NOTES ON TOR AND EXT

Contents

To make life perhaps a little easier for you, I thought I would bang out some notes. Be warned, these may be full of misprints. To ease texing, the first two sections are adapted from my book “A Concise Course in Algebraic Topology”.

1. Basic homological algebra

Let R be a commutative ring. The main example will be $R = \mathbb{Z}$. We develop some rudimentary homological algebra in the category of R-modules.

1.1. **Chain complexes.** A chain complex over R is a sequence of maps of R-modules

$$\cdots \longrightarrow X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \longrightarrow \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all i. We generally abbreviate $d = d_i$. A cochain complex over R is an analogous sequence

$$\cdots \longrightarrow Y^{i-1} \xrightarrow{d^{i-1}} Y^i \xrightarrow{d^i} Y^{i+1} \longrightarrow \cdots$$

with $d^i \circ d^{i-1} = 0$. In practice, we usually require chain complexes to satisfy $X_i = 0$ for $i < 0$ and cochain complexes to satisfy $Y^i = 0$ for $i < 0$. Without these restrictions, the notions are equivalent since a chain complex $\{X_i, d_i\}$ can be rewritten as a cochain complex $\{X^{-i}, d^{-i}\}$, and vice versa.

An element of the kernel of d_i is called a cycle and an element of the image of d_{i+1} is called a boundary. We say that two cycles are "homologous" if their difference is a boundary. We write $B_i(X) \subset Z_i(X) \subset X_i$ for the submodules of boundaries and cycles, respectively, and we define the ith homology group $H_i(X)$ to be the quotient module $Z_i(X)/B_i(X)$. We write $H_*(X)$ for the sequence of R-modules $H_i(X)$. We understand "graded R-modules" to be sequences of R-modules such as this (and we never take the sum of elements in different gradings).

1.2. **Maps and homotopies of maps of chain complexes.** A map $f : X \longrightarrow X'$ of chain complexes is a sequence of maps of R-modules $f_i : X_i \longrightarrow X'_i$ such that $d'_i \circ f_i = f_{i-1} \circ d_i$ for all i. That is, the following diagram commutes for each i:

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "d_i"'] & X'_i \arrow[d, "d'_i"] \\ X_{i-1} \arrow[r, "f_{i-1}"'] & X'_{i-1}. \end{tikzcd}$$

It follows that $f_i(B_i(X)) \subset B_i(X')$ and $f_i(Z_i(X)) \subset Z_i(X')$. Therefore f induces a map of R-modules $f_* = H_i(f) : H_i(X) \longrightarrow H_i(X')$.

A chain homotopy $s : f \simeq g$ between chain maps $f, g : X \longrightarrow X'$ is a sequence of homomorphisms $s_i : X_i \longrightarrow X'_{i+1}$ such that

$$d'_{i+1} \circ s_i + s_{i-1} \circ d_i = f_i - g_i$$

for all i. Chain homotopy is an equivalence relation since if $t : g \simeq h$, then $s + t = \{s_i + t_i\}$ is a chain homotopy $f \simeq h$.

Lemma 1.1. *Chain homotopic maps induce the same homomorphism of homology groups.*

Proof. Let $s : f \simeq g$, $f, g : X \longrightarrow X'$. If $x \in Z_i(X)$, then

$$f_i(x) - g_i(x) = d'_{i+1} s_i(x),$$

so that $f_i(x)$ and $g_i(x)$ are homologous. □

1.3. **Tensor products of chain complexes.** The tensor product (over R) of chain complexes X and Y is specified by letting

$$(X \otimes Y)_n = \sum_{i+j=n} X_i \otimes Y_j.$$

When X_i and Y_i are zero for $i < 0$, the sum is finite, but we don't need to assume this. The differential is specified by

$$d(x \otimes y) = d(x) \otimes y + (-1)^i x \otimes d(y)$$

for $x \in X_i$ and $y \in Y_j$. The sign ensures that $d \circ d = 0$. We may write this as

$$d = d \otimes \mathrm{id} + \mathrm{id} \otimes d.$$

The sign is dictated by the general rule that whenever two entities to which degrees m and n can be assigned are permuted, the sign $(-1)^{mn}$ should be inserted. In the present instance, when calculating $(\mathrm{id} \otimes d)(x \otimes y)$, we must permute the map d of degree -1 with the element x of degree i.

We regard R-modules M as chain complexes concentrated in degree zero, and thus with zero differential. For a chain complex X, there results a chain complex $X \otimes M$; $H_*(X \otimes M)$ is called the homology of X with coefficients in M.

Define a chain complex $\mathscr{I}$ by letting $\mathscr{I}_0$ be the free Abelian group with two generators $[0]$ and $[1]$, letting $\mathscr{I}_1$ be the free Abelian group with one generator $[I]$ such that $d([I]) = [0] - [1]$, and letting $\mathscr{I}_i = 0$ for all other i. We could just as well use free R-modules, but it is nice to have just the single complex $\mathscr{I}$. Observe that the tensor product $M \otimes A$ over $\mathbb{Z}$ of an R-module M and an Abelian group A is an R-module via $r(m \otimes a) = (ra) \otimes a$. Similarly, the tensor product over $\mathbb{Z}$ of an R-chain complex X and a $\mathbb{Z}$-chain complex Y is an R-chain complex.

Lemma 1.2. *A chain homotopy $s : f \simeq g$ between chain maps $f, g : X \longrightarrow X'$ determines and is determined by a chain map $h : X \otimes \mathscr{I} \longrightarrow X'$ such that $h(x, [0]) = f(x)$ and $h(x, [1]) = g(x)$.*

Proof. Let s correspond to h via $(-1)^i s(x) = h(x \otimes [I])$ for $x \in X_i$. The relation

$$d'_{i+1}(s_i(x)) = f_i(x) - g_i(x) - s_{i-1}(d_i(x))$$

corresponds to the relation $d'h = hd$ by the definition of our differential on $\mathscr{I}$. The sign in the correspondence would disappear if we replaced by $X \otimes \mathscr{I}$ by $\mathscr{I} \otimes X$. $\square$

1.4. **Short and long exact sequences.** A sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ of modules is exact if $\mathrm{im}\, f = \ker g$. If $M' = 0$, this means that g is a monomorphism; if $M'' = 0$, it means that f is an epimorphism. A longer sequence is exact if it is exact at each position. A short exact sequence of chain complexes is a sequence

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0$$

that is exact in each degree. Here 0 denotes the chain complex that is the zero module in each degree.

Proposition 1.3. *A short exact sequence of chain complexes naturally gives rise to a long exact sequence of R-modules*

$$\cdots \longrightarrow H_q(X') \xrightarrow{f_*} H_q(X) \xrightarrow{g_*} H_q(X'') \xrightarrow{\partial} H_{q-1}(X') \longrightarrow \cdots .$$

Proof. Write $[x]$ for the homology class of a cycle x. We define the "connecting homomorphism" $\partial : H_q(X'') \longrightarrow H_{q-1}(X')$ by $\partial[x''] = [x']$, where $f(x') = d(x)$ for some x such that $g(x) = x''$. There is such an x since g is an epimorphism, and there is such an x' since $gd(x) = dg(x) = 0$. It is a standard exercise in "diagram chasing" to verify that ∂ is well defined and the sequence is exact. Naturality means that a commutative diagram of short exact sequences of chain complexes gives rise to a commutative diagram of long exact sequences of R-modules. The essential point is the naturality of the connecting homomorphism, which is easily checked. □

1.5. **Dual cochain complexes and** Hom **complexes.** For a chain complex $X = X_*$, we define the dual cochain complex X^* by setting

$$X^n = \mathrm{Hom}(X_n, R) \quad \text{and} \quad d^n = (-1)^n \mathrm{Hom}(d_{n+1}, \mathrm{id}).$$

As with tensor products, we understand Hom to mean Hom_R when R is clear from the context. On elements, for an R-map $f : X_n \longrightarrow R$ and an element $x \in X_{n+1}$,

$$(d^n f)(x) = (-1)^n f(d_n(x)).$$

More generally, for an R-module M, we define a cochain complex $\mathrm{Hom}(X, M)$ in the same way. The sign is conventional. In analogy with the notation $H_*(X; M) = H_*(X \otimes M)$, we write

$$H^*(X; M) = H^*(\mathrm{Hom}(X, M)).$$

More generally, for a cochain complex Y, define

$$\mathrm{Hom}(X, Y)^n = \times_q \mathrm{Hom}(X_q, Y^{n-q})$$

with differential δ specified by

$$(\delta f)(x) = d(f(x)) - (-1)^n f(d(x))$$

More explicitly, writing $f = (f_q)$, $f_q \colon X_q \longrightarrow Y^{n-q}$, this means that $\delta(f) = (g_q)$, where $g_q \colon X_q \longrightarrow Y^{n+1-q}$ is given on $x \in X_q$ as the difference of $d^{n-q}(f_q(x))$ and $(-1)^n f_{q-1} d_q(x)$. When $Y = M$ is concentrated in degree 0, this agrees with the previous definition.

Note that if we take X to be just a chain complex of $\mathbb{Z}$-modules and take Hom's over $\mathbb{Z}$, then the definition still makes sense and gives a complex of R-modules. We have defined Hom's between chain and cochain complexes in the way that they are most frequently used, but, when thinking categorically it makes more sense to regrade all cochain complexes as chain complexes and redefine

$$\mathrm{Hom}(X, Y)_n = \times_q \mathrm{Hom}(X_q, Y_{q+n}).$$

1.6. **Relations between $\otimes$ and** Hom**.** We record a few observations relating $\otimes$ and Hom of complexes, starting with relations between $\otimes$ and Hom on the category of R-modules. For R-modules L, M, and N, we have an adjunction

$$\mathrm{Hom}(L \otimes M, N) \cong \mathrm{Hom}(L, \mathrm{Hom}(M, N)).$$

We also have a natural homomorphism

$$\mathrm{Hom}(L, M) \otimes N \longrightarrow \mathrm{Hom}(L, M \otimes N),$$

and this is an isomorphism if either L or N is a finitely generated projective R-module. Again, we have a natural map

$$\mathrm{Hom}(L, M) \otimes \mathrm{Hom}(L', M') \longrightarrow \mathrm{Hom}(L \otimes L', M \otimes M'),$$

which is an isomorphism if L and L' are finitely generated and projective or if L is finitely generated and projective and $M = R$.

We can replace L and L' by chain complexes and obtain similar maps, inserting signs where needed. For example, a chain homotopy $X \otimes \mathscr{I} \longrightarrow X'$ between chain maps $f, g : X \longrightarrow X'$ induces a chain map

$$\mathrm{Hom}(X', M) \longrightarrow \mathrm{Hom}(X \otimes \mathscr{I}, M) \cong \mathrm{Hom}(\mathscr{I}, \mathrm{Hom}(X, M)) \cong \mathrm{Hom}(X, M) \otimes \mathscr{I}^*,$$

where $\mathscr{I}^* = \mathrm{Hom}(\mathscr{I}, R)$. It should be clear that this implies that our original chain homotopy induces a homotopy of cochain maps

$$f^* \simeq g^* : \mathrm{Hom}(X', M) \longrightarrow \mathrm{Hom}(X, M).$$

2. The universal coefficient and Künneth theorems

We describe some classical results in homological algebra that explain how to calculate $H_*(X; M)$ from $H_*(X) \equiv H_*(X; R)$ and how to calculate $H_*(X \otimes Y)$ from $H_*(X) \otimes H_*(Y)$. We then describe how to calculate $H^*(X; M)$ from $H_*(X)$. We again work over a general commutative ring R, and $\otimes$ and Hom are implicitly understood to be taken over R.

2.1. **Universal coefficients in homology.** Let X and Y be chain complexes over R. We think of $H_*(X) \otimes H_*(Y)$ as a graded R-module which, in degree n, is $\sum_{p+q=n} H_p(X) \otimes H_q(Y)$. We define

$$\alpha : H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \otimes Y)$$

by $\alpha([x] \otimes [y]) = [x \otimes y]$ for cycles x and y that represent homology classes $[x]$ and $[y]$. As a special case, for an R-module M we have

$$\alpha : H_*(X) \otimes M \longrightarrow H_*(X \otimes M).$$

We omit the proof of the following standard result, but we shall shortly give the quite similar proof of a cohomological analogue. Recall that an R-module M is said to be flat if the functor $M \otimes N$ is exact (that is, preserves exact sequences in the variable N). We say that a graded R-module is flat if each of its terms is flat.

We shall treat torsion products, which measure the failure of tensor products to be exact functors, shortly. For a principal ideal domain (PID) R, the only torsion product is the first one, denoted $\mathrm{Tor}_1^R(M, N)$. It can be computed by constructing a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

and tensoring with N to obtain an exact seqence

$$0 \longrightarrow \mathrm{Tor}_1^R(M, N) \longrightarrow F_1 \otimes N \longrightarrow F_0 \otimes N \longrightarrow M \otimes N \longrightarrow 0,$$

where F_1 and F_0 are free R-modules. That is, we choose an epimorphism $F_0 \longrightarrow M$ and note that, since R is a PID, its kernel F_1 is also free.

Theorem 2.1 (Universal coefficient)**.** *Let R be a PID and let X be a flat chain complex over R. Then, for each n, there is a natural short exact sequence*

$$0 \longrightarrow H_n(X) \otimes M \xrightarrow{\alpha} H_n(X \otimes M) \xrightarrow{\beta} \mathrm{Tor}_1^R(H_{n-1}(X), M) \longrightarrow 0.$$

The sequence splits, so that

$$H_n(X \otimes M) \cong (H_n(X) \otimes M) \oplus \mathrm{Tor}_1^R(H_{n-1}(X), M),$$

but the splitting is not natural.

Remark 2.2. The result holds more generally when R is any ring and the chains and cycles of X are also flat R-modules.

Corollary 2.3. *If R is a field, then*

$$\alpha : H_*(X) \otimes M \longrightarrow H_*(X; M)$$

is a natural isomorphism.

2.2. **The Künneth theorem.** The universal coefficient theorem in homology is a special case of the Künneth theorem.

Theorem 2.4 (Künneth)**.** *Let R be a PID and let X be a flat chain complex and Y be any chain complex. Then, for each n, there is a natural short exact sequence*

$$0 \longrightarrow \sum_{p+q=n} H_p(X) \otimes H_q(Y) \xrightarrow{\alpha} H_n(X \otimes Y) \xrightarrow{\beta} \sum_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(Y)) \longrightarrow 0.$$

The sequence splits, so that

$$H_n(X \otimes Y) \cong (\sum_{p+q=n} H_p(X) \otimes H_q(Y)) \oplus (\sum_{p+q=n-1} \mathrm{Tor}_1^R(H_p(X), H_q(Y))),$$

but the splitting is not natural.

This applies directly to the computation of the homology of the Cartesian product of CW complexes X and Y in view of the isomorphism

$$C_*(X \times Y) \cong C_*(X) \otimes C_*(Y).$$

Corollary 2.5. *If R is a field, then*

$$\alpha : H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \otimes Y)$$

is a natural isomorphism.

We prove the corollary to give the idea. The general case is proved by an elaboration of the argument. There is a simple but important technical point to make here. Let us for the moment remember to indicate the ring over which we are taking tensor products. For chain complexes X and Y over $\mathbb{Z}$, we have

$$(X \otimes_{\mathbb{Z}} R) \otimes_R (Y \otimes_{\mathbb{Z}} R) \cong (X \otimes_{\mathbb{Z}} Y) \otimes_{\mathbb{Z}} R.$$

We can therefore use the corollary to compute $H_*(X \otimes_{\mathbb{Z}} Y; R)$ from $H_*(X; R)$ and $H_*(Y; R)$.

Proof of the corollary. Assume first that $X_i = 0$ for $i \neq p$, so that $X = X_p$ is just an R-module with no differential. The square commutes and the row and column are exact in the diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
0 \arrow[r] & B_q(Y) \arrow[r] & Z_q(Y) \arrow[r] \arrow[d] & H_q(Y) \arrow[r] & 0. \\
& Y_{q+1} \arrow[u, "d_{q+1}"] \arrow[r, "d_{q+1}"'] & Y_q \arrow[d, "d_q"] & & \\
& & Y_{q-1} & &
\end{tikzcd}$$

Since all modules over a field are free and thus flat, this remains true when we tensor the diagram with X_p. This proves that if $n = p + q$, then

$$Z_n(X_p \otimes Y) = X_p \otimes Z_q(Y), \quad B_n(X_p \otimes Y) = X_p \otimes B_q(Y),$$

and therefore

$$H_n(X \otimes Y) = X_p \otimes H_q(Y).$$

In the general case, regard the graded modules $Z(X)$ and $B(X)$ as chain complexes with zero differential. The exact sequences

$$0 \longrightarrow Z_p(X) \longrightarrow X_p \xrightarrow{d_p} B_{p-1}(X) \longrightarrow 0$$

of R-modules define a short exact seqence of chain complexes since $d_{p-1} \circ d_p = 0$. Define the suspension of a graded R-module N by $(\Sigma N)_{n+1} = N_n$. Tensoring with Y, we obtain a short exact sequence of chain complexes

$$0 \longrightarrow Z(X) \otimes Y \longrightarrow X \otimes Y \longrightarrow \Sigma B(X) \otimes Y \longrightarrow 0.$$

It follows from the first part and additivity that

$$H_*(Z(X) \otimes Y) = Z(X) \otimes H_*(Y) \quad \text{and} \quad H_*(\Sigma B(X) \otimes Y) = \Sigma B(X) \otimes H_*(Y).$$

Moreover, by inspection of definitions, the connecting homomorphism of the long exact sequence of homology modules associated to our short exact sequence of chain complexes is just the inclusion $B \otimes H_*(Y) \longrightarrow Z \otimes H_*(Y)$. In particular, the long exact sequence breaks up into short exact sequences

$$0 \longrightarrow B(X) \otimes H_*(Y) \longrightarrow Z(X) \otimes H_*(Y) \longrightarrow H_*(X \otimes Y) \longrightarrow 0.$$

However, since tensoring with $H_*(Y)$ is an exact functor, the cokernel of the inclusion $B \otimes H_*(Y) \longrightarrow Z \otimes H_*(Y)$ is $H_*(X) \otimes H_*(Y)$. The conclusion follows. $\square$

2.3. **Universal coefficients in cohomology.** We have a cohomological version of the universal coefficient theorem. We shall treat Ext modules, which measure the failure of Hom to be an exact functor, shortly. For a PID R, the only Ext module is the first one, denoted $\mathrm{Ext}^1_R(M, N)$. It can be computed by constructing a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

and applying Hom to obtain an exact seqence

$$0 \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(F_0, N) \longrightarrow \mathrm{Hom}(F_1, N) \longrightarrow \mathrm{Ext}^1_R(M, N) \longrightarrow 0,$$

where F_1 and F_0 are free R-modules.

For each n, define

$$\alpha : H^n(\mathrm{Hom}(X,M)) \longrightarrow \mathrm{Hom}(H_n(X),M)$$

by letting $\alpha[f]([x]) = f(x)$ for a cohomology class $[f]$ represented by a "cocycle" $f : X_n \longrightarrow M$ and a homology class $[x]$ represented by a cycle x. It is easy to check that $f(x)$ is independent of the choices of f and x since x is a cycle and f is a cocycle.

Theorem 2.6 (Universal coefficient)**.** *Let R be a PID and let X be a free chain complex over R. Then, for each n, there is a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_R(H_{n-1}(X),M) \xrightarrow{\beta} H^n(X;M) \xrightarrow{\alpha} \mathrm{Hom}(H_n(X),M) \longrightarrow 0.$$

The sequence splits, so that

$$H^n(X;M) \cong \mathrm{Hom}(H_n(X),M) \oplus \mathrm{Ext}^1_R(H_{n-1}(X),M),$$

but the splitting is not natural.

Corollary 2.7. *If R is a field, then*

$$\alpha : H^*(X;M) \longrightarrow \mathrm{Hom}(H_*(X),M)$$

is a natural isomorphism.

Again, there is a technical point to be made here. If X is a complex of free Abelian groups and M is an R-module, such as R itself, then

$$\mathrm{Hom}_{\mathbb{Z}}(X,M) \cong \mathrm{Hom}_R(X \otimes_{\mathbb{Z}} R, M).$$

One way to see this is to observe that, if B is a basis for a free Abelian group F, then $\mathrm{Hom}_{\mathbb{Z}}(F,M)$ and $\mathrm{Hom}_R(F \otimes_{\mathbb{Z}} R, M)$ are both in canonical bijective correspondence with maps of sets $B \longrightarrow M$. More algebraically, a homomorphism $f : F \longrightarrow M$ of Abelian groups determines the corresponding map of R-modules as the composite of $f \otimes \mathrm{id}$ and the action of R on M:

$$F \otimes_{\mathbb{Z}} R \longrightarrow M \otimes_{\mathbb{Z}} R \longrightarrow M.$$

2.4. **Proof of the universal coefficient theorem.** We need two properties of Ext in the proof. First, $\mathrm{Ext}^1_R(F,M) = 0$ for a free R-module F. Second, when R is a PID, a short exact sequence

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$$

of R-modules gives rise to a six-term exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(L'',M) &\longrightarrow \mathrm{Hom}(L,M) \longrightarrow \mathrm{Hom}(L',M) \\ &\xrightarrow{\delta} \mathrm{Ext}^1_R(L'',M) \longrightarrow \mathrm{Ext}^1_R(L,M) \longrightarrow \mathrm{Ext}^1_R(L',M) \longrightarrow 0. \end{aligned}$$

Proof of the universal coefficient theorem. We write $B_n = B_n(X)$, $Z_n = Z_n(X)$, and $H_n = H_n(X)$ to abbreviate notation. Since each X_n is a free R-module and R is a PID, each B_n and Z_n is also free. We have short exact sequences

$$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \xrightarrow{\pi_n} H_n \longrightarrow 0$$

and

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, "j_n"] & X_n \arrow[r, shift left, "d_n"] & B_{n-1} \arrow[l, shift left, dashed, "\sigma_n"] \arrow[r] & 0; \end{tikzcd}$$

we choose a splitting σ_n of the second. Writing $f^* = \mathrm{Hom}(f, M)$ consistently, we obtain a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(H_n, M) \arrow[r, "\pi_n^*"] & \mathrm{Hom}(Z_n, M) \arrow[u] \arrow[r, "i_n^*"] & \mathrm{Hom}(B_n, M) \arrow[d, "d_{n+1}^*"] & \\
\cdots \arrow[r] & \mathrm{Hom}(X_{n-1}, M) \arrow[r, "d_n^*"] \arrow[d, "j_{n-1}^*"'] & \mathrm{Hom}(X_n, M) \arrow[u, "j_n^*"'] \arrow[r, "d_{n+1}^*"] \arrow[d, dashed, shift right, "\sigma_n^*"'] & \mathrm{Hom}(X_{n+1}, M) \arrow[r] & \cdots \\
 & \mathrm{Hom}(Z_{n-1}, M) \arrow[r, "i_{n-1}^*"'] \arrow[d] & \mathrm{Hom}(B_{n-1}, M) \arrow[u, shift right, "d_n^*"'] \arrow[r, "\delta"'] \arrow[ur, dashed, "0"'] & \mathrm{Ext}_R^1(H_{n-1}, M) \arrow[r] & 0 \\
 & 0 & 0 \arrow[u] & &
\end{tikzcd}$$

By inspection of the diagram, we see that the canonical map α coincides with the composite

$$H^n(X; M) = \ker d_{n+1}^* / \operatorname{im} d_n^* = \ker i_n^* j_n^* / \operatorname{im} d_n^* i_{n-1}^* \xrightarrow{j_n^*} \operatorname{im} \pi_n^* \xrightarrow{(\pi_n^*)^{-1}} \mathrm{Hom}(H_n, M).$$

Since j_n^* is an epimorphism, so is α. The kernel of α is $\operatorname{im} d_n^* / \operatorname{im} d_n^* i_{n-1}^*$, and $\delta (d_n^*)^{-1}$ maps this group isomorphically onto $\mathrm{Ext}_R^1(H_{n-1}, M)$. The composite $\delta \sigma_n^*$ induces the required splitting. □

2.5. **Künneth relations for cochain complexes.** If Y and Y' are cochain complexes, then we have the natural homomorphism

$$\alpha : H^*(Y) \otimes H^*(Y') \longrightarrow H^*(Y \otimes Y')$$

given by $\alpha([y] \otimes [y']) = [y \otimes y']$, exactly as for chain complexes. In fact, by regrading, we may view this as a special case of the map for chain complexes. The Künneth theorem applies to this map. For its flatness hypothesis, it is useful to remember that, for any Noetherian ring R, the dual $\mathrm{Hom}(F, R)$ of a free R-module is a flat R-module.

As indicated in §1.6, if $Y = \mathrm{Hom}(X, M)$ and $Y' = \mathrm{Hom}(X', M')$ for chain complexes X and X' and R-modules M and M', then we also have the map of cochain complexes

$$\omega : \mathrm{Hom}(X, M) \otimes \mathrm{Hom}(X', M') \longrightarrow \mathrm{Hom}(X \otimes X', M \otimes M')$$

specified by the formula

$$\omega(f \otimes f')(x \otimes x') = (-1)^{(\deg f')(\deg x)} f(x) \otimes f'(x').$$

Also writing ω for the map it induces on cohomology, we then have the composite

$$\omega \circ \alpha : H^*(X; M) \otimes H^*(X'; M') \longrightarrow H^*(X \otimes X'; M \otimes M').$$

When $M = M' = A$ is a commutative R-algebra, we may compose with the map

$$H^*(X \otimes X'; A \otimes A) \longrightarrow H^*(X \otimes X'; A)$$

induced by the multiplication of A to obtain a map

$$H^*(X; A) \otimes H^*(X'; A) \longrightarrow H^*(X \otimes X'; A).$$

The cases most commonly used are when $R = \mathbb{Z}$ and A is either $\mathbb{Z}$ or a field. For example, this is a starting point for the definition of the cup product in topology.

3. Torsion products

We give a quick development of the basic theory of torsion products. Here we change our point of view and work more generally with non-commutative rings R and their right and left modules. Thus, in general, $M \otimes_R N$ is only an Abelian group. When R is commutative, all of our functors take values in the category of R-modules rather than just Abelian groups.

3.1. **Projective resolutions.** We work with right R-modules here, but we could equally well work with left R-modules. Recall that an R-module P is said to be projective if for each epimorphism $g\colon M \longrightarrow N$ and each map $f\colon P \longrightarrow N$, there exists a map $\tilde{f}\colon P \longrightarrow M$ such that $g \circ \tilde{f} = f$. This means that

$$\mathrm{Hom}(\mathrm{id}, g)\colon \mathrm{Hom}_R(P, M) \longrightarrow \mathrm{Hom}(P, N)$$

is an epimorphism. There is a standard characterization.

Lemma 3.1. *A module P is projective if and only if it is a direct summand of a free module. Any projective module is flat.*

Proof. It is clear that R itself is projective and any direct sum of projective modules is projective. Therefore every free module is projective. If there is a module Q such that $P \oplus Q = F$ is free, then, using the inclusion $P \longrightarrow F$ and projection $F \longrightarrow P$, we see that P is projective. If P is projective and $g\colon F \longrightarrow P$ is an epimorphism with F free, as can always be chosen, then application of projectivity to the identity map $P \longrightarrow P$ shows that P is a direct summand of F. Since direct sums and direct summands of flat modules are flat, the second statement follows. $\square$

Let M be an R-module. A complex of R-modules over M is a complex of the form

$$\cdots \longrightarrow X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \longrightarrow \cdots \longrightarrow X_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

It is a projective (or free or flat) complex over M if each X_i is projective (or free or flat). It is a resolution of M if the displayed sequence is exact. It is appropriate to think of M itself as a complex concentrated in degree 0 and $\varepsilon\colon X \longrightarrow M$ as a morphism of complexes. If X is a resolution of M, then ε induces an isomorphism on homology since Coker $d_1 = M$.

Lemma 3.2. *Every R-module M has a projective resolution.*

Proof. Every module is a quotient of a free module. Start with an epimorphism $\varepsilon\colon X_0 \longrightarrow M$ with X_0 free. Then choose an epimorphism $X_1 \longrightarrow \mathrm{Ker}\ \varepsilon$, with X_1 free and let d_1 be its composite with the inclusion $\mathrm{Ker}\ \varepsilon \subset X_0$. Continue inductively to construct epimorphisms $X_{i+1} \longrightarrow \mathrm{Ker}\ d_i$. $\square$

Projective resolutions are unique up to chain homotopy equivalence, as we see by taking $f = \mathrm{id}$ in the following result.

Lemma 3.3. *Let $f\colon M \longrightarrow N$ be a map of R-modules. Let $\varepsilon\colon X \longrightarrow M$ be a projective complex over M and let $\zeta\colon Y \longrightarrow N$ be a resolution of N. Then there is a map $\tilde{f}\colon X \longrightarrow Y$ of chain complexes such that $\zeta \circ \tilde{f} = f \circ \varepsilon$, and $\tilde{f}$ is unique up to chain homotopy.*

Proof. Consider the following diagram.

$$
\begin{tikzcd}
\cdots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\tilde{f}_{i+1}"'] & X_i \arrow[r, "d_i"] \arrow[d, "\tilde{f}_i"'] & X_{i-1} \arrow[r] \arrow[d, "\tilde{f}_{i-1}"] & \cdots \arrow[r] & X_0 \arrow[r, "\varepsilon"] \arrow[d, "\tilde{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Y_{i+1} \arrow[r, "d_{i+1}"'] & Y_i \arrow[r, "d_i"'] & Y_{i-1} \arrow[r] & \cdots \arrow[r] & Y_0 \arrow[r, "\zeta"'] & N \arrow[r] & 0
\end{tikzcd}
$$

Since X_0 is projective and ζ is an epimorphism, there exists a map $\tilde{f}_0$ such that $\zeta\tilde{f}_0 = f\varepsilon$. Inductively (thinking of f as f_{-1} and d_{-1} as ε or ζ), suppose given a map $\tilde{f}_{i-1}$, $i \geq 1$, such that $d_{i-1}\tilde{f}_{i-1} = \tilde{f}_{i-2}d_{i-1}$. Then $d_{i-1}\tilde{f}_{i-1}d_i = 0$, so that $\tilde{f}_{i-1}d_i$ maps X_i to $\operatorname{Ker} d_{i-1} = \operatorname{Im} d_i$. We obtain $\tilde{f}_i$ such that $d_i\tilde{f}_i = \tilde{f}_{i-1}d_i$ by the projectivity of X_i.

Now assume given another such map $\tilde{f}'$. Then $\zeta(\tilde{f}_0 - \tilde{f}'_0) = 0$, so by the projectivity of X_0 and surjectivity of ζ, we obtain $s_0 \colon X_0 \longrightarrow Y_1$ such that $d_1 s_0 = \tilde{f}_0 - \tilde{f}'_0$. Inductively (thinking of s_{-1} as zero), suppose given s_{i-1}, $i \geq 1$, such that $d_i s_{i-1} + s_{i-2}d_{i-1} = \tilde{f}_{i-1} - \tilde{f}'_{i-1}$. Then $d_i(\tilde{f}_i - \tilde{f}'_i - s_{i-1}d_i) = 0$, so there exists $s_i \colon X_i \longrightarrow Y_{i+1}$ such that $d_{i+1}s_i = \tilde{f}_i - \tilde{f}'_i - s_{i-1}d_i$ by the projectivity of X_i. □

We need another such lemma to deal with long exact sequences of Tor groups.

Lemma 3.4. *Let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence of R-modules. Let X' and X'' be projective resolutions of M' and M''. Then there is a projective resolution X of M and an exact sequence of resolutions over the given exact sequence:*

$$
\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "\tilde{f}"] \arrow[d, "\varepsilon'"'] & X \arrow[r, "\tilde{g}"] \arrow[d, "\varepsilon"] & X'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0.
\end{tikzcd}
$$

Proof. Define $X_i = X'_i \oplus X''_i$ and let $\tilde{f}$ and $\tilde{g}$ be the canonical inclusion and projection. We must construct an epimorphism $\varepsilon \colon X_0 \longrightarrow M$ and maps $d_i \colon X_i \longrightarrow X_{i-1}$ that give a complex over M such that the diagram commutes. By the long exact sequence associated to the resulting short exact sequence of chain complexes, it will follow that X is a projective resolution of M. The restriction of ε to X'_0 must be ε', and the restriction of d_i to X'_i must be d'_i. Choose the restriction of ε to X''_0 to be any map $\zeta \colon X''_0 \longrightarrow M$ such that $g\zeta = \varepsilon''$. There is such a map since X''_0 is projective, and a diagram chase shows that ε is then an epimorphism. For the diagram to commute, the restriction of d_i, $i \geq 1$, to X''_0 must be of the form (e_i, d''_i), $e_i \colon X''_i \longrightarrow X'_{i-1}$. For X to be a complex over M, we must have $\varepsilon' e_1 = -\zeta d''_1$ and, for $i \geq 2$, $d'_{i-1}e_i = -e_{i-1}d''_i$. These will ensure that $\varepsilon d_1 = 0$ and $d_{i-1}d_i = 0$. Since $\varepsilon'' d''_1 = 0$, $\zeta d''_1$ can be viewed as a map $X''_1 \longrightarrow M'$, and we can choose such an e_1. Given e_{i-1}, $e_{i-1}d''_i \colon X''_i \longrightarrow X'_{i-2}$ takes values in $\operatorname{Ker} d'_{i-2} = \operatorname{Im} d'_{i-1}$, and we can choose such an e_i. □

3.2. **The definition and properties of** Tor**.** For a short exact sequence of right R-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ and a left R-module N, the sequence of Abelian groups

$$M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \longrightarrow 0$$

is exact, but the left-most arrow need not be a monomorphism. We say that $\otimes_R$ is right exact. Torsion products measure the deviation from exactness. Here is an omnibus theorem that states the basic properties of torsion products.

Theorem 3.5. *There are Abelian group–valued functors* $\mathrm{Tor}_n^R(M,N)$ *of right* R*-modules* M *and left* R*-modules* N*, together with natural connecting homomorphisms*

$$\partial\colon \mathrm{Tor}_n^R(M'',N) \longrightarrow \mathrm{Tor}_{n-1}^R(M',N) \quad \textit{and} \quad \partial\colon \mathrm{Tor}_n^R(M,N'') \longrightarrow \mathrm{Tor}_{n-1}^R(M,N')$$

for short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0.$$

These satisfy the following properties.

(i) $\mathrm{Tor}_n^R(M,N) = 0$ *for* $n < 0$.
(ii) $Tor_0^R(M,N)$ *is naturally isomorphic to* $M \otimes_R N$.
(iii) $\mathrm{Tor}_n^R(M,N) = 0$ *for* $n > 0$ *if either* M *or* N *is projective.*
(iv) *The following sequences are exact.*

$$\cdots \longrightarrow \mathrm{Tor}_n^R(M',N) \longrightarrow \mathrm{Tor}_n^R(M,N) \longrightarrow \mathrm{Tor}_n^R(M'',N) \longrightarrow \mathrm{Tor}_{n-1}^R(M',N) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Tor}_n^R(M,N') \longrightarrow \mathrm{Tor}_n^R(M,N) \longrightarrow \mathrm{Tor}_n^R(M,N'') \longrightarrow \mathrm{Tor}_{n-1}^R(M,N') \longrightarrow \cdots$$

For each fixed N*, the functors* $\mathrm{Tor}_n^R(M,N)$ *of* M *together with the natural connecting homomorphisms* $\partial\colon \mathrm{Tor}_n^R(M'',N) \longrightarrow \mathrm{Tor}_{n-1}^R(M',N)$ *on exact sequences* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *are uniquely determined up to isomorphism by (i)—(iv), and similarly for each fixed* M.

Proof. The naturality statements imply that the long exact sequences of (iv) are functorial on short exact sequences. For the existence statement, let us fix N. Choose a projective resolution X of M. Define

$$\mathrm{Tor}_*^R(M,N) = H_*(X \otimes_R N)$$

Since chain homotopic maps of complexes induce the same map on homology, Lemma 3.3 shows that this is well-defined up to natural isomorphism and gives a functor of M. Manifestly, it also gives a functor of N. Since $\otimes_R$ is right exact, (ii) is clear. Since the identity map $X_0 = M \longrightarrow M$ is a projective resolution of a projective module M and since projective modules are flat, both parts of (iii) are also clear. We define the first map ∂ and derive the first long exact sequence of (iv) by use of Lemma 3.4. We define the second map ∂ and derive the second long exact sequence by use of the short exact sequence of chain complexes

$$0 \longrightarrow X \otimes N' \longrightarrow X \otimes N \longrightarrow X \otimes N'' \longrightarrow 0.$$

Here exactness holds by the projectivity and thus flatness of the X_i.

For the axiomatization, we proceed by induction on n, starting from (i) and (ii). If we have two systems of functors and natural connecting homomorphisms and we have proven they are isomorphic through stage $(n-1)$, then a diagram chase starting from short exact sequences $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ where M is free shows that they are isomorphic at the stage n.

We can reverse the roles of M and N in our original construction, starting from a projective resolution of N. We again have all of the properties (i) – (iv), so by the uniqueness we obtain the same sequence of functors and natural connecting homomorphisms. Moreover, we can also check that $H_*(X \otimes_R Y)$ gives functors and natural connecting homomorphisms that satisfy the axioms. Altenatively, we can observe that the morphisms of complexes $X \longrightarrow M$ and $Y \longrightarrow N$ induce morphisms of complexes

$$X \otimes_R M \longleftarrow X \otimes_R Y \longrightarrow Y \otimes_R N.$$

Either by a direct, hands on, Künneth type argument, or more elegantly by an application of the theory of spectral sequences if one knows about that, one can check directly that these give isomorphisms. □

3.3. **Change of rings.** In applications, it is very often the case that we use further structure on torsion products. We describe some of this here and in the next section. We take the opportunity to introduce and use some standard and important adjoint functors on the way.

In this section, we fix a homomorphism of rings $f\colon R \longrightarrow S$. Let M and P be a right R module and a right S-module, and let $g\colon M \longrightarrow P$ be an f-equivariant map, meaning that $g(mr) = g(m)f(r)$. Similarly, let N and Q be a left R-module and a left S-module and let $h\colon N \longrightarrow Q$ be f-equivariant, $h(rn) = f(r)h(n)$. We shall obtain a map

$$\mathrm{Tor}^f_*(g,h)\colon \mathrm{Tor}^R_*(M,N) \longrightarrow \mathrm{Tor}^S_*(P,Q). \tag{3.6}$$

This allows us to view Tor as a functor of all three variables.

Let us write f^* for the pullback of action functor $\mathscr{M}_S \longrightarrow \mathscr{M}_R$, where $\mathscr{M}_R$ denotes the category of right R-modules. Thus f^*P is P with the right R-action $pr = pf(r)$. Clearly g is just a map of right R-modules $M \longrightarrow f^*P$. The functor f^* has left and right adjoints, often denoted $f_!$ and f_*, that are given by extension and coextension of scalars. Explicitly,

$$f_!M = M \otimes_R S \quad \text{and} \quad f_*M = \mathrm{Hom}_R(S, M).$$

We are using that S is both a left and a right S-module and therefore a left and right R-module via f. In defining $M \otimes_R S$, we use the left action of R to define the tensor product and use the right action of S to give $M \otimes_R S$ an S-action, whereas $\mathrm{Hom}_R(S, M)$ is the Abelian group of maps of right R-modules $k\colon S \longrightarrow M$, with right S-action given by $(ks)(s') = k(ss')$. That is, it is induced by the left action of S on itself. The adjunctions read

$$\mathrm{Hom}_S(f_!M, P) \cong \mathrm{Hom}_R(M, f^*P) \quad \text{and} \quad \mathrm{Hom}_R(f^*P, M) \cong \mathrm{Hom}_S(P, f_*M).$$

The easy verifications are left to the reader.

The functor $f_!$ takes R to S and preserves direct sums, hence it takes R-projective modules to S-projective modules. If f^*S is flat as an R-module, in which case we say that f is a "flat extension" of R, then $f_!$ takes exact sequences of R-modules to exact sequences of S-modules. Now Lemma 3.3 gives the following result.

Lemma 3.7. *Let X be a projective R-resolution of M and Y be a projective S-resolution of P. Then there is a map $\tilde{g}\colon f_!X \longrightarrow Y$ of S-projective complexes over the adjoint $\hat{g}\colon f_!M \longrightarrow P$ of the map $g\colon M \longrightarrow f^*P$. It is a map of projective resolutions if S is a flat extension of R.*

We agree to write $f_!$ for both of the functors $M \otimes_R S$ and $S \otimes_R N$. We have the natural homomorphism of Abelian groups

$$\zeta\colon M \otimes_R N \longrightarrow f_!M \otimes_S f_!N \cong M \otimes_R S \otimes_R N$$

specified by $\zeta(m\otimes n) = m\otimes 1\otimes n$. With the notations of the previous lemma, and using the adjoint $\hat{h}\colon f_!N \longrightarrow Q$ of $h\colon N \longrightarrow f^*Q$, this gives a natural composite

$$X\otimes_R N \longrightarrow f_!X\otimes_S f_!N \longrightarrow Y\otimes_S Q.$$

Passing to homology, we obtain the promised map (3.6). A moment's reflection shows that it factors naturally as the composite

$$\mathrm{Tor}^R_*(M,N)\xrightarrow{\mathrm{Tor}^{\mathrm{id}}_*(\eta,\eta)}\mathrm{Tor}^R_*(f^*f_!M,f^*f_!N)\xrightarrow{\mathrm{Tor}^{f}_*(\mathrm{id},\mathrm{id})}\mathrm{Tor}^S_*(f_!M,f_!N)\xrightarrow{\mathrm{Tor}^{f}_*(\hat{g},\hat{h})}\mathrm{Tor}^S_*(P,Q)$$

where $\eta\colon M\longrightarrow f^*f_!M$ is the unit of the adjunction, $\eta(m) = m\otimes 1\in M\otimes_R S$.

Remember that when R is commutative, $\mathrm{Tor}^R_*(M,N)$ is a graded R-module. The following result could be obtained a little more directly but it illustrates ideas to derive it from what we have done.

Proposition 3.8. *Let T be a multiplicative subset of a commutative ring R and let $S = R_T = R[T^{-1}]$ be the localization of R at T. For all R-modules M and N, the canonical map*

$$\mathrm{Tor}^R_*(M,N)\longrightarrow \mathrm{Tor}^S_*(M\otimes_R S, N\otimes_R S)$$

is localization at T.

Proof. The localization functor sends M to $f_!M = M\otimes_R S$, where $f\colon R\longrightarrow S$ is the localization map. Of course, f is a flat extension. The target of our map is an S-module and is therefore T-local. After tensoring with S, our map becomes the isomorphism on homology induced by the canonical isomorphism of chain complexes

$$(X\otimes_R N)\otimes_R S\cong (X\otimes_R S)\otimes_S (N\otimes_R S)$$

where X is an R-projective resolution of M. □

3.4. **Pairings and products.** Again let R and S be rings. In applications, it is often the case that our rings are algebras over a commutative ring k. Taking tensor products over k rather than $\mathbb{Z}$ and writing $\otimes = \otimes_k$, the constructions of this section generalize directly. Let M and N be right and left R-modules, and let P and Q be right and left S-modules. Letting γ be the symmetry isomorphism, $R\otimes S$ is a ring with product

$$R\otimes S\otimes R\otimes S\xrightarrow{\mathrm{id}\otimes\gamma\otimes\mathrm{id}} R\otimes R\otimes S\otimes S\longrightarrow R\otimes S$$

induced by the products of R and S. Similarly, $M\otimes P$ is a right $(R\otimes S)$-module with action

$$M\otimes P\otimes R\otimes S\xrightarrow{\mathrm{id}\times\gamma\times\mathrm{id}} M\otimes R\otimes P\otimes S\longrightarrow M\otimes P$$

induced by the actions of R on M and S on P, and $N\otimes Q$ is a left $R\otimes S$-module.

Let X be an R-projective resolution of M and Y be a S-projective resolution of P. Then $X\otimes Y$ is an $R\otimes S$-projective complex over $M\otimes P$. Rather than use a Künneth argument to check that it is a resolution, we can map it to a projective $R\otimes S$-projective resolution of $M\otimes P$ by use of Lemma 3.3. Using the natural map α from the tensor product of homologies to the homology of a tensor product, the map

$$(X\otimes_R N)\otimes (Y\otimes_S Q)\longrightarrow (X\otimes Y)\otimes_{R\otimes S}(N\otimes Q)$$

induced by $\mathrm{id}\otimes\gamma\otimes\mathrm{id}$ gives rise to a natural pairing

$$\mathrm{Tor}^R_*(M,N)\otimes\mathrm{Tor}^S(P,Q)\longrightarrow\mathrm{Tor}^{R\otimes S}_*(M\otimes P, N\otimes Q). \tag{3.9}$$

Now recall that a ring R is commutative if and only if its product is a map of rings. In formulas, this means that $rr'ss' = ss'rr'$ for all r, r', s, s' if and only if $rs = sr$ for all r and s. Taking $r' = 1 = s'$ gives one implication, and the other is obvious. When we deal with graded rings with products $R_m \otimes R_n \longrightarrow R_{m+n}$, we understand commutativity to mean graded commutativity, $rs = (-1)^{mn}sr$. Formally, that means that we are defining the graded symmetry isomorphism γ with our usual sign convention on interchange.

We say that a ring R is augmented over a ring k if there is an epimorphism of rings $\varepsilon\colon R \longrightarrow k$. It is especially interesting to consider the quotient homomorphism $\varepsilon\colon R \longrightarrow R/m = k$ of a local ring R with maximal ideal m.

Theorem 3.10. *If R is a commutative ring with augmentation $\varepsilon\colon R \longrightarrow k$, then $\mathrm{Tor}_*^R(k,k)$ is a graded commutative k-algebra.*

Proof. We are regarding k as the R-module ε^*k. Writing ϕ for the products on R and on k, the required product on $\mathrm{Tor}_*^R(k,k)$ is the composite

$$\mathrm{Tor}_*^R(k,k) \otimes_k \mathrm{Tor}_*^R(k,k) \longrightarrow \mathrm{Tor}_*^{R\otimes R}(k\otimes k, k\otimes k) \longrightarrow \mathrm{Tor}_*^R(k,k).$$

While R is not a k-algebra, the construction just given works with $M = N = P = Q$ to give the first map, and the second map is $\mathrm{Tor}^{\phi}(\phi,\phi)$ of (3.6). Note that ϕ induces an isomorphism $k \otimes_R k \longrightarrow k$, so that $\mathrm{Tor}_0^R(k,k) \cong k$. It is a straightforward exercise, left to the reader, that the product on $\mathrm{Tor}_*^R(k,k)$ is associative, unital, and graded commutative. $\square$

3.5. **A sample computation.** It is all very well to define a product, but how do we compute it? A DGA (differential graded k-algebra) is a k-chain complex A with an associative and unital product $A \otimes_k A \longrightarrow A$ which is a map of chain complexes, meaning $d(ab) = d(a)b + (-1)^{\deg a} a d(b)$. It is (graded) commutative if $ab = (-1)^{\deg a \deg b} ba$ for all a and b. We can also define the weaker notion of a DGA up to chain homotopy. This is defined in the same way, except that we only require associativity and unitality up to chain homotopy. The homotopies ensure that $H_*(A)$ is a graded algebra. For example, the (normalized) singular cochains of a space with coefficients in k form a cohomologically graded DGA, but it is only commutative up to chain homotopy, and that is enough to ensure that the cohomology of a space is a commutative k-algebra.

Now return to a commutative ring R with augmentation $\varepsilon : R \longrightarrow k$, for example an augmented k-algebra. Let X be an R-projective resolution of k. Then Lemma 3.3 gives that there is a map of $(R\otimes R)$-chain complexes $\tilde{\phi}\colon X \otimes X \longrightarrow X$ over the product $\phi\colon k \otimes k \longrightarrow k$, where X is regarded as an $(R \otimes R)$-chain complex via the product $R \otimes R \longrightarrow R$. Since the product on k is associative, unital, and commutative, the same result applies to show that the product $\tilde{\phi}$ is associative, unital, and (graded) commutative up to chain homotopy. Then $X \otimes_R k$ is a DG k-algebra, and its homology is $\mathrm{Tor}_*^R(k,k)$ as a k-algebra. This is all that one can expect in general, but for especially nice rings R, one can find an X which is actually a DGA, with no need for chain homotopies.

To illustrate, let R be the polynomial algebra $k[x]$ with augmentation determined by $\varepsilon(x) = 0$. Let X be the graded k-algebra and free right R-module $E[y] \otimes_k R$, where $E[y]$ is the exterior algebra on one generator y of degree 1. Thus $y^2 = 0$, and $E[y]$ is the free k-module with one basis element 1 of degree 0 and one basis element y of degree 1. Define $\varepsilon\colon X \longrightarrow k$ by letting $\varepsilon(1\otimes 1) = 1$ and $\varepsilon(y \otimes 1) = 0 = \varepsilon(1 \otimes x)$.

Define a differential on X by letting $d(y\otimes 1)=1\otimes x$ and requiring d to be a map of R-modules. Then $d(y\otimes x^n)=1\otimes x^{n+1}$. Since X has the k-basis $\{1\times x^n\}\cup\{y\otimes x^n\}$ we see that X is an R-free resolution of k. Moreover, it is a differential graded R-algebra, and $X\otimes_R k$ is the k-algebra $E[y]$ with zero differential. This proves the case $n=1$ of the following result.

Theorem 3.11. *Let R be the polynomial algebra $k[x_1,\cdots,x_n]$. Then $Tor^R_*(k,k)$ is the exterior algebra $E[y_1,\cdots,y_n]$, where $deg(y_i)=1$.*

Proof. $X=E[y_1,\cdots,y_n]\otimes R$ is a DG R-algebra with differential determined by $d(y_i)=x_i$ on generators over R, and it is isomorphic to the tensor product of DG k-algebras

$$(E[y_1]\otimes k[y_1])\otimes_k\cdots\otimes_k(E[y_n]\otimes k[y_n]).$$

With the evident map to k, it follows that the DGA X is an R-free resolution of k. The differential on $X\otimes_R k=E[y_1,\cdots,y_n]$ is zero, and the conclusion follows. $\square$

Since the associated graded k-algebra of a regular local ring R with respect to the filtration given by the powers of its maximal ideal is a polynomial algebra on n generators, where n is the Krull dimension of R is n, this suggests that the conclusion applies equally well to $\mathrm{Tor}^R_*(k,k)$. In particular, this suggests that $\mathrm{Tor}^R_q(k,k)=0$ for $q>n$. In fact, we shall see later that much more is true. We state the result now and prove it later.

Theorem 3.12 (Serre)**.** *Let R be a (Noetherian) local ring of Krull dimension n. If $\mathrm{Tor}^R_q(k,k)$ is zero for any $q>n$, then R is regular. If R is regular, then $\mathrm{Tor}^R_q(M,N)=0$ for all R-modules M and N and all $q>n$ and, as a k-algebra,*

$$Tor^R_*(k,k)\cong E[y_1,\cdots,y_n].$$

4. Ext groups

We give a quick parallel development of the basic theory of ext groups. We again work with non-commutative rings R and their right and left modules. By default, modules mean left modules, and then $\mathrm{Hom}_R(M,N)$ is the Abelian group of maps of left R-modules $M\longrightarrow N$. Again, when R is commutative, everything we do works just as well in the category of R-modules.

4.1. Two identities between Hom functors.

For rings R and S, an (R,S)-bimodule M is a left R-module and right S-module with commuting actions, meaning that $(rm)s=r(ms)$. When R is commutative, any R-module may be considered as an (R,R)-bimodule with $rm=mr$. We record two standard identities. Their proofs are elementary, but it is fun to use the Yoneda lemma and analogous identities between tensor products to derive them.

Lemma 4.1. *Let M be a left R-module, N be a right S-module, and P be an (R,S)-bimodule. Then there is a natural isomorphism of Abelian groups*

$$\mathrm{Hom}_R(M,\mathrm{Hom}_S(N,P))\cong\mathrm{Hom}_S(N,\mathrm{Hom}_R(M,P)).$$

Here the R and S actions on P induce the R and S actions on $\mathrm{Hom}_S(N,P)$ and $\mathrm{Hom}_R(M,P)$.

Lemma 4.2. *Let M be a right R-module, N be an (R,S)-bimodule, and P be a left S-module. Then there is a natural isomorphism of Abelian groups*

$$\mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P)).$$

Here the S and R actions on N induce the S and R actions on $M \otimes_R N$ and $\mathrm{Hom}_S(N,P)$. The analogous result with left and right reversed also holds.

We shall have immediate use of the following special case of the reversed version. Here we specialize S to $\mathbb{Z}$. We take $N = R$, regarded as a right R-module, noting that any right R-module can be viewed as a $(\mathbb{Z}, R)$-bimodule.

Lemma 4.3. *Let M be a left R-module and P be an Abelian group. Then there is a natural isomorphism of Abelian groups*

$$\mathrm{Hom}_{\mathbb{Z}}(M,P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R,P)).$$

4.2. **Injective resolutions.** Recall that modules mean left modules. A module I is said to be injective if for each monomorphism $e\colon L \longrightarrow M$ and each map $f\colon L \longrightarrow I$, there exists a map $\tilde{f}\colon M \longrightarrow I$ such that $\tilde{f} \circ e = f$. This means that

$$\mathrm{Hom}(e, \mathrm{id})\colon\ \mathrm{Hom}_R(M,I) \longrightarrow \mathrm{Hom}(L,I)$$

is an epimorphism. There is no analogue of Lemma 3.1, but there is the following analogue of its key consequence.

Lemma 4.4. *Every R-module N embeds as a submodule of an injective R-module.*

Proof. By an exercise, an Abelian group is divisible if and only if it is an injective $\mathbb{Z}$-module. Clearly a direct sum of divisible Abelian groups is divisible, and so is a quotient of a divisible Abelian group. Since $\mathbb{Z}$ embeds in the divisible group $\mathbb{Q}$ and any Abelian group is a quotient of a free Abelian group, the result holds for modules over the ring $\mathbb{Z}$. Define $j\colon N \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,N)$ by $j(n)(r) = rn$. On the right, N is regarded just as an Abelian group. Clearly j is a homomorphism of Abelian groups, but in fact it is a map of R-modules. Indeed, for $s \in R$,

$$j(sn)(r) = r(sn) = (rs)n = j(n)(rs) = (sj(n))(r)$$

where the last equality holds by the definition of the left action of R on $\mathrm{Hom}_{\mathbb{Z}}(R,N)$. Moreover, j is a monomorphism since $j(n)(1) = n$. Now, ignoring the R-action on N, choose a monomorphism $i\colon N \longrightarrow D$, where D is a divisible Abelian group, and let $i_* = \mathrm{Hom}(\mathrm{id}, i)\colon\ \mathrm{Hom}_{\mathbb{Z}}(R,N) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R,D)$. Then i_* and therefore the composite $i_* \circ j$ is a monomorphism of R-modules. Using the natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M,D) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R,D))$$

of Lemma 4.3, we see that $\mathrm{Hom}_{\mathbb{Z}}(R,D)$ is injective as an R-module since D is injective as a $\mathbb{Z}$-module. □

Let N be an R-module. A (cochain) complex of R-modules under N is a complex of the form

$$0 \longrightarrow N \xrightarrow{\eta} Y^0 \longrightarrow \cdots \longrightarrow Y^i \longrightarrow Y^{i+1} \longrightarrow \cdots$$

It is an injective complex under N if each Y^i is injective. It is a resolution of N if the displayed sequence is exact. We think of N itself as a complex concentrated in degree 0 and $\eta\colon N \longrightarrow Y$ as a morphism of complexes. If Y is a resolution of N, then η induces an isomorphism on (co)homology since $\mathrm{Ker}\, d^0 = N$. Now the following three results are proven in exactly the same way as Lemmas 3.2, 3.3, and 3.4, except that we reverse all of the arrows and replace projectivity by injectivity.

Lemma 4.5. *Every R-module N has an injective resolution.*

Lemma 4.6. *Let $f\colon M \longrightarrow N$ be a map of R-modules. Let $\zeta\colon M \longrightarrow X$ be a resolution of M and let $\eta\colon N \longrightarrow Y$ be an injective complex under N. Then there is a map $\tilde{f}\colon X \longrightarrow Y$ of complexes such that $\tilde{f}\circ\zeta = \eta\circ f$, and $\tilde{f}$ is unique up to chain homotopy.*

Lemma 4.7. *Let $0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$ be an exact sequence of R-modules. Let Y' and Y'' be injective resolutions of N' and N''. Then there is an injective resolution Y of N and an exact sequence of resolutions under the given exact sequence:*

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r, "f"] \arrow[d, "\eta'"'] & N \arrow[r, "g"] \arrow[d, "\eta"] & N'' \arrow[r] \arrow[d, "\eta''"] & 0 \\
0 \arrow[r] & Y' \arrow[r, "\tilde{f}"'] & Y \arrow[r, "\tilde{g}"'] & Y'' \arrow[r] & 0.
\end{tikzcd}$$

4.3. **The definition and properties of** Ext**.** For a short exact sequence of R-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ and an R-module N, the sequence of Abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M', N)$$

is exact, but the right-most arrow need not be a epimorphism. Similarly, for a short exact sequence of R-modules $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ and an R-module M, the sequence of Abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(M, N') \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, N'')$$

is exact, but the right-most arrow need not be a epimorphism. We say that the functor Hom_R is left exact. Ext groups measure the deviation from exactness. Here is an omnibus theorem that states their basic properties.

Theorem 4.8. *There are Abelian group valued functors $\mathrm{Ext}^n_R(M, N)$ of (left) R-modules M and N, together with natural connecting homomorphisms*

$$\delta\colon \mathrm{Ext}^n_R(M', N) \longrightarrow \mathrm{Ext}^{n+1}_R(M'', N) \quad \textit{and} \quad \delta\colon \mathrm{Ext}^n_R(M, N'') \longrightarrow \mathrm{Ext}^{n+1}_R(M, N')$$

for short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0.$$

These satisfy the following properties.

(i) $\mathrm{Ext}^n_R(M, N) = 0$ *for* $n < 0$.
(ii) $Ext^0_R(M, N)$ *is naturally isomorphic to* $\mathrm{Hom}_R(M, N)$.
(iii) $\mathrm{Ext}^n_R(M, N) = 0$ *for* $n > 0$ *if either* M *is projective or* N *is injective.*
(iv) The following sequences are exact.

$$\cdots \longrightarrow \mathrm{Ext}^n_R(M'', N) \longrightarrow \mathrm{Ext}^n_R(M, N) \longrightarrow \mathrm{Ext}^n_R(M', N) \longrightarrow \mathrm{Ext}^{n+1}_R(M'', N) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Ext}^n_R(M, N') \longrightarrow \mathrm{Ext}^n_R(M, N) \longrightarrow \mathrm{Ext}^n_R(M, N'') \longrightarrow \mathrm{Ext}^{n+1}_R(M, N') \longrightarrow \cdots$$

For each fixed N, the functors $\mathrm{Ext}^n_R(M, N)$ of M together with the natural connecting homomorphisms $\delta\colon \mathrm{Ext}^n_R(M', N) \longrightarrow \mathrm{Ext}^{n+1}_R(M'', N)$ on exact sequences $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ are uniquely determined up to isomorphism by (i)—(iv), and similarly for each fixed M.

Proof. The naturality statements imply that the long exact sequences of (iv) are functorial on short exact sequences. For the existence statement, let us fix N. Choose a projective resolution X of M. Define

$$\mathrm{Ext}^*_R(M,N) = H^*(\mathrm{Hom}_R(X,N))$$

Since chain homotopic maps of complexes induce the same map on homology, Lemma 4.6 shows that this is well-defined up to natural isomorphism and gives a functor of M. Manifestly, it also gives a functor of N. Since Hom_R is left exact, (ii) is clear, and (iii) is also clear. We define the first map δ and derive the first long exact sequence of (iv) by use of Lemma 4.7. We define the second map δ and derive the second long exact sequence by use of the short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Hom}_R(X,N') \longrightarrow \mathrm{Hom}_R(X,N) \longrightarrow \mathrm{Hom}_R(X,N'') \longrightarrow 0.$$

Here exactness holds by the projectivity of the X_i.

For the axiomatization, we proceed by induction on n, starting from (i) and (ii). If we have two systems of functors and natural connecting homomorphisms and we have proven they are isomorphic through stage $(n-1)$, then a diagram chase starting from short exact sequences $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ where M is free shows that they are isomorphic at the stage n.

We can reverse the roles of M and N in our original construction, starting from an injective resolution Y of N and redefining

$$\mathrm{Ext}^*_R(M,N) = H^*(\mathrm{Hom}_R(M,Y)).$$

We again have all of the properties (i) – (iv), so by the uniqueness we obtain the same sequence of functors and natural connecting homomorphisms. Moreover, we can also check that $H_*(\mathrm{Hom}_R(X,Y))$ gives functors and natural connecting homomorphisms that satisfy the axioms. Alternatively, we can observe that the morphisms of complexes $X \longrightarrow M$ and $N \longrightarrow Y$ induce morphisms of complexes

$$\mathrm{Hom}_R(X,N) \longrightarrow \mathrm{Hom}_R(X,Y) \longleftarrow \mathrm{Hom}_R(M,Y)$$

and can check directly that these give isomorphisms. □

4.4. **Change of rings.** Just as for Tor, we can make Ext into a functor of three variables, allowing for change of rings. As in §3.3, let $f\colon R \longrightarrow S$ be a map of rings, let M and N be R-modules, and let P and Q be S-modules. Let $g\colon f^*P \longrightarrow M$ and $h\colon f_*N \longrightarrow Q$ be maps of R-modules. We assume that S is projective as an R-module, and under that hypothesis we shall obtain a map

$$\mathrm{Ext}^*_f(g,h)\colon\ \mathrm{Ext}^*_R(M,N) \longrightarrow \mathrm{Ext}^*_S(P,Q). \tag{4.9}$$

This allows us to view Ext as a functor of three variables.

Since S is R-projective, any projective S-module is projective as an R-module. Let Y be a projective resolution of the S-module P. Then f^*Y is a projective resolution of the R-module f^*P. If X is a projective resolution of M, there is a map $\tilde{g}\colon f^*Y \longrightarrow X$ over g. Using $\tilde{g}$ and h, there results a map

$$\mathrm{Hom}_R(X,N) \longrightarrow \mathrm{Hom}_R(f^*Y,N) \cong \mathrm{Hom}_S(Y,f_*N) \longrightarrow \mathrm{Hom}_S(Y,Q)$$

of chain complexes. Passage to homology gives the promised map (4.9).

4.5. **Extensions.** Expanding on an exercise, we explain the classical interpretation of $\mathrm{Ext}^1_R(M,N)$ in terms of extensions of modules. We omit details. An extension of N by M is a short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0. \tag{4.10}$$

A map of extensions is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "f"'] & E \arrow[r] \arrow[d, "g"] & M \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & N' \arrow[r] & E' \arrow[r] & M' \arrow[r] & 0. \end{tikzcd} \tag{4.11}$$

It is an equivalence of extensions if f and h are identity maps, in which case g must be an isomorphism. Define $\mathrm{Ext}(M,N)$ to be the set of equivalence classes of extensions of N by M.

For a map $f\colon N \longrightarrow N'$ and an extension E of N by M, as in (4.10), we obtain an extension E' of N' by M by taking E' to be the pushout of f and the inclusion $N \longrightarrow E$. By the universal property of pushouts applied to the quotient map $E \longrightarrow M$ and $0\colon N' \longrightarrow M$, we obtain a map (4.11) in which h is the identity map of M. Dually for a map $h\colon M \longrightarrow M'$ and an extension E' of N by M', we obtain an extension E of N by M by taking E to be the pullback of h and the quotient map $E' \longrightarrow M'$. With these constructions, $\mathrm{Ext}(M,N)$ is a contravariant functor of N and a covariant functor of M.

Given two extensions E and E' of N by M, we can take direct sums to obtain

$$0 \longrightarrow N \oplus N \longrightarrow E \oplus E' \longrightarrow M \oplus M \longrightarrow 0$$

Using functoriality with respect to the diagonal $N \longrightarrow N \oplus N$ and the codiagonal (or sum) $\nabla\colon M \oplus M \longrightarrow M$, we construct an extension of N by M, denoted $E+E'$ and called the Baer sum of E and E'. This gives $\mathrm{Ext}(M,N)$ a natural structure of Abelian group.

Theorem 4.12. *$Ext(M,N)$ is naturally isomorphic to $\mathrm{Ext}^1_R(M,N)$.*

Proof. If $X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$ is the start of a projective resolution of M and E is an extension of N by M, Lemma 3.3 gives a commutative diagram

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

The map $X_1 \longrightarrow N$ is a cocycle of $\mathrm{Hom}_R(X,N)$. Its cohomology class is independent of choices, and the resulting map $\mathrm{Ext}^1_R(M,N) \longrightarrow \mathrm{Ext}(M,N)$ is an isomorphism. $\square$

4.6. **Long extensions and the Yoneda product.** The description of Ext^1_R generalizes to Ext^n_R, starting from extensions of length n, namely long exact sequences

$$0 \longrightarrow N \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0. \tag{4.13}$$

Maps of extensions of length n are commutative diagrams

(4.14)
$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "f"'] & E_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r] & E'_{n-1} \arrow[r] & \cdots \arrow[r] & E'_0 \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

Taking f and h to be identity maps and requiring the other vertical arrows to be isomorphisms, we obtain the notion of an elementary equivalence of extensions of N by M. Elaborating from the previous section, but using a more complicated equivalence relation that we make precise in the next section, the resulting sets of equivalence classes of extensions of length n give well-defined Abelian group valued functors of M and N. One can elaborate the proof of Theorem 4.12 to obtain the following generalization.

Theorem 4.15. *For $n \geq 1$, $\mathrm{Ext}^n(M,N)$ is naturally isomorphic to the Abelian group of equivalence classes of extensions of N by M of length n.*

This result leads to a beautiful construction of a pairing between Ext groups, called the Yoneda product.

Definition 4.16. Define the Yoneda product

$$\mathrm{Ext}^n_R(N,P) \otimes \mathrm{Ext}^m_R(M,N) \longrightarrow \mathrm{Ext}^{m+n}_R(M,P)$$

as follows. Suppose given extensions

$$0 \longrightarrow N \longrightarrow E_{m-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow P \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0.$$

Rename F_i as E_{m+i} and splice the sequences using the evident composite map $F_0 \longrightarrow N \longrightarrow E_{m-1}$. This gives an extension

$$0 \longrightarrow P \longrightarrow E_{m+n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0,$$

which is the Yoneda product of the given extensions. We can extend the definition to allow $m = 0$ or $n = 0$ by using functoriality on maps. Then identity maps of modules act as identities for the pairing.

Theorem 4.17. *The Yoneda product passes to equivalence classes to give an associative and unital system of pairings of* Ext *groups.*

Categorically, we can say that the Ext groups specify a category enriched in graded Abelian groups whose objects are the R-modules M and whose graded Abelian group of morphisms $M \longrightarrow N$ is $\mathrm{Ext}^*_R(M,N)$. In particular, we see that each $\mathrm{Ext}^*_R(M,M)$ is a graded ring. Categories like this are often called "rings with many objects". This extra structure is central to many applications.

4.7. **Equivalences of long extensions.** We first need notations for pushout and pullback constructions on extensions. Consider an extension

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

and a map $\beta\colon N \longrightarrow N'$. Let $\beta_* E$ be the extension displayed in the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\beta"'] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & N' \arrow[r] & \beta_* E \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where the left square is a pushout. Similarly for a map $\alpha\colon M' \longrightarrow M$, let $\alpha^* E$ be the extension displayed in the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "="'] & \alpha^* E \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where the right square is a pullback. For extensions

$$0 \longrightarrow N \longrightarrow E \longrightarrow B \longrightarrow 0,$$

a map $\beta\colon B \longrightarrow B'$ and an extension

$$0 \longrightarrow B' \longrightarrow E' \longrightarrow M \longrightarrow 0$$

we have two generally different Yoneda composites $\beta^* E \circ E'$ and $\beta_* E' \circ E$.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "="'] & \beta^* E \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, "\beta"] & E' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & B \arrow[r] & \beta_* E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

We agree to say that these two length two extensions are equivalent. This is an equivalence of length two. We also have the elementary equivalences given by isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

This is an equivalence of length one.

Consider long exact sequences

$$S\colon 0 \longrightarrow N \xrightarrow{f_n} E_{n-1} \xrightarrow{f_{n-1}} E_{n-2} \longrightarrow \cdots \longrightarrow E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} M \longrightarrow 0.$$

Let B_i be the image of f_i; in particular, by abuse, let $B_n = N$ and $B_0 = M$. We have extensions

$$0 \longrightarrow B_{i+1} \longrightarrow E_i \longrightarrow B_i \longrightarrow 0$$

for $0 \leq i \leq n-1$, and S is their Yoneda composite $E_{n-1} \circ \cdots \circ E_0$. Say that two such sequences S and S' are equivalent if there is a chain of equivalences of length one or of length two connecting them. That is, we form the smallest equivalence relation that identifies equivalent subsequences of length one or length two. The set of equivalence classes admits an addition under which it gives an abelian group isomorphic to $\mathrm{Ext}^n(M, N)$.

4.8. **Pairings of** Ext **groups.** The Yoneda pairing also admits a direct construction in terms of our original definition of Ext groups. Let M, N, and P be R-modules and let X be a projective resolution of M and Y be an injective resolution of P. The composition pairing

$$\mathrm{Hom}_R(N,P)\otimes \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,P)$$

gives a map of chain complexes

$$\mathrm{Hom}_R(N,Y)\otimes \mathrm{Hom}_R(X,N) \longrightarrow \mathrm{Hom}_R(X,Y).$$

Passing to homology and using the pairing α of §2.1, we obtain a pairing

$$\mathrm{Ext}^n_R(N,P)\otimes \mathrm{Ext}^m_R(M,N) \longrightarrow \mathrm{Ext}^{m+n}_R(M,P).$$

This coincides with the Yoneda product of the previous section.

There is also an external pairing related to change of rings. Let R and S be rings, let M and N be R-modules, and let N and Q be S-modules. Then there is a pairing

$$\mathrm{Ext}^m_R(M,N)\otimes \mathrm{Ext}^n_S(P,Q) \longrightarrow \mathrm{Ext}^{m+n}_{R\otimes S}(M\otimes P, N\otimes Q) \tag{4.18}$$

If X is a projective resolution of M and Y is a projective resolution of N, we can apply the tensor pairing

$$\omega\colon\ \mathrm{Hom}_R(M,N)\otimes \mathrm{Hom}_S(P,Q) \longrightarrow \mathrm{Hom}_{R\otimes S}(M\otimes P, N\otimes Q)$$

specified by $\omega(f\otimes g)(m\otimes p)=f(m)\otimes g(n)$ to obtain a map of chain complexes

$$\omega\colon\ \mathrm{Hom}_R(X,N)\otimes \mathrm{Hom}_S(Y,Q) \longrightarrow \mathrm{Hom}_{R\otimes S}(X\otimes Y, N\otimes Q).$$

Here, as usual, we must insert the sign $(-1)^{\deg(g)\deg(m)}$ when interpreting the tensor pairing in order to obtain a map of chain complexes. Again using α from §2.1 and passing to homology, we obtain the pairing (4.18).

Remember that when R is commutative the Ext groups and all maps in sight between them take values in the category of R-modules. More generally, if R is an algebra over a commutative ring k, then the Ext groups and all maps in sight between them take values in the category of k-modules. When R is commutative and augmented over k, we have seen that $\mathrm{Tor}^R_*(k,k)$ is a graded k-algebra. We now see that $\mathrm{Ext}^*_R(k,k)$ is also a graded k-algebra, via the Yoneda product. If k is a field and X is an R-free resolution of k, then we have

$$\mathrm{Hom}_R(X,k)\cong \mathrm{Hom}_k(k\otimes_R X,k)$$

as k-chain complexes. Writing $M^*=\mathrm{Hom}_k(M,k)$ for the vector space dual of M, this implies that

$$\mathrm{Ext}^*_R(k,k)\cong (\mathrm{Tor}^R_*(k,k))^* \tag{4.19}$$

Therefore $\mathrm{Ext}^*_R(k,k)$ has both an algebra structure and the dual of an algebra structure, which is called a coalgebra structure. We shall return to consideration of such structures later, when we shall talk about bialgebras and Hopf algebras.

For now, we sum up by saying that for any k-algebra R, $\mathrm{Ext}^*_R(k,k)$ is a k-algebra under the Yoneda product. It is not necessarily commutative even when R is commutative, but then $\mathrm{Ext}^*_R(k,k)$ is a Hopf algebra.

We shall later return to this point and show that if R is a Hopf algebra over k, then $\mathrm{Ext}^*_R(k,k)$ is a commutative k-algebra, but not necessarily a Hopf algebra. As we shall see, when specialized to group algebras this is closely related to the fact that the cohomology of a space with coefficients in k is a commutative k-algebra.