LIMITS AND COLIMITS

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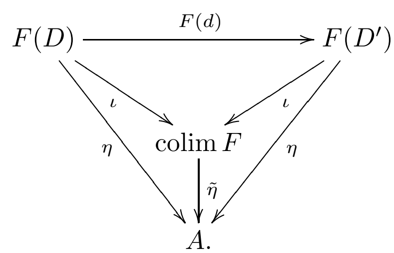
Let $\mathscr{D}$ be a small category and let $\mathscr{C}$ be any category. A $\mathscr{D}$-shaped diagram in $\mathscr{C}$ is a functor $F : \mathscr{D} \longrightarrow \mathscr{C}$. A morphism $F \longrightarrow F'$ of $\mathscr{D}$-shaped diagrams is a natural transformation, and we have the category $\mathscr{D}[\mathscr{C}]$ of $\mathscr{D}$-shaped diagrams in $\mathscr{C}$. Any object C of $\mathscr{C}$ determines the constant diagram $\underline{C}$ that sends each object of $\mathscr{D}$ to C and sends each morphism of $\mathscr{D}$ to the identity morphism of C.

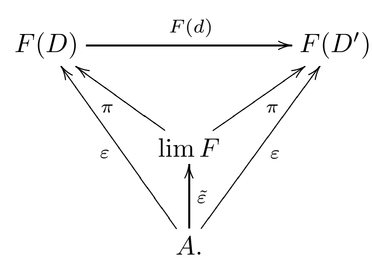
The colimit, $\operatorname{colim} F$, of a $\mathscr{D}$-shaped diagram F is an object of $\mathscr{C}$ together with a morphism of diagrams $\iota : F \longrightarrow \underline{\operatorname{colim} F}$ that is initial among all such morphisms. This means that if $\eta : F \longrightarrow \underline{A}$ is a morphism of diagrams, then there is a unique map $\tilde{\eta} : \operatorname{colim} F \longrightarrow A$ in $\mathscr{C}$ such that $\tilde{\eta} \circ \iota = \eta$. Diagrammatically, this property is expressed by the assertion that, for each map $d : D \longrightarrow D'$ in $\mathscr{D}$, we have a commutative diagram



This can be summarized by saying that there is an adjunction

$$\mathscr{D}[\mathscr{C}](F, \underline{A}) \cong \mathscr{C}(\operatorname{colim} F, A).$$

The limit of F is defined by reversing arrows: it is an object $\lim F$ of $\mathscr{C}$ together with a morphism of diagrams $\pi : \underline{\lim F} \longrightarrow F$ that is terminal among all such morphisms. This means that if $\varepsilon : \underline{A} \longrightarrow F$ is a morphism of diagrams, then there is a unique map $\tilde{\varepsilon} : A \longrightarrow \lim F$ in $\mathscr{C}$ such that $\pi \circ \tilde{\varepsilon} = \varepsilon$. Diagrammatically, this property is expressed by the assertion that, for each map $d : D \longrightarrow D'$ in $\mathscr{D}$, we have a commutative diagram



This can be summarized by saying that there is an adjunction

$$\mathscr{D}[\mathscr{C}](\underline{A}, F) \cong \mathscr{C}(A, \lim F).$$

If $\mathscr{D}$ is a set regarded as a discrete category (only identity morphisms), then colimits and limits indexed on $\mathscr{D}$ are coproducts and products indexed on the set $\mathscr{D}$. Coproducts are disjoint unions in the categories of sets or spaces, wedges (or one-point unions) in based spaces, free products in groups, and direct sums in Abelian groups. Products are Cartesian products in all of these categories; more precisely, they are Cartesian products of underlying sets, with additional structure. If $\mathscr{D}$ is the category displayed schematically as

$$e \longleftarrow d \longrightarrow f \qquad \text{or} \qquad d \rightrightarrows d',$$

where we have displayed all objects and all non-identity morphisms, then the colimits indexed on $\mathscr{D}$ are called pushouts or coequalizers, respectively. Similarly, if $\mathscr{D}$ is displayed schematically as

$$e \longrightarrow d \longleftarrow f \qquad \text{or} \qquad d \rightrightarrows d',$$

then the limits indexed on $\mathscr{D}$ are called pullbacks or equalizers, respectively.

A given category may or may not have all colimits, and it may have some but not others. A category is said to be cocomplete if it has all colimits, complete if it has all limits. The categories of sets, spaces, based spaces, groups, and Abelian groups are complete and cocomplete. If a category has coproducts and coequalizers, then it is cocomplete, and dually for completeness. The proof is an exercise.