

SYMMETRIC MONOIDAL CATEGORIES AND ENRICHED CATEGORIES

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In practice, categories come in nature with more structure than just sets of morphisms. This extra structure is central to all of category theory, homotopical or not. While every mathematician who ever makes use of categories should understand enrichment, this is not the place for a full exposition and we shall just give a cursory outline¹. The most thorough source is Kelly’s book [3], and an introduction can be found in Borceux [2, Ch. 6].

A monoidal structure on a category \mathcal{V} is a product, \otimes say, and a unit object U such that the product is associative and unital up to coherent natural isomorphisms; \mathcal{V} is symmetric if \otimes is also commutative up to coherent natural isomorphism². Informally, coherence means that diagrams that intuitively should commute do in fact commute. (The symmetry coherence admits a weakening that gives braided monoidal categories, but those will not concern us). A symmetric monoidal category \mathcal{V} is closed if it has internal hom objects $\underline{\mathcal{V}}(X, Y)$ in \mathcal{V} together with adjunction isomorphisms

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(X, \underline{\mathcal{V}}(Y, Z)).$$

These isomorphisms of hom sets imply isomorphisms of internal hom objects in \mathcal{V}

$$\underline{\mathcal{V}}(X \otimes Y, Z) \cong \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, Z)).$$

The proof is an exercise in the use of the Yoneda lemma: these two objects represent isomorphic functors.

From now on, we let \mathcal{V} be a bicomplete closed symmetric monoidal category. Such categories appear so often in nature that category theorists have invented a name for them: such a category is often called a “cosmos”. When \otimes is the cartesian product, we say that \mathcal{V} is cartesian monoidal, but the same category \mathcal{V} can admit other symmetric monoidal structures.

Example 1. We give examples of cosmoi \mathcal{V} .

- (i) The category \mathcal{U} of spaces is closed cartesian monoidal. The space $\underline{\mathcal{U}}(X, Y)$ is the function space of maps $X \rightarrow Y$ with the k -ification of the compact open topology.
- (ii) The category \mathcal{T} of based spaces is closed symmetric monoidal under the smash product. The smash product would not be associative if we used just spaces, rather than compactly generated spaces [6, §1.7]. The based space $\underline{\mathcal{T}}(X, Y)$ is the function space of based maps $X \rightarrow Y$.
- (iii) The category $sSet$ of simplicial sets is closed cartesian monoidal. The standard simplex simplicial sets $\Delta[n]$ give a covariant functor from Δ to sets, and $\underline{sSet}(X, Y)_n = sSet(X \times \Delta[n], Y)$ [5, I§6].

¹This is a brief extract from a book in preparation, coauthored by Kate Ponto.

²Mac Lane gave the first precise definitions; see [4].

- (iv) For a commutative ring R , the category \mathcal{M}_R of R -modules is closed symmetric monoidal under the functors \otimes_R and Hom_R .
- (v) For a commutative ring R , the category Ch_R of \mathbb{Z} -graded chain complexes of R -modules (with differential lowering degree) is closed symmetric monoidal under the graded tensor product and hom functors

$$(X \otimes_R Y)_n = \sum_{p+q=n} X_p \otimes_R Y_q; \quad d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y).$$

$$\text{Hom}_R(X, Y)_n = \prod_i \text{Hom}_R(X_i, Y_{i+n}); \quad d(f)_i = d \circ f_i - (-1)^n f_{i-1} \circ d.$$

Here the symmetry $\gamma: X \otimes Y \rightarrow Y \otimes X$ should be defined with a sign,

$$\gamma(x \otimes y) = (-1)^{pq} y \otimes x \quad \text{for } x \in X_p \text{ and } y \in Y_q.$$

Definition 2. Let \mathcal{M} be a category and \mathcal{V} be a symmetric monoidal category. We say that \mathcal{M} is enriched over \mathcal{V} , or that \mathcal{M} is a \mathcal{V} -category, if there are hom objects $\underline{\mathcal{M}}(X, Y)$ in \mathcal{V} and maps $U \rightarrow \underline{\mathcal{M}}(X, X)$ in \mathcal{V} for objects $X, Y \in \mathcal{M}$ together with associative and unital composition morphisms in \mathcal{V}

$$\underline{\mathcal{M}}(Y, Z) \otimes \underline{\mathcal{M}}(X, Y) \rightarrow \underline{\mathcal{M}}(X, Z).$$

Up to natural isomorphism, the original set-valued hom functions must be given by

$$\mathcal{M}(X, Y) = \mathcal{V}(U, \underline{\mathcal{M}}(X, Y)).$$

The last equation holds when $\mathcal{M} = \mathcal{V}$ and can be taken as a definition when \mathcal{M} is given only in enriched form. It is a nice fact that in practice this always does give the expected answer. This is often obvious. For example, for a space X , $\mathcal{U}(*, X)$ is the underlying set of X , and for an Abelian group A , $\mathcal{A}b(\mathbb{Z}, A)$ is the underlying set of A .

Example 3. Some types of enriched categories have standard names.

- (i) Categories enriched in $\mathcal{A}b$ are called additive categories.
- (ii) Categories enriched in \mathcal{U} are called topological categories.
- (iii) Categories enriched in $sSet$ are called simplicial categories.
- (iv) Categories enriched in any Ch_R are called DG-categories.

Examples of all four sorts are very common. For any ring R , not necessarily commutative, the category of left R -modules is additive. All categories of structured spaces, such as the categories of topological monoids and of topological groups, are topological categories. The letters DG stand for “differential graded”. If A is a DGA (differential graded algebra over some commutative ring R), then the category of chain complexes of left A -modules is enriched over Ch_R via the chain complexes of R -modules $\text{Hom}_A(X, Y)$.

There are many categorical notions that take account of the extra structure given by enrichment. In particular, there are weighted (or indexed) colimits and limits. The most important of these (in non-standard notation) are tensors $X \odot V$ (sometimes called copowers) and cotensors (sometimes called powers) $F(V, X)$ in \mathcal{M} for objects $X \in \mathcal{M}$ and $V \in \mathcal{V}$. These are characterized by natural isomorphisms

$$(0.4) \quad \mathcal{M}(X \odot V, Y) \cong \mathcal{V}(V, \underline{\mathcal{M}}(X, Y)) \cong \mathcal{M}(X, F(V, Y))$$

of hom sets, which again imply natural isomorphisms of objects in \mathcal{V}

$$(0.5) \quad \underline{\mathcal{M}}(X \odot V, Y) \cong \underline{\mathcal{V}}(V, \underline{\mathcal{M}}(X, Y)) \cong \underline{\mathcal{M}}(X, F(V, Y)).$$

Again, examples are very common. If R is a ring, X and Y are left R -modules, and V is an Abelian group, then $X \otimes_{\mathbb{Z}} V$ and $\text{Hom}_{\mathbb{Z}}(V, Y)$ are left R -modules that

give tensors and cotensors in the additive category of left R -modules. This works equally well if X and Y are chain complexes of R -modules and V is a chain complex of Abelian groups.

We say that the \mathcal{V} -category \mathcal{M} is \mathcal{V} -bicomplete if it has all weighted colimits and limits. We dodge the definition of these limits by noting that \mathcal{M} is \mathcal{V} -bicomplete if it is bicomplete in the ordinary sense and has all tensors and cotensors [3, 3.73]. The category \mathcal{V} is itself a \mathcal{V} -bicomplete \mathcal{V} -category. Its tensors and cotensors are given by its product \otimes and internal hom functor $\underline{\mathcal{V}}$.

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