=n} X_p \otimes_R Y_q; \quad d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y).$$

$$\mathrm{Hom}_R(X, Y)_n = \Pi_i \, \mathrm{Hom}_R(X_i, Y_{i+n}); \quad d(f)_i = d \circ f_i - (-1)^n f_{i-1} \circ d.$$

Here the symmetry $\gamma \colon X \otimes Y \longrightarrow Y \otimes X$ is defined with a sign,

$$\gamma(x \otimes y) = (-1)^{pq} y \otimes x \quad \text{for} \quad x \in X_p \quad \text{and} \quad y \in Y_q.$$

(vii) The category $\mathscr{C}at$ of small categories is cartesian closed.

Example (iii) generalizes from $\mathscr{U}$ to an arbitrary cartesian closed category $\mathscr{V}$.

EXAMPLE 1.2. Let $\mathscr{V}$ be cartesian closed and let $\mathscr{V}_* = */\mathscr{V}$ be the category of based objects in $\mathscr{V}$, with base maps denoted $i \colon * \longrightarrow V$. Note that the unit object $* \in \mathscr{V}$ is a terminal object, so that there is a unique map $t \colon V \longrightarrow *$ for any $V \in \mathscr{V}$. For $V, W \in \mathscr{V}$, define the smash product $V \wedge W$ and the function object $F(V, W)$ to be the pushout and pullback in $\mathscr{V}$ displayed in the diagrams

$$\begin{array}{ccc} V \amalg W & \xrightarrow{j} & V \times W \\ \downarrow & & \downarrow \\ * & \longrightarrow & V \wedge W \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F(V,W) & \longrightarrow & \underline{\mathscr{V}}(V,W) \\ \downarrow & & \downarrow{\scriptstyle i^*} \\ \underline{\mathscr{V}}(*,*) & \xrightarrow[i_*]{} & \underline{\mathscr{V}}(*,\mathscr{W}). \end{array}$$

Here $j$ has coordinates $(\mathrm{id}, i)$ on $V \cong V \times *$ and $(i, \mathrm{id})$ on $W \cong * \times W$, and the base map $i \colon * \longrightarrow F(V, W)$ is induced by the canonical isomorphism $* \longrightarrow \underline{\mathscr{V}}(*, *)$ and the map $\mathscr{V}(t, i) \colon * \cong \mathscr{V}(*, *) \longrightarrow \underline{\mathscr{V}}(V, W)$. The unit $S^0 = *_+$ in $\mathscr{V}_*$ is the coproduct of two copies of $*$, with one of them giving the base map $i \colon * \longrightarrow S^0$.

There are two ways of thinking about enriched categories. One can think of "enriched" as an adjective, in which case one thinks of enrichment as additional structure on a preassigned ordinary category. Alternatively, one can think of "enriched category" as a noun, in which case one thinks of a self-contained definition of a new kind of object. From that point of view, one constructs an ordinary category from an enriched category. Thinking from the two points of view simultaneously, it is essential that the constructed ordinary category be isomorphic to the ordinary category that one started out with. Either way, there is a conflict of notation between that preferred by category theorists and that in common use by "working mathematicians" (to whom [**4**] is addressed). We give the definition in its formulation as a noun, but we use notation that implicitly takes the working mathematician's point of view that we are starting with a preassigned category $\mathscr{M}$.

DEFINITION 1.3. Let $\mathscr{V}$ be a symmetric monoidal category. A $\mathscr{V}$-category $\mathscr{M}$, or a category $\mathscr{M}$ enriched over $\mathscr{V}$, consists of

(i) a class of objects, with typical objects denoted $X$, $Y$, $Z$;
(ii) for each pair of objects $(X, Y)$, a hom object $\underline{\mathscr{M}}(X, Y)$ in $\mathscr{V}$;
(iii) for each object $X$, a unit map $\mathrm{id}_X \colon \mathbf{I} \longrightarrow \mathscr{M}(X, X)$ in $\mathscr{V}$;
(iv) for each triple of objects $(X, Y, Z)$, a composition morphism in $\mathscr{V}$

$$\underline{\mathscr{M}}(Y, Z) \otimes \underline{\mathscr{M}}(X, Y) \longrightarrow \underline{\mathscr{M}}(X, Z).$$

The evident associativity and unity diagrams are required to commute.

$$\begin{array}{ccc}
\underline{\mathscr{M}}(Y,Z)\otimes\underline{\mathscr{M}}(X,Y)\otimes\underline{\mathscr{M}}(W,X) & \longrightarrow & \underline{\mathscr{M}}(Y,Z)\otimes\underline{\mathscr{M}}(W,Y) \\
\downarrow & & \downarrow \\
\underline{\mathscr{M}}(X,Z)\otimes\underline{\mathscr{M}}(W,X) & \longrightarrow & \underline{\mathscr{M}}(W,Z)
\end{array}$$

$$\begin{array}{ccccc}
\mathbf{I}\otimes\underline{\mathscr{M}}(X,Y) & \xleftarrow{\cong} & \underline{\mathscr{M}}(X,Y) & \xrightarrow{\cong} & \underline{\mathscr{M}}(X,Y)\otimes\mathbf{I} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\underline{\mathscr{M}}(Y,Y)\otimes\underline{\mathscr{M}}(X,Y) & \longrightarrow & \underline{\mathscr{M}}(X,Y) & \longleftarrow & \underline{\mathscr{M}}(X,Y)\otimes\underline{\mathscr{M}}(X,X)
\end{array}$$

The underlying category of the enriched category has the same objects and has morphism sets specified by

$$\mathscr{M}(X,Y)=\mathscr{V}(\mathbf{I},\underline{\mathscr{M}}(X,Y)). \tag{1.4}$$

The unit element of $\mathscr{M}(X,X)$ is $\mathrm{id}_X$. The composition is the evident composite

$$\begin{array}{c}
\mathscr{V}(\mathbf{I},\underline{\mathscr{M}}(Y,Z))\times\mathscr{V}(\mathbf{I},\underline{\mathscr{M}}(X,Y)) \\
\downarrow{\scriptstyle\otimes} \\
\mathscr{V}(\mathbf{I}\otimes\mathbf{I},\underline{\mathscr{M}}(Y,Z)\otimes\underline{\mathscr{M}}(X,Y)) \\
\downarrow \\
\mathscr{V}(\mathbf{I},\underline{\mathscr{M}}(X,Z)),
\end{array}$$

where we have used the unit isomorphism $\mathbf{I}\otimes\mathbf{I}\cong\mathbf{I}$.

As said, we have given the definition in its "noun" form. In its "adjectival" form, one starts with a preassigned ordinary category $\mathscr{M}$, prescribes the appropriate enrichment, and requires a canonical isomorphism between the original category $\mathscr{M}$ and the underlying category of the prescribed enriched category. Rigorously, equality must be replaced by isomorphism in (1.4), but one generally regards that canonical isomorphism as an identification. Less formally, we start with an ordinary category $\mathscr{M}$, construct the hom objects $\underline{\mathscr{M}}(X,Y)$ in $\mathscr{V}$, and check that we have the identification (1.4). For example, any cosmos is naturally enriched over itself via its internal hom objects. The reader is urged to think through the identifications (1.4) in Examples 1.1.

Examples 1.5. When $\mathscr{V}$ is one of the cosmoi specified in Examples 1.1, categories enriched in $\mathscr{V}$, or $\mathscr{V}$-categories, have standard names.

(i) Categories as usually defined are categories enriched in $\mathscr{S}et$.
(ii) Categories enriched in $\mathscr{A}b$ are called $\mathscr{A}b$-categories. They are called additive categories if they have zero objects and biproducts [**4**, p. 196]. They are called abelian categories if, further, all maps have kernels and cokernels, every monomorphism is a kernel, and every epimorphism is a cokernel [**4**, p. 198].
(iii) Categories enriched in $\mathscr{U}$ are called topological categories.
(iv) Categories enriched in $s\mathscr{S}et$ are called simplicial categories.
(v) Categories enriched in $Ch_R$ for some $R$ are called DG-categories.

(vi) Categories enriched in $\mathscr{C}at$ are called (strict) 2-categories and, inductively, categories enriched in the cartesian monoidal category of $(n-1)$-categories are called $n$-categories.

Examples of all six sorts are ubiquitous. For any ring $R$, not necessarily commutative, the category of left $R$-modules is abelian. All categories of structured spaces, such as the categories of topological monoids and of topological groups, are topological categories. The letters DG stand for "differential graded". We shall return to the last example in §**??**.

Most of the model category literature focuses on simplicial categories. Although there are technical reasons for this preference, we prefer to work with naturally occurring enrichments wherever possible, and these may or may not be simplicial. In our examples, we shall focus on $\mathscr{V} = \mathscr{U}$ and $\mathscr{V} = Ch_R$. These have features in common that are absent when $\mathscr{V} = s\mathscr{S}et$.

Of course, the definition of a $\mathscr{V}$-category is accompanied by the notions of a $\mathscr{V}$-functor $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ and a $\mathscr{V}$-natural transformation $\eta\colon F \longrightarrow G$ between two $\mathscr{V}$-functors $\mathscr{M} \longrightarrow \mathscr{N}$. For the former, we require maps

$$F\colon \underline{\mathscr{M}}(X,Y) \longrightarrow \underline{\mathscr{N}}(FX,FY)$$

in $\mathscr{V}$ which preserve composition and units. For the latter, we first observe that maps $f\colon X' \longrightarrow X$ and $g\colon Y \longrightarrow Y'$ in $\mathscr{M}$ induce maps

$$f^*\colon \mathscr{M}(X,Y) \longrightarrow \mathscr{M}(X',Y) \quad \text{and} \quad g_*\colon \mathscr{M}(X,Y) \longrightarrow \mathscr{M}(X,Y')$$

in $\mathscr{V}$, and we then require maps $\eta\colon FX \longrightarrow GX$ in $\mathscr{M}$ such that the following naturality diagrams commute in $\mathscr{V}$ for all objects $X,Y \in \mathscr{M}$.

$$\begin{array}{ccc}
\underline{\mathscr{M}}(Y,X) & \xrightarrow{F} & \underline{\mathscr{N}}(FX,FY) \\
{\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \eta_*} \\
\underline{\mathscr{N}}(GY,GX) & \xrightarrow[\eta^*]{} & \underline{\mathscr{N}}(FY,GX)
\end{array}$$

The general idea is that one first expresses categorical notions diagrammatically on hom sets, and one then sees how to reinterpret the notions in the enriched sense.

However, there are important enriched categorical notions that take account of the extra structure given by the enrichment and are not just reinterpretations of ordinary categorical notions. In particular, there are weighted (or indexed) colimits and limits. The most important of these (in non-standard notation) are tensors $X \odot V$ (sometimes called copowers) and cotensors (sometimes called powers) $\Phi(V,X)$ in $\mathscr{M}$ for objects $X \in \mathscr{M}$ and $V \in \mathscr{V}$. These are characterized by natural isomorphisms

$$\mathscr{M}(X \odot V, Y) \cong \mathscr{V}(V, \underline{\mathscr{M}}(X,Y)) \cong \mathscr{M}(X, \Phi(V,Y)) \tag{1.6}$$

of hom sets, which again imply natural isomorphisms of objects in $\mathscr{V}$

$$\underline{\mathscr{M}}(X \odot V, Y) \cong \underline{\mathscr{V}}(V, \underline{\mathscr{M}}(X,Y)) \cong \underline{\mathscr{M}}(X, \Phi(V,Y)). \tag{1.7}$$

Again, examples are ubiquitous. If $R$ is a ring, $X$ and $Y$ are left $R$-modules, and $V$ is an abelian group, then $X \otimes V$ and $\mathrm{Hom}(V,Y)$ are left $R$-modules that give tensors and cotensors in the abelian category of left $R$-modules. This works equally well if $X$ and $Y$ are chain complexes of $R$-modules and $V$ is a chain complex of abelian groups. We shall return to this example in Chapter **??**.

We say that the $\mathscr{V}$-category $\mathscr{M}$ is $\mathscr{V}$-bicomplete if it has all weighted colimits and limits. We dodge the definition of these limits by noting that $\mathscr{M}$ is $\mathscr{V}$-bicomplete if it is bicomplete in the ordinary sense and has all tensors and cotensors [**3**, 3.73]. The category $\mathscr{V}$ is itself a $\mathscr{V}$-bicomplete $\mathscr{V}$-category. Its tensors and cotensors are given by its product $\otimes$ and internal hom functor $\underline{\mathscr{V}}$.

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