PROOFS OF THE MODEL AXIOMS FOR TOP

Theorem 1 (Quillen model structure)**.** *The category of spaces is a model category with the usual weak equivalences, q-fibrations the Serre fibrations, and q-cofibrations the maps that satisfy the RLP with respect to the acyclic q-fibrations.*

Lemma 2 (Small object argument)**.** *Let $\mathscr{I}$ be a set of maps of spaces with compact domain. Then any map $f : X \longrightarrow Y$ of spaces factors as a composite*

$$X \xrightarrow{\ i\ } X' \xrightarrow{\ p\ } Y,$$

where p satisfies the RLP with respect to each map in $\mathscr{I}$ and i satisfies the LLP with respect to any map that satisfies the RLP with respect to each map in $\mathscr{I}$.

Proof. Let $X = X_0$. We construct a commutative diagram

$$(1)\qquad \begin{array}{ccccccccc} X_0 & \xrightarrow{i_0} & X_1 & \longrightarrow \cdots \longrightarrow & X_n & \xrightarrow{i_n} & X_{n+1} & \longrightarrow & \cdots \\ {\scriptstyle f=p_0}\downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_{n+1}} & & \\ Y & \xrightarrow[\mathrm{id}]{} & Y & \longrightarrow \cdots \longrightarrow & Y & \xrightarrow[\mathrm{id}]{} & Y & \longrightarrow & \cdots \end{array}$$

as follows. Suppose inductively that we have constructed p_n. Consider all maps from a map in $\mathscr{I}$ to p_n. Each such map is a commutative diagram of the form

$$(2)\qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & X_n \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_n} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

with $i \in \mathscr{I}$. Summing over such diagrams, we construct a pushout diagram

$$\begin{array}{ccc} \coprod A & \xrightarrow{\sum \alpha} & X_n \\ \downarrow & & \downarrow{\scriptstyle i_n} \\ \coprod B & \longrightarrow & X_{n+1}. \end{array}$$

The maps β and p_n induce a map $p_{n+1} : X_{n+1} \longrightarrow Y$ such that $p_{n+1} \circ i_n = p_n$. Let $X' = \operatorname{colim} X_n$, let $i : X \longrightarrow X'$ be the canonical map, and let $p : X' \longrightarrow Y$ be obtained by passage to colimits from the p_n. Constructing lifts by passage to coproducts, pushouts, and colimits of sequences, we see that each i_n and therefore also i satisfies the LLP with respect to maps that satisfy the RLP with respect to maps in $\mathscr{I}$. Assume given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\alpha'} & X' \\ {\scriptstyle i}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\beta]{} & Y, \end{array}$$

where i is in $\mathscr{I}$. To verify that p satisfies the RLP with respect to i, we must construct a map g that makes the diagram commute. Clearly X' is constructed as the colimit of a sequence of inclusions. Since A is compact, the natural map

$$\operatorname{colim} Top(A, X_n) \longrightarrow Top(A, X') \tag{3}$$

is a bijection. Therefore $\alpha' : A \longrightarrow X'$ factors through some X_n, giving one of the commutative squares used in the construction of X_{n+1}. By construction, there is a map $B \longrightarrow X_{n+1}$ whose composite with the natural map to X' gives a map g. □

We use this to prove a refined version of one of the factorization axioms.

Lemma 3. *Any map $f : X \longrightarrow Y$ factors as $p \circ i$, where i is an acyclic q-cofibration that satisfies the LLP with respect to any q-fibration and p is a q-fibration.*

Proof. Let $\mathscr{J}$ be the set of maps $i_0 : D^n \longrightarrow D^n \times I$, $n \geq 0$. A map is a q-fibration if and only if it satisfies the RLP with respect to every map in $\mathscr{J}$ and every map in $\mathscr{J}$ is an acyclic q-cofibration. Use the lemma to factor f. Then p is a q-fibration and i satisfies the LLP with respect to all q-fibrations. In particular, i is a q-cofibration. Since each i_n is the inclusion of a deformation retract, so is i. Therefore i is an acyclic q-cofibration. □

By definition, the q-cofibrations satisfy the LLP with respect to the acyclic q-fibrations. The other lifting axiom is now formal.

Lemma 4. *The q-fibrations satisfy the RLP with respect to the acyclic q-cofibrations.*

Proof. Let $f : X \longrightarrow Y$ be any acyclic q-cofibration. We must show that f satisfies the LLP with respect to q-fibrations. By the previous lemma, we may factor f as $f = p \circ i$, where $i : X \longrightarrow X'$ is an acyclic q-cofibration that does satisfy the LLP with respect to q-fibrations and $p : X' \longrightarrow Y$ is a q-fibration. Since f and i are weak equivalences, so is p. Since f satisfies the LLP with respect to acyclic q-fibrations, there exists $g : Y \longrightarrow Y'$ such that $g \circ f = i$ and $p \circ g = \mathrm{id}_Y$. Clearly p and g, together with the identity map on X, express f as a retract of i. Since i satisfies the LLP with respect to q-fibrations, so does f. □

Finally, here is the proof of the other factorization axiom.

Lemma 5. *Any map $f : X \longrightarrow Y$ factors as $p \circ i$, where i is a q-cofibration and p is an acyclic q-fibration.*

Proof. This is another application of the first lemma. Let $\mathscr{I}$ be the set of inclusions $i : S^{n-1} \longrightarrow D^n$, $n \geq 0$, where S^{-1} is empty. A map is an acyclic q-fibration if and only if it satisfies the RLP with respect to all maps in $\mathscr{I}$. The proof of this statement is an exercise in the meaning of homotopy groups and weak equivalences. Each map in $\mathscr{I}$ is thus a q-cofibration. In the factorization $f = p \circ i$ that we now obtain from the lemma, p is an acyclic q-fibration and i is a q-cofibration. □

We define cell $\mathscr{I}$-complexes and relative cell $\mathscr{I}$-complexes in the evident way. With $\mathscr{I}$ and $\mathscr{J}$ as in the previous two lemmas, we deduce the following corollary. It says that Top is a compactly generated model category with generators $\mathscr{I}$ for the q-cofibrations and $\mathscr{J}$ for the acyclic q-cofibrations.

Lemma 6. *The q-cofibrations are the retracts of the relative $\mathscr{I}$-cell complexes. The acyclic q-cofibrations are the retracts of the relative $\mathscr{J}$-cell complexes.*