COHEN-MACAULEY AND REGULAR LOCAL RINGS

Contents

While everything I'll talk about can be found in one or another of Eisenbud, Matsumura, or Weibel, I'll bang out selected notes to avoid being encyclopedic. With a few exceptions, rings are understood to be commutative and Noetherian. We shall be explicit when we assume R to be local, and we then write $\mathfrak{m}$ for the maximal ideal and k for the residue field $R/\mathfrak{m}$.

1. Flat, projective, and free modules

Since I recently mentioned Serre's problem, I'll give a brief sketch proof of the following weak analogue for local rings.

Theorem 1.1 (Matsumura, 7.10)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Then all finitely generated flat R-modules are free.*

Sketch Proof. A set of generators for a finitely generated R-module M is minimal if and only if its image in $M/\mathfrak{m}M$ is a k-basis. If M is flat, then such a set is linearly independent over R and forms an R-basis for M. □

The following result is the only one in which we drop the Noetherian hypothesis.

Theorem 1.2 (Weibel, 3.2.7)**.** *A finitely presented flat R-module is projective.*

Corollary 1.3. *If R is Noetherian, a finitely generated flat R-module is projective.*

2. Review of regular local rings

I'm assuming that the material of this section was covered in the Winter quarter, but I'll try to supply proofs of anything that was not covered.

Let R be a local ring of Krull dimension n with maximal ideal $\mathfrak{m}$ and residue field k. Write $\bar{a}$ for the image in k of an element a in R. Recall that there is at least one ideal I with radical $\mathfrak{m}$, so that $\mathfrak{m}$ is minimal over I, that can be generated by a set $\{a_1, \cdots, a_n\}$ of n elements, which is then called a system of parameters. Recall that $\dim R/(a_1, \cdots, a_i) = n - i$, and we can choose a system of parameters such that $\mathrm{ht}(a_1, \cdots, a_i) = i$ for $1 \leq i \leq n$.

A minimal set of generators of $\mathfrak{m}$ has q generators, where $q = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Then $q \geq n$, and equality holds if and only if R is regular. In that case, an n element generating set for $\mathfrak{m}$ is called a regular system of parameters.

Theorem 2.1. *Let R be a regular local ring of dimension n and let $\{a_1, \cdots, a_i\}$ be a set of elements of $\mathfrak{m}$. Then the following are equivalent.*

(i) $\{a_1, \cdots, a_i\}$ *is a subset of a regular system of parameters.*
(ii) $\{\bar{a}_1, \cdots, \bar{a}_i\}$ *is a subset of a k-basis for $\mathfrak{m}/\mathfrak{m}^2$.*
(iii) $R/(a_1, \cdots, a_i)$ *is a regular local ring of dimension $n - i$.*

Theorem 2.2. *R is regular of dimension n if and only if*

$$gr_{\mathfrak{m}}(R) \cong k[x_1, \cdots, x_n].$$

Corollary 2.3. *A regular local ring is an integral domain.*

Theorem 2.4. *A completion of a regular local ring is regular.*

3. Cohen-Macauley rings

We give the following definition in general, but we are interested primarily in the local case, with $I = \mathfrak{m}$.

Definition 3.1. Let M be an R-module and $\{a_1, \cdots, a_q\}$ a set of elements in an ideal I such that $M/IM \neq 0$. The set $\{a_1, \cdots, a_q\}$ is said to be a regular (or normal) M-sequence in I if a_i is not a zero divisor for $M/(a_1, \cdots, a_{i-1})M$ for $1 \leq i \leq q$. The I-depth of M is the maximal length of an M-sequence in I.

Definition 3.2. Let R be local and abbreviate $\mathfrak{m}$-depth to depth. A finitely generated R-module M is a Cohen-Macauley module if $M = 0$ or the depth of M is equal to the dimension of M, which by definition is the dimension of the ring $R/\mathrm{ann}(M)$. R is a Cohen-Macauley ring (= CM ring) if $\mathrm{depth}(R) = \dim(R)$.

Definition 3.3. A ring R is CM if $R_{\mathfrak{m}}$ is a CM ring for all maximal ideals $\mathfrak{m}$.

Let R be local in the rest of this section. We recall the following consequence of Krull's Hauptidealsatz.

Lemma 3.4. *If $a \in \mathfrak{m}$ is not a zero divisor, then*

$$dim\, R = dim\, R/(a) + 1$$

Proposition 3.5. *$depth(R) \leq dim\,(R)$.*

Proof. If $\{a_1, \cdots, a_q\}$ is a regular R-sequence and $R_i = R/(a_1, \cdots, a_i)$, the lemma gives that $\dim R_i = \dim R_{i+1} + 1$. Therefore $\dim R = \dim R_q + q \geq q$. □

Theorem 3.6. *If R is a regular local ring, then any regular system of parameters is a regular R-sequence and R is therefore a CM ring.*

Proof. If $\{a_1, \cdots, a_n\}$ is a regular system of parameters, then $R/(a_1, \cdots, a_i)$ is a regular local ring and thus an integral domain. Therefore a_{i+1} is not a zero divisor for $R/(a_1, \cdots, a_i)$ and $\mathrm{depth}(R) \geq \dim(R)$. □

The converse fails in general.

4. The Koszul complex

If $\{a_1, \cdots, a_q\}$ is a regular M-sequence in I and $M_i = M/(a_1, \cdots, a_i)M$, then we have short exact sequences

$$0 \longrightarrow M_{i-1} \xrightarrow{a_i} M_{i-1} \longrightarrow M_i \longrightarrow 0.$$

These give rise to long exact sequences of Tor and Ext groups and suggest the relevance of homological methods. A basic starting point is the elementary notion of a Koszul complex. We define it and give its basic properties here, and we show that it detects depth in the next section.

For $a \in R$, define $K(a)$ to be the complex

$$0 \longrightarrow R \xrightarrow{a} R \longrightarrow 0,$$

where the first and second copies of R are in degrees 1 and 0. The alternative cohomological grading is generally used in algebraic geometry. For a set of elements $\mathbf{a} = \{a_1, \cdots, a_n\}$ of R, define

$$K(\mathbf{a}) = K(a_1) \otimes \cdots \otimes K(a_n).$$

We may think of $K(a)$ as the exterior algebra over R on one generator e of degree 1 and $K(\mathbf{a})$ as the exterior algebra on n generators e_i, $1 \leq i \leq n$, all of degree 1. Then, as a graded R-module, $K(\mathbf{a})$ is free on the basis

$$\{e_{i_1} \cdots e_{i_q} | 1 \leq i_1 < \cdots < i_q \leq n\},$$

where the empty product is interpreted as the basis element $1 = e_0 \in K_0(\mathbf{a})$. The differential is given explicitly on basis elements by

$$d(e_{i_1} \cdots e_{i_q}) = \sum_{p=1}^{q} (-1)^{p-1} a_{i_p} e_{i_1} \cdots e_{i_{p-1}} e_{i_{p+1}} \cdots e_{i_q}.$$

Viewed as the exterior algebra, $K(\mathbf{a})$ is a DGA (differential graded R-algebra), meaning that $d(xy) = d(x)y + (-1)^{\deg x} x d(y)$ for all $x, y \in K(\mathbf{a})$.

For an R-module M, define $K(\mathbf{a}; M) = K(\mathbf{a}) \otimes_R M$ and define

$$H_*(\mathbf{a}; M) = H_*(K(\mathbf{a}; M)).$$

Let $I = (a_1, \cdots, a_n)$. Visibly from the definition,

$$H_0(\mathbf{a}; M) = M/IM \quad \text{and} \quad H_n(\mathbf{a}; M) = \{m | Im = 0\}.$$

For a chain complex C, define the suspension or shift $\Sigma C = C[1]$ to be the chain complex specified by $C_{p+1}[1] = C_p$, with the same differentials as in C. Consider the chain complex $C \otimes K(a)$. We have the short exact sequence

$$0 \longrightarrow C \xrightarrow{i} C \otimes K(a) \xrightarrow{p} C[1] \longrightarrow 0,$$

where $i(c) = c \otimes e_0$, $p(c \otimes e_0) = 0$, and $p(c \otimes e_1) = c$, viewed as an element of $C[1]$. This gives rise to a long exact sequence

$$\cdots \longrightarrow H_q(C) \xrightarrow{i_*} H_q(C \otimes K(a)) \xrightarrow{p_*} H_{q-1}(C) \xrightarrow{\partial} H_{q-1}(C) \longrightarrow \cdots$$

We have used that $H_q(C[1]) = H_{q-1}(C)$. Note for inductive arguments that we can take C to be a Koszul complex. The identification of ∂ is elementary but crucial.

Lemma 4.1. *The connecting homomorphism ∂ is multiplication by $(-1)^{q-1}a$, and a annihilates $H_*(C \otimes K(a))$.*

Proof. Let $z \in C_{q-1} = C_q[1]$ be a cycle. Then $z = p(z \otimes e_1)$ and

$$d(z \otimes e_1) = (-1)^{q-1}(z \otimes ae_0) = (-1)^{q-1}ai(z).$$

This gives the first statement. For the second, let $z = x \otimes e_1 + y \otimes e_0$ be a cycle in $(C \otimes K(a))_q$. We then have

$$d(z) = d(x) \otimes e_1 + (-1)^{q-1}ax \otimes e_0 + d(y) \otimes e_0 = 0.$$

This implies that $d(x) = 0$ and $d(y) = (-1)^q ax$ and therefore $az = (-1)^q d(y \otimes e_1)$, so that $a[z] = 0$. □

Corollary 4.2. *The ideal $I = (a_1, \cdots, a_n)$ annihilates $H_*(\mathbf{a}; M)$.*

Proof. Proceed inductively, taking $a = a_n$ and $C = K(a_1, \cdots, a_{n-1}) \otimes_R M$ in the lemma. □

Theorem 4.3. *Let M be a finitely generated R-module, $M \neq 0$. Let $\mathbf{a} = \{a_1, \cdots, a_n\}$, and let $I = (a_1, \cdots, a_n)$, $I \neq R$.*

(i) If $\mathbf{a}$ is a regular M-sequence, then

$$H_0(\mathbf{a}; M) = M/IM \quad \text{and} \quad H_p(\mathbf{a}; M) = 0 \quad \text{for} \quad p > 0.$$

(ii) Conversely, if R is local and $H_1(\mathbf{a}; M) = 0$, then $\mathbf{a}$ is a regular M-sequence and therefore $H_p(\mathbf{a}; M) = 0$ for $p > 0$.

Proof. Let $\mathbf{a}' = \{a_1, \cdots, a_{n-1}\}$ and take $C = K(\mathbf{a}') \otimes_R M$ and $a = a_n$ above. We obtain the following end part of a long exact sequence.

$$H_1(\mathbf{a}'; M) \longrightarrow H_1(\mathbf{a}; M) \longrightarrow H_0(\mathbf{a}'; M) \xrightarrow{a_n} H_0(\mathbf{a}'; M) \longrightarrow H_0(\mathbf{a}; M) \longrightarrow 0.$$

We proceed by induction on n, the case $n = 1$ being clear in both parts. For (i), earlier parts of the long exact sequence and the induction hypothesis give $H_q(\mathbf{a}; M) = 0$ for $q > 1$. Since $H_0(\mathbf{a}'; M) = M/I'M$, where $I' = (a_1, \cdots, a_{n-1})$, multiplication by a_n in the sequence above is a monomorphism and $H_1(\mathbf{a}; M) = 0$ since $H_1(\mathbf{a}'; M) = 0$. For (ii), since $H_1(\mathbf{a}; M) = 0$ the beginning part of the exact sequence above gives that $a_n \colon H_1(\mathbf{a}'; M) \longrightarrow H_1(\mathbf{a}'; M)$ is an epimorphism and therefore $H_1(\mathbf{a}'; M) = 0$ by Nakayama's lemma. By the induction hypothesis, $\mathbf{a}'$ is a regular M-sequence. Again by the exact sequence above $a_n \colon H_0(\mathbf{a}'; M) \longrightarrow H_0(\mathbf{a}'; M)$ is a monomorphism, which means that $\mathbf{a}$ is a regular M-sequence. □

Up to isomorphism, $K(\mathbf{a})$ is independent of how the sequence $\mathbf{a}$ is ordered, hence Theorem 4.3(ii) has the following consequence. By an exercise, the conclusion is false if R is not local.

Corollary 4.4. *If R is local, any permutation of a regular R-sequence is regular.*

Theorem 4.3 has the following interpretation in terms of Tor.

Corollary 4.5. *If* $\mathbf{a}$ *is a regular* R*-sequence, then* $K(\mathbf{a})$ *is a free* R*-resolution of* R/I*,* $I = (a_1, \cdots, a_n)$*, and therefore*

$$H_*(K(\mathbf{a}); M) = \mathrm{Tor}_*^R(R/I, M).$$

In particular, if R *is local and* $I = \mathfrak{m}$*, then*

$$H_*(K(\mathbf{a}); M) = \mathrm{Tor}_*^R(k, M).$$

5. The Koszul resolution detects depth

There is an alternative way of reaching much the same conclusion as Theorem 4.3 but which gives precise information on the depth and shows that any two maximal regular M-sequences in I have the same length.

Theorem 5.1. *Let* $I = (b_1, \cdots, b_n)$ *be a proper ideal of* R *and let* M *be a finitely generated* R*-module such that* $M/IM \neq 0$*. Let* s *be the largest integer for which* $H_s(\mathbf{b}; M)$ *is non-zero. Then any maximal regular* M*-sequence in* I *has length* $n-s$*, so that*

$$depth_I(M) = n - s.$$

Proof. Note that n is quite arbitrary, since we could throw in more generators; the result would be to increase both n and s without changing $n - s$. Let $\{a_1, \cdots, a_r\}$ be a maximal regular M-sequence in I. It suffices to prove that $H_p(\mathbf{b}; M) = 0$ for $p > n - r$ and $H_{n-r}(\mathbf{b}; M) \neq 0$. We prove this by induction on r. If $r = 0$, then all elements of I are zero-divisors for M. The set of zero divisors for M is the union of its associated prime ideals P, hence $I \subset P$ for some associated prime P of M. By the definition of an associated prime, P is the annihilator of some non-zero element $x \in M$, and thus $Ix = 0$. Since $H_n(\mathbf{b}; M) = \{m | Im = 0\}$, $H_n(\mathbf{b}; M) \neq 0$, and the conclusion holds in this case. Now let $r > 0$ and set $M_1 = M/a_1M$, so that we have the short exact sequence

$$0 \longrightarrow M \xrightarrow{a_1} M \longrightarrow M_1 \longrightarrow 0.$$

Tensoring with $K(\mathbf{b})$, we obtain a short exact sequence of chain complexes and thus a long exact sequence of homology groups. Since $IH_*(\mathbf{b}; M) = 0$, a_1 annhilates $H_*(\mathbf{b}; M)$ and therefore the long exact sequence breaks into short exact sequences

$$0 \longrightarrow H_p(\mathbf{b}; M) \longrightarrow H_p(\mathbf{b}; M_1) \longrightarrow H_{p-1}(\mathbf{b}; M) \longrightarrow 0.$$

Since $\{a_2, \cdots, a_r\}$ is a maximal M_1-sequence in I, the induction hypothesis gives $H_p(\mathbf{b}; M_1) = 0$ for $p > n+1-r$ and $H_{n+1-r}(\mathbf{b}; M_1) \neq 0$. The conclusion follows. □

Specializing this result, we obtain a more precise version of Theorem 4.3(i).

Corollary 5.2. *Let* $I = (a_1, \cdots, a_n)$*,* $I \neq R$*. Of the following statements, (i) implies (ii), (ii) and (iii) are equivalent, and, if* R *is local, (iii) implies (i).*

(i) $\{a_1, \cdots, a_n\}$ *is a regular* M*-sequence.*
(ii) $depth_I(M) = n$*.*
(iii) $H_p(\mathbf{a}; M) = 0$ *if* $p > 0$*.*

Proof. Theorem 5.1 implies that $n \geq \mathrm{depth}_I(M)$, hence (i) implies equality. With $b_i = a_i$, (ii) and (iii) are each equivalent to the statement that $s = 0$ in Theorem 5.1. When R is local, the implication (ii) implies (i) is part of Theorem 4.3(ii). □

Remark 5.3. The implication (iii) implies (i) is said to hold in general in Matsumura (Corollary, p. 131), but that would imply that any permutation of the sequence $\{a_1, \cdots, a_n\}$ is regular, which an exercise shows to be false.

6. The detection of depth by use of Ext

Let $(R, \mathfrak{m}, k)$ be a local ring (for simplicity only) and let M be a finitely generated R-module. We give another way to detect depth, starting with the following observation.

Lemma 6.1. *The depth of M is zero if and only if* $\mathrm{Hom}_R(k, M) \neq 0$.

Proof. The depth of M is zero if and only if every $a \in \mathfrak{m}$ is a zero divisor for M. Since the set of zero divisors for M is the union of its associated prime ideals, this holds if and only if $\mathfrak{m}$ is contained in that union, hence is contained in and therefore equal to an associated prime of M. This means that there is an $x \neq 0$ in M such that $\mathfrak{m}$ is the annihilator of x. A non-zero map of R-modules $f\colon k \longrightarrow M$ is a choice of a non-zero element $x = f(1) \in M$ such that $\mathfrak{m}x = 0$, and the conclusion follows. □

Theorem 6.2. *The depth of M is the smallest number d such that* $\mathrm{Ext}^d_R(k, M)$ *is non-zero.*

Proof. Let $\{a_1, \cdots, a_n\}$ be a maximal regular M-sequence. If $n = 0$, the conclusion holds by the lemma. Let $n > 0$ and proceed by induction on n. Let $M_1 = M/a_1M$ and note that the depth of M_1 is $n-1$, so that $\mathrm{Ext}^j_R(k, M_1) = 0$ for $j < n-1$ and $\mathrm{Ext}^{n-1}_R(k, M_1) \neq 0$. The short exact sequence

$$0 \longrightarrow M \xrightarrow{a_1} M \longrightarrow M_1 \longrightarrow 0$$

gives a long exact sequence of Ext groups, in which multiplication by a_1 is zero since $a_1k = 0$. Thus we have short exact sequences

$$0 \longrightarrow \mathrm{Ext}^j_R(k, M) \longrightarrow \mathrm{Ext}^j_R(k, M_1) \longrightarrow \mathrm{Ext}^{j+1}_R(k, M) \longrightarrow 0.$$

Taking $j < n-1$, these give that $\mathrm{Ext}^j_R(k, M) = 0$ for $j \leq n-1$, and, taking $j = n-1$, we then see that $\mathrm{Ext}^n_R(k, M) \cong \mathrm{Ext}^{n-1}_R(k, M_1) \neq 0$. □

7. Global dimension

The projective dimension of an R-module M, denoted $pd_R(M)$, is the smallest q such that M admits a projective resolution P of length q, meaning that $P_r = 0$ for $r > q$; $pd_R(M) = \infty$ if there is no such q. The injective dimension of M, $id_R(M)$ is defined similarly. The global dimension of R, $\mathrm{gldim}(R)$, is the smallest q such that $pd_R(M) \leq q$ for all R-modules M. It is a kind of measure of the complexity of R. Wedderburn theory concerns (non-commutative) rings of global dimension zero. Dedekind rings are examples of rings of global dimension one. The following characterization is the starting point for the understanding of global dimension.

Proposition 7.1. *The following statements are equivalent.*

(i) $gldim(R) \leq n$*; that is,* $pd_R(M) \leq n$ *for all* M*.*
(ii) $pd_R(M) \leq n$ *for all finitely generated* M*.*
(iii) $pd_R(R/I) \leq n$ *for all ideals* I*.*
(iv) $id_R(N) \leq n$ *for all* N*.*

(v) $\mathrm{Ext}^q_R(M,N)=0$ *for all* $q>n$ *and all* M *and* N.

If R *is Noetherian, the following statement is also equivalent to these.*

(vi) $\mathrm{Tor}^R_q(M,N)=0$ *for all* $q>n$ *and all finitely generated* M *and* N.

Proof. Trivially, (i) implies (ii), (ii) imples (iii), and either (i) or (iv) implies (v). To see that (iii) implies (iv), let

$$0\longrightarrow N\longrightarrow I_0\longrightarrow\cdots\longrightarrow I_{n-1}\longrightarrow Q\longrightarrow 0$$

be an exact sequence, where the I_j are injective. We must show that Q is injective. Using cokernels to break the displayed long exact sequence into a sequence of short exact sequences and using their associated long exact sequences of Ext groups, we see that

$$\mathrm{Ext}^1_R(R/I,Q)\cong\mathrm{Ext}^{n+1}_R(R/I,N)=0.$$

This implies that $\mathrm{Hom}(R,Q)\longrightarrow\mathrm{Hom}(I,Q)$ is an epimorphism for all I, and an exercise shows that this is equivalent to the injectivity of Q. Similarly, to show that (v) implies (i), let

$$0\longrightarrow T\longrightarrow P_{n-1}\longrightarrow\cdots\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

be an exact sequence, where the P_j are projective. We must show that T is projective. Breaking this long exact sequence into short exact sequences and using long exact sequences of Ext groups, we see that

$$\mathrm{Ext}^1_R(T,N)\cong\mathrm{Ext}^{n+1}_R(M,N)=0$$

for all N. This implies that the functor $\mathrm{Hom}_R(T,-)$ preserves epimorphims, which is equivalent to the projectivity of T. In general, (i) implies (vi). When R is Noetherian, (vi) implies (ii) by a proof similar to that of (v) implies (i), using Corollary 1.3. The point is that the argument applies to detect the length of flat rather than projective resolutions, but these can be used interchangeably for finitely generated modules over Noetherian rings. □

8. Minimal resolutions and global dimension

Let $(R,\mathfrak{m},k)$ be a local ring in this section. Looking at the third characterization of global dimension, it seems plausible that $\mathrm{gldim}(R)=pd_R(k)$. This means that $pd_R(k)\geq pd_R(R/I)$ for all ideals I. We shall prove this by the use of minimal resolutions. Note that since finitely generated R-modules detect global dimension and finitely generated flat R-modules are projective and indeed free, we may as well work with Tor, which in principle measures flat dimension, rather than Ext.

Definition 8.1. A free resolution

$$\cdots\longrightarrow F_n\longrightarrow F_{n-1}\longrightarrow\cdots\longrightarrow F_0\longrightarrow M\longrightarrow 0$$

is said to be minimal if the image of $d_n\colon F_n\longrightarrow F_{n-1}$ is contained in $\mathfrak{m}F_{n-1}$ for each n. This implies that the induced differential on $F_*\otimes_R k$ is zero and therefore

$$\mathrm{Tor}^R_*(M,k)=F_*\otimes_R k=F_*/\mathfrak{m}F_*.$$

We usually assume that M is finitely generated, and then $F_n=0$ if and only if $\mathrm{Tor}^R_n(M,k)=F_n/\mathfrak{m}F_n=0$.

Proposition 8.2. *Any finitely generated* R*-module* M *admits a minimal resolution, and any two minimal resolutions are isomorphic.*

Proof. Choose a minimal set of generators for M and let F_0 be free with one basis element for each generator. Construct $\varepsilon\colon F_0 \longrightarrow M$ by mapping basis elements to generators. Then $\mathrm{Ker}(\varepsilon) \subset \mathfrak{m}F_0$ because the minimality means that ε induces an isomorphism $F_0 \otimes_R k \longrightarrow M \otimes_R k$. To construct d_1, choose a minimal set of generators for $\mathrm{Ker}(\varepsilon)$, let F_1 be free with one basis element for each generator, and let d_1 take basis elements to generators. Clearly we can continue inductively. If we have two minimal resolutions F and F', we can construct a map $F \longrightarrow F'$ of resolutions of M. It is an isomorphism since it induces an isomorphism $F \otimes_R k \longrightarrow F' \otimes_R k$, both of these being $\mathrm{Tor}^R_*(M, k)$. □

Proposition 8.3. *Let R be local. Then $gldim(R) = pd_R(k)$.*

Proof. Let $pd_R(k) = n$. It suffices to show that $pd_R(M) \leq n$ for any finitely generated R-module M. For a minimal resolution F of M, $F_{n+1}/\mathfrak{m}F_{n+1} = \mathrm{Tor}^R_{n+1}(M, k) = 0$ and therefore $F_{n+1} = 0$. □

Use of minimal resolutions gives the following non-vanishing result. This is only the tip of the iceberg. There are many related results that I may mention at the end of these notes.

Proposition 8.4. *Let M be a non-zero finitely generated R-module and let $pd_R(M) = r < \infty$. Then $\mathrm{Ext}^r_R(M, N) \neq 0$ for all finitely generated R-modules N.*

Proof. Let F be a minimal resolution of M, so that $F_{r+1} = 0$ and $F_r \neq 0$. Then $\mathrm{Ext}^r_R(M, N)$ is computed by the exact sequence

$$\mathrm{Hom}(F_{r-1}, N) \xrightarrow{d_r^*} \mathrm{Hom}(F_r, N) \longrightarrow \mathrm{Ext}^r_N(M, N) \longrightarrow 0.$$

The two Hom R-modules are finite direct sums of copies of N, and the map $d_R^* = \mathrm{Hom}_R(d_r, \mathrm{id}_N)$ is given by a matrix with coefficients in $\mathfrak{m}$ since $d_r(F_r) \subset \mathfrak{m}F_{r-1}$. Therefore the image of d_r^* is contained in $\mathfrak{m}\,\mathrm{Hom}(F_r, N)$, and d_r^* cannot be an epimorphism since $\mathrm{Hom}(F_r, N)$ is non-zero. □

9. Serre's characterization of regular local rings

Again, let $(R, \mathfrak{m}, k)$ be local. We saw in Theorem 3.6 that regular systems of parameters are regular R-sequences. By Corollary 4.5, this implies that the Koszul complex of a regular system of parameters is a free resolution of k. In fact, it is a minimal resolution, and we have the following conclusion.

Proposition 9.1. *If R is regular and $dim(R) = n$, then $\mathrm{Tor}^R_*(k, k)$ is the exterior algebra over k on n generators of degree 1 and thus $pd_r(k) = gldim(R) = dim(R)$.*

This proves one direction of the following beautiful theorem.

Theorem 9.2 (Serre)**.** *R is a regular local ring if and only if $gldim(R)$ is finite, and then $gldim(R) = dim(R)$.*

Proof. Assume that $\mathrm{gldim}(R) = n < \infty$ and let $\mathbf{a} = \{a_1, \cdots, a_q\}$ be a minimal set of generators for $\mathfrak{m}$. By the Hauptidealsatz, $\dim(R) \leq q$. In this section, we shall prove that $q \leq n$ and in the next section we shall prove that $\mathrm{depth}(R) = n$. Since $\mathrm{depth}(R) \leq \dim(R)$, this will give

$$n = \mathrm{depth}(R) \leq \dim(R) \leq q \leq n.$$ □

Proof that $q \le n$. Since $K(\mathbf{a})$ is a free complex, there is a map ϕ from it to any minimal resolution F of k, as displayed in the diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & K_q(\mathbf{a}) & \longrightarrow & K_{q-1}(\mathbf{a}) & \longrightarrow & \cdots & \longrightarrow & K_0(\mathbf{a}) & \longrightarrow & k & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi_q} & & \Big\downarrow{\scriptstyle \phi_{q-1}} & & & & \Big\downarrow{\scriptstyle \phi_0} & & \Big\| & & \\ \cdots & \longrightarrow & F_q & \longrightarrow & F_{q-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & k & \longrightarrow & 0. \end{array}$$

We claim that each ϕ_p is a split monomorphism of R-modules. We have $K_0(\mathbf{a}) = R$ and $K_1(\mathbf{a}) = R^q$; d_1 sends the i^{th} basis element to a_i. Since $K_*(\mathbf{a})$ is minimal this far, ϕ_0 and ϕ_1 are isomorphisms. Assume inductively that ϕ_{p-1} is a split monomorphism and consider the diagram

$$\begin{array}{ccc} K_p(\mathbf{a}) & \xrightarrow{d_p} & K_{p-1}(\mathbf{a}) \\ \Big\downarrow{\scriptstyle \phi_p} & & \Big\downarrow{\scriptstyle \phi_{p-1}} \\ F_p & \xrightarrow[d_p]{} & F_{p-1}. \end{array}$$

The images of the differentials d_p are contained in $\mathfrak{m}K_{p-1}(\mathbf{a})$ and $\mathfrak{m}F_{p-1}$. By Nakayama's lemma, it suffices to prove that the induced map

$$\bar{\phi}_p \colon K_p(\mathbf{a})/\mathfrak{m}K_p(\mathbf{a}) \longrightarrow F_p/\mathfrak{m}F_p$$

is a monomorphism. Indeed, any map $\phi\colon F \longrightarrow F'$ of finitely generated free R-modules that induces a monomorphism mod $\mathfrak{m}$ is a split monomorphism since a basis for F must map to a set that can be extended to a basis for F'. The map $\bar{\phi}_p$ fits into the following diagram, which is induced by the one above.

$$\begin{array}{ccc} K_p(\mathbf{a}) \otimes_R (R/\mathfrak{m}) & \xrightarrow{\bar{d}_p} & K_{p-1}(\mathbf{a}) \otimes_R (\mathfrak{m}/\mathfrak{m}^2) \\ \Big\downarrow{\scriptstyle \bar{\phi}_p} & & \Big\downarrow{\scriptstyle \bar{\phi}_{p-1}} \\ F_p \otimes_R (R/\mathfrak{m}) & \xrightarrow[\bar{d}_p]{} & F_{p-1} \otimes_R (\mathfrak{m}/\mathfrak{m}^2) \end{array}$$

By the induction hypothesis, $\bar{\phi}_{p-1}$ is a monomorphism since ϕ_{p-1} is a split monomorphism. We claim that the top map $\bar{d}_p$ is a monomorphism by direct linear algebra from the definition of $K(\mathbf{a})$, and it will follow that $\bar{\phi}_p$ is a monomorphism. For the claim, consider a typical element

$$x = \sum a_{i_1,\cdots,i_p} e_{i_1} \cdots e_{i_p}$$

of $K_p(\mathbf{a})$, where $1 \le i_1 < \cdots < i_p \le q$. To prove our claim, it suffices to show that if $d_p(x) \in \mathfrak{m}^2 K_{p-1}(\mathbf{a})$, then $x \in \mathfrak{m}K_p(\mathbf{a})$. We consider the coefficient of $d_p(x)$ of a typical basis element $y = e_{j_1} \cdots e_{j_{p-1}}$, $1 \le j_1 < \cdots < j_{p-1} \le q$. Using the exterior algebra multiplication to order the basis elements of $K(\mathbf{a})$, we see that, up to sign, for each k not in the set $\{j_1, \cdots, j_{p-1}\}$, the element ye_k is one of the basis elements $e_{i_1} \cdots e_{i_p}$. Again up to sign, this basis element contributes $a_{i_1,\cdots,i_p} a_k$ to the coefficient of y in $d_p(x)$, which is then obtained by summing over such k. Since $\{a_i\}$ is a minimal generating set for $\mathfrak{m}$, we see that each non-zero $a_{i_1,\cdots,i_p}$ must be in $\mathfrak{m}$, which proves our claim. □

10. The Auslander-Buchsbaum theorem

We must still prove that $\text{depth}(R) = \text{gldim}(R)$ if the latter is finite, and we have the following more general result. Observe that $\text{depth}(k) = 0$ since k is annihilated by $\mathfrak{m}$.

Theorem 10.1 (Auslander–Buchsbaum)**.** *Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a finitely generated R-module such that $pd_R(M) = r < \infty$. Then*

$$pd_R(M) + depth(M) = depth(R).$$

Proof. Proceed by induction on r. If $r = 0$, then M is free and $\text{depth}(M) = \text{depth}(R)$. Assume that $r > 0$. Choose a free module F of minimal dimension for which there is an epimorphism $\varepsilon \colon F \longrightarrow M$. Then $L = \text{Ker}(\varepsilon)$ is contained in $\mathfrak{m}F$ and $pd_R(L) = r-1$. Using Theorem 6.2 and the induction hypothesis, we find that

$$\inf\{i | \,\text{Ext}^i_R(k, L) \neq 0\} = \text{depth}(L) = \text{depth}(R) - pd_R(L) = \text{depth}(F) + 1 - r.$$

Call this number d. We claim that $\text{depth}(M) = d - 1$. The conclusion will follow. Since $L \subset \mathfrak{m}F$, we see that the map $\text{Ext}^i_R(k, L) \longrightarrow \text{Ext}^i_R(k, F)$ is zero. Therefore the long exact sequence of Ext groups breaks up into short exact sequences

$$0 \longrightarrow \text{Ext}^i_R(k, F) \longrightarrow \text{Ext}^i_R(k, M) \longrightarrow \text{Ext}^{i+1}_R(k, L) \longrightarrow 0.$$

Noting that $d \leq \text{depth}(F)$, we see that $\text{Ext}^i_R(k, M) = 0$ for $i < d-1$ and $\text{Ext}^{d-1}(k, M) \neq 0$. By Theorem 6.2, this proves the claim. □

11. Localizations of regular rings

Here we no longer assume that R is local, but we still insist that it be commutative and Noetherian.

Definition 11.1. A ring R is regular if all of its localizations at prime ideals are regular.

Before the introduction of homological methods, it was unclear that the localization of a regular local ring is regular, but Serre's theorem implies the following stronger conclusion.

Theorem 11.2. *R is regular if and only if all of its localizations at maximal ideals are regular. If R has finite Krull dimension and finite global dimension, then R is regular and these dimensions are equal. If R is regular and the dimensions of its localizations are all at most n, then R has finite global dimension at most n.*

Proof. Let P be a prime ideal of R such that $pd_R(R/P) < \infty$. This always holds if R is a regular local ring or if R has finite global dimension. Let k be the residue field R_P/P_P of R_P and observe that $k \cong R/P \otimes_R R_P$. If X is a projective R-resolution of R/P of finite length, then $X \otimes_R R_P$ is a flat and therefore projective R_P-resolution of k of finite length. Therefore R_P has finite global dimension and is thus regular. By the definition of the Krull dimension, if $\dim(R)$ is finite, then $\dim(R_\mathfrak{m}) \leq \dim(R)$ for all maximal ideals $\mathfrak{m}$, and there is at least one $\mathfrak{m}$ for which equality holds. For the last statement, recall that we have proven that localization commutes with Tor, in the sense that

$$(\text{Tor}^R_q(M, N)_P \cong \text{Tor}^{R_P}_q(M_P, N_P).$$

This is zero for all maximal ideals P when $q > n$. Since a module is zero if and only if all of its localizations at maximal ideals are zero, this implies that $\mathrm{Tor}_q^R(M,N) = 0$ for $q > n$. By Proposition 7.1(vi), this implies that $\mathrm{gldim}(R) \leq n$. □

The Hilbert syzygy theorem, in a generalized form, says that if R has finite global dimension d, then the polynomial algebra $R[x_1, \cdots, x_n]$ has finite global dimension $d+n$. In particular, the theorem shows that all localizations of $k[x_1, \cdots, x_n]$ are regular.

12. A theorem about maps of rings

We are headed for some other results, also due to Auslander and Buchsbaum, that are of intrinsic algebraic interest and have interesting consequences for characteristic classes in algebraic topology. We return to local rings, but our exposition will no longer be entirely self-contained.

Lemma 12.1. *Let $f\colon R \longrightarrow S$ be a map of local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, where R is regular. Let $J = f(\mathfrak{m})S$. Assume that S (that is, f^*S) is finitely generated as an R-module. Then $dim(S) = ht(J)$.*

Proof. Since S is non-zero and finitely generated over R, we cannot have $\mathfrak{m}S = S$ and therefore cannot have $f(\mathfrak{m})S = S$. Thus $f(\mathfrak{m}) \subset \mathfrak{n}$. Let $k = R/\mathfrak{m}$. Then the ring S/J is a finite dimensional vector space over k and is therefore Artinian. Thus there is some q such that $\mathfrak{n}^q \subset J \subset \mathfrak{n}$. The conclusion follows. □

Theorem 12.2. *Let $f\colon R \longrightarrow S$ be a map of regular local rings of the same dimension such that S is finitely generated as an R-module. Then S is free as an R-module and, in particular, f is a monomorphism.*

Proof. Let $J = f(\mathfrak{m})S$. Then $\dim(S) = ht(J)$ by Lemma 12.1. By assumption, this is equal to $\dim(R)$. Let n be the common dimension. If $\mathfrak{m} = (a_1, \cdots, a_n)$, then $\{a_1, \cdots, a_n\}$ is a regular sequence in R. The ideal J in the regular local ring S is generated by the n elements $f(a_i)$. By an exercise, this implies that $\{f(a_1), \cdots, f(a_n)\}$ is a regular sequence in S and thus $\{a_1, \cdots, a_n\}$ is a regular sequence for S in $\mathfrak{m}$. Thus $\mathrm{depth}_R(S) \geq n$. Since $\mathrm{depth}_R(S) + pd_R(S) = n$, this implies that $pd_R(S) = 0$, so that S is free over R. □

Everything we have done for commutative rings works just as well for graded commutative algebras over a field k. Such an algebra R is given by vector spaces R_n over k together with products $R_m \otimes_k R_n \longrightarrow R_{m+n}$ and a unit element $1 \in R_0$ such that the product is associative with 1 as a two-sided identity element. We can understand commutativity in either the classical or the graded sense with the standard sign convention, and we take the latter. Then, unless k has characteristic two, a polynomial algebra must be understood to have generators in even degree. We emphasize that we are thinking homogeneously, not allowing addition of elements of different degree. We say that R is connected if $R_n = 0$ for $n < 0$ and $R_0 = k$. Then R is local with maximal ideal consisting of the graded sub vector space consisting of all positive degree elements, that is, all elements of all R_n for $n > 0$. The Hilbert syzygy theorem applies: a polynomial algebra R on n generators (of even degreee unless $\mathrm{char}(k) = 2$) is a regular local ring of dimension n.

Corollary 12.3. *Let $R = k[x_1, \cdots, x_n]$ and $S = k[y_1, \cdots, y_n]$ be connected polynomial algebras and let $f\colon R \longrightarrow S$ be a map of rings such that S is finitely generated as an R-module. Then S is free as an R-module and f is a monomorphism.*

This result has beautiful applications to the theory of characteristic classes in algebraic topology. This will be sketched in class, but not included in these notes.

13. Unique factorization

The goal of this section is to explain the proof of the following remarkable theorem, a triumph of homological techniques.

Theorem 13.1. *A regular local ring is a UFD.*

We need a few preliminaries. Let R be an integral domain throughout this section. Recall that any PID is a UFD and that $k[x_1, \cdots, x_n]$, k a field, is a UFD. Recall too that an element $p \in R$ is prime if (p) is a prime ideal and is irreducible if $p = xy$ implies that x or y is a unit. Prime elements are irreducible, and R is a UFD if and only if every irreducible element is prime and the ascending chain condition on principal ideals is satisfied. We omit the easy verifications of these statements. The following two results are a little less obvious. In both, we assume that the integral domain R is Noetherian.

Proposition 13.2. *R is a UFD if and only if every prime ideal of height one is principal.*

Proof. First, assume that R is a UFD and let P be a prime ideal of height one. Let $a \in P$, $a \neq 0$, and write a as the product of finitely many prime elements b_i. Since P is prime, one of the b_i, denoted b, must be in P. Then $(b) \subset P$, and since $ht(P) = 1$, $(b) = P$. For the converse, since R is Noetherian it suffices to prove that every irreducible element a is prime. Let P be a minimal prime over (a). By the Hauptidealsatz, $ht(P) \leq 1$, hence $ht(P) = 1$. By assumption $P = (b)$. Since $a \in P$, $a = bc$. Since a is irreducible and b is not a unit, c is a unit. Therefore $(a) = (b) = P$, so (a) is prime. □

Proposition 13.3. *Let S be the multiplicative subset of R generated by a prime element x. If the localization $R_S = R[x^{-1}]$ is a UFD, then R is a UFD.*

Proof. Let $P \subset R$ be a prime ideal of height one. If $P \cap S$ is nonempty, then P contains an element $x^i \in S$, hence $x \in P$ and $(x) \subset P$. Since $ht(P) = 1$, $(x) = P$ is principal. Thus assume that $P \cap S$ is empty. Then PR_S is a height one prime ideal of the UFD R_S, hence is principal, say $PR_S = aR_S$. We may choose a generator $a \in P$ such that (a) is maximal among those principal ideals of R that are generated by an element of P that generates PR_S. Then a is not divisible by x, since if $a = bx$ then $(a) \subset (b)$ and b also generates PR_S. We claim that $P = (a)$. If $z \in P$, then $z \in aR_S$ and there is an element $x^i \in S$ such that $x^i z = ab$ for some $b \in R$. Since x does not divide a, a is not in (x). Since x is prime, b is in (x), say $b = xc$, and $x^{i-1} z = ac$. Repeating inductively, we see that z is in (a) and therefore $P = (a)$. □

Definition 13.4. An R module M is stably free if $M \oplus F$ is free for some finitely generated free R-module F. An ideal I is stably free if $I \oplus R^n \cong R^{n+1}$ for some n.

Lemma 13.5. *A stably free ideal I is principal.*

Proof. If $I \oplus R^n \cong R^{n+1}$, then, applying the $(n+1)^{\text{st}}$ exterior power, we see that I is isomorphic to R. □

Lemma 13.6. *If a projective module P has a FFR (finite free resolution)*

$$0 \to F_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} P \to 0,$$

then P is stably free.

Proof. If $n = 0$, $F_0 \cong P$. Assume the result for $n-1$. Then $F_0 \cong P \oplus \operatorname{Ker}(\varepsilon)$ and $\operatorname{Ker}(\varepsilon)$ is stably free. Therefore P is stably free. □

Proof of Theorem 13.1. We proceed by induction on the dimension of R. If $\dim(R) = 0$, then R is a field, and if $\dim(R) = 1$, R is a DVR. The result is clear in these cases. Assume $\dim(R) > 1$ and let $x \in \mathfrak{m} - \mathfrak{m}^2$. Then x is a prime element since $R/(x)$ is again regular and is therefore an integral domain.

By Proposition 13.3, it suffices to prove that $R[x^{-1}]$ is a UFD. Let Q be a prime ideal of $R[x^{-1}]$ of height one. By Proposition 13.2, it suffices to prove that Q is principal. By Lemma 13.5, it suffices to prove that Q is stably free. Let $P = Q \cap R$. Then $Q = PR[x^{-1}]$. Since R is regular, P has a FFR

$$0 \to F_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} P \to 0.$$

Tensoring this with $R[x^{-1}]$, we obtain a FFR of the $R[x^{-1}]$-module Q. By Lemma 13.6, it suffices to prove that Q is projective.

For any maximal ideal (indeed any prime ideal) $\mathfrak{n}$ of $R[x^{-1}]$, $R[x^{-1}]_{\mathfrak{n}}$ is isomorphic to $R_{\mathfrak{n}\cap R}$, which is a regular local ring of dimension less than $\dim(R)$, so is a UFD by induction. Therefore $Q_{\mathfrak{n}}$ is a principal ideal and thus a free $R[x^{-1}]_{\mathfrak{n}}$-module. For an epimorphism $M \longrightarrow N$ of $R[x^{-1}]$-modules, let C be the cokernel of $\operatorname{Hom}_{R[x^{-1}]}(Q, M) \longrightarrow \operatorname{Hom}_{R[x^{-1}]}(Q, N)$. Looking at localizations, we see that $C_{\mathfrak{n}}$ is zero for all $\mathfrak{n}$ and therefore C is zero. Thus Q is projective. □

14. Stably free does not imply free

The Serre conjecture, now a theorem of Quillen and Suslin (1976), says that projective modules over $k[x_1, \cdots, x_n]$ are free. They are stably free because they have FFR's. However, it is not always true that stably free modules are free. We give some details of a counterexample (Eisenbud, p. 485). Let

$$A = \mathbb{R}[x_1, \cdots, x_n]/(1 - \sum x_i^2).$$

We think of this as a sphere, or more precisely as the affine coordinate ring of the real $(n-1)$-sphere S^{n-1}. We have the following split short exact sequence.

$$0 \longrightarrow A \underset{\psi}{\overset{\xi}{\rightleftarrows}} A^n \longrightarrow T \longrightarrow 0$$

Here $\xi(a) = (ax_1, \cdots, ax_n)$ and $\psi(a_1, \cdots, a_n) = a_1x_1 + \cdots + a_nx_n$. clearly $(\psi\xi)(a) = a\sum x_i^2 = a$. Therefore $A^n \cong A \oplus T$ and thus T is stably free. Note that T may be identified with the kernel of ψ, which consists of those $t = (t_1, \cdots, t_n)$ such that $\sum t_i x_i = 0$. We may think of T as the module of polynomial sections of the tangent bundle of S^{n-1}. To see this, let $t \in T \subset A^n$ and let $z \in S^{n-1}$, so that $\sum z_i^2 = 1$. Let $P_z = (x_1 - z_1, \cdots, x_n - z_n) \subset A$. Then P_z is a maximal ideal. Since $\sum t_i x_i = 0$, the image of t in the quotient $A^n/P_zA^n \cong \mathbb{R}^n$ is a vector orthogonal to

z and is thus a tangent vector to S^{n-1} at z. Since this holds for all z, t corresponds to a tangent vector field. If T were free, then it would have $n-1$ basis elements v_i since it clearly has rank $n-1$. The images of the v_i in T/P_zT would form a basis of the tangent space for all $z \in S^{n-1}$, and we would therefore have $n-1$ linearly independent vector fields. This would give us a trivialization of the tangent bundle of S^{n-1}. By Adams' solution of the Hopf invariant one problem, this is possible if and only if n is 1, 2, 4, or 8, corresponding to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. For example, for $n = 3$, we cannot comb the hair of a tennis ball, hence a rank 1 free summand of T does not exist.