TWO GAMES

Here are two games invented by John Conway and Michael Paterson at Cambridge in 1967. I can attest that these games were still corrupting the place, along with Conway's game of life, in 1970. An account of them is given in Martin Gardner's "Mathematical Carnival" (1976). I found them in P.A. Firby and C.F. Gardiner's "Surface topology" (1982).

The game of sprouts

Fix a compact surface C and a set of n vertices on it. This is the starting point of this two person game. Players take turns moving, and we let $m(C, n)$ be the maximal number of moves to the end of the game. The winner is the last player able to make an allowable move.

An allowable move is to draw an edge, called a "sprout", connecting two vertices (which may be the same vertex) and mark an interior point of the edge as a new vertex. There are two rules.

1. The edge must not intersect a previously drawn edge, or itself, and it must not pass through any vertex.

2. No vertex may have more than three edges meeting it.

Thus each of the initial n vertices can have three edges drawn to meet it. Each new vertex that is introduced already comes with two edges meeting it, and so only one more can be added. Thus each move reduces by 1 the number of meetings that can be made between edges and vertices. Therefore the game ends after at most $3n - 1$ moves, so that $m(C, n) \leq 3n - 1$.

Claim: if $C = S$ and $n = 2$, the second player to move always wins if she plays carefully, but if $n = 3$, 4, or 5, the player to move first has the advantage. However, according to Firby and Gardiner, "Even on such a mild surface as S, the analysis becomes overwhelming when the game begins with 6 or more moves". Anybody want to try?

The game of Brussell sprouts

Fix a compact surface C and a set of n crosses $\times$ drawn on it. They must not intersect. This is the starting point of another two person game, and we again let $m(C, n)$ be the maximal number of moves to the end of the game. The four arms of the crosses are called "sprouts". An allowable move is to join two sprouts (which may be on the same cross) by an edge that does not cross another edge or itself and that does not pass through another cross, and to draw a cross on the new edge, with two of its sprouts on itself and connecting to the ends of the previously drawn sprouts that this one is joining.

This game appears to be more complex than sprouts, but it is not. Firby and Gardiner state the following results, but give no proofs. For any C,

$$5n - 2 \leq m(C, n) \leq 5n - \chi(C).$$

Of course, $\chi(S) = 2$, so these are equalities in that case. If C is orientable, the first player to move wins if and only if n is odd. If C is not orientable, the first player has the advantage unless both n and $\chi(M)$ are even, in which case the second player has the advantage.