QUASI-FROBENIUS RINGS

I'll sketch a solution to the following interesting problem, where R is commutative and Noetherian. (A full treatment would work with non-commutative rings.) I once assigned this problem. I misremembered at the time and imagined that it was easier than it is. I think that it *should* be easier, and I invite somebody to come up with a simpler proof, probably making use of homological algebra.

(1) R is said to be quasi-Frobenius if R is injective as an R-module. Prove that the following are equivalent.
(a) R is quasi-Frobenius.
(b) Every projective R-module is injective.
(c) Every injective R-module is projective.

Trivially, (b) implies (a), and (a) implies (b) since direct sums of injectives in a Noetherian ring are injective, as are direct summands of injectives. The equivalence of these conditions with (c) is a non-trivial result of Carl Faith and Elbert Walker, *Direct–sum representations of injective modules*, J. Algebra 5(1967), 203–221.

(a) $\Longrightarrow$ (c): R is Artinian (you can prove this) and any module M is a direct sum of indecomposable modules, injective if M is injective. Thus to show that injectives are projective, it suffices to show that an indecomposable injective module M is projective. Any module N has an injective hull (or envelope) $\hat{N}$. This is an injective module containing N such that if the intersection of N and a submodule N' is zero, then N' is zero; $\hat{N}$ embeds in any other injective module containing N. If C is the submodule of M generated by a nonzero element, then $\hat{C} = M$. A less hard characterization of R being quasi–Frobenius is that every ideal I is the annihilator of another ideal J. (The annilator of an annihilator is the ideal you started with). A cyclic module is isomorphic to R/I for some I, and if $I = ann(J)$, where J is generated by a subset $\{x_1, \cdots, x_n\}$ of R, then $r \mapsto (rx_1, \ldots, rx_n)$ induces an embedding $R/I \longrightarrow R^n$ since I is the annhilator of $\{x_i\}$. Thus our C embeds in some R^n. Since R is injective, so is the free R-module R^n, and the embedding extends to an embedding of $\hat{C} = M$. Since R^n is injective, the inclusion splits and M is a direct summand of R^n. Therefore M is projective.

(c) $\Longrightarrow$ (a): Since any module embeds in an injective module and injective modules are projective, any module embeds in a free module. It is a fact — not too difficult to prove — that if every module embeds in a direct sum of cyclic modules, then R is Artinian. A module C is said to be a "cogenerator" if every module embeds in a Cartesian product of copies of C. The main theorem of Faith and Walker is that if a ring admits a finitely generated cogenerator C, and if $R/rad(R)$ is semi-simple, then C and R are injective. In our case, R itself is a cogenerator (direct sums embed in direct products) and $R/rad(R)$ is semi-simple (since R is Artinian). Therefore R is injective, which means that R is quasi-Frobenius.