LOW DIMENSIONAL EXAMPLES

NOTES FOR REU BY J.P. MAY

Drawing posets, and thinking about them, leads to lots of eliminations from the list of finite T_0-spaces that might not be (weakly) contractible (we are unconcerned here with the difference between contractible and weakly contractible spaces).

Lemma 0.1. *If X has a unique maximal element or a unique minimal element, then X is contractible.*

Proof. If X has a unique maximal element, then the only open set containing that point is X and therefore X is contractible. Replacing open by closed gives the conclusion when X has a unique minimal point. $\square$

Proposition 0.2. *The only finite space X with at most four points that is not a disjoint union of contractible spaces is the four point circle $\mathscr{S}S^0$*

Proof. We may assume that X is a minimal finite space that has at least two minimal and two maximal points. This implies that it has four points. Unless both minimal points are less than both maximal points, X contains an upbeat or downbeat point and is thus not minimal. $\square$

Proposition 0.3. *There is only one minimal five point space X that is not a disjoint union of contractible spaces*

Proof. A counterexample would have at least two maximal and at least two minimal points. If it has exactly two minimal and two maximal points, then it has only one intermediate point y. But then a point connected to y must be upbeat or downbeat. By antisymmetry, we can assume that there are exactly two minimal and three maximal elements. By the minimality of X, each maximal element must be connected to both minimal elements. $\square$

Remark 0.4. The space $|\mathscr{K}(X)|$ associated to this X is homeomorphic to the union of three longitudes connecting the poles of a two sphere, as we see by thinking of the two minimal points as the north and south pole and the three maximal points as points on the equator.

We leave open the full statement and proof of the following result.

Proposition 0.5. *There are only ? minimal six point spaces X that are not disjoint unions of contractible spaces. One is the six point two sphere $\mathbb{S}^2S^0$. Another is homeomorphic to the union of three longitudes connecting the poles of a two sphere. The others are ?.*

Proof. We must have at least two minimal and at least two maximal points. If we have just one intermediate point y, any point greater or less than it is upbeat or downbeat. If we have two intermediate points, they cannot be comparable without again contradicting minimality, and if they are incomparable we arrive by minimality at $\mathbb{S}^2S^0$. The only remaining cases have all points either minimal or maximal, and we can assume there are either two or three minimal points, by antisymmetry. By the minimality of X, if there are two minimal points, each maximal point must be connected to both. Thinking of the minimal points as the north and south pole, we arrive at the second possibility. Finally, suppose that X has three minimal and three maximal points. The remaining analysis is left to you, at least for now. $\square$

The height $h(X)$ of a poset X is the maximal length h of a chain $x_1 < \cdots < x_h$ in X. It is one more than the dimension $d(X)$ of the space $|\mathscr{K}(X)|$. In the analysis just given, we noticed that if X has six elements then $h(X)$ is 2 or 3. Barmak and Minian [1] observed the following related inequality.

Proposition 0.6. *Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)$ points. It has exactly $2h(X)$ points if and only if it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$.*

Proof. Let $x_1 < \cdots < x_h$ be a maximal chain in X. Since X cannot have a minimimum point, there is a y_1 which is not greater than x_1. Since no x_i is an upbeat point, $1 \leq i < h$, there must be some $y_{i+1} > x_i$ such that y_{i+1} is not greater than x_{i+1}. The points y_i are easily checked to be distinct from each other and from the x_j. Now suppose that X has exactly these $2h$ points. By the maximality of our chain, the x_i and y_j are incomparable. For $i < j$, we started with $x_i < x_j$, and we check by cases from the absence of upbeat and downbeat points that $y_i < x_j$, $y_i < y_j$, and $x_i < y_j$. Comparing with the interated suspension, we see that this implies that X is homeomorphic to $\mathbb{S}^{h-1}S^0$. $\square$

Corollary 0.7. *If $|\mathscr{K}(X)|$ is homotopy equivalent to a sphere S^n, then X has at least $2n+2$ points, and if it has exactly $2n+2$ points it is homeomorphic to $\mathbb{S}^n S^0$.*

Proof. The dimension $h(X) - 1$ of $|\mathscr{K}(X)|$ must be at least n, so $h(X) \geq n+1$. The conclusion is immediate from the previous result. $\square$

References

[1] J.A. Barmak and E.G. Minian. Minimal finite models. J. Homotopy Relat. Struct. 2(2007), 127–140.