NOTES ON FILTRATIONS, TOPOLOGIES, AND COMPLETIONS

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Contents

1. Filtered groups

1. Let G be a group with a decreasing filtration by normal subgroups G_i. Then G is a topological group. The G_i form a fundamental system of neighborhoods of the identity. The open subsets are the arbitrary unions of finite intersections of cosets gG_i.

2. The following are equivalent.

(i) G is Hausdorff.
(ii) Points in G are closed.
(iii) The G_i intersect in $\{e\}$.

Of course, (i) implies (ii) by general topology. For (ii) implies (i), the diagonal in $G \times G$ is $\mu^{-1}(e)$, where $\mu(g,h) = gh^{-1}$. Since $G - G_i$ is the union of the cosets gG_i with $g \notin G_i$, $G - G_i$ is open, hence $G - G_i$ is both open and closed, hence so is G_i. Now (iii) clearly implies (ii). If x is in all G_i and is not e, then there are no open neighborhoods separating e and x, so G is not Hausdorff.

3. If $H \supset G_i$, then $H - G_i$ is open since it is the union of the cosets hG_i, $h \in H - G_i$ and, similarly, H is also closed.

4. $G/\cap G_i$ is the associated Hausdorff group of G.

5. Consider the canonical map $\gamma\colon G \longrightarrow \lim G/G_i$, obtained from the quotient homomorphisms $\gamma_i\colon G \longrightarrow G/G_i$. Give the target the inverse limit topology, where the G/G_i are discrete. Then γ is continuous since $\gamma_i^{-1}(eG_i) = G_i$. If γ is a bijection, then it is a homeomorphism. Indeed, the G_i then give a fundamental system of neighborhoods of the identity in both. We say that G is complete when this holds.

6. Define the completion of G to be $\hat{G} = \lim G/G_i$; it is more accurate to view the map $\gamma\colon G \longrightarrow \hat{G}$ as the completion of G. Let $\hat{G}_i$ be the kernel of $\hat{G} \longrightarrow G/G_i$. This gives $\hat{G}$ a decreasing filtration, and the topology on $\hat{G}$ is the same as the topology associated to this filtration. Moreover, $G/G_n \cong \hat{G}/\hat{G}_n$.

7. Two filtrations of G give the same topology if for each m and n there exist p and q such that $G_m \subset G'_p$ and $G'_n \subset G_q$. By cofinality, the completions are isomorphic as topological groups (homeomorphic via an isomorphism of groups).

2. Filtered rings and modules

Let R be a commutative ring and M an R-module. We consider decreasing filtrations $R = R_0 \supset R_1 \supset R_2 \supset \cdots$ by ideals such that $R_i \cdot R_j \subset R_{i+j}$. Similarly, we consider decreasing filtrations $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ by sub R-modules such that $R_i \cdot M_j \subset M_{i+j}$. In favorable cases $\cap R_i = \{0\}$ and $\cap M_i = \{0\}$. The notations $F_iR = R_i$ and $F_iM = M_i$ are frequently used. In the most important example, we take an ideal I of R and define $R_i = I^i$ and $M_i = I^iM$. These are called I-adic filtrations.

We can apply the constructions of the previous section to the underlying filtered Abelian groups of R and M. The completion $\hat{R}$ inherits a multiplication from R, and the completion $\hat{M}$ becomes an $\hat{R}$-module. We are interested in understanding the exactness properties of these constructions. We start in the general case in this section and specialize to I-adic completions in the next.

We start over with a filtered R-module M, with no filtration given on $R = R_0$, so that each M_i is an R-module. Of course each M_i is open and closed in the resulting "linear topology" on M. Let N be a sub R-module of M and let $P = M/N$. Then N has the filtration given by $N_i = N \cap M_i$, P has the filtration given by letting P_i be the image of M_i, and N and P have associated linear topologies.

1. The subspace topology on N coincides with the linear topology. Indeed, a subset $X \subset N$ is open in the subspace topology iff $X = N \cap U$ for some open subset U of M, while X is open in the linear topology iff X is a union of finite intersections of subsets of the form $x + N_i = x + N \cap M_i$, $x \in N$. Here U is a union of finite intersections of the form $y + M_i$, but if $N \cap (y + M_i)$ is non-empty, then $y + M_i = x + M_i$ for some $x \in N$.

2. The closure $\bar{N}$ of N in M is given by $\bar{N} = \cap_i(N + M_i)$. Indeed, $x \in \bar{N}$ iff $(x + M_i) \cap N \neq \emptyset$ for all i, and that holds if and only of $x \in N + M_i$ for all i. Therefore N is closed in M iff $\cap(N + M_n) = N$, and this holds iff $\cap P_i = 0$, that is, iff P is Hausdorff in the linear topology.

3. The quotient topology on P coincides with the linear topology. Indeed, let $X \subset P$. Then X is open in the quotient topology iff the inverse image, Y say, of X in M is open. This means that if $y \in Y$ then $y + M_i \subset Y$ for some i. Reducing mod N, this means that if $x \in X$, then $x + P_i \subset X$ for some i, which means that X is open in the linear topology on P.

4. Since $P/P_i \cong M/N + M_i$, we have the short exact sequences

$$0 \longrightarrow N/N \cap M_i \longrightarrow M/M_i \longrightarrow P/P_i \longrightarrow 0.$$

On passage to limits , there results a short exact sequence

$$0 \longrightarrow \hat{N} \longrightarrow \hat{M} \longrightarrow \hat{P} \longrightarrow 0.$$

That is, $\widehat{M/N} \cong \hat{M}/\hat{N}$. Here $\hat{N}$ is the closure of the image of N in $\hat{M}$. Exactness at the left is a general fact on inverse sequences. Exactness at the right uses that the maps in our inverse systems are epimorphisms. In detail, let $(p_j) \in \hat{P}$. Inductively, suppose chosen m_i, $i < j$, such that $m_i \longrightarrow p_i$ and $m_i \longrightarrow m_{i-1}$ for each i. Choose m'_j that maps to p_j. Then $m'_j - m_{j-1}$ is in $N + M_{j-1}$, say $m'_j - m_{j-1} = n + m$.

Let $m_j = m'_j - n = m_{j-1} + m$. Inductively, this gives an element (m_j) of $\hat{M}$ that maps to (p_j).

3. The Artin–Rees Lemma and Krull intersection theorem

Let I be a proper ideal in R and $N \subset M$ be R-modules. Filtering M by the I^iM, we obtain two filtrations on N, namely the I-adic filtration given by the I^iN and the submodule filtration given by the $N \cap I^iM$. Clearly $I^iN \subset N \cap I^iM$. The opposite inclusion fails, but the two linear topologies are sometimes the same. Let R be Noetherian and M be finitely generated throughout this section. Then we have the following result.

Lemma 3.1 (Artin–Rees)**.** *There is an m such that*

$$N \cap I^nM = I^{n-m}(N \cap I^mM)$$

and therefore $N \cap I^nM \subset I^{n-m}N$ for all $n > m$.

Proof. Define a graded ring $B_I(R) = R \oplus I \oplus I^2 \oplus \cdots$; it is called the Rees ring of (R, I). Think of $B_I(R)$ as $R[It] \subset R[t]$ for an indeterminate t. Write E_I^0R (or $gr_I(R)$) for the associated graded ring of R with respect to the I-adic filtration, that is, $\oplus_{i\geq 0} I^i/I^{i+1}$. Observe that $B_I(R)/IB_I(R) \cong E_I^0R$. Thus the construction replaces the associated graded by a simple quotient. Let $M_* = \{M_i\}$ be any decreasing I-filtration of M, meaning that $IM_i \subset M_{i+1}$. Say that the I-filtration is I-stable if $IM_n = M_{n+1}$ for all sufficiently large n. The filtration $\{I^iM\}$ is certainly I-stable, and the claim is that the filtration $\{N_i = N \cap I^iM\}$ is I-stable. Define $B(M_*) = M \oplus M_1 \oplus M_2 \oplus \cdots$ and observe that $B(M_*)$ is a graded $B_I(R)$-module. The second of the following two lemmas is a generalized version of the result we are after. □

Lemma 3.2. *The $B_I(R)$-module $B(M_*)$ is finitely generated iff the I-filtration M_* is I-stable.*

Proof. Suppose that $B(M_*)$ is finitely generated. Its generators lie in the first m terms for some m. Replace the generators by their homogeneous components (or work homogeneously from the start). These components are still finite in number and still generate $B(M_*)$. Thus $B(M_*)$ is generated by the elements of the M_i for $i \leq m$. This implies that $M_m \oplus M_{m+1} \oplus \cdots$ is generated as a $B_I(R)$-module by M_m. This means that $M_{i+m} = I^iM_m$ for $i \geq 0$ or, equivalently, that the filtration is I-stable. Conversely, if $M_{i+m} = I^iM_m$ for some m and all $i \geq 0$, then $B(M_*)$ is generated by the union of the sets of generators of the M_i for $i \leq m$, which is a finite set. □

Lemma 3.3. *Let M_* be any I-stable filtration of M, such as $\{I^iM\}$, and let $N_i = N \cap M_i$. Then N_* is an I-stable filtration of N.*

Proof. Clearly $B(N_*)$ is a sub $B_I(R)$-module of $B(M_*)$. Since M_* is I-stable, $B(M_*)$ is finitely generated. Since I is finitely generated, $B_I(R)$ is finitely generated as an R-algebra. Therefore, by the Hilbert basis theorem, $B_I(R)$ is a Noetherian ring. But then $B(N_*)$ is finitely generated and therefore N_* is I-stable. □

Corollary 3.4 (Krull intersection theorem)**.** *Let $N = \cap I^iM$. Then there exists $r \in R$ such that $1 - r$ is in I and $rN = 0$. If $I \subset \sqrt{R}$, then $N = 0$.*

Proof. By the Artin-Rees theorem and $N \subset I^n M$, there exists m such that

$$N = N \cap I^{m+1} M = I(N \cap I^m M) = IN.$$

The existence of r is now either a standard lemma in the proof of Nakayama's lemma (which says that $N = 0$ if $I \subset \sqrt{R}$) or is sometimes itself referred to as Nakayama's lemma. One proof is by induction on the number of generators of modules N such that $IN = N$. The last statement follows since r is a unit if $I \subset \sqrt{R}$. □

4. I-adic completion

Let I be a proper ideal of a commutative ring R and let M be an R-module. We have the completion $\gamma\colon R \longrightarrow \hat{R}_I = \lim R/I^i$, which is a continuous homomorphism of topological rings. We say that R is complete at I if γ is an isomorphism. We also have the completion $\gamma\colon M \longrightarrow \hat{M}_I = \lim M/I^i M$, which is a continuous homomorphism of topological R-modules. Let $\hat{I} = \{(r_i) | r_0 = 0\}$. Then $\hat{I}$ is an ideal of $\hat{R}_I$ such that $\hat{I}^n = \{(r_i) | r_i = 0 \text{ if } i < n\}$. The associated graded rings $E^0_I R$ and $E^0_{\hat{I}} \hat{R}$ are the same. The Artin–Rees lemma gives the following fundamental result.

Lemma 4.1. *If R is Noetherian, then completion is an exact functor on the category of finitely generated R-modules. Therefore $\hat{R}_I$ is a flat R-module.*

Algebraic topologists must often work with rings that are not Noetherian and modules that are not finitely generated even when R is Noetherian. For them, the "right" notion of completion is not I-adic completion, but rather its zeroth left derived functor (and its first left derived functor is also relevant).

Lemma 4.2. *Let $\mathfrak{m}$ be a maximal ideal of R. Then $\hat{R}_{\mathfrak{m}}$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$, and $R/\mathfrak{m} = \hat{R}_{\mathfrak{m}}/\hat{\mathfrak{m}}$. The completion $R \longrightarrow \hat{R}_{\mathfrak{m}}$ is the composite of the localization $R \longrightarrow R_{\mathfrak{m}}$ and the completion $R_{\mathfrak{m}} \longrightarrow \hat{R}_{\mathfrak{m}}$.*

Proof. We must show that an element (r_i) of $\hat{R}_I$ that is not in $\hat{\mathfrak{m}}$ is a unit. Now $(r_i) \notin \hat{\mathfrak{m}}$ if and only if $r_0 \neq 0$ in $R/\mathfrak{m}$. Since r_i maps to r_0 under $R/\mathfrak{m}^i \longrightarrow R/\mathfrak{m}$, r_i is not in $\mathfrak{m}R/\mathfrak{m}^i$, so is a unit in $R/\mathfrak{m}^i$. The sequence (r_i^{-1}) is $(r_i)^{-1}$ in $\hat{R}_{\mathfrak{m}}$. □

A complete local ring R is a Noetherian local ring which is complete at its maximal ideal $\mathfrak{m}$. Such rings are central to number theory and algebraic geometry.

Example 4.3. The p-adic integers $\hat{\mathbb{Z}}_{(p)}$ are usually denoted $\mathbb{Z}_p$ (or sometimes $\hat{\mathbb{Z}}_p$). They can be represented in terms of "infinite p-adic expansions" $\sum a_i p^i$, where $0 \leq a_i < p$.

Example 4.4. The completion of the polynomial ring $R[x_1, \cdots, x_n]$ at the ideal $I = (x_1, \cdots, x_n)$ is isomorphic to the power series ring $R[[x_1, \cdots, x_n]]$. Explicitly, send a formal power series f to the element $(f \bmod I^i)$ of the completion. For the inverse, consider an element (f_i) of the completion. Here f_i can be represented (mod I^i) as a polynomial of degree less than i in the x_q, and then $f_i = f_{i+1}$ plus terms of degree $i+1$. The formal power series $f_0 + (f_1 - f_0) + (f_2 - f_1) + \cdots$ gives the corresponding element of the power series ring.

It is left as an exercise to prove that if R is I-adically complete, then I is contained in the radical of R. If M is I-adically complete, then it is an $\hat{R}_I$-module and therefore multiplication by $1 + a$, $a \in I$ is an automorphism of M.

It is also left as an exercise to prove that if $I = (a_1, \cdots, a_n)$ is an ideal in a Noetherian ring R, then $\hat{R}_I$ is isomorphic to

$$\hat{R}_I \cong R[[x_1, \ldots, x_n]]/(x_1 - a_1, \cdots, x_n - a_n).$$

This has the following basic consequence.

Corollary 4.5. *If R is Noetherian, then $\hat{R}_I$ is Noetherian.*

One way to work the excercise just cited is to use the following result, which shows that ideal theory passes nicely to completions.

Proposition 4.6. *Let I and J be ideals in a commutative Noetherian ring R and let M be a finitely generated R-module. Then*

$$\widehat{(JM)}_I = J \cdot \widehat{M}_I \quad \text{and} \quad \widehat{(M/JM)}_I \cong \hat{M}_I/J\hat{M}_I.$$

Moreover, $\widehat{(JM)}_I$ is the closure of JM in $\hat{M}_I$.

Proof. By Artin–Rees, the short exact sequence

$$0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow \widehat{(JM)}_I \longrightarrow \widehat{M}_I \longrightarrow \widehat{(M/JM)}_I \longrightarrow 0$$

on passage to I-adic completion, and this implies that $\widehat{(JM)}_I$ is the closure of JM in $\widehat{M}_I$. Certainly $J \cdot \widehat{M}_I \subset \widehat{(JM)}_I$. Let $J = (a_1, \cdots, a_r)$ and define $\phi\colon M^r \longrightarrow M$ by $\phi(m_1, \cdots, m_r) = \sum a_i m_i$. The image of ϕ is JM, so we have an exact sequence

$$M^r \xrightarrow{\phi} M \xrightarrow{\pi} M/JM \longrightarrow 0.$$

On passage to limits , there results an exact sequence

$$(\widehat{M}_I)^r = \widehat{(M^r)}_I \xrightarrow{\hat{\phi}} \widehat{M}_I \xrightarrow{\hat{\pi}} \widehat{(M/JM)}_I \longrightarrow 0.$$

Since the kernel of $\hat{\pi}$ must be $\widehat{(JM)}_I$, this gives $\widehat{(M/JM)}_I \cong \widehat{M}_I/\widehat{(JM)}_I$. Here again, $\hat{\phi}(m_1, \cdots, m_r) = \sum a_i m_i$, where now $m_i \in \widehat{M}_I$. The image of $\hat{\phi}$ is $J \cdot \widehat{M}_I$, and this is equal to the kernel, $\widehat{JM}_I$, of $\hat{\pi}$. □

Corollary 4.7. *Let $\widehat{M}_i$ denote the kernal of the projection $\widehat{M}_I \longrightarrow M/I^iM$. Then $\widehat{M}_i = I^i \cdot \widehat{M}_I$. Thus the linear topology of $\widehat{M}_I$ coincides with its I-adic topology as an R-module, which in turn coincides with its $I \cdot \hat{R}_I$-adic topology as an $\hat{R}_I$-module.*

Proof. $M/I^iM = \widehat{(M/I^iM)}_I$, and the kernel of $\widehat{M}_I \longrightarrow \widehat{(M/I^iM)}_I$ is $I^i \cdot \widehat{M}_I$ by the previous result. □

5. Faithfully flat R-modules

An R-module N is said to be *faithfully flat* if a sequence of R-modules is exact if and only if it becomes exact on tensoring with N. We shall relate this notion to completions. We record the following general result.

Proposition 5.1. *An R-module N is flat if and only if the canonical map*

$$I \otimes_R N \longrightarrow R \otimes_R N \cong N$$

is a monomorphism for all finitely generated ideals I, so that $I \otimes_R N \cong IN$.

Proof. The forward implication is clear. Assume the condition on ideals. Any ideal is the colimit of its finitely generated ideals, and tensoring with N commutes with colimits. We conclude that $I \otimes_R N \longrightarrow N$ is a monomorphism for any ideal I. Let $M' \longrightarrow M$ be a monomorphism. We must show that $M' \otimes_R N \longrightarrow M \otimes_R N$ is a monomorphism. Clearly M is the colimit of the sums $M' + M''$, where $M'' \subset M$ is finitely generated. By induction on the number of generators and passage to colimits, it suffices to show the required monomorphism when $M = M' + Rx$ for some $x \in M$. Let $I = \{r | rx \in M'\}$. We then have a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow R/I \longrightarrow 0.$$

Since we have a short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

and since $I \otimes_R N \longrightarrow R \otimes_R N$ is a monomorphism, we see that $\mathrm{Tor}_1^R(R/I, N) = 0$. Therefore $M' \otimes_R N \longrightarrow M \otimes_R N$ is a monomorphism. □

The following result, together with Nakayama's lemma, shows that flat modules are often necessarily faithfully flat.

Proposition 5.2. *The following conditions on an R-module N are equivalent.*

(i) *N is faithfully flat.*
(ii) *N is flat and $M \otimes_R N = 0$ implies $M = 0$.*
(iii) *N is flat and $\mathfrak{m}N \neq N$ if $\mathfrak{m}$ is a maximal ideal.*

Proof. (i) $\Longrightarrow$ (ii): If $M \otimes_R N = 0$, then $0 \longrightarrow M \longrightarrow 0$ becomes exact after tensoring with N, hence is exact, and $M = 0$.
(ii) $\Longrightarrow$ (iii): $N/\mathfrak{m}N \cong R/\mathfrak{m} \otimes_R N$, so this is clear.
(iii) $\Longrightarrow$ (ii): Assume $M \otimes_R N = 0$ and $x \in M$ is non-zero. Then $Rx \cong R/I$, where I is the annihilator of x. Embed I in a maximal ideal $\mathfrak{m}$. Since $IN \subset \mathfrak{m}N \neq N$, $Rx \otimes_R N \cong N/IN \neq 0$. Since N is flat $Rx \otimes_R N \longrightarrow M \otimes_R N$ is a monomorphism, which contradicts the assumption that $M \otimes_R N = 0$.
(ii) $\Longrightarrow$ (i): Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a sequence such that

$$M' \otimes_R N \xrightarrow{f \otimes \mathrm{id}} M \otimes_R N \xrightarrow{g \otimes \mathrm{id}} M'' \otimes_R N$$

is exact. Then, using that N is flat, $\mathrm{Im}(g \circ f) \otimes_R N = 0$. Therefore $\mathrm{Im}(g \circ f) = 0$ and $g \circ f = 0$. Let $H = \mathrm{Ker(g)}/\mathrm{Im(f)}$. Again, $H \otimes_R N = 0$, hence $H = 0$. □

A ring homomorphism $f \colon R \longrightarrow S$ is said to be faithfully flat if S is faithfully flat as an R-module.

Proposition 5.3. *Let $f \colon R \longrightarrow S$ be a faithfully flat ring homomorphism.*

(i) *For any R-module M, extension of scalars*

$$\mathrm{id} \otimes f \colon M = M \otimes_R R \longrightarrow M \otimes_R S$$

is a monomorphism. In particular, f is a monomorphism.
(ii) *Regard f as an inclusion. If I is an ideal in R, then $IS \cap R = I$.*

Proof. For (i), let $x \in M$ be non-zero. The monomorphism $Rx \otimes_R S \longrightarrow M \otimes_R S$ has image $(x \otimes 1)S$, hence $x \otimes 1$ is non-zero. To see (ii), apply (i) to $M = R/I$, noting that $M \otimes_R S \cong S/IS$. If $r \in R$ is not in I, then it is also not in IS. □

6. Zariski rings

A pair (R, I) consisting of a commutative Noetherian ring R and an ideal $I \subset \sqrt{R}$ is called a *Zariski ring.* The interest of this notion comes from the following result.

Theorem 6.1. *The following conditions are equivalent for an ideal I in a commutative Noetherian ring R.*

(i) $I \subset \sqrt{R}$.
(ii) *Every ideal of R is closed in the I-adic topology.*
(iii) $\hat{R}_I$ *is faithfully flat over R.*

Proof. (i) $\Longrightarrow$ (ii): More generally, $N \subset M$ is closed for any submodule of a finitely generated R-module M since M/N is Hausdorff by the Krull intersection theorem and therefore $\{0\}$ is closed in M/N.
(ii) $\Longrightarrow$ (iii): It suffices to show that $\mathfrak{m}\hat{R}_I \neq \hat{R}_I$ for every maximal ideal $\mathfrak{m}$. Since $\{0\}$ is closed in R, $\cap I^i = 0$ and the completion $\gamma\colon R \longrightarrow \hat{R}_I$ is a monomorphism. Since $\mathfrak{m}$ is closed in R and $\mathfrak{m}\hat{R}_I$ is the closure of $\mathfrak{m}$ in $\hat{R}_I$, $\mathfrak{m}\hat{R}_I \cap R = \mathfrak{m}$ and therefore $\mathfrak{m}\hat{R}_I \neq \hat{R}_I$.
(iii) $\Longrightarrow$ (ii): $\mathfrak{m}\hat{R}_I \neq \hat{R}_I$ for any maximal ideal $\mathfrak{m}$. Since $\hat{R}_I$ is I-adically complete, $\hat{I}_I$ is contained in the radical of $\hat{R}_I$. As in (i) $\Longrightarrow$ (ii), if N is a submodule of a finitely generated $\hat{R}_I$-module M, then N is closed in M. Since γ is continuous, $\mathfrak{m} = \mathfrak{m}\hat{R}_I \cap R$ is closed in R. If I is not contained in $\mathfrak{m}$, then $I^i + \mathfrak{m} = R$ for all $i > 0$, contradicting that $\mathfrak{m}$ is closed in R. Therefore $I \subset \sqrt{R}$. □

Consider a Noetherian local ring R with maximal ideal $\mathfrak{m}$. Obviously, $\mathfrak{m} = \sqrt{R}$. We have proven the following results. Recall that R is said to be complete if it is $\mathfrak{m}$-adically complete.

1. $\cap \mathfrak{m}^i = 0$.
2. If N is a submodule of a finitely generated R-module M, then N is closed in the $\mathfrak{m}$-adic topology. That is, $N = \cap(N + \mathfrak{m}^i N)$.
3. Let $\hat{R} = \hat{R}_\mathfrak{m}$. Then $\hat{R}$ is faithfully flat over R, $R \subset \hat{R}$, and $I = I\hat{R} \cap R$ for any ideal I.
4. $\hat{R}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}\hat{R}$, and $\hat{R}/\mathfrak{m}^i\hat{R} \cong R/\mathfrak{m}^i$ for $i > 0$. In particular, R and $\hat{R}$ have the same residue field.
5. If R is a complete local ring and I is a proper ideal, then R/I is a complete local ring.

7. The I-adic metric on R

Let I be an ideal in R and define $d(x, y)$ to be $1/n$ if $x - y$ is in I^n and not in I^{n+1} and to be 0 if $x - y \in \cap I^n$. Then $d(x, y) = d(y, x)$ and

$$d(x, z) \leq \max(d(x, y), d(y, z)) \leq d(x, y) + d(y, z).$$

Thus d is a pseudo-metric on R, and it is a metric if $\cap I^n = 0$.

Any pseudo-metric space X is normal. If A and B are disjoint closed subset of X, let $U = \{x | d(A, x) < d(B, x)\}$ and $V = \{x | d(B, x) < d(A, x)\}$. Then U and V are disjoint open subsets that contain A and B. Of course, X need not be Hausdorff since points need not be closed.

We assume the reader knows what a Cauchy sequence is and what it means for two Cauchy sequences to be equivalent. We say that X is complete if every Cauchy sequence converges, and every Cauchy sequence then converges to a unique point if X is Hausdorff. We define the completion of X to be the set of equivalence classes of Cauchy sequences with the induced metric topology, where $d((x_n),(y_n))$ is the limit of the $d(x_n,y_n)$. The completion $\gamma\colon X \longrightarrow \hat{X}$ sends x to the constant sequence at x, and it is a continuous map with dense image.

Proposition 7.1. *The completion of R at I is canonically homeomorphic to its completion in the I-adic metric.*

Indeed, the metric topology is the same as the I-adic topology on R. More explicitly, an element (r_i) of $\lim R/I^i$ can be viewed as an equivalence class of Cauchy sequences in R.

8. Hensel's lemma

Here is one fundamental and beautiful reason to care about complete rings, and especially complete local rings.

Lemma 8.1 (Hensel's lemma)**.** *Let R be an I-adically complete Noetherian ring and let $k = R/I$ be the residue ring. Use small letters for polynomials in $k[x]$ and capital letters for polynomials in $R[x]$. Let F be a polynomial in $R[x]$ that reduces mod I to a polynomial $f = gh$ in $k[x]$, where g and h are relatively prime and g is monic. Then there is a factorization $F = GH$ in $R[x]$ such that G and H reduce mod I to g and h and G is monic. If h is also monic, then H can be chosen to be monic and the resulting factorization is unique.*

Sketch Proof. Choose any polynomials G_1 and H_1 that reduce mod I to g and h, taking G_1 to be monic and taking $\deg(G_1) = \deg(g)$ and $\deg(H_1) = \deg(h)$. Proceeding inductively, suppose given G_n and H_n that reduce mod I^n to g and h, where G_n is monic, $\deg(G_n) = \deg(g)$ and $\deg(H_n) = \deg(h)$. Write $F - G_nH_n = \sum a_iJ_i$, where $a_i \in I^n$ and $\deg(J_i) < \deg(F)$. Since $(g,h) = 1$, there are polynomials u_i and v_i such that $j_i = gu_i + hv_i$, and we can arrange that $\deg(u_i) < \deg(h)$ by replacing u_i by its remainder after division by h and adjusting v_i accordingly. Then $\deg(hv_i) = \deg(j_i - gu_i) < \deg(f)$ and therefore $\deg(v_i) < \deg(g)$. Choose U_i and V_i that reduce mod I to u_i and v_i, with $\deg(U_i) = \deg(u_i)$ and $\deg(V_i) = \deg(v_i)$. Set $G_{n+1} = G_n + \sum a_iV_i$ and $H_{n+1} = H_n + \sum a_iU_i$. A quick check shows that $F \equiv G_{n+1}H_{n+1} \bmod I^{n+1}$, G_{n+1} is monic, $\deg(G_{n+1}) = \deg(g)$ and $\deg(H_{n+1}) = \deg(h)$. Then (G_n) and (H_n) are Cauchy sequences (coefficientwise) and we can pass to limits to obtain polynomials G and H as required. When h is monic, we can choose the H_n to be monic, and comparison shows that different choices of the sequences G_n and H_n give equivalent Cauchy sequences. □

Corollary 8.2. *If $F \in R[x]$ and $a \in R$ are such that $F'(a)$ is a unit in R and $F(a) \equiv 0 \bmod I$, then there exists $b \in R$ such that $F(b) = 0$ and $b \equiv a \bmod I$.*

Proof. Reducing mod I, $f(x) = (x - \bar{a})g(x)$. Since

$$f'(x) = g(x) + (x - \bar{a})g'(x),$$

$$g(x) \equiv f'(x) \equiv f'(\bar{a}) \bmod (x - \bar{a}).$$

Since $f'(\bar{a})$ is a unit in k, $g(x)$ and $x - \bar{a}$ generate $k[x]$ and thus are relatively prime. Hensel's lemma gives $F(x) = (x - b)G(x)$, where G reduces to g and $x - b$ reduces to $x - \bar{a}$ mod I. □