THE AXIOMS FOR TRIANGULATED CATEGORIES

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Contents

This is an edited extract from my paper [11]. We define triangulated categories and discuss homotopy pushouts and pullbacks in such categories in §1 and §2. We focus on Verdier's octahedral axiom, since the axiom that is usually regarded as the most substantive one is redundant: it is implied by Verdier's axiom and the remaining, less substantial, axioms. Strangely, since triangulated categories have been in common use for over thirty years, this observation seemed to be new in [11]. We explain intuitively what is involved in the verification of the axioms in §3.

1. Triangulated categories

We recall the definition of a triangulated category from [15]; see also [2, 7, 10, 16]. Actually, one of the axioms in all of these treatments is redundant. The most fundamental axiom is called *Verdier's axiom*, or the *octahedral axiom* after one of its possible diagrammatic shapes. However, the shape that I find most convenient, a braid, does not appear in the literature of triangulated categories. It does appear in Adams [1, p. 212], who used the term "sine wave diagram" for it. We call a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \tag{1.1}$$

a "triangle" and use the notation (f, g, h) for it.

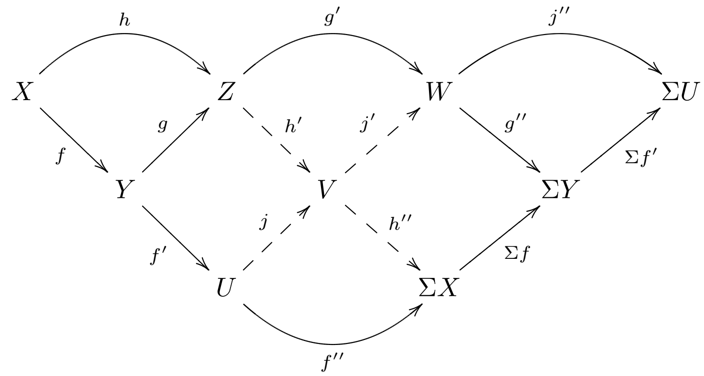
Definition 1.2. A triangulation on an additive category $\mathscr{C}$ is an additive self-equivalence $\Sigma : \mathscr{C} \longrightarrow \mathscr{C}$ together with a collection of triangles, called the *distinguished triangles*, such that the following axioms hold.

Axiom T 1. *Let X be any object and $f : X \longrightarrow Y$ be any map in $\mathscr{C}$.*

(a) *The triangle $X \xrightarrow{\text{id}} X \longrightarrow * \longrightarrow \Sigma X$ is distinguished.*
(b) *The map $f : X \longrightarrow Y$ is part of a distinguished triangle (f, g, h).*
(c) *Any triangle isomorphic to a distinguished triangle is distinguished.*

Axiom T 2. *If (f, g, h) is distinguished, then so is $(g, h, -\Sigma f)$.*

Axiom T 3 (Verdier's axiom)**.** *Consider the following diagram.*



Assume that $h = g \circ f$, $j'' = \Sigma f' \circ g''$, *and* (f, f', f'') *and* (g, g', g'') *are distinguished. If* h' *and* h'' *are given such that* (h, h', h'') *is distinguished, then there are maps* j *and* j' *such that the diagram commutes and* (j, j', j'') *is distinguished. We call the diagram a* braid of distinguished triangles generated by $h = g \circ f$ *or a* braid cogenerated by $j'' = \Sigma f' \circ g''$.

We have labeled our axioms (T?), and we will compare them with Verdier's original axioms (TR?). Our (T1) is Verdier's (TR1) [15], our (T2) is a weak form of Verdier's (TR2), and our (T3) is Verdier's (TR4). We have omitted Verdier's (TR3), since it is exactly the conclusion of the following result.

Lemma 1.3 (TR3)**.** *If the rows are distinguished and the left square commutes in the following diagram, then there is a map* k *that makes the remaining squares commute.*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

Proof. This is part of the 3×3 lemma, which we state and prove below. The point is that the construction of the commutative diagram in that proof requires only (T1), (T2), and (T3), not the conclusion of the present lemma; compare [2, 1.1.11]. □

Verdier's (TR2) includes the converse, (T2′) say, of (T2). That too is a consequence of our (T1), (T2), and (T3). A standard argument using only (T1), (T2), (TR3), and the fact that Σ is an equivalence of categories shows that, for any object A, a distinguished triangle (f, g, h) induces a long exact sequence upon application of the functor $\mathscr{C}(A, -)$. Here we do not need the converse of (T2) because we are free to replace A by $\Sigma^{-1}A$. In turn, by the five lemma and the Yoneda lemma, this implies the following addendum to the previous lemma.

Lemma 1.4. *If* i *and* j *in (TR3) are isomorphisms, then so is* k.

Lemma 1.5 (T2′)**.** *If* $(g, h, -\Sigma f)$ *is distinguished, then so is* (f, g, h).

Proof. Choose a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X$. By (T2), the triangles $(-\Sigma f, -\Sigma g', -\Sigma h')$ and $(-\Sigma f, -\Sigma g, -\Sigma h)$ are distinguished. By Lemmas 1.3 and 1.4, they are isomorphic. By desuspension, (f, g, h) is isomorphic to (f', g', h'). By (T1), it is distinguished. □

Similarly, we can derive the converse version, (T3′) say, of Verdier's axiom (T3).

Lemma 1.6 (T3′)**.** *In the diagram of (T3), if j and j' are given such that (j, j', j'') is distinguished, then there are maps h' and h'' such that the diagram commutes and (h, h', h'') is distinguished.*

Proof. Desuspend a braid of distinguished triangles generated by $j'' = \Sigma f' \circ g''$. □

Lemma 1.7 (The 3×3 lemma)**.** *Assume that $j \circ f = f' \circ i$ and the two top rows and two left columns are distinguished in the following diagram.*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \Sigma i} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \\
{\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle \Sigma i'} \\
X'' & \xrightarrow{f''} & Y'' & \xrightarrow{g''} & Z'' & \xrightarrow{h''} & \Sigma X'' \\
{\scriptstyle i''}\downarrow & & \downarrow{\scriptstyle j''} & & \downarrow{\scriptstyle k''} & & \downarrow{\scriptstyle -\Sigma i''} \\
\Sigma X & \xrightarrow[\Sigma f]{} & \Sigma Y & \xrightarrow[\Sigma g]{} & \Sigma Z & \xrightarrow[-\Sigma h]{} & \Sigma^2 X
\end{array}$$

Then there is an object Z'' and there are dotted arrow maps f'', g'', h'', k, k', k'' such that the diagram is commutative except for its bottom right square, which commutes up to the sign -1, and all four rows and columns are distinguished.

Proof. The bottom row is isomorphic to the triangle $(-\Sigma f, -\Sigma g, -\Sigma h)$ and is thus distinguished by (T2); similarly the right column is distinguished. Applying (T1), we construct a distinguished triangle

$$X \xrightarrow{j \circ f} Y' \xrightarrow{p} V \xrightarrow{q} \Sigma X.$$

Applying (T3), we obtain braids of distinguished triangles generated by $j \circ f$ and $f' \circ i$. These give distinguished triangles

$$Z \xrightarrow{s} V \xrightarrow{t} Y'' \xrightarrow{\Sigma g \circ j''} \Sigma Z$$

$$X'' \xrightarrow{s'} V \xrightarrow{t'} Z' \xrightarrow{\Sigma i' \circ h'} \Sigma X''$$

such that

$$p \circ j = s \circ g, \quad t \circ p = j', \quad q \circ s = h, \quad j'' \circ t = \Sigma f \circ q$$
$$p \circ f' = s' \circ i', \quad t' \circ p = g', \quad q \circ s' = i'', \quad h' \circ t' = \Sigma i \circ q.$$

Define $k = t' \circ s : Z \longrightarrow Z'$. Then $k \circ g = g' \circ j$ and $h' \circ k = \Sigma i \circ h$, which already completes the promised proof of Lemma 1.3. Define $f'' = t \circ s'$ and apply (T1) to construct a distinguished triangle

$$X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} \Sigma X''.$$

Applying (T3), we obtain a braid of distinguished triangles generated by $f'' = t \circ s'$. Here we start with the distinguished triangles $(s', t', \Sigma i' \circ h')$ and $(t, \Sigma g \circ j'', -\Sigma s)$, where the second is obtained by use of (T2). This gives a distinguished triangle

$$Z' \xrightarrow{k'} Z'' \xrightarrow{k''} V \xrightarrow{-\Sigma k} \Sigma Z`$$

such that the squares left of and above the bottom right square commute and

$$g'' \circ t = k' \circ t' \quad \text{and} \quad -\Sigma s \circ k'' = \Sigma s' \circ h''.$$

The commutativity (and anti-commutativity of the bottom right square) of the diagram follow immediately. It also follows immediately that (f'', g'', h'') and $(k', k'', -\Sigma k)$ are distinguished. Lemma 1.5 implies that (k, k', k'') is distinguished. □

2. Weak pushouts and weak pullbacks

In any category, weak limits and weak colimits satisfy the existence but not necessarily the uniqueness in the defining universal properties. They need not be unique and need not exist. When constructed in particularly sensible ways, they are called homotopy limits and colimits and are often unique up to non-canonical isomorphism. As we recall here, there are such homotopy pushouts and pullbacks in triangulated categories. Homotopy colimits and homotopy limits of sequences of maps in triangulated categories are studied in [3, 13], but a complete theory of homotopy limits and colimits in triangulated categories is not yet available. The material in this section is meant to clarify ideas and to describe a strengthened form of Verdier's axiom that is important in the applications.

Definition 2.1. A *homotopy pushout* of maps $f : X \longrightarrow Y$ and $g : X \longrightarrow Z$ is a distinguished triangle

$$X \xrightarrow{(f,-g)} Y \vee Z \xrightarrow{(j,k)} W \xrightarrow{i} \Sigma X.$$

A *homotopy pullback* of maps $j : Y \longrightarrow W$ and $k : Z \longrightarrow W$ is a distinguished triangle

$$\Sigma^{-1} W \xrightarrow{-\Sigma^{-1} i} X \xrightarrow{(f,g)} Y \vee Z \xrightarrow{(j,-k)} W.$$

The sign is conventional and ensures that in the isomorphism of extended triangles

$$\begin{array}{ccccccccc}
\Sigma^{-1}W & \xrightarrow{-\Sigma^{-1}i} & X & \xrightarrow{(f,-g)} & Y \vee Z & \xrightarrow{(j,k)} & W & \xrightarrow{i} & \Sigma X \\
\| & & \| & & \downarrow{\scriptstyle (\mathrm{id}, -\mathrm{id})} & & \| & & \| \\
\Sigma^{-1}W & \xrightarrow{-\Sigma^{-1}i} & X & \xrightarrow{(f,g)} & Y \vee Z & \xrightarrow{(j,-k)} & W & \xrightarrow{i} & \Sigma X,
\end{array}$$

the top row displays a homotopy pushout if and only if the bottom row displays a homotopy pullback.

At this point we introduce a generalization of the distinguished triangles.

Definition 2.2. A triangle (f, g, h) is *exact* if it induces long exact sequences upon application of the functors $\mathscr{C}(-, W)$ and $\mathscr{C}(W, -)$ for every object W of $\mathscr{C}$.

The following is a standard result in the theory of triangulated categories [15].

Lemma 2.3. *Every distinguished triangle is exact.*

If (f, g, h) is distinguished, then $(f, g, -h)$ is exact but generally not distinguished. These exact triangles $(f, g, -h)$ give a second triangulation of $\mathscr{C}$, which we call the negative of the original triangulation.

Problem 2.4. The relationship between distinguished and exact triangles has not been adequately explored in the literature. An additive category with a given Σ can admit several triangulations [5]. To see this, define a *global automorphism* of $\mathscr{C}$ to be a collection of automorphisms $\alpha_X \colon X \to X$ for all $X \in \mathscr{C}$ which commute with all morphisms, $\alpha_Y f = f\alpha_X$ for all $f \colon X \to Y$, and satisfy $\alpha_{\Sigma X} = \Sigma(\alpha_X)$. The collection of triangles (f, g, h) such that $(f\alpha, g\alpha, h\alpha)$ is distinguished in the original triangulation gives a new triangulation on $\mathscr{C}$. The negative triangulation is an example. Different automorphisms can give the same triangulation, but there are triangulated categories for which this construction gives infinitely many different triangulations. It is an open question whether or not $(\mathscr{C}, \Sigma)$ can admit two different triangulations that are not obtained from each other by a global automorphism.

The fact that the triangles in Definition 2.1 give rise to weak pushouts and weak pullbacks depends only on the fact that they are exact, not on the assumption that they are distinguished. This motivates the following definition.

Definition 2.5. For exact triangles of the form displayed in Definition 2.1, we say that the following commutative diagram, which displays both a weak pushout and a weak pullback, is a *pushpull square*.

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ Y & \xrightarrow[j]{} & W \end{array}$$

Lemma 2.6. *The central squares in any braid of distinguished triangles generated by $h = g \circ f$ are pushpull squares. More precisely, with the notations of (T3), the following triangles are exact.*

$$Y \xrightarrow{(f', g)} U \vee Z \xrightarrow{(j, h')} V \xrightarrow{-g'' \circ j'} \Sigma Y$$

$$Y \xrightarrow{j \circ f'} V \xrightarrow{(j', h'')} W \vee \Sigma X \xrightarrow{(g'', -\Sigma f)} \Sigma Y$$

Proof. Although rather lengthy, this is an elementary diagram chase. □

Remark 2.7. We would like to conclude that the triangles displayed in the lemma are distinguished and not just exact. Examples in [12] imply that this is not true for all choices of j and j'. The braid in (T3) gives rise to a braid of distinguished triangles that is cogenerated by $-g'' \circ j'$ or, equivalently, generated by $\Sigma^{-1}(j'' \circ g')$. Here $\Sigma^{-1}(j'' \circ g') = 0$ since $j'' = \Sigma f' \circ g''$. This implies that the central term in the braid splits as $U \vee Z$. Application of (T3) gives a distinguished triangle

$$Y \xrightarrow{\alpha} U \vee Z \xrightarrow{\beta} V \xrightarrow{-g'' \circ j'} \Sigma Y.$$

inspecting the relevant braid, we see that $\alpha = (\overline{f}', g)$ and $\beta = (j, \overline{h}')$. However, we cannot always replace $\overline{f}'$ and $\overline{h}'$ by f' and h' and still have a distinguished triangle.

This leaves open the possibility that the triangles displayed in Lemma 2.6 are distinguished for some choices of j and j'. It was stated without proof in [2, 1.1.13] that j and j' can be so chosen in the main examples, and we shall explain why that is true in §3. It was suggested in [2, 1.1.13] that this conclusion should be incorporated in Verdier's axiom if the conclusion were needed in applications. This course was taken in [10], and we believe it to be a sensible one. However, rather than try to change established terminology, we offer the following modified definition.

Definition 2.8. A triangulation of $\mathscr{C}$ is *strong* if the maps j and j' asserted to exist in (T3) can be so chosen that the two exact triangles displayed in Lemma 2.6 are distinguished.

Remark 2.9. Neeman has given an alternative definition of a triangulated category that is closely related to our notion of a strong triangulated category; compare [12, 1.8] and [13, §1.4]. It is based on the existence of particularly good choices of the map k in (TR3).

3. How to prove Verdier's axiom

We here recall the standard procedure for proving Verdier's axiom (T3). The exposition we give is general, thinking in terms of model categories, but we also discuss the elementary down to earth version. The reader unfamiliar with model category theory will be given pointers to more direct proofs in the literature.

We assume that our given category $\mathscr{C}$ is the "derived category" or "homotopy category" obtained from some Quillen model category $\mathscr{B}$. One can give general formal proofs of our axioms that apply to the homotopy categories associated to "simplicial", "topological", or "homological" model categories that are enriched over based simplicial sets, based spaces, or chain complexes, respectively. We shall be informal, but we shall give arguments in forms that should make it apparent that they apply equally well to any of these contexts. An essential point is to be careful about the passage from arguments in the point-set level model category $\mathscr{B}$, which is complete and cocomplete, to conclusions in its homotopy category $\mathscr{C}$, which generally does not have limits and colimits.

We assume that $\mathscr{B}$ is tensored and cotensored over the category in which it is enriched. We then have canonical cylinders, cones, and suspensions, together with their Eckmann-Hilton duals. The duals of cylinders are usually called "path objects" in the model theoretic literature (although in based contexts that term might more sensibly be reserved for the duals of cones). When we speak of homotopies, we are thinking in terms of the canonical cylinder $X \otimes I$ or path object Y^I, and we need not concern ourselves with left versus right homotopies in view of the adjunction

$$\mathscr{B}(X \otimes I, Y) \longrightarrow \mathscr{B}(X, Y^I).$$

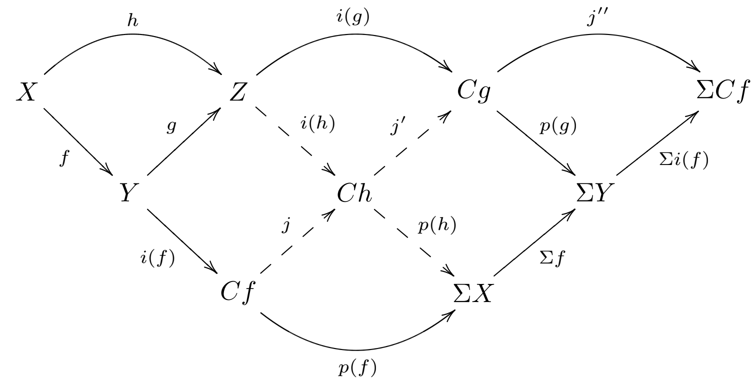
Hovey [6] gives an exposition of much of the relevant background material on simplicial model categories. Discussions of topological model categories appear in [4] and [9]. Homological model categories appear implicitly in [6] and [8, III§1]. Of course, we assume that the functor Σ on $\mathscr{B}$ induces a self-equivalence of $\mathscr{C}$.

The distinguished triangles in $\mathscr{C}$ are the triangles that are isomorphic in $\mathscr{C}$ to a canonical distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{i(f)} Cf \xrightarrow{p(f)} \Sigma X \tag{3.1}$$

in $\mathscr{B}$. Here $Cf = Y \cup_f CX$, where CX is the cone on X, and $i(f)$ and $p(f)$ are the evident canonical maps. Then (T1) is clear and (T2) is a standard argument with cofiber sequences. One uses formal comparison arguments (as in [15, II.1.3.2]) to reduce the verification of (T3) in $\mathscr{C}$ to consideration of canonical cofiber sequences in $\mathscr{B}$. In $\mathscr{B}$, one writes down the following version of the braid in (T3).

(3.2)

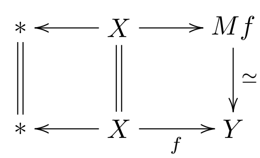


Here $h = g \circ f$, j and j' are evident canonical induced maps, $j'' = \Sigma i(f) \circ p(g)$, and the diagram commutes in $\mathscr{B}$. One proves (T3) by writing down explicit inverse homotopy equivalences

$$\xi : Cg \longrightarrow Cj \quad \text{and} \quad \nu : Cj \longrightarrow Cg$$

such that $j' = \nu \circ i(j)$ and $j'' = p(j) \circ \xi$. Details of the algebraic argument are in [15, pp. 75–77] and [16, p. 376], and the analogous topological argument is an illuminating exercise.

There is a standard and useful reformulation of the original triangulation. Assuming, as can be arranged by cofibrant approximation, that f is a cofibration between cofibrant objects, the quotient Y/X is cofibrant. Let Mf be the mapping cylinder of f. Passage to pushouts from the evident commutative diagram



gives a quotient map $q(f) : Cf \longrightarrow Y/X$. By [6, 5.2.6], we have the following standard result. It is central to our way of thinking about triangulated categories.

Lemma 3.3. *Let $f : X \longrightarrow Y$ be a cofibration between cofibrant objects. Then the quotient map $q(f) : Cf \longrightarrow Y/X$ is a weak equivalence.*

Now define

$$\delta(f) : Y/X \longrightarrow \Sigma X$$

to be the map in $\mathscr{C}$ represented by the formal "connecting map"

$$Y/X \xleftarrow{q(f)} Cf \xrightarrow{p(f)} \Sigma X \tag{3.4}$$

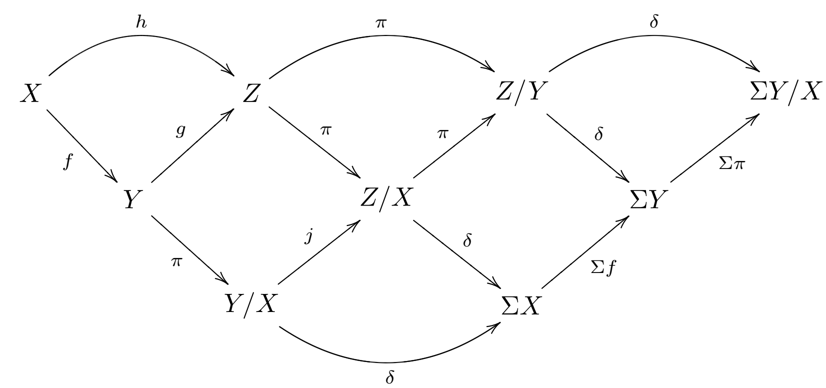
in $\mathscr{B}$. Observe that (3.4) gives a functor from cofibrations in $\mathscr{B}$ to diagrams in $\mathscr{B}$. The composite $q(f) \circ i(f) : Y \longrightarrow Y/X$ is the evident quotient map, which we denote by $\pi(f)$. Therefore, when we pass to $\mathscr{C}$, our canonical distinguished triangle (3.1) is isomorphic to the triangle represented by the diagram

$$X \xrightarrow{f} Y \xrightarrow{\pi(f)} Y/X \xrightarrow{\delta(f)} \Sigma X \tag{3.5}$$

in $\mathscr{B}$, and our triangulation consists of all triangles in $\mathscr{C}$ that are isomorphic to one of this alternative canonical form. This reformulation has distinct advantages.

Returning to Verdier's axiom, we can replace the given maps f, g, and thus $h = g \circ f$ by cofibrations between cofibrant objects, and then the quotient objects Y/X, Z/X and Z/Y are cofibrant. The point of Verdier's axiom now reduces to just the observation that Z/Y is canonically isomorphic in $\mathscr{B}$ to $(Z/X)/(Y/X)$. Using our new canonical cofibrations (3.5) starting from f, g, h, and the cofibration $j : Y/X \longrightarrow Z/X$, we obtain the following braid.

(3.6)



Expanding the arrows δ as in (3.4), we find that this braid in $\mathscr{C}$ is represented by an actual commutative diagram in $\mathscr{B}$, but of course with some wrong way arrows. With this proof of Verdier's axiom, there is no need to introduce the explicit homotopies ξ and ν of our first proof. Modulo equivalences, the two central braids in (3.6) are as follows. Here and later, we generally write $C(Y, X)$ instead of Cf for a given cofibration $f : X \longrightarrow Y$.

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y/X & \longrightarrow & Z/X \end{array} \qquad \begin{array}{ccc} C(Z,X) & \longrightarrow & C(Z,Y) \\ \downarrow & & \downarrow \\ \Sigma X & \longrightarrow & \Sigma Y \end{array}$$

These are both pushouts in which the horizontal arrows are cofibrations and all objects are cofibrant. By the following lemma, this implies that, in $\mathscr{C}$, these two squares give pushpull diagrams that arise from distinguished triangles. We conclude that $\mathscr{C}$ is strongly triangulated in the sense of Definition 2.8.

Lemma 3.7. *Suppose given a pushout diagram in $\mathscr{B}$,*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle j} \\ Z & \xrightarrow[k]{} & W, \end{array}$$

in which f and therefore k are cofibrations and all objects are cofibrant. Then there is a distinguished triangle

$$X \xrightarrow{(-f,g)} Y \vee Z \xrightarrow{(j,k)} W \longrightarrow \Sigma X.$$

in $\mathscr{C}$. Thus the original square gives rise to a pushpull square in $\mathscr{C}$.

Proof. Standard topological arguments work model theoretically to give a weak pushout (double mapping cylinder) $M(f,g)$ in $\mathscr{B}$ which fits into a canonical triangle

$$X \vee Y \xrightarrow{(j',k')} M(f,g) \xrightarrow{\pi} \Sigma X \xrightarrow{\delta} \Sigma X \vee \Sigma Y$$

as in (3.5). It is easy to check that $\delta = (f,-g)$ in $\mathscr{C}$ and that there is a weak equivalence $M(f,g) \longrightarrow W$ under $Y \vee Z$ in $\mathscr{B}$. The conclusion follows. □

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