FINITE SPACES AND SIMPLICIAL COMPLEXES

NOTES FOR REU BY J.P. MAY

1. Statements of results

Finite simplicial complexes provide a general class of spaces that is sufficient for most purposes of basic algebraic topology. There are more general classes of spaces, in particular the finite CW complexes, that are more central to the modern development of the subject, but they give exactly the same collection of homotopy types. The relevant background on simplicial complexes will be recalled as we go along and can be found in most textbooks in algebraic topology (but not in my own book [7]). We write $|K|$ for the geometric realization of K.

We recall the definition of the homotopy groups $\pi_n(X, x)$ of a space X at $x \in X$. When $n = 0$, this is just the set of path components of X, with the component of x taken as a basepoint (and there is no group structure). When $n = 1$ it is the fundamental group of X at the point x. For all $n \geq 0$, it can be described most simply by considering the standard sphere S^n with a chosen basepoint $*$. One considers all maps $\alpha\colon S^n \longrightarrow X$ such that $f(*) = x$. One says that two such maps α and β are based homotopic if there is a based homotopy $h\colon \alpha \simeq \beta$. Here a homotopy h is based if $h(*, t) = x$ for all $t \in I$. If $n = 1$, the map α is a loop at x, and we can compose loops to obtain a product which makes $\pi_1(X, x)$ a group. The homotopy class of the constant loop at x gives the identity element, and the loop $\alpha^{-1}(t) = \alpha(1-t)$ represents the inverse of the homotopy class of α. There is a similar product on the higher homotopy groups, but, in contrast to the fundamental group, the higher homotopy groups are Abelian.

A map $f\colon X \longrightarrow Y$ induces a function $f_*\colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$. One just composes maps α and homotopies h as above with the map f. If $n \geq 1$, f_* is a homomorphism.

Definition 1.1. A map $f\colon X \longrightarrow Y$ is a *weak homotopy equivalence* if

$$f_*\colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$. If $n = 0$, this means that components are mapped bijectively. Two spaces X and Y are weakly homotopy equivalent if there is a finite chain of weak homotopy equivalences $Z_i \longrightarrow Z_{i+1}$ or $Z_{i+1} \longrightarrow Z_i$ starting at $X = Z_1$ and ending at $Z_q = Y$.

The definition may seem strange at first sight, but it has gradually become apparent that the notion of a weak homotopy equivalence is even more important in algebraic topology than the notion of a homotopy equivalence. The notions are related. We state some theorems that the reader can take as reference points. Proofs can be found in [7].

Date: Summer, 2003; revised summer, 2008.

Theorem 1.2. *A homotopy equivalence is a weak homotopy equivalence. Conversely, a weak homotopy equivalence between CW complexes (for example, between simplicial complexes) is a homotopy equivalence.*

Theorem 1.3. *Spaces X and Y are weakly homotopy equivalent if and only if there is a space Z (in fact a CW complex Z) and weak homotopy equivalences $Z \longrightarrow X$ and $Z \longrightarrow Y$.*

That is, the chains that appear in the definition need only have length two. For those who know about homology and cohomology, we record the following result.

Theorem 1.4. *A weak homotopy equivalence induces isomorphisms of all singular homology and cohomology groups.*

Following McCord [8], we are going to relate finite spaces with finite simplicial complexes, explaining the following two theorems. Since any finite space is homotopy equivalent to a T_0-space, there is no loss of generality if we restrict attention to finite T_0-spaces. McCord actually deals more generally with A-spaces, but the arguments are no different.

Theorem 1.5. *For a finite T_0-space X, there is a finite simplicial complex $\mathscr{K}(X)$ with vertex set X, and there is a weak homotopy equivalence*

$$\psi = \psi_X : |\mathscr{K}(X)| \longrightarrow X.$$

A map $f: X \longrightarrow Y$ of finite spaces induces a map

$$\mathscr{K}(f): \mathscr{K}(X) \longrightarrow \mathscr{K}(Y)$$

of simplicial complexes such that $f \circ \psi_X = \psi_Y \circ |f|$.

The essential point in the proof, which we will take for granted, is that weak homotopy equivalence is a local notion in the sense of the following theorem. McCord [8] relies on point-by-point comparison with arguments in the early paper [2], which doesn't prove the result but comes close. He uses the second alternative hypothesis in (i). More modern references are [6, 12], which use the first alternative.

Theorem 1.6. *Let $p: E \longrightarrow B$ be a continuous map. Suppose that B has an open cover $\mathscr{O}$ with the following two properties.*

(i) *Either finite intersections of sets in $\mathscr{O}$ are also in $\mathscr{O}$ or $\mathscr{O}$ is a basis for a possibly smaller topology than that originally given on B, so that if b is in the intersection of sets U and V in $\mathscr{O}$, then there is some $W \in \mathscr{O}$ with $x \in W \subset U \cap V$.*

(ii) *For each $U \in \mathscr{O}$, the restriction $p: p^{-1}U \longrightarrow U$ is a weak homotopy equivalence.*

Then p is a weak homotopy equivalence.

Theorem 1.5 is itself used to obtain the following complementary result.

Theorem 1.7. *For a finite simplicial complex K, there is a finite T_0-space $\mathscr{X}(K)$ whose points are the barycenters of the simplices of K, and there is a weak homotopy equivalence*

$$\phi = \phi_K : |K| \longrightarrow \mathscr{X}(K).$$

A map $g: K \longrightarrow L$ of simplicial complexes induces a map

$$\mathscr{X}(g): \mathscr{X}(K) \longrightarrow \mathscr{X}(L)$$

of finite spaces such that $\mathscr{X}(g) \circ \phi_K \simeq \phi_L \circ |g|$.

Remark 1.8. Writing K' for the barycentric subdivision of K, so that $|K| = |K'|$, we will have $\mathscr{K}\mathscr{X}(K) = K'$. The map ϕ_K will be $\psi_{\mathscr{X}(K)}$, and Theorem 1.5 will apply to show that it is a weak homotopy equivalence.

As a warm-up exercise, we will consider suspensions of spaces and give a finite model for the n-sphere before turning to the proofs of these general results.

2. Problems

Also before turning to the proofs, we list a few problems that spring immediately to mind. To the best of my knowledge, none of them have been studied.

Problem 2.1. *For small n, determine all homotopy types and weak homotopy types of spaces with at most n elements.*

Addendum 2.2. This has been done in class or by students in the cases $n \leq 6$. Nearly all finite spaces with so few points are disjoint unions of (weakly) contractible spaces.

Problem 2.3. *Is there an effective algorithm for computing the homotopy groups of X in low degrees in terms of the increasing chains in X?*

Remark 2.4. The dimension of the simplicial complex $\mathscr{K}(X)$ is the maximal length of a sequence $x_0 < \cdots < x_n$ in X. A map $g\colon K \longrightarrow L$ of simplicial complexes of dimension less than n is a homotopy equivalence if and only if it induces an isomorphism of homotopy groups in dimension less than n and an epimorphism of homotopy groups in dimension n.

Problem 2.5. *Let X be a minimal finite space. Give a descriptive interpretation of what this says about $|\mathscr{K}(X)|$.*

Addendum 2.6. There is a nice paper of Osaki [10] that interprets Stong's process of passing from a finite T_0-space X to its core Y. He shows that $\mathscr{K}(Y)$ is obtained from $\mathscr{K}(X)$ by a sequence of elementary simplicial collapses, so that $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$ have the same simple homotopy type. It follows that if X and Y are homotopy equivalent finite T_0-spaces, then $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ have the same simple homotopy type. If K is not collapsible, then $\mathscr{X}(K)$ is a minimal finite space. He displays non-collapsible triangulations K_1 and K_2 of S^1 such that $\mathscr{X}(K_1)$ and $\mathscr{X}(K_2)$ are not homeomorphic and therefore, being minimal, not homotopy equivalent.

Remark 2.7. Finite spaces can be weak homotopy equivalent but not homotopy equivalent. For example, we shall shortly give a minimal finite space $\mathbb{S}S^0$ which is weakly equivalent to the circle S^1, while the finite space associated to the boundary of a geometric 2-simplex is another minimal finite space that is weakly equivalent to S^1. Since one of these minimal finite spaces has 4 points and the other has 6 points, they cannot homotopy equivalent.

Problem 2.8. *Suppose that two finite spaces X and Y are weakly homotopy equivalent. Are they weakly homotopy equivalent via a chain in which all of the Z_i are again finite spaces?*

Addendum 2.9. The answer is yes, as an application of the simplicial approximation theorem for finite spaces of Hardie and Vermeulen [3]. It is discussed below.

Problem 2.10. *Are there computationally effective algorithms for enumerating the homotopy types and, presumably much harder, the weak homotopy types of finite spaces?*

Addendum 2.11. Osaki [10] has given two theorems that describe when one can shrink a finite T_0-space, possibly minimal, to a smaller weakly homotopy equivalent space. He asks whether all weak homotopy equivalences are generated by the simple kinds that he describes. The question has since been answered in the negative, by Barmak and Minian [1].

Problem 2.12. *Is there a combinatorial way of determining when a weak homotopy equivalence of finite spaces is a homotopy equivalence?*

Problem 2.13. *Rather than restricting to finite simplicial complexes, can we model the world of finite CW complexes in the world of finite spaces. The discussion of spheres and cones in the next section gives a starting point.*

3. The non-Hausdorff suspension

The suspension is one of the most basic constructions in all of topology. Following McCord [8], we show that it comes in two weakly equivalent versions, the classical one and a non-Hausdorff analogue that preserves finite spaces. For the purposes of these notes, we shall use the following unbased variant of the classical suspension.

Definition 3.1. The *suspension* SX of a space X is the quotient space obtained from $X \times [-1, 1]$ by identifying $X \times \{-1\}$ to a single point $-$ and identifying $X \times \{1\}$ to a single point $+$. Thus SX can be thought of as obtained by gluing together the bases of two cones on X. For a map $f\colon X \longrightarrow Y$, define $Sf\colon SX \longrightarrow SY$ by $(Sf)(x,t) = (f(x), t)$.

We defined the non-Hausdorff cone $\mathbb{C}X$ by adjoining a new cone point $*$ and letting the proper open subsets of $\mathbb{C}X$ be all of the open subsets of X, and we saw that $\mathbb{C}X$ is contractible. We now change notation and call the added point $+$.

Definition 3.2. Define the *non-Hausdorff suspension* $\mathbb{S}X$ by adjoining two new points, denoted $+$ and $-$, and letting the proper open subsets be the open sets in X and the sets $X \cup +$ and $X \cup -$. Again, $\mathbb{S}X$ can be thought of as obtained by gluing together two copies of $\mathbb{C}X$. If $f\colon X \longrightarrow Y$ is a map, define maps $\mathbb{C}f$ and $\mathbb{S}f$ by using f on X and sending $+$ to $+$ and $-$ to $-$.

Observe that if X is a T_0-space, then so are $\mathbb{C}X$ and $\mathbb{S}X$.

Definition 3.3. Define a comparison map

$$\gamma = \gamma_X \colon SX \longrightarrow \mathbb{S}X$$

by $\gamma(x,t) = x$ if $-1 < t < 1$, $\gamma(+) = +$ and $\gamma(-) = -$. Observe that, for a map $f\colon X \longrightarrow Y$, $\gamma_Y \circ Sf = \mathbb{S}f \circ \gamma_X$. Inductively, define $S^n X = SS^{n-1}X$ and $\mathbb{S}^n X = \mathbb{S}\mathbb{S}^{n-1}X$ and let $\gamma^n\colon S^n X \longrightarrow \mathbb{S}^n X$ be the common composite displayed in the commutative diagram

$$\begin{array}{ccc} S^n X & \xrightarrow{S\gamma^{n-1}} & S\mathbb{S}^{n-1}X \\ {\scriptstyle\gamma}\downarrow & \searrow^{\gamma^n} & \downarrow{\scriptstyle\gamma} \\ \mathbb{S}S^{n-1}X & \xrightarrow[\mathbb{S}\gamma^{n-1}]{} & \mathbb{S}^n X \end{array}$$

Theorem 3.4. *For any space X, the map $\gamma\colon SX \longrightarrow \mathbb{S}X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f\colon X \longrightarrow Y$, the maps $Sf\colon SX \longrightarrow SY$ and $\mathbb{S}f\colon \mathbb{S}X \longrightarrow \mathbb{S}Y$ are weak homotopy equivalences. Therefore $\gamma^n\colon S^nX \longrightarrow \mathbb{S}^nX$ is a weak homotopy equivalence for any space X.*

Proof. This is an application, or rather several applications, of Theorem 1.6. Take the three subspaces X, $X\cup\{+\}$, and $X\cup\{-\}$ as our open cover of $\mathbb{S}X$ and observe that the latter two subspaces are copies of $\mathbb{C}X$ and are therefore contractible. The respective inverse images under γ of these three subsets are the images in SX of $X\times(-1,1)$, $X\times(-1,1]$, and $X\times[-1,1)$. The restrictions of γ on these three subspaces are homotopy equivalences, hence weak homotopy equivalences. Similarly, taking the three subspaces Y, $Y\cup\{+\}$, and $Y\cup\{-\}$ as our open cover of $\mathbb{S}Y$, their inverse images under $\mathbb{S}f$ are X, $X\cup\{+\}$, and $X\cup\{-\}$, and the restrictions of $\mathbb{S}f$ on these three subspaces are weak homotopy equivalences. Finally, take the images in SY of $Y\times(-1/2,1/2)$, $Y\times[-1,1/2)$, and $Y\times(-1/2,1]$ as our open cover of SY. Their inverse images under Sf are the corresponding subspaces of SX, and the restrictions of Sf to these subspaces are weak homotopy equivalences. □

Example 3.5. Let $X = S^0$, a two-point discrete space. Then S^nX is homeomorphic to the n-sphere S^n, while $\mathbb{S}^nX$ is a T_0-space with $2n+2$ points. Thus we have a weak homotopy equivalence γ^n from S^n to a finite space with $2n+2$ points.

Proposition 3.6. *Each $\mathbb{S}^nS^0$ is a minimal finite space.*

Proof. Certainly $\mathbb{S}^nS^0$ is T_0, and it has no upbeat or downbeat points since each point has incomparable points above or below it in the partial ordering. □

Problem 3.7. *Is $\mathbb{S}^nS^0$ the finite space with the smallest number of points that is weakly homotopy equivalent to an n-sphere? The answer is probably yes and probably known, but I don't know how to prove it and can't find it in the literature.*

Addendum 3.8. The answer is yes, as shown by Barmak and Minian [1].

4. Recollections about simplicial complexes

Definition 4.1. An *abstract simplicial complex* K is a set $V = V(K)$, whose elements are called *vertices*, together with a set K of (non-empty) finite subsets of V, whose elements are called *simplices*, such that every vertex is an element of some simplex and every subset of a simplex is a simplex; such a subset is called a *face* of the given simplex. We say that K is finite if V is a finite set. The *dimension* of a simplex is one less than the number of vertices in it. A map $g\colon K \longrightarrow L$ of abstract simplicial complexes is a function $g\colon V(K) \longrightarrow V(L)$ that takes simplices to simplices. We sat that K is a *subcomplex* of L if the vertices and simplices of K are some of the vertices and simplices of L. We say that K is a *full subcomplex* of L if, further, every simplex of L whose vertices are in K is a simplex of K.

Definition 4.2. A set $\{v_0,\cdots,v_n\}$ of points of $\mathbb{R}^N$ is *geometrically independent* if the equations $\sum t_iv_i = 0$ and $\sum t_i = 0$ for real numbers t_i imply $t_1 = \cdots = t_n = 0$. It is equivalent that the vectors $v_i - v_0$, $1 \le i \le n$, are linearly independent. The *n-simplex* σ spanned by $\{v_0,\cdots,v_n\}$ is then the set of all points $x = \sum t_iv_i$, where $0 \le t_i \le 1$ and $\sum t_i = 1$. The t_i are called the *barycentric coordinates* of the point x. When each $t_i = 1/n+1$, the point x is called the *barycenter* of σ. The points v_i are the *vertices* of σ. A simplex spanned by a subset of the vertices is a *face* of σ; it is a proper face if the subset is proper. The *standard* n-simplex Δ_n is the n-simplex spanned by the standard basis of $\mathbb{R}^{n+1}$.

Definition 4.3. A *simplicial complex*, or *geometric simplicial complex*, K is a set of simplices in some $\mathbb{R}^N$ such that every face of a simplex in K is a simplex in K and the intersection of two simplices in K is a simplex in K. The set of vertices of K is the union of the sets of vertices of its simplexes. The notions of subcomplex and full subcomplex are evident. Note that although we require all vertices to lie in some $\mathbb{R}^N$ and we require each set of vertices that spans a simplex of K to be geometrically independent, we do not require the entire set of vertices to be geometrically independent. For example, we can have three vertices on a single line in $\mathbb{R}^N$, as long as the two vertices furthest appart do not span a 1-simplex of K.

Definition 4.4. The *geometric realization* $|K|$ is the union of the simplices of K, each regarded as a subspace of $\mathbb{R}^N$, with the topology whose closed sets are the sets that intersect each simplex in a closed subset. If K is finite, but not in general otherwise, this is the same as the topology of $|K|$ as a subspace of $\mathbb{R}^N$. The open simplices of $|K|$ are the interiors of its simplices (where a vertex is an interior point of its 0-simplex), and every point of $|K|$ is an interior point of a unique simplex. The *boundary* $\partial\sigma$ of a simplex σ is the subcomplex given by the union of its proper faces. The *closure* of a simplex is the union of its interior and its boundary.

Definition 4.5. A map $g\colon K \longrightarrow L$ of simplicial complexes is a function from the vertex set $V(K)$ to the vertex set $V(L)$ such that, for each subset S of $V(K)$ that spans a simplex, the set $g(S)$ is the set of vertices of a simplex of L. Then g determines the continuous map $|g|\colon |K| \longrightarrow |L|$ that sends $\sum t_i v_i$ to $\sum t_i g(v_i)$. Note that we do not require g to be one–to–one on vertices, but $|g|$ is nevertheless well-defined and continuous. If g is a bijection on vertices and simplices, we say that it is an isomorphism, and then $|g|$ is a homeomorphism.

It is usual to abbreviate $|g|$ to g and to refer to it as a simplicial map.

Definition 4.6. The abstract simplicial complex aK determined by a geometric simplicial complex K has vertex set the union of the vertex sets of the simplices of K. Its simplices are the subsets that span a simplex of K. An abstract finite simplicial complex K determines a geometric finite simplicial complex gK by choosing any bijection between the vertices of K and a geometrically independent subset of some $\mathbb{R}^N$. For specificity, we can take the standard basis elements of $\mathbb{R}^N$ where N is the number of points in the vertex set $V(K)$. The geometric simplices are spanned by the images under this bijection of the simplices of K. For an abstract simplicial complex K, agK is isomorphic to K, the isomorphism being given by the chosen bijection. Similarly, for a finite geometric simplicial complex K, gaK is isomorphic to K.

We could remove the word finite from the previous definition by defining geometric simplicial complexes more generally, without reference to a finite dimensional ambient space $\mathbb{R}^N$, but there is no point in going into that here. We also note that we do not have to realize in such a high dimensional Euclidean space. The following result holds no matter how many vertices there are. It is rarely used, but is conceptually attractive. A proof can be found in [5, 1.9.6]

Theorem 4.7. *Any simplicial complex K of dimension n can be geometrically realized in $\mathbb{R}^{2n+1}$.*

In view of the discussion above, abstract and geometric finite simplicial complexes can be used interchangeably. In particular, the geometric realization of an abstract simplicial complex is K is understood to mean the geometric realization of any gK.

We need a criterion for when the geometric realizations of two simplicial maps are homotopic.

Proposition 4.8. *Let f and g be maps from a topological space X to $|K| \subset \mathbb{R}^N$. Say that f and g are* simplicially close *if, for each $x \in X$, both $f(x)$ and $g(x)$ are in the closure of some simplex $\sigma(x)$ of L. If f and g are simplicially close, then they are homotopic.*

Proof. Define $h\colon X \times I \longrightarrow \mathbb{R}^N$ by

$$h(x,t) = (1-t)f(x) + tg(x).$$

Since $h(x,t)$ is in the closure of $\sigma(x)$ and therefore in $|K|$, it specifies a homotopy as required. □

5. Cones and subdivisions of simplicial complexes

Let K be a finite geometric simplicial complex in $\mathbb{R}^N$.

Definition 5.1. Define the cone CT of a topological space T to be the quotient space $T \times I/T \times \{1\}$.

Definition 5.2. Let x be a point of $\mathbb{R}^N - K$ such that each ray starting at x intersects $|K|$ in a single point. Observe that the union of $\{x\}$ and the set of vertices of a simplex of K is a geometrically independent set. Define the cone $K * x$ on K with vertex x to be the geometric simplicial complex whose simplices are all of the faces of the simplices spanned by such unions. Then K is a subcomplex of $K * x$, x is the only vertex not in K, and $|K * x|$ is homeomorphic to $C|K|$.

Example 5.3. A simplex is the cone of any one of its vertices with the subcomplex spanned by the remaining vertices (the opposite face).

Definition 5.4. A *subdivision* of K is a simplicial complex L such that each simplex of L is contained in a simplex of K and each simplex of K is the union of finitely many simplices of L.

Lemma 5.5. *If L is a subdivision of K, then $|L| = |K|$ (as spaces).*

The n-skeleton K^n of K is the union of the simplices of K of dimension at most n. It is a subcomplex.

Construction 5.6. We construct the *(first) barycentric subdivision* K' of K. We subdivide the skeleta of K inductively. Let $L_0 = K^0$. Suppose that a subdivision L_{n-1} of K^{n-1} has been constructed. Let b_σ be the barycenter of an n-simplex σ of K. The space $|\partial\sigma|$ coincides with $|L_\sigma|$ for a subcomplex L_σ of L_{n-1}, and we can define the cone $L_\sigma * b_\sigma$. Clearly $|L_\sigma * b_\sigma| = |\sigma|$ and $|L_\sigma * b_\sigma| \cap |L_{n-1}| = |L_\sigma| = |\partial\sigma|$. If τ is another n-simplex, then $|L_\sigma * b_\sigma| \cap |L_\tau * b_\tau| = |\sigma \cap \tau|$, which is the realization of a subcomplex of L_{n-1} and therefore of both L_σ and L_τ. Define L_n to be the union of L_{n-1} and the complexes $L_\sigma * b_\sigma$, where σ runs over all n-simplices of K. Our observations about intersections show that L_n is a simplicial complex which contains L_{n-1} as a subcomplex. The union of the L_n is denoted K' and called the barycentric subdivision of K.

The second barycentric subdivision of K is the barycentric subdivision of the first barycentric subdivision, and so on inductively. We can enumerate the simplices of K' explicitly rather than inductively.

Proposition 5.7. *Define $\sigma < \tau$ if σ is a proper face of τ. Then K' is the simplicial complex whose simplices σ' are the spans of the geometrically independent sets $\{b_{\sigma_1}, \cdots, b_{\sigma_n}\}$, where $\sigma_1 > \cdots > \sigma_n$. In particular, the barycenters b_σ are the vertices of K'. The vertex b_{σ_1} is called the* leading vertex *of the simplex σ'.*

Proof. We show this inductively for the subcomplexes L_n. Since $L_0 = K^0$, this is clear for L_0. Assume that it holds for L_{n-1}. If τ is a simplex of L_n such that $|\tau|$ is contained in $|K^n|$ but not contained in K^{n-1}, then τ is a simplex in $L_\sigma * b_\sigma$ for some n-simplex σ. By the induction hypothesis and the definition of L_σ, each simplex of L_σ is the span of a set $\{b_{\sigma_1}, \cdots, b_{\sigma_m}\}$, where $\sigma > \sigma_1 > \cdots > \sigma_m$. Therefore τ is the span of a set $\{b_\sigma, b_{\sigma_1}, \cdots, b_{\sigma_m}\}$. □

Proposition 5.8. *There is a simplicial map $\xi = \xi_K \colon K' \longrightarrow K$ whose realization is simplicially close to the identity map and hence homotopic to the identity map.*

Proof. Let ξ map each vertex b_σ of K' to any chosen vertex of σ. If σ' is a simplex of K' with leading vertex b_{σ_1}, then all other vertices of σ' are barycenters of faces of σ_1, hence are mapped under ξ to vertices of σ_1. Therefore the images under ξ of the vertices of σ' span a face of σ_1, so that ξ is a simplicial map. If $x \in |K'|$ is an interior point of the simplex σ', then it is mapped under ξ to a point of $\sigma_1 \supset \sigma'$, and we let $\sigma(x) = \sigma_1$. Since ξ maps every vertex of σ' to a vertex of σ_1, x and $\xi(x)$ are both in the closure of σ_1. □

Remark 5.9. In the cases of interest to us, there is a partial order on the vertices of K that restricts to a total order on the vertices of each simplex of K. In that case, we have the *standard simplicial map* $\xi \colon K' \longrightarrow K$ specified by letting $\xi(b_\sigma)$ be the maximal vertex x_n of the simplex $\sigma = \{x_0, \cdots, x_n\}$.

Proposition 5.10. *A simplicial map $g \colon K \longrightarrow L$ induces a subdivided simplicial map $g' \colon K' \longrightarrow L'$ whose realization is simplicially close to $|g|$ and hence homotopic to $|g|$.*

Proof. The images under g of the vertices of a simplex σ of K span a simplex $g(\sigma)$, of possibly lower dimension than σ, and we define $g'(b_\sigma) = b_{g(\sigma)}$ on vertices. If b_{σ_1} is the leading vertex of a simplex σ' of K', then all other vertices of σ' are barcenters of faces of σ_1. Their images under g' are barycenters of vertices of $g(\sigma_1)$. If x is an interior point of σ', then both $g(x)$ and $g'(x)$ are in the closure of $g(\sigma_1)$. □

6. The definition and properties of $\mathscr{K}(X)$ and $\mathscr{X}(K)$

Let X be a finite T_0-space.

Definition 6.1. Define $\mathscr{K}(X)$ to be the abstract simplicial complex whose vertex set is X and whose simplices are the finite totally ordered subsets of the poset X. Since a map $f \colon X \longrightarrow Y$ is an order–preserving function, it may be regarded as a simplicial map $\mathscr{K}(f) \colon \mathscr{K}(X) \longrightarrow \mathscr{K}(Y)$.

Lemma 6.2. *If V is a subspace of X, then $\mathscr{K}(V)$ is a full subcomplex of $\mathscr{K}(X)$.*

Proof. The ordering $\leq$ on V is the restriction of the ordering $\leq$ on X. Every totally ordered subset of X whose points are in V is a totally ordered subset of V. □

Definition 6.3. Define $\psi\colon |\mathscr{K}(X)| \longrightarrow X$ as follows. Each point $u \in |\mathscr{K}(X)|$ is an interior point of a simplex σ spanned by some strictly increasing sequence $x_0 < x_1 < \cdots < x_n$ of points of X. Define $\psi(u) = x_0$.

It is convenient to start with the proof of the last statement of Theorem 1.5.

Lemma 6.4. *If $f\colon X \longrightarrow Y$ is a map of finite T_0-spaces, then $f \circ \psi_X = \psi_Y \circ |f|$.*

Proof. With u as in the previous definition, $|f|(u)$ is in the simplex spanned by the $f(x_i)$, and $f(x_0) \leq f(x_1) \leq \cdots \leq f(x_n)$. Omitting repetitions, we see that $\psi(f(u)) = f(x_0)$. □

Proposition 6.5. *Let V be an open subset of X. Then*

$$\psi^{-1}(V) = \cup\{star(v)|v \in V\},$$

where star(v) is the union of the open simplices of $|\mathscr{K}(X)|$ that have v as a vertex. Therefore ψ is continuous.

Proof. If $\psi(u) = v \in V$, then u is an interior point of a simplex σ of which v is the initial vertex x_0. Thus $u \in \text{star}(v)$. Conversely, suppose that $u \in \text{star}(v)$ with $v \in V$. Then u is an interior point of a simplex σ determined by an increasing sequence $x_0 < x_1 < \cdots < x_n$ such that some $x_i = v \in V$. Since $x_0 \leq v$, $x_0 \in U_v$. Since V is open, $U_v \subset V$. Thus $\psi(u) = x_0$ is in V. □

Corollary 6.6. *$|\mathscr{K}(V)|$ is a deformation retract of $\psi^{-1}(V)$.*

Proof. By Lemma 6.2, $\mathscr{K}(V)$ is a full subcomplex of $\mathscr{K}(X)$. It follows that $|\mathscr{K}(V)|$ is a deformation retract of its open star in $|\mathscr{K}(X)|$. This is a standard fact in the theory of simplicial complexes, and a more detailed proof is given in [9, 70.1 and p. 427]. Consider a simplex σ that is in the open star of V but is not contained in V. Then σ has vertex set the disjoint union of a set of vertices in V and a set of vertices in $X - V$. Each point u of σ that is neither in the span s of the vertices in V nor in the span t of the vertices not in V is on a unique line segment joining a point in t to a point in s. Define the required retraction r by sending u to the end point in $s \subset V$ of this line segment, letting r be the identity map on V and thus on s. Deformation along such line segments gives the required homotopy showing that $i \circ r$ is the identity, where i is the inclusion of $|\mathscr{K}(V)|$ in its open star. □

Recall that each open subset U_x of X is contractible.

Proposition 6.7. *For $x \in X$, $\psi^{-1}(U_x)$ is contractible.*

Proof. By the previous corollary, it suffices to show that $|\mathscr{K}(U_x)|$ is contractible. Let $V_x = U_x - x$. We claim that $|\mathscr{K}(U_x)|$ is isomorphic to the cone $\mathscr{K}(V_x) * x$. Indeed, a simplex of $\mathscr{K}(V_x)$ is given by an increasing sequence $x_0 < x_1 < \cdots < x_n$. The increasing sequence $x_0 < x_1 < \cdots < x_n < x$ gives a simplex of $\mathscr{K}(U_x)$, and every simplex of $\mathscr{K}(U_x)$ not in $\mathscr{K}(V_x)$ is of this form. □

The proof that ψ is a weak homotopy equivalence. Theorem 1.6 applies to the minimal open cover of X. If x is in $U_y \cap U_z$, then x is in both U_y and U_z, so that U_x is contained in both U_y and U_z. This verifies the first hypothesis of the cited theorem, and the second hypothesis holds by the previous result. □

Now let K be a finite geometric simplicial complex with first barycentric subdivision K'.

Definition 6.8. Define a finite T_0-space $\mathscr{X}(K)$ as follows. The points of $\mathscr{X}(K)$ are the barycenters b_σ of the simplices of K, that is, the vertices of K'. The required partial order $\leq$ is defined by $b_\sigma \leq b_\tau$ if $\sigma \subset \tau$. The open subspace U_{b_σ} coincides with $\mathscr{X}(\sigma)$, where σ (together with its faces) is regarded as a subcomplex of K.

The proof of Theorem 1.7. Using the barycenters themselves to realize the vertices geometrically, we find that $\mathscr{K}\mathscr{X}(K) = K'$, by Proposition 5.7. Define

$$\phi_K = \psi_{\mathscr{X}(K)} \colon |K| = |K'| = |\mathscr{K}\mathscr{X}(K)| \longrightarrow \mathscr{X}(K).$$

Then ϕ_K is a weak homotopy equivalence by Theorem 1.5. For a simplicial map $g\colon K \longrightarrow L$, define $\mathscr{X}(g) = g'$ on barycenters and note that this function is order–preserving and therefore continuous. Clearly $|\mathscr{X}(g)| = |g'|$ and therefore, by Theorem 1.5 and Proposition 5.10, $\mathscr{X}(g) \circ \phi_K = \phi_L \circ |g'| \simeq \phi_L \circ |g|$. □

7. Mapping spaces

For completeness, we record results of Stong [11, §6] that were obtained about the same time as the results of McCord recorded above and that give a quite different approach to the relationship between finite simplicial complexes and finite spaces. Since the proofs are fairly long and combinatorial in flavor, and since the statements do not have quite the same immediate impact as those in McCord's work, we shall not work through the details here.

Rather than constructing finite models for finite simplicial complexes, Stong studies all maps from the geometric realizations of simplicial complexes K into finite spaces X by studying the properties of the function space $X^K \equiv X^{|K|}$. More generally, he fixes a subcomplex L of K and a basepoint $* \in X$ and studies the subspace $(X,*)^{(K,L)}$ of maps $f\colon |K| \longrightarrow X$ such that $f(|L|) = *$. Homotopies relative to $|L|$ between such maps are homotopies h such that $h(p,t) = *$ for $p \in |L|$.

Theorem 7.1. *Let L be a subcomplex of a finite simplicial complex K, let X be a finite space with basepoint $*$, and let $F = (X,*)^{(K,L)}$ denote the subspace of X^K consisting of those maps $f\colon |K| \longrightarrow X$ such that $f(|L|) = *$.*

(i) *For any $f \in F$, there is a map $g \in F$ such that the set $V = \{h | h \leq g\} \subset F$ is a neighborhood of f in F; that is, there is an open subset U such that $f \in U \subset V$.*
(ii) *If $f \simeq f'$ relative to L, then there is a sequence of elements $\{g_1, \cdots, g_s\}$ in F such that $g_1 = f$, $g_s = f'$, and either $g_i \leq g_{i+1}$ or $g_{i+1} \leq g_i$ for $1 \leq i < s$.*

The essential point of this analysis is the following consequence.

Corollary 7.2. *The path components and components of F coincide. That is, the homotopy classes of maps $f\colon (K,L) \longrightarrow (X,*)$ are in bijective correspondence with the components of F.*

8. The simplicial approximation theorem

The classical point of barycentric subdivision is its use in the simplicial approximation theorem, which in its simplest form reads as follows. Starting with $K^{(0)} = K$, let $K^{(n)} = K'K^{(n-1)}$ be the nth barycentric subdivision of a simplicial complex K. By iteration of $\xi\colon K' \longrightarrow K$, we obtain a simplicial map $\xi^{(n)}\colon K^{(n)} \longrightarrow K$ whose geometric realization is a homotopy equivalence.

Theorem 8.1. *Let K be a finite simplicial complex and L be any simplicial complex. Let $f\colon |K| \longrightarrow |L|$ be any continuous map. Then, for some sufficiently large n, there is a simplicial map $g\colon K^{(n)} \longrightarrow L$ such that f is homotopic to $|g|$.*

This means that, for the purposes of homotopy theory, general continuous maps may be replaced by simplicial maps. We shall explain the proof shortly.

There are two papers, [3, 4], that start with the simplicial approximation theorem and take up where McCord and Stong leave off. In view of the explicit constructions of $\mathscr{K}(X)$ and $\mathscr{X}(K)$, the following definition is reasonable.

Definition 8.2. Define the *(first) barycentric subdivision* of a finite T_0-space X to be $X' = \mathscr{X}\mathscr{K}(X)$. For a map $f\colon X \longrightarrow Y$, define $f'\colon X' \longrightarrow Y'$ to be $\mathscr{X}\mathscr{K}(f)$. Iterating the construction, define $X^{(n)} = (X^{(n-1)})'$. where $X^{(0)} = X$. Observe inductively that $\mathscr{K}(X^{(n)}) = (\mathscr{K}(X))^{(n)}$ since $\mathscr{K}\mathscr{X}(K) = K'$.

Proposition 8.3. *There is a map $\zeta = \zeta_X\colon X' \longrightarrow X$ that makes the following diagram commute, and ζ is a weak homotopy equivlalence.*

$$\begin{array}{ccccc}
|\mathscr{K}\mathscr{X}\mathscr{K}(X)| & = & |\mathscr{K}(X)'| & \xrightarrow{|\xi_{\mathscr{K}(X)}|} & |\mathscr{K}(X)| \\
\downarrow{\scriptstyle \psi_{\mathscr{X}\mathscr{K}(X)}} & & & & \downarrow{\scriptstyle \psi_X} \\
X' = \mathscr{X}\mathscr{K}(X) & & \xrightarrow[\zeta_X]{} & & X.
\end{array}$$

The simplicial map $\xi_{\mathscr{K}(X)}$ coincides with $\mathscr{K}(\zeta_X)\colon \mathscr{K}(X') \longrightarrow \mathscr{K}(X)$. For any map $f\colon X \longrightarrow Y$, $\zeta_Y \circ f' = f \circ \zeta_X$.

Proof. The points of $\mathscr{X}\mathscr{K}(X)$ are the barycenters of the simplices of $\mathscr{K}(X)$. These simplices σ are spanned by increasing sequences $x_0 < \cdots < x_n$ of elements of X. Let $\zeta(b_\sigma) = x_n$. Since $b_\sigma \leq b_\tau$ implies $\sigma \subset \tau$ and thus $\zeta(b_\sigma) \leq \zeta(b_\tau)$, ζ is continuous. Inspection of definitions shows that $\xi_{\mathscr{K}(X)} = \mathscr{K}(\zeta_X)$, and the commutativity of the diagram follows from the "naturality" of ψ with respect to the map ζ_X. That ζ_X is a weak homotopy equivalence now follows from the diagram, since all other maps in it are weak homotopy equivalences. The last statement is clear by inspection of definitions. □

Theorem 8.4. *Let X and Y be finite T_0-spaces and let $f\colon |\mathscr{K}(X)| \longrightarrow |\mathscr{K}(Y)|$ be any continuous map. Then, for some sufficiently large n there is a continuous map $g\colon X^{(n)} \longrightarrow Y$ such that f is homotopic to $|\mathscr{K}(g)|$.*

Proof. By the classical simplicial approximation theorem above, for a sufficiently large n there is a simplicial approximation

$$j\colon \mathscr{K}(X^{(n-1)}) = (\mathscr{K}(X))^{(n-1)} \longrightarrow \mathscr{K}(Y)$$

to f. Let $g = \zeta_Y \circ \mathscr{X}(j)$. Since $\mathscr{K}(g) = \mathscr{K}(\zeta_Y) \circ \mathscr{K}\mathscr{X}(j) = \mathscr{K}(\zeta_Y) \circ j'$ and since $|j'| \simeq |j| \simeq f$ and $|\mathscr{K}(\zeta_Y)| = |\xi_{\mathscr{K}(Y)}| \simeq \mathrm{id}$, we have $|\mathscr{K}(g)| \simeq f$, as required. □

The point is that finite models for spaces have far too few maps between them. For example, $\pi_n(S^n, *) = \mathbb{Z}$, but there are only finitely many distinct maps from any finite model for S^n to itself. The theorem says that, after subdividing the domain sufficiently, we can realize any of these homotopy classes in terms of maps between (different) finite models for S^n.

Sketch proof of the simplicial approximation theorem. We are given $f\colon |K| \longrightarrow |L|$. Give $|K|$ the open cover by the sets $f^{-1}(\text{star}(w))$, where w runs over the vertices of L. Since $|K|$ is a compact subspace of a metric space, the "Lebesque lemma" ensures that there is a number δ such that any subset of $|K|$ of diameter less than δ is contained in one of the open sets $\text{star}(w)$. The diameter of a (closed) simplex is easily seen to be the maximal length of a one-dimensional face. Each barycentric subdivision therefore has the effect of cutting the maximal diameter of a simplex in half, so that there is an n such that every simplex of $K^{(n)}$ has diameter less than $\delta/2$. Then each $\text{star}(v)$ for a vertex v of $K^{(n)}$ has diameter less than δ, and we conclude that $f(\text{star}(v)) \subset \text{star}(w)$ for some vertex w of L. Define $g\colon V(K^{(n)}) \longrightarrow V(L)$ by letting $g(v) = w$ for some w such $f(\text{star}(v)) \subset \text{star}(w)$. One checks that g maps simplices to simplices and so specifies a map of simplicial complexes. If u is an interior point of a simplex σ of K, then $f(x)$ is an interior point of some simplex τ of L. One can check that g maps each vertex of σ to a vertex of τ. This implies that $|g|$ is simplicially close to f and therefore homotopic to f. □

9. Really finite H-spaces

The circle is a *topological group*. If we regard it as a the subspace of the complex plane consisting of points of norm one, then complex multiplication gives the product $S^1 \times S^1 \longrightarrow S^1$. How can we model such a basic structure in terms of a map of finite spaces?

Stong proved a rather amazing *negative* result about this problem. We will not go into the combinatorial details of his proof, contenting ourselves with the statement.

Definition 9.1. Let (X, e) be a finite space with a basepoint e. Suppose given a map $\phi\colon X \times X \longrightarrow X$. Say that X is an H-space of type I if multiplication by e on either the right or the left is homotopic to the identity. That is, the maps $x \to \phi(e, x)$ and $x \to \phi(x, e)$ are each homotopic to the identity. Say that X is an H-space of type II if the *shearing maps* $X \times X \longrightarrow X \times X$ defined by sending (x, y) to either $(x, \phi(x, y))$ or $(y, \phi(x, y))$ are homotopy equivalences.

A topological group is an H-space of both types, but it is much less restrictive for a space to be an H-space than for a space to be a topological group. By definition, the notion of H-space is homotopy invariant in the sense that if one defines an H-space structure on (X, e) to be a homotopy class of products ϕ, then one has the following result.

Proposition 9.2. *If (X, e) and (Y, f) are homotopy equivalent, then H-space structures on (X, e) correspond bijectively to H-space structures on (Y, f).*

This motivates Stong to study H-space structures on minimal finite spaces. Here it is easy to see the following result.

Proposition 9.3. *Let (X, e) be a finite H-space of either type. Then the maps $X \longrightarrow X$ that send x to either $\phi(x, e)$ or $\phi(x, e)$ are homeomorphisms.*

Examining the combinatorial relationship of general points of X to the point e, Stong then arrives at the following striking conclusion.

Proposition 9.4. *If (X, e) is an H-space of either type, then the point e is both maximal and minimal under $\leq$.*

This means that e is a component of X. Stong shows that this implies the following conclusions for general finite H-spaces.

Theorem 9.5. *Let X be a finite space and let $e \in X$. Then there is a product ϕ making (X, e) an H-space of type I if and only if e is a deformation retract of its component in X. Therefore X is an H-space for some basepoint e if and only if some component of X is contractible.*

Theorem 9.6. *Let X be a finite space. Then there is a product ϕ making X an H-space of type II if and only if every component of X is contractible.*

Corollary 9.7. *A connected finite space X is an H-space of either type if and only if X is contractible.*

So there is no way that we can model the product on S^1 by means of an H-space structure on some finite space X. Our standard model $\mathbb{T} = \mathbb{S}S^0$ of S^1 can be embedded in $\mathbb{C}$ as the four point subgroup $\{\pm 1, \pm i\}$, but then the complex multiplication is not continuous. However, the multiplication can be realized as a map $(\mathbb{T} \times \mathbb{T})^{(n)} \longrightarrow \mathbb{T}$ for some finite n, by the simplicial approximation theorem for finite spaces. It is natural to expect that some small n works here. The following result is proven in [4].

Theorem 9.8. *Choosing minimal points e in $\mathbb{T}$ and $f \in \mathbb{T}'$ as basepoints, there is a map*

$$\phi\colon \mathbb{T}' \times \mathbb{T}' \longrightarrow \mathbb{T}$$

such that $\phi(f, f) = e$ and the maps $x \longrightarrow \phi(x, f)$ and $x \longrightarrow \phi(f, x)$ from $\mathbb{T}'$ to $\mathbb{T}$ are weak homotopy equivalences.

That is, we can realize a kind of H-space structure after barycentric subdivision. The proof is horribly unilluminating. The space $\mathbb{T}'$ has eight elements, the space $\mathbb{T}$ has four elements. One writes down an 8×8 matrix with values in $\mathbb{T}$, choosing it most carefully so that when the 8 point and 4 point spaces are given the appropriate partial order, and the 64 point product space the product order, the function represented by the matrix is order preserving. Then one checks the row and column corresponding to multiplication by the basepoint.

Several other interesting spaces and maps are modelled similarly in the cited paper, for example $\mathbb{R}P^2$ and $\mathbb{C}P^2$.

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