1. Enriching confusion

Let me try to clarify the example. Let \( \mathcal{V} \) be the category of chain complexes over \( \mathbb{Z} \). The set of morphisms \( \mathcal{V}(X,Y) \) is the set of maps of chain complexes \( f: X \to Y \); thus \( f \) is given by maps \( f_q: X_q \to Y_q \) that commute with the differentials. We can of course add maps of chain complexes, so that we may regard \( \mathcal{V}(X,Y) \) as an abelian group. Thus \( \mathcal{V} \) is a category enriched over the category \( \mathcal{Ab} \) of abelian groups. That is not the main point however.

The category \( \mathcal{V} \) is closed symmetric monoidal. Its tensor product is the tensor product of chain complexes, and its internal hom \( \mathcal{V} \) is given by

\[
\mathcal{V}(X,Y) = \text{Hom}(X,Y),
\]

the chain complex we have defined twice which in degree \( n \) is \( \prod_q \text{Hom}(X_q, Y_{q+n}) \). Such internal hom objects are objects of the original category \( \mathcal{V} \). You may think of them as having elements if your habits require you to do so, but that misses the point. You should not regard the elements as morphisms in the category; that really misses the point: the morphisms in \( \mathcal{V} \) remain the chain maps. The unit of \( \mathcal{V} \) is \( \mathbb{Z} \), regarded as a chain complex with differential 0, concentrated in degree 0.

For a ring \( R \), we have the category \( \text{Ch}_R \) of chain complexes of (right, say) \( R \)-modules. Its set of morphisms \( \text{Ch}_R(X,Y) \) is the set of maps of chain complexes of \( R \)-modules. We can again add, so that we may regard \( \text{Ch}_R \) as enriched over \( \mathcal{Ab} \). If \( R \) is commutative, we also have multiplication by elements of \( R \), so that \( \text{Ch}_R \) is enriched over the category \( \mathcal{M}_R \) of \( R \)-modules. Again, that’s not the point.

The category \( \text{Ch}_R \) is enriched over \( \mathcal{V} = \text{Ch}_\mathbb{Z} \). The internal hom object \( \text{Ch}_R \) is the chain complex \( \text{Hom}_R(X,Y) \) of Abelian groups. In degree \( n \), it is \( \prod_q \text{Hom}_R(X_q, Y_{q+n}) \). Here, for \( R \)-modules \( M \) and \( N \), \( \text{Hom}_R(M,N) \) is just the abelian group of homomorphisms \( M \to N \) of \( R \)-modules.

What is \( \mathcal{V}(\mathbb{Z}, \text{Hom}_R(X,Y)) \)? Why, it is just the set of maps of chain complexes from \( \mathbb{Z} \) to \( \text{Hom}_R(X,Y) \). What is such a map? Well, it is certainly determined by where the element 1 goes. It must go to an element of degree 0, so an element of \( \prod \text{Hom}_R(X_q, Y_q) \). Since \( d(1) = 0 \), that element must be a cycle. That means precisely that the image of 1 is an ordinary map \( X \to Y \) of chain complexes, that is an element of the set \( \text{Ch}_R(X,Y) \) of morphisms \( X \to Y \) in the ordinary category \( \text{Ch}_R \).

Now let \( f: R \to S \) be a homomorphism of rings. We have the functors

\[
f^*: \mathcal{M}_S \longrightarrow \mathcal{M}_R \quad f_!: \mathcal{M}_R \longrightarrow \mathcal{M}_S \quad f_*: \mathcal{M}_R \longrightarrow \mathcal{M}_S.
\]

For an \( S \)-module \( N \), \( f^*N \) is \( N \) regarded as an \( R \)-module via \( nr = nf(r) \). For an \( R \)-module \( M \), \( f_!M = M \otimes_R S \) and \( f_*M = \text{Hom}_R(S,M) \). Do not forget the obvious: this defines these functors on objects, but they are defined the same way on morphisms. We have the adjunctions displayed in class, and they can be written in two ways. If we just think about categories with sets of morphisms between objects, we have the natural bijections

\[
\mathcal{M}_S(f_!M, N) \cong \mathcal{M}_R(M, f^*N) \quad \text{and} \quad \mathcal{M}_R(f^*N, M) \cong \mathcal{M}_S(N, f_*M)
\]

for \( R \)-modules \( M \) and \( S \)-modules \( N \). But if we remember that the categories \( \mathcal{M}_R \) and \( \mathcal{M}_S \) are enriched over \( \mathcal{Ab} \), we see that we have isomorphisms of abelian groups

\[
\text{Hom}_S(f_!M, N) \cong \text{Hom}_R(M, f^*N) \quad \text{and} \quad \text{Hom}_R(f^*N, M) \cong \text{Hom}_S(N, f_*M).
\]
Now consider chain complexes of $R$-modules and $S$-modules. We can apply the functors $f^*$, $f_!$, and $f_*$ to the modules in degree $q$ that comprise a chain complex, and we see that the result inherits a differential from the differentials we start with. That is, if $X$ is a chain complex of $R$-modules and $Y$ is a chain complex of $S$-modules, then $f^*Y$ is a chain complex of $R$-modules and $f_!X$ and $f_*X$ are chain complexes of $S$-modules. The adjunctions, applied degreewise, give us bijections

$$
\mathscr{M}^C_S(f_!X, Y) \cong \mathscr{M}^C_R(X, f^*Y) \quad \text{and} \quad \mathscr{M}^C_R(f_*Y, X) \cong \mathscr{M}^C_S(Y, f_*X).
$$

These can also be viewed as giving isomorphisms of the abelian groups of morphisms of chain complexes, using that our categories are enriched over $\text{Ab}$.

However, more is true. We can look at our internal hom objects in $\mathcal{V}$. Applying $f^*$, $f_!$ and $f_*$ to the morphisms of modules in the coordinate abelian groups $\text{Hom}_S(Y_q, Y'_q + n)$ or $\text{Hom}_R(X_q, X'_q + n)$, we obtain maps of chain complexes of abelian groups

$$
\begin{align*}
  f^* : & \text{Hom}_S(Y, Y') \longrightarrow \text{Hom}_R(f^*Y, f^*Y') \\
  f_! : & \text{Hom}_R(X, X') \longrightarrow \text{Hom}_S(f_!Y, f_!Y') \\
  f_* : & \text{Hom}_R(X, X') \longrightarrow \text{Hom}_S(f_*Y, f_*Y').
\end{align*}
$$

This says that $f^*$, $f_!$, and $f_*$ are enriched functors. We obtain enriched adjunctions, given by natural isomorphisms between objects of $\mathcal{V}$, that is, between chain complexes of abelian groups:

$$
\text{Hom}_S(f_!X, Y) \cong \text{Hom}_R(X, f^*Y) \quad \text{and} \quad \text{Hom}_R(f_*Y, X) \cong \text{Hom}_S(Y, f_*X).
$$

Writing these out in terms of infinite products, you will see instantly that they really are isomorphisms. There is a lot of information packaged in these formulations.