

# FINITE TOPOLOGICAL SPACES

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## 1. INTRODUCTION: FINITE SPACES AND PARTIAL ORDERS

Standard saying: One picture is worth a thousand words.

In mathematics: One good definition is worth a thousand calculations.

But, to quote a slogan from a T-shirt worn by one of my students:

Calculation is the way to the truth.

The intuitive notion of a set in which there is a prescribed description of nearness of points is obvious. Formulating the “right” general abstract notion of what a “topology” on a set should be is not. Distance functions lead to metric spaces, which is how we usually think of spaces. Hausdorff came up with a much more abstract and general notion that is now universally accepted.

**Definition 1.1.** A *topology* on a set  $X$  consists of a set  $\mathcal{U}$  of subsets of  $X$ , called the “open sets of  $X$  in the topology  $\mathcal{U}$ ”, with the following properties.

- (i)  $\emptyset$  and  $X$  are in  $\mathcal{U}$ .
- (ii) A finite intersection of sets in  $\mathcal{U}$  is in  $\mathcal{U}$ .
- (iii) An arbitrary union of sets in  $\mathcal{U}$  is in  $\mathcal{U}$ .

A complement of an open set is called a closed set. The closed sets include  $\emptyset$  and  $X$  and are closed under finite unions and arbitrary intersections.

It is very often interesting to see what happens when one takes a standard definition and tweaks it a bit. The following tweaking of the notion of a topology is due to Alexandroff [1], except that he used a different name for the notion.

**Definition 1.2.** A topological space is an *A-space* if the set  $\mathcal{U}$  is closed under arbitrary intersections.

A space is *finite* if the set  $X$  is finite, and the following observation is clear.

**Lemma 1.3.** *A finite space is an A-space.*

It turns out that a great deal of what can be proven for finite spaces applies equally well more generally to *A-spaces*. However, the finite spaces have recently captured people’s attention. Since digital processing and image processing start from finite sets of observations and seek to understand pictures that emerge from a notion of nearness of points, finite topological spaces seem a natural tool in many such scientific applications. There are many papers on the subject, but few of any mathematical depth, dating from the 1980’s and 1990’s. There was a brief early flurry of beautiful mathematical work on this subject. Two independent papers, by McCord and Stong [11, 15], both published in 1966, are especially interesting. We will work through them. We are especially interested in questions raised by

the union of these papers that are answered in neither and have not been pursued since. We are also interested in calculational questions about the enumeration of finite topologies.

There is a hierarchy of “separation properties” on spaces, and intuition about finite spaces is impeded by too much habituation to the stronger of them.

**Definition 1.4.** Let  $(X, \mathcal{U})$  be a topological space.

- (i)  $X$  is a  $T_0$ -space if for any two points of  $X$ , there is an open neighborhood of one that does not contain the other.
- (ii)  $X$  is a  $T_1$ -space if each point of  $X$  is a closed subset.
- (iii)  $X$  is a  $T_2$ -space, or *Hausdorff space*, if any two points of  $X$  have disjoint open neighborhoods.

**Lemma 1.5.**  $T_2 \implies T_1 \implies T_0$ .

In most of topology, the spaces considered are Hausdorff. For example, metric spaces are Hausdorff. Intuition gained from thinking about such spaces is rather misleading when one thinks about finite spaces.

**Definition 1.6.** The *discrete topology* on  $X$  is the topology in which all sets are open. The *trivial* or *coarse topology* on  $X$  is the topology on  $X$  in which  $\emptyset$  and  $X$  are the only open sets. We write  $D_n$  and  $C_n$  for the discrete and coarse topologies on a set with  $n$  elements. They are the largest and the smallest possible topologies (in terms of the number of open subsets).

**Lemma 1.7.** *If a finite space is  $T_1$ , then it is discrete.*

*Proof.* Every subset is a union of finitely many points, hence is closed. Therefore every set is open.  $\square$

In contrast,  $T_0$  finite spaces are very interesting.

**Exercise 1.8.** *Show (by induction) that a finite  $T_0$  space has at least one point which is a closed subset.*

Finite spaces have canonical minimal “bases”, which we describe next.

**Definition 1.9.** A *basis*  $\mathcal{B}$  for a topological space  $X$  is a set of open sets, called *basic open sets*, with the following properties.

- (i) Every point of  $X$  is in some basic open set.
- (ii) If  $x$  is in basic open sets  $B_1$  and  $B_2$ , then  $x$  is in a basic set  $B_3 \subset B_1 \cap B_2$ .

If  $\mathcal{B}$  is a set satisfying these two properties, the *topology generated by  $\mathcal{B}$*  is the set  $\mathcal{U}$  of subsets  $U$  of  $X$  such that, for each point  $x \in U$ , there is a set  $B$  in  $\mathcal{B}$  such that  $x \in B \subset U$ .

**Example 1.10.** The set of singleton sets  $\{x\}$  is a basis for the discrete topology on  $X$ . The set of disks  $D_r(x) = \{y \mid d(x, y) < r\}$  is a basis for the topology on a metric space  $X$ .

**Lemma 1.11.**  *$\mathcal{B}$  is a basis for  $\mathcal{U}$  if and only if, for each  $x \in U \in \mathcal{U}$ , there is a  $B \in \mathcal{B}$  such that  $x \in B \subset U$ .*

**Definition 1.12.** Let  $X$  be a finite space. For  $x \in X$ , define  $U_x$  to be the intersection of the open sets that contain  $x$ . Define a relation  $\leq$  on the set  $X$  by  $x \leq y$  if  $x \in U_y$  or, equivalently,  $U_x \subset U_y$ . Write  $x < y$  if the inclusion is proper.

From now on  $X$  is a finite space. We write  $|X|$  for the number of points in  $X$ .

**Lemma 1.13.** *The set of open sets  $U_x$  is a basis for  $X$ . Indeed, it is the unique minimal basis for  $X$ .*

*Proof.* The first statement is clear. If  $\mathcal{C}$  is another basis, there is a  $C \in \mathcal{C}$  such that  $x \in C \subset U_x$ . This implies  $C = U_x$ , so that  $U_x \in \mathcal{C}$  for all  $x \in X$ .  $\square$

**Lemma 1.14.** *The relation  $\leq$  is transitive and reflexive. It is a partial order if and only if  $X$  is  $T_0$ .*

*Proof.* The first statement is clear. For the second,  $x \leq y$  and  $y \leq x$  means that  $U_x = U_y$ . This holds if and only if every open set that contains either  $x$  or  $y$  also contains the other.  $\square$

**Lemma 1.15.** *A finite set  $X$  with a reflexive and transitive relation  $\leq$  determines a topology with basis the set of all sets  $U_x = \{y \mid y \leq x\}$ .*

We put things together to obtain the following conclusion.

**Proposition 1.16.** *For a finite set  $X$ , the topologies on  $X$  are in bijective correspondence with the reflexive and transitive relations  $\leq$  on  $X$ . The topology corresponding to  $\leq$  is  $T_0$  if and only if the relation  $\leq$  is a partial order.*

At first sight, one might conclude that finite spaces are uninteresting, but that turns out to be far from the case.

## 2. CONTINUOUS MAPS AND HOMEOMORPHISMS

**Definition 2.1.** Let  $X$  and  $Y$  be topological spaces. A function  $f: X \rightarrow Y$  is continuous if  $f^{-1}(V)$  is open in  $X$  for each open set  $V$  in  $Y$ . We call continuous functions “maps”. A map  $f$  is a *homeomorphism* if  $f$  is one-to-one and onto and its inverse function is continuous.

Topologists are only interested in spaces up to homeomorphism, and we proceed to classify finite spaces up to homeomorphism. Let  $X$  and  $Y$  be finite spaces in what follows.

**Lemma 2.2.** *A function  $f: X \rightarrow Y$  is continuous if and only if it is order preserving:  $x \leq y$  in  $X$  implies  $f(x) \leq f(y)$  in  $Y$ .*

*Proof.* Let  $f$  be continuous and suppose that  $x \leq y$ . Then  $x \in U_y \subset f^{-1}U_{f(y)}$  and thus  $f(x) \in U_{f(y)}$ , which means that  $f(x) \leq f(y)$ . For the converse, let  $V$  be open in  $Y$ . If  $f(y) \in V$ , then  $U_{f(y)} \subset V$ . If  $x \in U_y$ , then  $x \leq y$  and thus  $f(x) \leq f(y)$  and  $f(x) \in U_{f(y)} \subset V$ , so that  $x \in f^{-1}(V)$ . Thus  $f^{-1}(V)$  is the union of these  $U_y$  and is therefore open.  $\square$

**Lemma 2.3.** *A map  $f: X \rightarrow X$  is a homeomorphism if and only if  $f$  is either one-to-one or onto.*

*Proof.* By finiteness, one-to-one and onto are equivalent. Assume they hold. Then  $f$  induces a bijection  $2^f$  from the set  $2^X$  of subsets of  $X$  to itself. Since  $f$  is continuous, if  $f(U)$  is open, then so is  $U$ . Therefore the bijection  $2^f$  must restrict to a bijection from the topology  $\mathcal{U}$  to itself.  $\square$

The previous lemma fails if we allow different topologies on  $X$ : there are continuous bijections between different topologies. We proceed to describe how to enumerate the distinct topologies up to homeomorphism. There are quite a few papers on this enumeration problem in the literature, although some of them focus on enumeration of all topologies, rather than homeomorphism classes of topologies [3, 4, 6, 5, 9, 7, 8, 10, 13, 14]. The furthest out<sup>1</sup> precise calculation that I have found gives that there are 35,979 topologies (up to homeomorphism) on a set with eight elements and 363,083 topologies on a set with nine elements [9]. However, this is not the kind of enumeration problem for which one expects to obtain a precise answer for all  $n$ . Rather, one expects bounds and asymptotics.

It is useful to describe minimal bases without reference to their enumeration by elements of the set, since the latter is redundant.

**Lemma 2.4.** *A set  $\mathcal{B}$  of nonempty subsets of  $X$  is the minimal base for a topology if and only if*

- (i) *Every point of  $X$  is in some set  $B$  in  $\mathcal{B}$ .*
- (ii) *The intersection of two sets in  $\mathcal{B}$  is a union of sets in  $\mathcal{B}$ .*
- (iii) *If a union of sets  $B_i$  in  $\mathcal{B}$  is again in  $\mathcal{B}$ , then the union is equal to one of the  $B_i$ .*

*Proof.* Conditions (i) and (ii) are equivalent to saying that  $\mathcal{B}$  is a basis, and then the minimal basis is contained in  $\mathcal{B}$ . If (iii) also holds, then each  $B$  in  $\mathcal{B}$ , being a union of sets of the form  $U_x$ , must be one of the  $U_x$ . Conversely, if  $\mathcal{B}$  is the minimal basis and  $U_x \in \mathcal{B}$  is the union of sets  $U_y$ , then  $x \in U_y$  and  $U_x = U_y$  for some  $y$ , so (iii) holds.  $\square$

The following three definitions apply to all spaces, not necessarily finite.

**Definition 2.5.** The *subspace topology* on  $A \subset X$  is the set of all intersections  $A \cap U$  for open sets  $U$  of  $X$ .

**Definition 2.6.** The *topology of the union* on  $X \amalg Y$  has as open sets the unions of an open set of  $X$  and an open set of  $Y$ .

**Definition 2.7.** The *product topology* on  $X \times Y$  is the topology with basis the products  $U \times V$  of an open set  $U$  in  $X$  and an open set  $V$  in  $Y$ .

Returning to finite spaces, the previous lemma gives the following results.

**Lemma 2.8.** *If  $A$  is a subspace of  $X$ , the minimal basis of  $A$  consists of the intersections  $A \cap U$ , where  $U$  is in the minimal basis of  $X$ .*

**Lemma 2.9.** *The minimal basis of  $X \amalg Y$  is the union of the minimal basis of  $X$  and the minimal basis of  $Y$ .*

**Lemma 2.10.** *The minimal basis of  $X \times Y$  is the set of products  $U \times V$ , where  $U$  and  $V$  are in the minimal bases of  $X$  and  $Y$ .*

**Definition 2.11.** Consider square matrixes  $M = (a_{i,j})$  with integer entries that satisfy the following properties.

- (i)  $a_{i,i} \geq 1$ .
- (ii)  $a_{i,j}$  is  $-1$ ,  $0$ , or  $1$  if  $i \neq j$ .

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<sup>1</sup>That was written in 2003; Wikipedia goes further now, giving 4,717,687 for  $n = 10$ .

- (iii)  $a_{i,j} = -a_{j,i}$  if  $i \neq j$ .
- (iv)  $a_{i_1, i_s} = 0$  if there is a sequence of distinct indices  $\{i_1, \dots, i_s\}$  such that  $s > 2$  and  $a_{i_k, i_{k+1}} = 1$  for  $1 \leq k \leq s-1$ .

Say that two such matrices  $M$  and  $N$  are equivalent if there is a permutation matrix  $T$  such that  $T^{-1}MT = N$  and let  $\mathcal{M}$  denote the set of equivalence classes of such matrices.

**Theorem 2.12.** *The homeomorphism classes of finite spaces are in bijective correspondence with  $\mathcal{M}$ . The number of sets in a minimal basis for  $X$  determines the size of the corresponding matrix, and the trace of the matrix is the number of elements of  $X$ .*

*Proof.* We work with minimal bases for the topologies rather than with elements of the set. For a minimal basis  $U_1, \dots, U_r$  of a topology  $\mathcal{U}$  on a finite set  $X$ , define an  $r \times r$  matrix  $M = (a_{i,j})$  as follows. If  $i = j$ , let  $a_{i,i}$  be the number of elements  $x \in X$  such that  $U_x = U_i$ . Define  $a_{i,j} = 1$  and  $a_{j,i} = -1$  if  $U_i \subset U_j$  and there is no  $k$  (other than  $i$  or  $j$ ) such that  $U_i \subset U_k \subset U_j$ . Define  $a_{i,j} = 0$  otherwise. Clearly (i)–(iv) hold, and a reordering of the basis results in a permutation matrix that conjugates  $M$  into the matrix determined by the reordered basis. Thus  $X$  determines an element of  $\mathcal{M}$ .

If  $f: X \rightarrow Y$  is a homeomorphism, then  $f$  determines a bijection from the basis for  $X$  to the basis for  $Y$  that preserves inclusions and the number of elements that determine corresponding basic sets, hence  $X$  and  $Y$  determine the same element of  $\mathcal{M}$ . Conversely, suppose that  $X$  and  $Y$  have minimal bases  $\{U_1, \dots, U_r\}$  and  $\{V_1, \dots, V_r\}$  that give rise to the same element of  $\mathcal{M}$ . Reordering bases if necessary, we can assume that they give rise to the same matrix. For each  $i$ , choose a bijection  $f_i$  from the set of elements  $x \in X$  such that  $U_x = U_i$  and the set of elements  $y \in Y$  such that  $V_y = V_i$ . We read off from the matrix that the  $f_i$  together specify a homeomorphism  $f: X \rightarrow Y$ . Therefore our mapping from homeomorphism classes to  $\mathcal{M}$  is one-to-one.

To see that our mapping is onto, consider an  $r \times r$ -matrix  $M$  of the sort under consideration and let  $X$  be the set of pairs of integers  $(u, v)$  with  $1 \leq u \leq r$  and  $1 \leq v \leq a_{i,i}$ . Define subsets  $U_i$  of  $X$  by letting  $U_i$  have elements those  $(u, v) \in X$  such that either  $u = i$  or  $u \neq i$  but  $u = i_1$  for some sequence of distinct indices  $\{i_1, \dots, i_s\}$  such that  $s \geq 2$ ,  $a_{i_k, i_{k+1}} = 1$  for  $1 \leq k \leq s-1$ , and  $i_s = i$ . We see that the  $U_i$  give a minimal basis for a topology on  $X$  by verifying the conditions specified in Lemma 2.4. Condition (i) is clear since  $(u, v) \in U_u$ . To verify (ii) and (iii), we observe that if  $(u, v) \in U_i$  and  $u \neq i$ , then  $U_u \subset U_i$ . Indeed, we certainly have  $(u, v) \in U_i$  for all  $v$ , and if  $(k, v) \in U_u$  with  $k \neq u$ , we must have a sequence connecting  $k$  to  $u$  and a sequence connecting  $u$  to  $i$  which can be concatenated to give a sequence connecting  $k$  to  $i$  that shows that  $(k, v)$  is in  $U_i$ . To see (ii), if  $(u, v) \in U_i \cap U_j$ , then  $U_u \subset U_i \cap U_j$ , which implies that  $U_i \cap U_j$  is a union of sets  $U_u$ . To see (iii), if a union of sets  $U_i$  is a set  $U_j$ , there is an element of  $U_j$  in some  $U_i$  and then  $U_j \subset U_i$ , so that  $U_j = U_i$ . A counting argument for the diagonal entries and consideration of chains of inclusions show that the matrix associated to the topology whose minimal basis is  $\{U_i\}$  is the matrix  $M$  that we started with.  $\square$

## 3. SPACES WITH AT MOST FOUR POINTS

We describe the homeomorphism classes of spaces with at most four points, with just a start on taxonomy.

There is a unique space with one point, namely  $C_1 = D_1$ .

There are three spaces with two points, namely  $C_2, P_2 = \mathbb{C}D_1, D_2$ .

Proper subsets of  $X$  are those not of the form  $\emptyset$  or  $X$ . We often restrict to proper subsets when specifying topologies. The following definitions prescribe the two names for the second space in the short list just given.

**Definition 3.1.** For a set with  $n$  elements, let  $P_n = P_{1,n}$  be the space which has only one proper open set, containing only one point; for  $1 < m < n$ , let  $P_{m,n}$  be the space whose proper open subsets are the non-empty subsets of a given subset with  $m$  elements.

**Definition 3.2.** For a space  $X$  define the *non-Hausdorff cone*  $\mathbb{C}X$  by adjoining a new point  $*$  and letting the proper open subsets of  $\mathbb{C}X$  be the non-empty open subsets of  $X$ .

Note that  $\mathbb{C}X$  is contractible by Lemma 6.2 below.

Here is a table of the nine homeomorphism classes of topologies on a three point set  $X = \{a, b, c\}$ .

Proper open sets	Name	$T_0$ ?	connected?
all	$D_3$	yes	no
a, b, (a,b), (b,c)	$D_1 \amalg P_2$	yes	no
a, b, (a,b)	$P(2,3) = \mathbb{C}D_2$	yes	yes
a	$P_3$	no	yes
a, (a,b)	$\mathbb{C}P_2$	yes	yes
a, (b,c)	$D_1 \amalg C_2$	no	no
a, (a,b), (a,c)	?	yes	yes
(a,b)	$\mathbb{C}C_2$	no	yes
none	$C_3$	no	yes

Here is a tabulation of *bases* for the proper open subsets of the 33 homeomorphism classes of topologies on a four point space  $X = \{a, b, c, d\}$ . That is, the topologies are obtained by adding in the empty set, the whole set, and all unions of the listed sets. The list is ordered by decreasing number of singleton sets in the topology, and, when that is fixed, by increasing number of two-point subsets and then by increasing number of three-point subsets.

1	all
2	a, b, c, (a,b), (a,c), (b,c), (a,b,c)
3	a, b, c, (a,b), (a,c), (b,c), (a,b,c), (a,b,d)
4	a, b, c, (a,b), (a,c), (b,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
5	a, b, (a,b)
6	a, b, (a,b), (a,b,c)
7	a, b, (a,b), (a,c,d)
8	a, b, (a,b), (a,b,c), (a,b,d)
9	a, b, (a,b), (a,c), (a,b,c)
10	a, b, (a,b), (a,c), (a,b,c), (a,c,d)
11	a, b, (a,b), (a,c), (a,b,c), (a,b,d)
12	a, b, (a,b), (c,d), (a,c,d), (b,c,d)
13	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d)
14	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
15	a
16	a, (a,b)
17	a, (a,b), (a,b,c)
18	a, (b,c), (a,b,c)
19	a, (a,b), (a,c,d)
20	a, (a,b), (a,b,c), (a,b,d)
21	a, (b,c), (a,b,c), (b,c,d)
22	a, (a,b), (a,c), (a,b,c)
23	a, (a,b), (a,c), (a,b,c), (a,b,d)
24	a, (c,d), (a,b), (a,c,d)
25	a, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
26	a, (a,b,c)
27	a, (b,c,d)
28	(a,b)
29	(a,b), (c,d)
30	(a,b), (a,b,c)
31	(a,b), (a,b,c), (a,b,d)
32	(a,b,c)
33	none

**Problem 3.3.** *Determine which of these spaces are  $T_0$  and which are connected. Give a taxonomy in terms of explicit general constructions that accounts for all of these topologies. That is, determine appropriate “names” for all of these spaces.*

#### 4. CONNECTIVITY AND PATH CONNECTIVITY

We begin the exploration of homotopy properties of finite spaces by discussing connectivity and path connectivity. We recall the general definitions.

**Definition 4.1.** A space  $X$  is *connected* if it is not the disjoint union of two non-empty open subsets. Equivalently,  $X$  is connected if the only open and closed subsets of  $X$  are  $\emptyset$  and  $X$ . Define an equivalence relation  $\sim$  on  $X$  by  $x \sim y$  if  $x$  and  $y$  are elements of some connected subspace of  $X$ . An equivalence class under  $\sim$  is called a *component* of  $X$ .

**Lemma 4.2.** *The components of  $X$  are connected,  $X$  is the disjoint union of its components, and any connected subspace of  $X$  is contained in a component.*

*Proof.* Left as an exercise. (Or see Munkres [12, 3.3.1].) □

**Lemma 4.3.** *If  $f: X \rightarrow Y$  is a map and  $X$  is connected, then  $f(X)$  is a connected subspace of  $Y$ .*

*Proof.* Left as an exercise. (Or see Munkres [12, 3.1.5].) □

Let  $I = [0, 1]$  with its usual metric topology as a subspace of  $\mathbb{R}^n$ . It is a connected space, hence so is its image under any map. A map  $p: I \rightarrow X$  is called a *path* from  $p(0)$  to  $p(1)$  in  $X$ .

**Definition 4.4.** A space  $X$  is *path connected* if any two points can be connected by a path. Define a second equivalence relation  $\simeq$  on  $X$  by  $x \simeq y$  if there is a path connecting  $x$  to  $y$ . An equivalence class under  $\simeq$  is a *path component* of  $X$ . An equivalence class under  $\simeq$  is called a *path component* of  $X$ . Note that  $x \simeq y$  implies  $x \sim y$ , but not conversely in general.

**Lemma 4.5.** *The path components of  $X$  are path connected,  $X$  is the disjoint union of its path components, and any path connected subspace of  $X$  is contained in a path component. Each path component is contained in a component.*

*Proof.* Left as an exercise. (Or see Munkres [12, 3.3.2].) □

Now return to finite spaces  $X$ . At first sight, one might imagine that there are no continuous maps from  $I$  to a finite space, but that is far from the case. The most important feature of finite spaces is that they are surprisingly richly related to the “real” spaces that algebraic topologists care about.

**Lemma 4.6.** *Each  $U_x$  is connected. If  $X$  is connected and  $x, y \in X$ , there is a sequence of points  $z_i$ ,  $1 \leq i \leq s$ , such that  $z_1 = x$ ,  $z_s = y$  and either  $z_i \leq z_{i+1}$  or  $z_{i+1} \leq z_i$  for  $i < s$ .*

*Proof.* If  $U_x = A \amalg B$ ,  $A$  and  $B$  open, say  $x \in A$ , then  $U_x \subset A$  and therefore  $B = \emptyset$ . Fix  $x$  and consider the set  $A$  of points  $y$  that are connected to  $x$  by some sequence  $z_i$ . We see that  $A$  is open since  $z \leq z'$  implies  $U_z \subset U_{z'}$ . We see that  $A$  is closed since if  $y$  is not so connected to  $x$ , then neither is any point of  $U_y$ , so that the complement of  $A$  is open. Since  $X$  is connected, it follows that  $A = X$ . □

**Lemma 4.7.** *If  $x \leq y$ , then there is a path  $p$  connecting  $x$  and  $y$ .*

*Proof.* Define  $p(t) = x$  if  $t < 1$  and  $p(1) = y$ . We claim that  $p$  is continuous. Let  $V$  be an open set of  $X$ . If  $x \in V$  and  $y \notin V$ , then  $p^{-1}(V) = [0, 1)$ . If  $x \in V$  and  $y \in V$ , then  $p^{-1}(V) = I$ . If  $y \in V$ , then  $x \in V_y \subset V$  since  $x \leq y$ . Therefore  $f^{-1}(V) = I$ . □

**Proposition 4.8.** *A finite space is connected if and only if it is path connected.*

*Proof.* The previous two lemmas imply that  $x \sim y$  if and only if  $x \simeq y$ . □



## 5. FUNCTION SPACES AND HOMOTOPIES

**Definition 5.1.** A space is *compact* if every open cover has a finite subcover.

**Definition 5.2.** Let  $X$  and  $Y$  be spaces and consider the set  $Y^X$  of maps  $X \rightarrow Y$ . The *compact-open topology* on  $Y^X$  is the topology in which a subset is open if and only if it is a union of finite intersections of sets  $W(C, U) = \{f \mid f(C) \subset U\}$ , where  $C$  is compact in  $X$  and  $U$  is open in  $Y$ . This means that the set of all  $W(C, U)$  is a *subbasis* for the topology.

We insert a small but non-standard technical condition. Experts will recognize that it is a standard property of locally compact Hausdorff spaces, but it is also true trivially for all finite spaces.

**Definition 5.3.** A space  $X$  is *locally compact* if for each  $x \in X$ , there is a compact subspace  $C$  of  $X$  and an open subspace  $U$  such that  $x \in U \subset C$ ;  $X$  is *smallly compact* if every open subset  $V$  is locally compact. When  $X$  is finite,  $X$  is smallly compact since every subset is compact and we can take  $U = C = V$ .

Ignoring topology, for sets  $X, Y$ , and  $Z$ , functions  $f: X \times Y \rightarrow Z$  are in bijective correspondence with functions  $\hat{f}: X \rightarrow Z^Y$  via the relation  $f(x, y) = \hat{f}(x)(y)$ . Returning to topology, and so restricting  $Z^Y$  to consist of the continuous maps  $Y \rightarrow Z$ , inspection of the proof of similar statements in any standard text, for example [12, 7.5.4], shows that the following result holds.

**Proposition 5.4.** *For spaces  $X, Y$ , and  $Z$  such that  $X$  is smallly compact, a function  $f: X \times Y \rightarrow Z$  is continuous if and only if  $\hat{f}: X \rightarrow Z^Y$  is continuous.*

**Definition 5.5.** A *homotopy*  $h: f \simeq g$  is a map  $h: X \times I \rightarrow Y$  such that  $h(x, 0) = f(x)$  and  $h(x, 1) = g(x)$ . Two maps are homotopic, written  $f \simeq g$  if there is a homotopy between them.

**Lemma 5.6.** *If  $X$  is smallly compact, then homotopies  $h: X \times I \rightarrow Y$  correspond bijectively to continuous maps  $j: X \rightarrow Y^I$  via  $h \leftrightarrow j$  if  $h(x, t) = j(x)(t)$ . The homotopy classes of maps  $X \rightarrow Y$  are in canonical bijective correspondence with the path components of  $Y^X$ .*

**Definition 5.7.** If  $Y$  is finite, define the pointwise ordering of maps  $X \rightarrow Y$  by  $f \leq g$  if  $f(x) \leq g(x)$  for all  $x \in X$ .

**Proposition 5.8.** *If  $Y$  is finite and  $X$  is smallly compact, then the intersection of the open sets in  $Y^X$  that contain a map  $g$  is  $\{f \mid f \leq g\}$ .*

*Proof.* Let  $V_g$  be the cited intersection and let  $Z_g = \{f \mid f \leq g\}$ . Let  $f \in V_g$  and  $x \in X$ . Since  $g \in W(\{x\}, U_{g(x)})$ ,  $f \in W(\{x\}, U_{g(x)})$ , so  $f(x) \in U_{g(x)}$  and  $f(x) \leq g(x)$ . Since  $x$  was arbitrary,  $f$  is in  $Z_g$ . Conversely, let  $f \leq g$ . Consider any  $W(C, U)$  which contains  $g$ . Then  $g(x) \in U$  for  $x \in C$ . Since  $f(x) \leq g(x)$ ,  $f(x) \in U_{g(x)} \subset U$ . Therefore  $f \in W(C, U)$  and  $f$  is in all open subsets of  $Y^X$  that contain  $g$ .  $\square$

**Corollary 5.9.** *If  $X$  and  $Y$  are finite, then the pointwise ordering on  $Y^X$  coincides with the ordering given by the compact open topology.*

**Proposition 5.10.** *If  $Y$  is finite and  $f \leq g$ , then  $f \simeq g$  by a homotopy  $h$  such that  $h(x, t) = f(x)$  for all  $t$  if  $f(x) = g(x)$ .*

*Proof.* We have the path  $p$  connecting  $f$  to  $g$  in  $Y^X$  specified by  $p(t) = f$  if  $t < 1$  and  $p(1) = g$ . Indeed, with  $V = W(C, U)$ , the proof that  $p$  is continuous is a direct adaptation of the proof of Lemma 4.7, the key point being that if  $g \in V$ , then  $f \in V$  by Proposition 5.8.  $\square$

## 6. HOMOTOPY EQUIVALENCES

We have seen that enumeration of finite sets with reflexive and transitive relations  $\leq$  amounts to enumeration of the topologies on finite sets. We have refined this to consideration of homeomorphism classes of finite spaces. We are much more interested in the enumeration of the homotopy types of finite spaces. We will come to a still weaker and even more interesting enumeration problem later.

**Definition 6.1.** Two spaces  $X$  and  $Y$  are *homotopy equivalent* if there are maps  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  such that  $g \circ f \simeq \text{id}_X$  and  $f \circ g \simeq \text{id}_Y$ . A space is *contractible* if it is homotopy equivalent to a point.

This relationship can change the number of points. We have a first example.

**Lemma 6.2.** *If  $X$  is a space containing a point  $y$  such that the only open (or only closed) subset of  $X$  containing  $y$  is  $X$  itself, then  $X$  is contractible. In particular, the non-Hausdorff cone  $\mathbb{C}X$  is contractible for any  $X$ .*

*Proof.* This is a variation on a theme we have already seen twice. Define  $r: X \rightarrow *$  by  $r(x) = *$  for all  $x$  and define  $i: * \rightarrow X$  by  $i(*) = y$ . Clearly  $r \circ i = \text{id}$ . Define  $h: X \times I \rightarrow X$  by  $h(x, t) = x$  if  $t < 1$  and  $h(x, 1) = y$ . Then  $h$  is continuous. Indeed, let  $U$  be open in  $X$ . If  $y \in U$ , then  $U = X$  and  $h^{-1}(U) = X \times I$ , while if  $y \notin U$ , then  $h^{-1}(U) = U \times [0, 1)$ . The argument when  $X$  is the only closed subset containing  $y$  is the same. Clearly  $h$  is a homotopy  $\text{id} \simeq i \circ r$ .  $\square$

**Corollary 6.3.** *If  $X$  is finite, then  $U_x$  is contractible.*

*Proof.* The only open subset of  $U_x$  that contains  $x$  is  $U_x$  itself.  $\square$

The following result of McCord [11, Thm. 4] says that, when studying finite spaces up to homotopy type, there is no loss of generality if we restrict attention to  $T_0$ -spaces, that is, to finite posets (poset = partially ordered set).

**Theorem 6.4.** *Let  $X$  be a finite space. There is a quotient  $T_0$ -space  $X_0$  such that the quotient map  $q_X: X \rightarrow X_0$  is a homotopy equivalence. For a map  $f: X \rightarrow Y$  of finite spaces, there is a unique map  $f_0: X_0 \rightarrow Y_0$  such that  $q_Y \circ f = f_0 \circ q_X$ .*

*Proof.* Define  $x \sim y$  if  $U_x = U_y$ , or, equivalently, if  $x \leq y$  and  $y \leq x$ . Let  $X_0$  be the set of equivalence classes and let  $q = q_X$  send  $x$  to its equivalence class  $[x]$ . Give  $X_0$  the quotient topology. This means that a subset  $V$  of  $X_0$  is open if and only if  $q^{-1}(V)$  is open in  $X$ . Clearly  $q$  is continuous. The relation  $\leq$  on  $X$  induces a relation  $\leq$  on  $X_0$ . Since  $X_0$  is finite, we have the open set  $U_{q(x)}$  for  $x \in X$ . Observe that  $q^{-1}q(U_x) = U_x$  since if  $q(y) = q(z)$  where  $z \in U_x$ , then  $y \in U_y = U_z \subset U_x$ . Therefore  $q(U_x)$  is open, hence contains  $U_{q(x)}$ . Conversely,  $U_x \subset q^{-1}(U_{q(x)})$  by continuity and thus  $q(U_x) \subset U_{q(x)}$ . This proves that  $q(U_x) = U_{q(x)}$ . It follows that  $[x] \leq [y]$  if and only if  $x \leq y$ . Indeed,  $q(x) \leq q(y)$  implies  $q(x) \in U_{q(y)} = q(U_y)$ . Thus  $q(x) = q(z)$  for some  $z \in U_y$  and  $U_x = U_z \subset U_y$ , so that  $x \leq y$ . Conversely, if  $x \leq y$ , then  $U_x \subset U_y$  and therefore  $U_{q(x)} \subset U_{q(y)}$ , so that  $q(x) \leq q(y)$ . It follows that  $\leq$  is antisymmetric on  $X_0$ , so that  $X_0$  is a  $T_0$ -space.

We must prove that  $q$  is a homotopy equivalence. Let  $f: X_0 \rightarrow X$  be any function such that  $q \circ f = \text{id}$ . That is, we choose a point from each equivalence class. By what we have just proven,  $f$  preserves  $\leq$  and is therefore continuous. Let  $g = f \circ q$ . We must show that  $g$  is homotopic to the identity. We see that  $g$  is obtained by first choosing one  $x_u$  with  $U_{x_u} = U$  for each  $U$  in the minimal basis for  $X$  and then letting  $g(x) = x_u$  if  $U_x = U$ . Thus  $U_{g(x)} = U_x$  and  $g(x) \in U_x$ , which means that  $g \leq \text{id}$ . Now Proposition 5.10 gives the required homotopy  $h: \text{id} \simeq g$ . Note that  $h(g(x), t) = g(x)$  for all  $t$ .

For the last statement, a map  $f: X \rightarrow Y$  is a function that preserves  $\leq$ , and it follows that it induces a unique function  $f_0: X_0 \rightarrow Y_0$  such that  $q_Y \circ f = f_0 \circ q_X$ . Clearly  $f_0$  preserves  $\leq$  and is thus continuous.  $\square$

The space  $T_0$  is called the Kolmogorov quotient of  $T$ . The construction is classical and has many other applications. We conclude that to classify finite spaces up to homotopy equivalence, it suffices to classify  $T_0$ -spaces up to homotopy equivalence. Stong [15, §4] has given an interesting way of studying this. We change his language a bit in the following exposition.

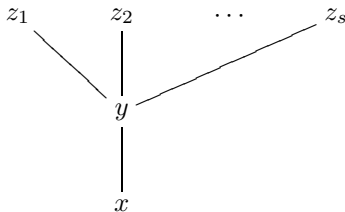
**Definition 6.5.** Let  $X$  be a finite space.

- (a) A point  $x \in X$  is *upbeat* if there is a  $y > x$  such that  $z > x$  implies  $z \geq y$ .
- (b) A point  $x \in X$  is *downbeat* if there is a  $y < x$  such that  $z < x$  implies  $z \leq y$ .

$X$  is a minimal finite space if it is a  $T_0$ -space and has no upbeat or downbeat points. If  $(X, *)$  is a based finite space, that is, a finite space with a chosen basepoint  $*$ , then  $(X, *)$  is minimal if it is a  $T_0$ -space and has no upbeat or downbeat points except possibly  $*$ . A *core* of a finite space  $X$  is a subspace  $Y$  that is a minimal finite space and a deformation retract of  $X$ . That is, if  $i: Y \rightarrow X$  is the inclusion, there is a map  $r: X \rightarrow Y$  such that  $r \circ i = \text{id}$  together with a homotopy  $h: X \times I \rightarrow X$  from  $\text{id}$  to  $i \circ r$  such that  $h(y, t) = y$  if  $y \in Y$ . A core  $(Y, *)$  of a finite based space  $(X, *)$  is defined similarly.

The importance of working with based spaces and keeping track of basepoints will emerge later. The arguments and results to follow work equally well with or without basepoints.

*Remark 6.6.* If we draw a graph of a poset by drawing a line upwards from  $x$  to  $y$  if  $x < y$ , we see that, above an upbeat point  $x$ , the graph looks like



Turning the picture upside down, we see what the graph below a downbeat point looks like.

Intuitively, identifying  $x$  and  $y$  and erasing the line between them should not change the homotopy type. We say this another way in the proof of the following result, looking at inclusions rather than quotients in accordance with our definition of a core.

**Theorem 6.7.** *Any finite (or finite based) space  $X$  has a core.*

*Proof.* With the notations of the proof of Theorem 6.4, identify  $X_0$  with its image  $g(X_0) \subset X$ . The proof of Theorem 6.4 shows that  $X_0$ , so interpreted, is a deformation retract of  $X$ . It is based if we choose  $*$  as one of the  $x_u$ . Thus we may as well assume that  $X$  is  $T_0$ . Suppose that  $X$  has an upbeat point  $x$ . We claim that the subspace  $X - \{x\}$  is a deformation retract of  $X$ . To see this define  $f: X \rightarrow X - \{x\} \subset X$  by  $f(z) = z$  if  $z \neq x$  and  $f(x) = y$ , where  $y > x$  is such that  $z > x$  implies  $z \geq y$ . Clearly  $f \geq \text{id}$ . We claim that  $f$  preserves order and is therefore continuous. Thus suppose that  $u \leq v$ . We must show that  $f(u) \leq f(v)$ . If  $u = v = x$  or if neither  $u$  nor  $v$  is  $x$ , there is nothing to prove. When  $u = x < v$ ,  $f(u) = y$  and  $f(v) = v \geq y$ . When  $u < x = v$ ,  $f(u) = u < x < y = f(v)$ . Now Proposition 5.10 gives the required deformation. A similar argument applies to show that  $X - \{x\}$  is a deformation retract of  $X$  if  $x$  is a downbeat point. Starting with  $X_0$ , define  $X_i$  from  $X_{i-1}$  by deleting one upbeat or downbeat point. After finitely many stages, there are no more upbeat or downbeat points left (except possibly  $*$ ), and we arrive at the required core.  $\square$

**Theorem 6.8.** *If  $X$  is a minimal finite space and  $f: X \rightarrow X$  is homotopic to the identity, then  $f$  is the identity.*

*Proof.* First suppose that  $f \geq \text{id}$ . For all  $x$ ,  $f(x) \geq x$ . If  $x$  is a maximal point, then  $f(x) = x$ . Let  $x$  be any point of  $X$  and suppose inductively that  $f(z) = z$  for all  $z > x$ . Then, by continuity,  $z > x$  implies  $z = f(z) \geq f(x) \geq x$ . If  $f(x) \neq x$ , this implies that  $x$  is an upbeat point. However, by hypothesis, either there are no upbeat points, or the basepoint is the only upbeat point and we require  $f(*) = *$ . In either case, we have a contradiction, so we conclude that  $f(x) = x$ . By induction,  $f(x) = x$  for all  $x$ . A similar argument shows that  $f \leq \text{id}$  implies  $f = \text{id}$ . By Lemma 4.6, it now follows that the component of the identity map in the finite space  $X^X$  consists only of the identity map. That is, any map homotopic to the identity is the identity.  $\square$

**Corollary 6.9.** *If  $f: X \rightarrow Y$  is a homotopy equivalence of minimal finite spaces, then  $f$  is a homeomorphism.*

*Proof.* If  $g: Y \rightarrow X$  is a homotopy inverse, then  $g \circ f \simeq \text{id}$  and  $f \circ g \simeq \text{id}$ . By the theorem,  $g \circ f = \text{id}$  and  $f \circ g = \text{id}$ .  $\square$

**Corollary 6.10.** *Finite spaces  $X$  and  $Y$  are homotopy equivalent if and only if they have homeomorphic cores. In particular, the core of  $X$  is unique up to homeomorphism.*

*Proof.* This is immediate since the cores of  $X$  and  $Y$  are minimal finite spaces that are homotopy equivalent to  $X$  and  $Y$ .  $\square$

*Remark 6.11.* In any homotopy class of finite spaces, there is a representative with the least possible number of points. This representative must be a minimal finite space, since its core is a homotopy equivalent subspace. The minimal representative is homeomorphic to a core of any finite space in the given homotopy class.

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