

SOME NOTES ON NOETHERIAN LOCAL RINGS

J.P. MAY

These notes may serve as a very brief outline sketch of a few things that every algebraist and algebraic geometer should eventually learn and that I have sometimes overambitiously tried to cover in the course. The theme is to intertwine ring theoretical concepts and homological algebra, both to explain the ring theory and to develop familiarity with homological algebra. Let R be a commutative Noetherian local ring with maximal ideal m and residue field $k = R/m$. There are a number of successively weaker conditions that one can impose on R . I will list them and then explain the definitions. Each of the following properties of R implies the next.

- (1) R is regular.
- (2) R is a complete intersection ring.
- (3) R is Gorenstein.
- (4) R is Cohen Macaulay.
- (5) Any quotient ring of R is universally catenary.

Let $n = \dim(R)$ be the Krull dimension of R and let r (alias the “embedding dimension of R ”) be the dimension of the k vector space m/m^2 . A *regular sequence* $\{a_1, \dots, a_d\}$ in m is a sequence of elements such that a_i is not a zero divisor for $R/(a_1, \dots, a_{i-1})$, and then (a_1, \dots, a_d) is called a *regular ideal*. Let $d = \text{depth}(R)$ be the maximal d for which there exists such a sequence. It is the smallest d such that $\text{Ext}_R^d(k, R) \neq 0$. The *associated primes* of an ideal I are the prime ideals that are annihilators of some element of R/I . The *height* and *coheight* of a prime ideal P are the maximal lengths of chains of prime ideals contained in and containing P , respectively. Thus $ht(m) = n$, $coht(P) = \dim(R/P)$, and $ht(P) + coht(P) \leq n$. The height of an ideal I is the minimal height of an associated prime. R satisfies the *unmixedness property* if every ideal I that is generated by q elements and has height q is unmixed, in the sense that all of its associated primes have height q .

Here are definitions of the properties of R in the implications above.

- (1) It is always true that $n \leq r$. R is *regular* if equality holds. It is equivalent that $\text{Ext}_R^i(M, N) = 0$ for all $i > n$ and all R -modules M and N .
- (2) There is a certain chain complex $K(R)$, called the Koszul complex of R . Its homology groups are k -vector spaces. The dimensions e_i of the $H_i(K(R))$, $i \geq 1$, are invariants of R ($e_0 = 1$). If R is regular, then $e_i = 0$ for $i \geq 1$ and $K(R)$ is a projective resolution of k . In general $e_1 \geq r - n$. R is a *complete intersection* if equality holds. It is equivalent that the completion of R is the quotient of a regular local ring by a regular ideal.
- (3) R is *Gorenstein* if $\text{Ext}_R^i(k, R) = 0$ for $i \neq n$ and $\text{Ext}_R^n(k, R) \cong k$. It is equivalent that $\text{Ext}_R^i(k, R) = 0$ for any one $i > n$. It is also equivalent that R be Cohen-Macaulay (see below) and $\text{Ext}_R^n(k, R) \cong k$.
- (4) It is always true that $d \leq n$. R is *Cohen-Macaulay* if $d = n$. It is equivalent that R satisfies the unmixedness property.

- (5) R (Noetherian but not necessarily local) is *catenary* if all maximal chains of primes between primes $P \subset Q$ have the same length. R is *universally catenary* if every finite dimensional commutative R -algebra is catenary. A local domain R is catenary iff $ht(P) + coht(P) = n$ for all primes P .

For details and a great deal more information on these matters, see, for example: H. Matsumura. Commutative ring theory. Cambridge Univ. Press. 1986.

D. Eisenbud. Commutative algebra with a view toward algebraic geometry. Springer Verlag. 1999.

A very deep and interesting part of recent research in algebraic topology is to see to what extent one can generalize some of these notions, and various other algebraic notions, to ring-like objects in algebraic topology, called “ring spectra”, and even to use these esoteric objects to prove new results in algebra . . . but that is a digression. These kinds of rings are of quite sufficient interest in their own right.