NOTES ON PICARD GROUPOIDS

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We consider connective spectra $X$. If the only nonzero homotopy group of $X$ is the zero$^\text{th}$, then $X$ is an Eilenberg–Mac Lane spectrum $K(\pi_0(X), 0)$, and $\pi_0$ establishes an equivalence between the homotopy category of such spectra and the category of Abelian groups. The same holds for the category of spaces $X$ with a unique non-zero homotopy group for any fixed $n \geq 2$. That is, we are in the stable range for such spaces at dimension 2. We want to understand the analogous statements for connective spectra $X$ such that $\pi_i(X) = 0$ for all $i > q$ for some fixed $q$. It is an exercise to determine the relevant stable range for spaces with non-zero homotopy groups between $n$ and $n + q$ which can be read off from known results. When $q = 1$, it is $n \geq 3$.

In these notes, I’ll discuss the case $q = 1$, which is used non-trivially by Sasha Beilinson and Deepam Patel, to whom the basic result is due.

Definition 0.1. A Picard groupoid is a small symmetric monoidal groupoid $(\mathcal{P}, \otimes, I, c)$ in which every object is invertible under $\otimes$.

Standard categorical yogi (e.g. [4, 4.2], [5, VI.3.2]) shows that $\mathcal{P}$ is equivalent as a symmetric monoidal category to a permutative category, in which $\otimes$ is strictly associative and unital, with unit object $I$. Moreover, any skeleton of a permutative category is again permutative. Passage to the skeleton of an equivalent permutative category preserves the property of being a groupoid and of having invertible objects, so it takes Picard groupoids to permutative and skeletal Picard groupoids. Since the term strict permutative groupoid is sometimes used for the strictly commutative case (which is uninteresting to us), we call these “elementary Picard groupoids”. We have indicated the proof of the following result.

Proposition 0.2. Any Picard groupoid is equivalent (as such) to an elementary Picard groupoid.

We abbreviate $\pi_0$ for the group of objects of $\mathcal{P}$ under $\otimes$ and we let $\pi_1$ denote the group of automorphisms of the object $I$. As in any monoidal category (symmetric not being needed for this), $\pi_1$ is an Abelian group. Since $\mathcal{P}$ is skeletal, $A \otimes B$ and $B \otimes A$ for objects $A$ and $B$ are the same element of $\pi_0$, but we must remember and encode the commutativity isomorphism $c$.

We give a way to think about this.

Think of $\pi_0$ as a discrete category (only identity maps). It still makes sense to think of it as a group, the product and inverse being functors between discrete categories. This makes it a discrete monoidal category, and we could view it as a strict permutative groupoid. Think of the Abelian group $\pi_1$ as a category with

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a single object *. Using \( I \otimes I = I \), we see as usual that the tensor product and composite of automorphisms of \( I \) agree. Form the product category \( \pi_1 \times \pi_0 \). Give it the evident product \( \otimes \):

\[(\ast, A) \otimes (\ast, B) = (\ast, A \otimes B) \]
on objects, and

\[(f, A) \otimes (g, B) = (fg, A \otimes B) \]
on morphisms, where we agree to also write \( A \) for the identity morphism of \( \pi_0 \). The unit object is \((\ast, e)\). This product is strictly associative and unital, and it makes sense for any pair of Abelian groups \((\pi_0, \pi_1)\). We have just perversely denoted the group structure on \( \pi_0 \) as \( \otimes \).

We can ask for a commutativity isomorphism. It assigns to each pair \((A, B)\) a morphism

\[(c_{I}(A, B), A \otimes B)\]
where \( c_{I}(A, B) \in \pi_1 \). Thus \( c_{I} \) is a function

\[c_{I}: \pi_0 \times \pi_0 \rightarrow \pi_1.\]

Since \( \pi_1 \) is an Abelian group, we automatically have the “naturality condition” \( fc_{I}(A, B) = c_{I}(A, B)f \) for all \( f \in \pi_1 \). We must impose the symmetry condition

\[c_{I}(A, B)^{-1} = c_{I}(B, A)\]
and the coherence conditions

\[c_{I}(I, A) = id = c_{I}(A, I)\]
for all objects \( A \) and the triple permutation condition

\[c_{I}(A \otimes B, C) = c_{I}(A, C)c_{I}(B, C)\]
for all triples of objects \( A, B, C \). In the presence of the first of these three conditions, if we change to additive notation for the product in both \( \pi_0 \) and \( \pi_1 \), we see that the second and third conditions amount to bilinearity, so that \( c_{I} \) is really nothing but a homomorphism

\[c_{I}: \pi_0 \otimes \pi_0 \rightarrow \pi_1\]
of Abelian groups satisfying \( c_{I}(A \otimes B) = c_{I}(B \otimes A) \).

The trivial choice \( c_{I}(A, B) = id \) for all pairs \( A, B \) is available to us, giving us a strict Picard groupoid, but there are other choices with the same underlying monoidal category. Any choice of a commutativity isomorphism \( c_{I} \), meaning a function \( c_{I}: \pi_0 \times \pi_0 \rightarrow \pi_1 \) satisfying the specified algebraic properties, gives an example

\[\mathcal{P}(\pi_0, \pi_1, c_{I})\]
of an elementary Picard groupoid. Up to isomorphism these are the only examples.

**Proposition 0.3.** Any Picard groupoid \( \mathcal{P} \) is isomorphic to an elementary Picard groupoid \( \mathcal{P}(\pi_0, \pi_1, c_{I}) \).

**Proof.** We have already described how to construct \( \pi_0 \) and \( \pi_1 \) from \( \mathcal{P} \). We begin by defining inverse isomorphisms of categories

\[F: \mathcal{P} \rightarrow \pi_1 \times \pi_0\]
\[F^{-1}: \pi_1 \times \pi_0 \rightarrow \mathcal{P}.\]
We then define $F_g$ for a map $f : A \to A$, define
$$g_t = g \otimes A^{-1} : I = A \otimes A^{-1} \to A \otimes A^{-1} = I.$$  
For a map $f : I \to I$, define
$$f_A = f \otimes A : A = I \otimes A \to I \otimes A = A.$$  
We then define $F$ by $F(A) = (*, A)$ on objects and $F(g) = (g_t, A)$ on morphisms $g : A \to A$. We define $F^{-1}$ by $F^{-1}(*, A) = A$ on objects and $F^{-1}(f, A) = f_A$ on morphisms $f : I \to I$. This already gives an isomorphism of monoidal categories. What about $c$? We are given isomorphisms
$$c(A,B) : A \otimes B \to B \otimes A$$
in $\mathcal{P}$. Applying $F$ to this morphism gives a morphism
$$(c_1(A,B), A \otimes B) : A \otimes B \to A \otimes B,$$
where $c_1(A,B) \in \pi_1$. Coherence in $\mathcal{P}$ says that these elements satisfy the algebraic properties of a commutativity isomorphism that we prescribed above. Conversely, we can reconstruct the coherence isomorphism $c$ from a commutativity isomorphism $c_1 : \pi_0 \times \pi_0 \to \pi_1$. \hfill \Box

By the two results above, any Picard groupoid is equivalent to an elementary Picard groupoid $\mathcal{P}(\pi_1, \pi_0, c_1)$. Consider the classifying space functor $B$. As a space, and ignoring $c_1$, $B(\pi_1 \times \pi_0)$ is just the disjoint union over the elements $A \in \pi_0$ of copies of the group $\pi_1$, so that $B(\pi_1 \times \pi_0)$ is the disjoint union over $A$ of copies of the classifying space $B\pi_1 = K(\pi_1, 1)$. Therefore the homotopy groups of $B(\pi_1 \times \pi_0)$ are $\pi_0$ (an Abelian group by definition) and $\pi_1$. We need to understand what $c_1$ corresponds to on the classifying space level, and it is clear conceptually that it must correspond to the unique $k$-invariant of a two-stage spectrum or space (in a stable range).

Form the associated spectrum $E \mathcal{P}(\pi_1, \pi_0, c_1)$, as characterized axiomatically in [6]. The unique axiom on such a functor $E$ from permutative categories $\mathcal{C}$ to spectra is that there is a natural group completion map $\eta : B\mathcal{C} \to E_{0}\mathcal{C}$. In our case, $B\mathcal{P}(\pi_1, \pi_0, c_1)$ is grouplike ($\pi_0$ is a group), and this implies that $\eta$ is a weak homotopy equivalence. Therefore $\pi_0$ and $\pi_1$ are the only non-vanishing homotopy groups of $E\mathcal{P}(\pi_1, \pi_0, c_1)$.

Conversely, consider spectra $X$ with two non-vanishing homotopy groups $\pi_0$ and $\pi_1$. Such an $X$ fits into an exact triangle (cofiber sequence)
$$K(\pi_1, 1) \to X \to K(\pi_0, 0) \to K(\pi_1, 2)$$
of spectra; call the last map $k$: it is the unique $k$-invariant of this two-stage Postnikov system of spectra. Working stably, we can think of this as a two-stage Postnikov system
$$K(\pi_1, n + 1) \to X \to K(\pi_0, n) \to K(\pi_1, n + 2)$$
of spaces for $n \geq 3$.

We know how to describe such spaces purely algebraically, for example by work of Loday [3] and, especially relevant, [2]; see also [1, 7]. For $n = 1$, this goes all the way back to very old work of J.H.C. Whitehead, but we are interested only in the simpler case of large $n$. We need only compare definitions to see that one of the several known algebraic descriptions of such two-stage spaces is equivalent to
the description given by a commutativity isomorphism \( c_I \), as above. This amounts to describing algebraically how to calculate \( H^{n+2}(K(\pi_0, n); \pi_1) \). In fact, the triple \((\pi_0, \pi_1, C_I)\) is a very special case of a stable crossed module \([2, 3.1]\) or stable quadratic module \([1, IV.C.1]\), and one way to proceed is to check that every stable quadratic module is suitably equivalent to one of this form. This exercise is carried out, not very explicitly, in \([2, 3.10]\).

An alternative, purely stable, verification that \( E \) specifies an equivalence from the category of Picard groupoids to the homotopy category of spectra whose only non-vanishing homotopy groups are \( \pi_0 \) and \( \pi_1 \) can be obtained by using the Segal machine to construct \( E \) and then reconstructing from such a two-stage spectrum a very special \( \Gamma \)-space that is equivalent to one that comes from a Picard groupoid. This approach is worked out in Deepam Patel’s thesis \([8]\).

What happens for larger \( q \) might make a good thesis topic. We know the connection between symmetric monoidal categories and spectra. But making \( \pi_0 \) an Abelian group necessarily kills all \( \pi_i \) with \( i > 2 \), so it doesn’t seem that an answer in terms of symmetric monoidal categories with special properties makes sense. Rather, we should think in terms of higher categorical versions of grouplike symmetric monoidal categories, starting with the appropriate formulation of the notion of a Picard 2-groupoid.

Interesting!

References