CATEGORICAL GROTHENDIECK RINGS AND PICARD GROUPS

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Contents

1. The ring $K(R)$ and the group $\mathrm{Pic}(R)$

We assume familiarity with the Grothendieck construction which assigns to an Abelian monoid M an Abelian group $G(M)$ and a homomorphism of monoids $M \longrightarrow G(M)$ such that any homomorphism of monoids from M to an Abelian group A factors uniquely through a homomorphim $G(M) \longrightarrow A$. The construction is just like the construction of the integers from the natural numbers. We leave as an exercise that if M is a semi-ring (which satisfies all of the axioms for a ring except the existence of additive inverses), then $G(M)$ is a ring and $M \longrightarrow G(M)$ is a map of semi-rings which satisfies an analogous universal property. We then call $G(M)$ the *Grothendieck ring* of M.

Definition 1.1. For a commutative ring R, define $K(R)$, or $K_0(R)$, to be the Grothendieck ring of the semi-ring (under $\oplus$ and $\otimes$) of isomorphism classes of finitely generated projective R-modules. Define $\tilde{K}(R)$ to be the quotient of $K(R)$ by the subgroup generated by the free R-modules; for an integral domain R, $\tilde{K}(R)$ can be identified with the kernel of the rank homomorphism $K(R) \longrightarrow \mathbb{Z}$.

Definition 1.2. An R-module M is said to be invertible if there is an R-module M^{-1} such that $M \otimes_R M^{-1}$ is isomorphic to R. Define the Picard group $\mathrm{Pic}(R)$ to be the group under $\otimes$ of isomorphism classes of invertible R-modules.

Actually, $\mathrm{Pic}(R)$ can be defined more generally, without commutativity, but we shall focus for simplicity on commutative rings. The definition looks reminiscent of the definition of the class group $C(R)$ of an integral domain, and this is no accident.

Lemma 1.3. *For fractional ideals A and B in an integral domain R, multiplication defines an isomorphism $A \otimes_R B \longrightarrow AB$. Therefore A is invertible as a fractional ideal if and only if it is invertible under $\otimes$ with inverse A^{-1}.*

Proof. If $I = dA \subset R$ and $J = eB \subset R$, multiplication by de sends $I \otimes_R J$ isomorphically onto $A \otimes_R B$ and sends IJ isomorphically onto AB, so it suffices to prove this for the ideals I and J. Here the flatness of the torsion free R-module I gives that the top row is exact in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \otimes_R J & \longrightarrow & I \otimes_R R & \longrightarrow & I \otimes_R R/J & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & IJ & \longrightarrow & R & \longrightarrow & R/IJ & \longrightarrow & 0
\end{array}$$

Since the right two arrows are isomorphisms, so is the left arrow. □

Proposition 1.4. *For an integral domain R, $C(R)$ is a subgroup of $Pic(R)$. At least if R is Dedekind, these groups are equal.*

Proof. If M is invertible, then M has rank one since extension of scalars from R to K commutes with tensor products. If R is Dedekind, this means that M is isomorphic to a fractional ideal. □

Lemma 1.5. *For a fractional ideal in an integral domain R, define an R-map $\xi\colon A^{-1} \longrightarrow Hom(A,R)$ by $\xi(b)(a) = ba$. Then ξ is a monomorphism and the following diagram commutes, where ε is the evaluation homomorphism.*

$$\begin{array}{ccc}
A^{-1} \otimes A & \xrightarrow{\xi \otimes id} & Hom_R(A,R) \otimes_R A \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \varepsilon} \\
A^{-1}A & \xrightarrow{\subset} & R.
\end{array}$$

If A is invertible, then ξ is an isomorphism.

Proof. The diagram commutes since $\varepsilon(\xi(b)\otimes a) = \xi(b)(a) = ba$, and ξ is a monomorphism since $\xi(b) = 0$ implies $ba = 0$ for all a. If A is invertible, say $\sum b_i a_i = 1$, define $g_i(a) = b_i a$. Then each g_i is in the image of ξ. Any g is the linear combination $\sum g(a_i) g_i(a)$ of the g_i, hence is also in the image of ξ. Thus the monomorphism ξ is an epimorphism. □

This suggests that the inverse of an R-module A in the Picard group of a commutative ring R must be $\hom_R(A,R)$. That is true, but it is true in vastly greater generality than just ring theory. It is time to express our thoughts in appropriate categorical language.

2. Symmetric monoidal categories, $K(\mathscr{C})$, and $\mathrm{Pic}(\mathscr{C})$

The following notions are fundamental to many fields of mathematics.[1]

Definition 2.1. A *symmetric monoidal category* $\mathscr{C}$ is a category $\mathscr{C}$ with a product $\otimes$ and a unit object U such that $\otimes$ is unital, associative, and commutative up to coherent natural isomorphism; $\mathscr{C}$ is *closed* if it also has an internal Hom functor, so that we have a natural isomorphism

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B,C)).$$

[1]This section and the next are adapted from my paper "Picard groups, Grothendieck rings, and Burnside rings of categories", Advances in Mathematics 163, 1–16(2001), which is available on my web page and which gives further constructions and examples.

There is then a natural map

$$\nu\colon \mathrm{Hom}(A,B)\otimes C \longrightarrow \mathrm{Hom}(A,B\otimes C), \tag{2.2}$$

namely, modulo a use of the commutativity isomorphism, the adjoint of

$$\varepsilon\otimes \mathrm{id}\colon \mathrm{Hom}(A,B)\otimes A\otimes C \longrightarrow B\otimes C$$

where ε is the evaluation map (adjoint to the identity map of $\mathrm{Hom}(A,B)$). Similarly, there is a natural map

$$\otimes\colon \mathrm{Hom}(X,Y)\otimes \mathrm{Hom}(X',Y') \longrightarrow \mathrm{Hom}(X\otimes X', Y\otimes Y'). \tag{2.3}$$

Define the dual of A to be $DA = \mathrm{Hom}(A,U)$. Then ν specializes to

$$\nu\colon DA\otimes A \longrightarrow \mathrm{Hom}(A,A).$$

Say that A is *dualizable* or *rigid* if ν is an isomorphism. When A is dualizable, define the *coevaluation map* $\eta : U \longrightarrow A\otimes DA$ to be the composite

$$U \xrightarrow{\ \iota\ } \mathrm{Hom}(A,A) \xrightarrow{\nu^{-1}} DA\otimes A \xrightarrow{\ \gamma\ } A\otimes DA,$$

where ι is adjoint to the identity map of A and γ is the natural commutativity isomorphism given by the symmetric monoidal structure. Note that we have an evaluation map $\varepsilon : DA\otimes A \longrightarrow U$ for any object A. Say that A is *invertible* if there is an object A^{-1} such that $A^{-1}\otimes A$ is isomorphic to U.

For example, a vector space is dualizable if and only if it is finite dimensional. More generally, a corollary of the dual basis theorem can be stated as follows.

Proposition 2.4. *For a commutative ring R, an R-module is dualizable if and only if it is finitely generated and projective.*

Definition 2.5. Assume that $\mathscr{C}$ is a symmetric monoidal category that has coproducts, denoted $\oplus$. Assume too that $\mathscr{C}$ has only a set, $\mathscr{D}$ say, of isomorphism classes of dualizable objects. Then $\mathscr{D}$ is a semi-ring under $\oplus$ and $\otimes$. Define $K(\mathscr{C})$, or $K_0(\mathscr{C})$, to be the Grothendieck ring of $\mathscr{D}$ and let $\alpha\colon \mathscr{D} \longrightarrow K(\mathscr{C})$ be the canonical map of semi-rings. Define $\mathrm{Pic}(\mathscr{C})$ to be the group of isomorphism classes of invertible objects of $\mathscr{C}$.

Theorem 2.6. *Fix objects X and Y of $\mathscr{C}$. The following are equivalent.*

(i) *X is dualizable and Y is isomorphic to DX.*

(ii) *There are maps $\eta : U \longrightarrow X\otimes Y$ and $\varepsilon : Y\otimes X \longrightarrow U$ such that the composites*

$$X \cong U\otimes X \xrightarrow{\eta\otimes\mathrm{id}} X\otimes Y\otimes X \xrightarrow{\mathrm{id}\otimes\varepsilon} X\otimes U \cong X$$

and

$$Y \cong Y\otimes U \xrightarrow{\mathrm{id}\otimes\eta} Y\otimes X\otimes Y \xrightarrow{\varepsilon\otimes\mathrm{id}} U\otimes Y \cong Y$$

are identity maps.

(iii) *There is a map $\eta : U \longrightarrow X\otimes Y$ such that the composite*

$$\mathscr{C}(W\otimes X, Z) \xrightarrow{(-)\otimes Y} \mathscr{C}(W\otimes X\otimes Y, Z\otimes Y) \xrightarrow{(\mathrm{id}\otimes\eta)^*} \mathscr{C}(W, Z\otimes Y)$$

is a bijection for all objects W and Z of $\mathscr{C}$.

(iv) *There is a map* $\varepsilon : Y \otimes X \longrightarrow U$ *such that the composite*

$$\mathscr{C}(W, Z \otimes Y) \xrightarrow{(-)\otimes X} \mathscr{C}(W \otimes X, Z \otimes Y \otimes X) \xrightarrow{(\mathrm{id}\otimes\varepsilon)_*} \mathscr{C}(W \otimes X, Z)$$

is a bijection for all objects W *and* Z *of* $\mathscr{C}$.

Here the adjoint $\tilde{\varepsilon} : Y \longrightarrow DX$ of a map ε satisfying (ii) or (iv) is an isomorphism under which the given map ε corresponds to the canonical evaluation map $\varepsilon : DX \otimes X \longrightarrow U$. We also have the following general categorical observations.

Proposition 2.7. *If* X *and* Y *are dualizable, then* DX *and* $X \otimes Y$ *are dualizable and the canonical map* $\rho : X \longrightarrow DDX$ *is an isomorphism. Moreover, the map* ν *of (2.2) is an isomorphism if either* X *or* Z *is dualizable, and the map* $\otimes$ *of (2.3) is an isomorphism if both* X *and* X' *are dualizable or if both* X *and* Y *are dualizable.*

It should be clear that invertible objects are dualizable, and we have the following more detailed statement.

Corollary 2.8. *An object* X *is invertible if and only if the functor* $(-) \otimes X : \mathscr{C} \longrightarrow \mathscr{C}$ *is an equivalence of categories. If* X *is invertible, then the canonical maps* $\iota : U \longrightarrow Hom(X, X)$, $\eta : U \longrightarrow X \otimes DX$, *and* $\varepsilon : DX \otimes X \longrightarrow U$ *are isomorphisms. Conversely, if* ε *is an isomorphism or if* X *is dualizable and* η *or* ι *is an isomorphism, then* X *is invertible.*

Proof. The first statement is clear. If X is invertible, the map

$$\mathscr{C}(-, U) \longrightarrow \mathscr{C}(-, \mathrm{Hom}(X, X)) \cong \mathscr{C}(- \otimes X, X)$$

induced by ι is the isomorphism $(-) \otimes X$ given by smashing maps with X, hence ι is an isomorphism by the Yoneda lemma. When X is dualizable, the definition of η in terms of ι shows that ι is an isomorphism if and only if η is an isomorphism; in turn, it is easy to check that η is an isomorphism if and only if ε is an isomorphism. Trivially, if ε is an isomorphism, then X is invertible. □

Definition 2.9. Objects A and B of $\mathscr{D}$ are stably isomorphic if there is an object C of $\mathscr{D}$ such that $A \oplus C \cong B \oplus C$. The *cancellation property* holds if stably isomorphic objects are isomorphic.

Remark 2.10. In the category of R-modules, where $\mathscr{D}$ is the collection of finitely generated projective R-modules, we may as well insist that C be free.

Proposition 2.11. *Dualizable objects* X *and* Y *are stably isomorphic if and only if* $\alpha[X] = \alpha[Y]$, *hence* $\alpha : Iso(\mathscr{C}) \longrightarrow K(\mathscr{C})$ *is an injection if and only if* $\mathscr{C}$ *satisfies the cancellation property.*

Corollary 2.12. $\alpha[X]$ *is a unit of* $K(\mathscr{C})$ *if and only if there is a dualizable object* Y *such that* $X \otimes Y$ *is stably isomorphic to* U.

Recall that $R^\times$ denotes the group of units of a commutative ring R.

Proposition 2.13. α *restricts to a homomorphism* $\beta : Pic(\mathscr{C}) \longrightarrow K(\mathscr{C})^\times$, *and* β *is a monomorphism if stably isomorphic invertible objects are isomorphic.*

The last condition is much weaker than the general cancellation property. For example, cancellation usually does not hold in $\mathscr{M}_R$, but it does hold on invertible R-modules. We have already seen this for Dedekind rings.

Proposition 2.14. *Stably isomorphic invertible modules M and N over a commutative ring R are isomorphic.*

Proof. Adding a suitable finitely generated projective module to a given isomorphism if necessary, we have $M \oplus F \cong N \oplus F$ for some finitely generated free R-module F. Applying the determinant functor gives an isomorphism $M \cong N$. □

We have the following commutative diagram, in which the horizontal arrows are inclusions:

$$\begin{array}{ccc} \mathrm{Pic}(\mathscr{C}) & \longrightarrow & \mathrm{Iso}(\mathscr{C}) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ K(\mathscr{C})^{\times} & \longrightarrow & K(\mathscr{C}). \end{array}$$

Proposition 2.15. *Let $\mathscr{C} = \mathscr{M}_R$ for a commutative ring R. Then the diagram just displayed is a pullback in which β is a monomorphism.*

Proof. Here $K(\mathscr{C}) = K(R)$. To show that the diagram is a pullback, we must show that if P is a finitely generated projective R-module such that $\alpha[P]$ is a unit, then P is invertible. There are finitely generated projective R-modules P' and Q such that $(P \otimes P') \oplus Q \cong R \oplus Q$. This implies that the localization of $P \otimes P'$ at any prime ideal is free of rank one, which means that $P \otimes P'$ has rank one. But then $P \otimes P'$, hence also P, is invertible. Proposition 2.14 gives that β is a monomorphism. □

The proofs above don't generalize, but the results might.

Problem 2.16. Find general conditions on $\mathscr{C}$ that ensure that the diagram above is a pullback in which β is a monomorphism.

The discussion above puts the following remarkable fact about Dedekind rings into proper perspective. What is remarkable is that it relates Pic(R) not just to the units of $K(R)$ under $\otimes$, but to $\tilde{K}(R)$ with its structure as an Abelian group under $\oplus$.

Proposition 2.17. *If R is a Dedekind ring, then $\beta\colon Pic(R) \longrightarrow K(R)^{\times}$ is an isomorphism, and there is also an isomorphism*

$$Pic(R) = C(R) \cong \tilde{K}(R).$$

Proof. The units of $K(R)$ are the isomorphism classes of rank one finitely generated projective R-modules, alias $C(R)$, so β is an identification. In $\tilde{K}(R)$, where we have quotiented out the subgroup of free R-modules under $\oplus$, the elements represented by $R^n \oplus A$ and A are equal, where A is a fractional ideal, and in particular the elements represented by R and 0 are equal. Since every finitely generated projective R-module is isomorphic to one of the form $R^n \oplus A$ this gives an identification of the elements of $\tilde{K}(R)$ with the elements of $C(R)$. Since $A \oplus B$ is isomorphic to $R \oplus AB$, this is an identification of groups. □

3. The unit endomorphism ring $R(\mathscr{C})$

For the interested reader, we give some further categorical observations in our general context. They relate duality for objects in a suitable symmetric monoidal category $\mathscr{C}$ to duality for modules over an associated commutative ring. We assume that the category $\mathscr{C}$ is additive, so that $\oplus$ is its biproduct; it follows that the functor $\otimes$ is bilinear. The relevant commutative ring is the unit endomorphism ring $R(\mathscr{C})$.

Definition 3.1. Define $R(\mathscr{C})$ to be the commutative ring $\mathscr{C}(U,U)$ of endomorphisms of U, with multiplication given by the $\otimes$-product of maps or, equivalently, by composition of maps. Then $\mathscr{C}(X,Y)$ is an $R(\mathscr{C})$-module and composition is $R(\mathscr{C})$-bilinear, so that $\mathscr{C}$ is "enriched" over $\mathscr{M}_{R(\mathscr{C})}$.

Definition 3.2. Define a functor $\pi_0 : \mathscr{C} \longrightarrow \mathscr{M}_{R(\mathscr{C})}$ by letting $\pi_0(X) = \mathscr{C}(U,X)$, so that $\pi_0(U) = R(\mathscr{C})$, and observe that $\otimes$ induces a natural map

$$\phi : \pi_0(X) \otimes_{R(\mathscr{C})} \pi_0(Y) \longrightarrow \pi_0(X \otimes Y).$$

It may or may not be an isomorphism. Say that X is a *Künneth object* of $\mathscr{C}$ if X is dualizable and ϕ is an isomorphism when $Y = DX$.

The adjoint of $\pi_0(\varepsilon) \circ \phi : \pi_0(DX) \otimes_{R(\mathscr{C})} \pi_0(X) \longrightarrow \pi_0(S)$ is a natural map $\delta : \pi_0(DX) \longrightarrow D(\pi_0(X))$ of $R(\mathscr{C})$-modules. The following result relates Künneth objects of $\mathscr{C}$ to dualizable $R(\mathscr{C})$-modules.

Proposition 3.3. *Let X be a Künneth object of $\mathscr{C}$. Then $\pi_0(X)$ is a finitely generated projective $R(\mathscr{C})$-module, $\delta : \pi_0(DX) \longrightarrow D(\pi_0(X))$ is an isomorphism, and $\phi : \pi_0(X) \otimes_{R(\mathscr{C})} \pi_0(Y) \longrightarrow \pi_0(X \otimes Y)$ is an isomorphism for all objects Y.*

4. Some examples of Grothendieck rings

Example 4.1. Let G be a finite group and let $\mathscr{C}$ be the category of G-sets. It is cartesian monoidal, meaning symmetric monoidal under cartesian product, and its coproduct is given by disjoint union. It is closed, the Hom being set maps with group action given by conjugation. The Grothendieck ring of the semi-monoid of finite G-sets is called the *Burnside ring* of G, denoted $A(G)$ or $B(G)$. Are the finite sets the dualizable objects here?

Example 4.2. Let G be a finite group and let K be a field say. An action of G on a finite dimensional vector space over K is a representation of G in K, and can be thought of as a homomorphism from G into the groups of K-linear isomorphisms of K. The set $V(G,K)$ of isomorphism classes of representations of G is a semi-ring under $\oplus$ and $\otimes$. Its Grothendieck ring is the representation ring of G with respect to K, denoted $R(G;K)$. Are representations the dualizable objects in the category of (possibly infinite dimensional) vector spaces with G action? The special cases $K = \mathbb{R}$ and $K = \mathbb{C}$ are most important, and here it is natural to focus on vector spaces with inner product and actions by linear isometries. The standard notations for the resulting real and complex representation rings are $RO(G)$ and $R(G)$.

Example 4.3. The extension of scalars functor $(-) \otimes_{\mathbb{R}} \mathbb{C}$ gives rise to a ring homomorphism $RO(G) \longrightarrow R(G)$. The free K-module functor gives rise to a ring homomorphism $A(G) \longrightarrow R(G;K)$ for any K, the image of which consists of "permutation representations". Similarly, permutation representations give a ring homomorphism $A(G) \longrightarrow RO(G)$.

Example 4.4. Let X be a compact space and let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of real or complex vector bundles over X. Then $\mathrm{Vect}(X)$ is a semigroup under $\oplus$ and $\otimes$. Its Grothendieck ring is called the real or complex K-theory of X, denoted $KO(X)$, or sometimes $KO^0(X)$, and $K(X)$, or sometimes $K^0(X)$. The alternative notations suggest that these are the zeroth terms of cohomology theories, and that is in fact true, the essential ingredient in the proof being the Bott periodicity theorem.