FINITE GROUPS AND FINITE SPACES

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We shall explain some of the results and questions in a beautiful 1978 paper [3] by Daniel Quillen. He relates properties of groups to homotopy properties of the simplicial complexes of certain posets constructed from the group. He does not explicitly think of these posets as finite topological spaces. He seems to have been unaware of the earlier papers of McCord [2] and Stong [4] that we have studied, and it is interesting to look at his work from their perspective. Stong himself first looked at Quillen's work this way [5], and we will include his results on the topic. We usually work with a finite group G, but the basic definitions apply more generally.

1. Equivariance and finite spaces

We begin with some general observations about equivariance and finite T_0 topological spaces, largely following Stong [5].

A topological group G is a group and a space whose product $G \times G \longrightarrow G$ and inverse map $G \longrightarrow G$ are continuous. An action of G on a topological space X is a continuous map $G \times X \longrightarrow X$, written $(g, x) \mapsto gx$, such that $g(hx) = (gh)x$ and $ex = x$, where e is the identity element of G. A map $f\colon X \longrightarrow Y$ of G-spaces is a continuous map f such that $f(gx) = gf(x)$ for $g \in G$ and $x \in X$. We say that G acts trivially on X if $gx = x$ for all g and x. We let G act diagonally on products $X \times Y$, $g(x, y) = (gx, gy)$. In particular, with G acting trivially on I, we have the notion of a G-homotopy, namely a G-map $h\colon X \times I \longrightarrow Y$. There is a large subject of equivariant algebraic topology, in which one studies the algebraic invariants of G-spaces.

We begin with some basic ideas of equivalence in this context. We say that a G-map $f\colon X \longrightarrow Y$ is a G-homotopy equivalence if there is a G-map $f'\colon Y \longrightarrow X$ and there are G-homotopies $f \circ f' \simeq \mathrm{id}$ and $f' \circ f \simeq \mathrm{id}$. For a subgroup H of G, define the H-fixed point space X^H of X to be $\{x | hx = x \text{ for } h \in H\}$. Say that a G-map f is an H-equivalence if $f^H\colon X^H \longrightarrow Y^H$ is a nonequivariant homotopy equivalence. For nice G-spaces, the sort one usually encounters in classical algebraic topology, which are called G-CW complexes, a map f is a G-homotopy equivalence if and only if it is an H-equivalence for all subgroups H. Note that we have the much weaker notion of an e-equivalence, namely a G-map which is a homotopy equivalence of underlying spaces, forgetting the action of G.

We also have weak notions. A G-map f is a weak G-homotopy equivalence if each $f^H\colon X^H \longrightarrow Y^H$ is a weak equivalence in the nonequivariant sense. We also have the notion of a weak e-equivalence, meaning a G-map that is a weak equivalence of underlying spaces, forgetting the action of G.

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In general, the notions of G-equivalence are very much stronger than the notions of e-equivalence. There are lots of G maps that are e-equivalences but are not G-equivalences. We show that cannot happen when G acts on a finite space.

Let us note first that if a topological group G is a finite T_0-space, then it is discrete. Indeed, if $h \leq g$, then, by the continuity of the inverse map, $h^{-1} \leq g^{-1}$. By the continuity of left multiplication by h^{-1}, $e \leq h^{-1}g$, and then, by the continuity of right multiplication by g^{-1}, $g^{-1} \leq h^{-1}$. Since G is T_0, $g^{-1} = h^{-1}$ and therefore $g = h$. Thus $U_g = g$ is open for all g and therefore every subset is open.

Recall the notion of upbeat and downbeat points in a finite T_0-space X. Note that if x is downbeat, so that there is a $y > x$ such that $z > x$ implies $z \geq y$, then y is uniquely determined by x.

Theorem 1.1. *Let X be a finite T_0-space with an action by a group G. Then there is a core $C \subset X$ such that C is a sub G-space and equivariant deformation retract of X. We call C an equivariant core of X.*

Proof. The orbit Gx of an element x is $\{gx | g \in G\}$. If x is upbeat, then gx is also upbeat, with gy playing the role of y. The inclusion $X - Gx \subset X$ is the inclusion of a sub G-space. Define $f\colon X \longrightarrow X - Gx \subset X$ by $f(z) = z$ if $z \notin Gx$ and $f(gx) = gy$, where $y > x$ is such that $z > x$ implies $z \geq y$. Clearly $f \geq \mathrm{id}$ and thus $f \simeq \mathrm{id}$. An explicit homotopy used to show this is given by $h(z,t) = z$ if $t < 1$ and $h(z,1) = f(z)$, and this homotopy is a G-map. Removing upbeat and downbeat orbits successively until none are left, we reach an equivariant core. □

Corollary 1.2. *If X is a contractible finite T_0-space with an action by a group G, then X is equivariantly contractible.*

Proof. The core of X is a point, so the equivariant core must be a point with the trivial action by G. □

Corollary 1.3. *If X is a contractible finite T_0-space, then X has a point that is fixed by every homeomorphism of X.*

Proof. The finite group G of homeomorphisms of X acts on X, and the equivariant core is a fixed point. □

Theorem 1.4. *Let X and Y be finite T_0-spaces with actions by G and let $f\colon X \longrightarrow Y$ be a G-map. If f is an e-homotopy equivalence, then f is a G-homotopy equivalence.*

Proof. Let C and D be cores of X and Y. Write $i_X\colon C \longrightarrow X$ and $r_X\colon X \longrightarrow C$ for the inclusion and retraction, and similarly for Y. Let p be the composite

$$C \xrightarrow{i_X} X \xrightarrow{f} Y \xrightarrow{r_Y} D.$$

Then p is a G-map and a homotopy equivalence between minimal finite spaces. The latter property implies that p is a homeomorphism, and p^{-1} is necessarily also a G-map. Define $f^{-1}\colon Y \longrightarrow X$ to be the composite

$$Y \xrightarrow{r_Y} D \xrightarrow{p^{-1}} C \xrightarrow{i_X} X.$$

Then $f^{-1} \circ f$ and $f \circ f^{-1}$ are equivariantly homotopic to the respective identity maps. Indeed, we have the homotopies

$$f^{-1}f = f^{-1}f\, id_X \simeq f^{-1}f\, i_X r_X = i_X p^{-1}(r_Y f i_X) r_X = i_X r_X \simeq id_X$$

and

$$ff^{-1} = id_Y ff^{-1} \simeq i_Y r_Y ff^{-1} = i_Y(r_Y f i_X)p^{-1} r_Y = i_Y r_Y \simeq id_Y. \qquad \square$$

2. The basic posets and Quillen's conjecture

Fix a finite group G and a prime p. We define two posets.

Definition 2.1. Let $\mathscr{S}_p(G)$ be the poset of non-trivial p-subgroups of G, ordered by inclusion. A p-group is *elementary Abelian* if every element has order 1 or p. This means that it is a vector space over the field of p elements. Define $\mathscr{A}_p(G)$ to be the poset of non-trivial elementary p-subgroups of G, ordered by inclusion and let $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ be the inclusion.

Remark 2.2. Quillen calls a non-trivial elementary Abelian p-group a *p-torus*, and he defines its rank to be its dimension as a vector space.

The reason these posets are interesting is that G acts on them in such a way that their topological properties relate nicely to algebraic properties of G. The action of G is by conjugation. If H is a subgroup of G and $g \in G$, write $H^g = gHg^{-1}$. The function f_g that sends P to P^g gives an automorphism of the posets $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$. Clearly $f_e = \mathrm{id}$, where e is the identity element of G, and $f_{g'g} = f_{g'} \circ f_g$. These automorphisms are what give these posets their interest: the poset together with its group action describe how the different p-subgroups are related under subconjugation in G.

In particular, a point P in $\mathscr{A}_p(G)$ is fixed under the action of G if and only if $P^g = P$ for all $g \in G$, and this means that P is a normal subgroup of G. Thus the poset $(\mathscr{A}_p(G))^G$ of fixed points is the poset of normal p-tori of G. We can therefore relate algebraic questions about the presence of normal subgroups to topological questions about the existence of fixed points. Of course, we may regard these posets as finite T_0-spaces with G actions, and the theory of the previous section applies.

Remark 2.3. Some of Quillen's language for studying these posets is similar to the language we have been using, but it can be quite confusing. For example, he says that a subset S of a poset X is *closed* if $x \in S$ and $y \leq x$ implies $y \in S$. In our language, this means that $x \in S$ implies $U_x \subset S$, which says that S is *open*.

The posets $\mathscr{S}_p(G)$ and $\mathscr{A}_p(G)$ are both empty if p does not divide the order of G. At first sight, it might seem that $\mathscr{S}_p(G)$ is a lot more interesting and complicated than $\mathscr{A}_p(G)$, but that is not the case. To understand the discussion to follow, it is helpful to keep the following commutative diagram of spaces in mind, remembering that its vertical arrows are weak equivalences.

$$\begin{array}{ccc} |\mathscr{K}\mathscr{A}_p(G)| & \xrightarrow{|\mathscr{K}(i)|} & |\mathscr{K}\mathscr{S}_p(G)| \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathscr{A}_p(G) & \xrightarrow[i]{} & \mathscr{S}_p(G) \end{array}$$

We first consider p-groups.

Proposition 2.4. *If P is a non-trivial p-group, then $\mathscr{A}_p(P)$ and $\mathscr{S}_p(P)$ are both contractible.*

Proof. There is a central subgroup B of P of order p. We will be accepting as known some basic facts in the theory of finite groups, such as this one. The proof is just an easy counting argument after breaking P into conjugacy classes of elements. For any subgroup A of P, we have $A \subset AB \supset B$. If A is a p-torus, then so is AB since B is central. Define three maps $\mathscr{A}_p(P) \longrightarrow \mathscr{A}_p(P)$: the identity map id, the map f that sends A to AB, and the constant map c_B that sends A to B. These are all continuous, and our inclusions say that $\mathrm{id} \leq f \geq c_B$. This implies that $\mathrm{id} \simeq f \simeq c_B$. Since the identity is homotopic to the constant map, $\mathscr{A}_p(G)$ is contractible. The proof for $\mathscr{S}_p(G)$ is the same. □

Quillen calls a poset X *conically contractible* if there is an $x_0 \in X$ and a map of posets $f\colon X \longrightarrow X$ such that $x \leq f(x) \geq x_0$ for all x. He was thinking in terms of associated simplicial complexes, but we are thinking in terms of finite T_0-spaces. The previous proof says that $\mathscr{A}_p(P)$ and $\mathscr{S}_p(P)$ are conically contractible. It is to be emphasized that conically contractible finite spaces are genuinely and not just weakly contractible. As we shall see, the difference is profound in the case at hand. In contrast with the previous result, we emphasize the word "weak" in the following result.

Theorem 2.5. *The inclusion $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is a weak homotopy equivalence, hence so is the induced map $|\mathscr{K} i|\colon |\mathscr{K}\mathscr{A}_p(G)| \longrightarrow |\mathscr{K}\mathscr{S}_p(G)|$.*

Proof. We have the open cover of $\mathscr{S}_p(G)$ given by the U_P, where P is a non-trivial finite p-group. Clearly $i^{-1}U_P$ is the poset of p-tori of G that are contained in P, and this is the contractible space $\mathscr{A}_p(P)$. Our general theorem that weak homotopy equivalence is a local notion applies. □

Definition 2.6. Define the p-rank of G, denoted $r_p(G)$, to be the maximal rank of a p-torus in G. Observe that this is one greater than the dimension of the simplicial complex $\mathscr{K}\mathscr{A}_p(G)$ (interpreting the dimension of the empty complex to be -1).

Example 2.7. If the p-Sylow subgroups of G are cyclic of order p and there are q of them, then $\mathscr{A}_p(G)$ is a discrete space with q points. This holds, for example, if G is the symmetric group of order n, where p is a prime and $p \leq n < 2p$.

Remark 2.8. Sylow's third theorem is relevant. The number of Sylow p-subgroups of G is congruent to 1 mod p and divides the order of G.

Theorem 2.9. *If G has a non-trivial normal p-subgroup, then $\mathscr{S}_p(G)$ is conically contractible and therefore contractible, hence $\mathscr{A}_p(G)$ is weakly contractible. Conversely, if either $\mathscr{A}_p(G)$ or $\mathscr{S}_p(G)$ is contractible (not just weakly!), then G has a non-trivial normal p-subgroup.*

Proof. Suppose G has a normal p-subgroup P. Then, for any p-subgroup Q of G, $Q \subset QP \supset P$, where QP denotes the subgroup generated by P and Q. This means that $\mathrm{id} \leq f \geq c_P$, where $f(Q) = QP$ and $c_P(Q) = P$, hence $\mathscr{S}_p(G)$ is conically contractible. Conversely, suppose that $\mathscr{S}_p(G)$ is contractible. It is then G-contractible to a G-fixed point P, which means that P is a normal p-subgroup. This argument works equally well with $\mathscr{S}_p(G)$ replaced by $\mathscr{A}_p(G)$, and then of course the conclusion gives a normal elementary Abelian p-subgroup. □

In the converse, the hypotheses are different because $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is not generally a homotopy equivalence. To see this, we use the following observation.

Lemma 2.10. *Let $\mathscr{Q}_p(G) \subset \mathscr{S}_p(G)$ be the subposet of intersections of Sylow p-subgroups. Then $\mathscr{Q}_p(G)$ is an equivariant deformation retract of $\mathscr{S}_p(G)$.*

Proof. For $P \in \mathscr{S}_p(G)$, let $f(P)$ be the intersection of the Sylow p-subgroups that contain P and observe that $f\colon \mathscr{S}_p(G) \longrightarrow \mathscr{Q}_p(G) \subset \mathscr{S}_p(G)$ is continuous and G-equivariant. Since $Q \le f(Q)$, $\mathrm{id} \simeq f$ via an equivariant homotopy. □

Example 2.11. Let $G = \Sigma_5$ be the symmetric group on five letters. Then $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$ are not homotopy equivalent. There are 6 conjugacy classes of 2-subgroups of G, as follows.

(i) Dihedral groups D_8 of order 8, the Sylow 2-subgroups.
(ii) Cyclic groups C_4 of order 4.
(iii) Elementary 2-groups $C_2 \times C_2$ generated by transpositions (ab) and (cd).
(iv) Elementary 2-groups $C_2 \times C_2$ generated by products of transpositions $(ab)(cd)$, $(ac)(bd)$, whose product in either order is $(ad)(bc)$.
(v) Cyclic groups C_2 generated by a transposition.
(vi) Cyclic groups C_2 generated by a product of two transpositions.

Of course, each $C_2 \times C_2$ contains three C_2's. Each C_2 of type (v) is contained in three $C_2 \times C_2$'s of type (iii) and each C_2 of type (vi) is contained in one $C_2 \times C_2$ of type (iii) and one $C_2 \times C_2$ of type (iv). This information shows that $\mathscr{A}_2(G)$ is minimal, hence not homotopy equivalent to any space with fewer points. The intersections of Sylow 2-subgroups of G are the dihedral groups in (i), the groups $C_2 \times C_2$ of type (iv) and the subgroups C_2 of type (v) (and in fact $\mathscr{Q}_2(G)$ is a core of $\mathscr{S}_2(G)$). Counting, one checks that there are fewer points in $\mathscr{Q}_2(G)$ than there are in $\mathscr{A}_p(G)$, so there can be no homotopy equivalence between them.

Quillen conjectured the following stronger version of the converse, and he proved the conjecture for solvable groups.

Conjecture 2.12. *If $\mathscr{A}_p(G)$ is weakly contractible, then G contains a non-trivial normal p-subgroup.*

The hypothesis holds if and only if $|\mathscr{K}\mathscr{A}_p(G)|$ or, equivalently, $|\mathscr{K}\mathscr{S}_p(G)|$ is weakly contractible and therefore contractible.

In particular, if G is simple, then it has no non-trivial normal subgroups and the conjecture implies that $\mathscr{A}_p(G)$ cannot be weakly contractible. This has been verified for many but not all finite simple groups, using the classification theorem. A conceptual proof would be a wonderful achievement!

3. Joins and products

As a preliminary to an example to be given later and an illustration of the translation of algebra to topology, we explain how to compute $\mathscr{A}_p(G \times H)$ for finite groups G and H.

Definition 3.1. Let X and Y be posets. As dictated by the product topology on $X \times Y$, the *product poset* $X \times Y$ is the set $X \times Y$ with the induced partial order specified by $(x, y) \le (x', y')$ if $x \le x'$ and $y \le y'$. The *join poset* $X * Y$ is the poset given by the disjoint union of the posets X and Y, together with the additional relations $x < y$ if $x \in X$ and $y \in Y$.

If Y is a single point, then $X * Y$ is the cone CX as we defined it earlier. Quillen defines cones by taking X rather than Y to be a point.

Remark 3.2. It is perhaps illuminating to use both choices, and we write C^+X for the first choice and C^-X for the second choice. They give rise to simplicial complexes with homeomorphic realizations. There is a canonical map i from $X*Y$ to the poset $C^+X \times C^-Y - \{(c_X, c_Y)\}$, where c_X and c_Y denote the cone points. Indeed, we set $i(x) = (x, c_Y)$ and $i(y) = (c_X, y)$. Since $x < c_X$ and $c_Y < y$, $i(x) < i(y)$ for all x and y, while $i(x) \leq i(x')$ if and only $x \leq x'$ and $i(y) \leq i(y')$ if and only if $y \leq y'$. This map does not figure explicitly in Quillen's work, but a similar comparison below will show why we care about it.

Proposition 3.3. *The poset $\mathscr{A}_p(G \times H)$ is homotopy equivalent to the poset $C^-\mathscr{A}_p(G) \times C^-\mathscr{A}_p(H) - \{(c_G, c_H)\}$.*

Proof. Let T be the subposet of $\mathscr{A}_p(G\times H)$ whose points are the p-tori in $G = G\times e$, the p-tori in $H = e \times H$, and the products $A \times B$ of p-tori A in G and B in H. (Remember that p-tori are non-trivial elementary Abelian p-groups). Visibly, thinking of trivial groups as conepoints and therefore $<$ non-trivial subgroups, T is isomorphic to $C^-\mathscr{A}_p(G) \times C^-\mathscr{A}_p(G) - \{(c_G, c_H)\}$. Let $i\colon T \longrightarrow \mathscr{A}_p(G \times H)$ be the inclusion. The projections $G \times H \longrightarrow G$ and $G \times H \longrightarrow H$ induce a map $r\colon \mathscr{A}_p(G\times H) \longrightarrow T$ such that $r\circ i = \text{id}$. For $C \in \mathscr{A}_p(G\times H)$, we have $i(r(C)) \supset C$, which means that $i \circ r \geq \text{id}$ and thus $i \circ r \simeq \text{id}$. □

We consider products and joins of simplicial complexes and then return to relate joins geometrically to the posets of the proposition.

Definition 3.4. The product $K \times L$ of two abstract simplicial complexes K and L has $V(K \times L) = V(K) \times V(L)$ and has simplices all subsets of products $\sigma \times \tau$ of sets σ and τ that prescribe simplices of K and L. We must take subsets here since a general subset of $\sigma \times \tau$ is not a product of subsets of σ and τ. The product of geometric simplicial complexes in $\mathbb{R}^M$ and $\mathbb{R}^N$ is defined similarly as a geometric simplicial complex in $\mathbb{R}^{M+N} = \mathbb{R}^M \times \mathbb{R}^N$.

Proposition 3.5. *Let X and Y be posets. Then $\mathscr{K}(X) \times \mathscr{K}(Y)$ is a subdivision of $\mathscr{K}(X \times Y)$, hence both have the same geometric realization, and their common realization is homeomorphic to $|\mathscr{K}(X)| \times |\mathscr{K}(Y)|$.*

Proof. Since $\mathscr{K}(X) \times \mathscr{K}(Y)$ and $\mathscr{K}(X \times Y)$ have the same finite set of vertices, it is clear that every simplex of $\mathscr{K}(X) \times \mathscr{K}(Y)$ is a union of finitely many simplices of $\mathscr{K}(X\times Y)$. If $(x_0, y_0) < \cdots < (x_n, y_n)$, then the set of pairs (x_i, y_i) is a subset of $\sigma\times\tau$, where $\sigma = \{x_i\}$ and $\tau = \{y_i\}$. Thus every simplex of $\mathscr{K}(X\times Y)$ is contained in a simplex of $\mathscr{K}(X) \times \mathscr{K}(Y)$. The projections $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$ induce the coordinates of a map

$$|\mathscr{K}(X \times Y)| \longrightarrow |\mathscr{K}(X)| \times |\mathscr{K}(Y)|.$$

A point on the right is a pair (u, v) where u is an interior point of some simplex σ of $g\mathscr{K}(X)$ and v is an interior point of some simplex τ of $g\mathscr{K}(Y)$. Since all simplices on the left are subsimplices of some $\sigma \times \tau$, this map is a homeomorphism. □

The last statement works more generally for simplicial complexes that are not given in the form $\mathscr{K}(X)$ for a poset X.

Proposition 3.6. *Let K and L be geometric simplicial complexes. Then the projections induce a homeomorphism*

$$|K \times L| \longrightarrow |K| \times |L|.$$

Intuitively, the point is that the product of two geometric simplices is not a geometric simplex (a square is not a triangle) but can be subdivided into geometric simplices. In effect, the displayed homeomorphism carries out this subdivision consistently over all of the simplices of a product of simplicial complexes.

We have analogous statements for joins. Just as for products, the precise definition of which is different when we consider products of posets, of simplicial complexes, and of topological spaces, we have different meanings of the notion of join, all of which are denoted by the symbol $*$. However, unlike products, which are characterized by a universal property, the different definitions of the join are largely motivated by the comparisons among them.

Definition 3.7. The *join* $K * L$ of abstract simplicial complexes K and L has vertex set $V(K * L)$ the disjoint union of $V(K)$ and $V(L)$ and has simplices the simplices of K, the simplices of L, and all disjoint unions of simplices of K and L. The join of geometric simplicial complexes is defined similarly, requiring the disjoint union of $V(K)$ and $V(L)$ to be a linearly independent set.

Conceptually, it is helpful to note that, just like the product, where $X \times Y$ is not literally the same as $Y \times X$ but only isomorphic to it, we should think of disjoint union as an operation only commutative up to isomorphism. Then the choice of order corresponds to the analogous choice we had when defining the join of posets in Definition 3.1.

Definition 3.8. The join of topological spaces X and Y is the quotient space of $X \times I \times Y$ obtained by identifying $(x, 0, y)$ with $(x', 0, y)$ and $(x, 1, y)$ with $(x, 1, y')$ for all $x, x' \in X$ and $y, y' \in Y$. It is the space of lines connecting X to Y. If X and Y are subspaces of some large Euclidean space, $X * Y$ is defined geometrically as the subspace of points $tx + (1 - t)y$ for $x \in X$, $y \in Y$, and $0 \le t \le 1$, noting that the point is independent of x if $t = 0$ and of y if $t = 1$.

Proposition 3.9. *For posets X and Y,*

$$\mathscr{K}(X * Y) \cong \mathscr{K}(X) * \mathscr{K}(Y).$$

For abstract simplicial complexes K and L,

$$g(K * L) \cong gK * gL.$$

For geometric simplicial complexes K and L,

$$|K * L| \cong |K| * |L|.$$

We give another way to think about the join $|K| * |L|$ in $\mathbb{R}^N$, where K and L are geometric simplicial complexes. The notion of $X - \{x\}$, $x \in X$, is clear for a poset. For a simplicial complex K, $K - \{v\}$ for $v \in V(K)$ means the simplicial complex that is obtained from K by deleting all simplices which have v as a vertex, and $\mathscr{K}(X - \{x\}) = \mathscr{K}(X) - \{x\}$. However, $|K - \{v\}|$ is quite different from $|K| - v$.

Proposition 3.10. *For simplicial complexes K and L, $|K| * |L|$ is homeomorphic to $|CK \times CL - \{(c_K, c_L)\}|$*

Proof. The simplices of $CK \times CL$ that do not have (c_K, c_L) as a vertex are the simplices in either $CK \times L$ or $K \times CL$. Thus

$$CK \times CL - \{(c_K, c_L)\} = (CK \times L) \cup (K \times CL)$$

as subcomplexes of $CK \times CL$. Geometric realization commutes up to homeomorphism with cones, products and unions, so that

$$|(CK \times L) \cup (K \times CL)| \cong (C|K| \times |L|) \cup (|K| \times C|L|).$$

Now the following observation gives the conclusion. $\square$

Lemma 3.11. *For spaces X and Y, $X*Y$ is homeomorphic to $(CX\times Y)\cup(X\times CY)$.*

Proof. We identify $X * Y$ and $(CX \times Y) \cup (X \times CY)$ as homeomorphic quotients of subspaces of $X \times Y \times I \times I$. Let J be the diagonal $\{(s,t)|s+t=1\}$. Then $X * Y$ is homeomorphic to the quotient of $X \times Y \times J$ obtained from the equivalence relation given by

$$(x, y, (1,0)) \sim (x', y, (1,0)) \quad \text{and} \quad (x, y, (0,1)) \sim (x, y', (0,1)).$$

Think of the cone coordinates of CX and CY as the edges $I_1 = [(0,0),(1,0)]$ and $I_2 = [(0,0),(0,1)]$ of $I \times I$. Let $K = I_1 \cup I_2 \subset I \times I$. Then $(CX \times Y) \cup (X \times CY)$ is homeomorphic to the quotient of $X \times Y \times K$ obtained from precisely the same equivalence relation. Radial projection from $(1,1)$ gives a deformaion $I \times I - \{1,1\} \longrightarrow K$ that restricts to a homeomorphism $J \longrightarrow K$ and thus induces the claimed homeomorphism. $\square$

Corollary 3.12. *$|\mathscr{K}(\mathscr{A}_p(G \times H))|$ is homeomorphic to $|\mathscr{K}(\mathscr{A}_p(G)| * |\mathscr{K}(\mathscr{A}_p(H)|$.*

4. Group actions and Quillen's conjecture

We have proven that if G has a non-trivial normal p-subgroup, then $\mathscr{A}_p(G)$ is weakly contractible, and Quillen's conjecture is that, conversely, if $\mathscr{A}_p(G)$ is weakly contractible, then G has a non-trivial normal p-subgroup. We have seen that the conjecture can be thought of as a problem in the equivariant homotopy theory of finite T_0-spaces.

Proposition 4.1. *The conjecture holds if $r_p(G) \leq 2$.*

Proof. The hypothesis cannot hold if $r_p(G) = 0$, since $\mathscr{A}_p(G)$ is then empty and hence not contractible. If $r_p(G) = 1$, then the space $\mathscr{A}_p(G)$ is discrete since there are no proper inclusions. It is weakly contractible if and only if it consists of a single point, and then its single point must be fixed by the action of G. This means that there is a unique p-torus in G, and it is a normal subgroup of order p. If $r_p(G) = 2$, then $|\mathscr{K}(\mathscr{A}_p(G))|$ is one dimensional and contractible, which means that it is a tree. According to Quillen, "one knows (Serre) that a finite group acting on a tree always has a fixed point". This means that G has a normal p-torus. The trees here are of a particularly elementary sort, but the conclusion is still not altogether obvious. The following problem gives a way of thinking about it. $\square$

Problem 4.2. *Consider a T_0 space X such that $|\mathscr{K}(X)|$ is a tree (a contractible graph). Clearly X is weakly contractible. Prove that X is contractible. (Search for upbeat or downbeat points). It follows that if a finite group G acts on X, then X is G-contractible and therefore has a G-fixed point.*

Much of Quillen's paper is devoted to proving that the conjecture holds for *solvable* groups G. This means that there is a decreasing chain of subgroups of G, each normal in the next, such that the subquotients are cyclic of prime order.

Following Quillen, we work out the structure of $\mathscr{A}_p(G)$ when $G = \Sigma_{2p}$ is the symmetric group on $2p$ letters for an odd prime p. First note that $\mathscr{A}_p(\Sigma_n)$ is empty if $n < p$ and is a discrete space with one element for each cyclic subgroup of order p if $p \leq n < 2p$. There are $n!/(n-p)!p(p-1)$ such subgroups.

Let $g \in G = \Sigma_{2p}$ have order p. It has an orbit R with p elements in the set $S = \{1, \cdots, 2p\}$ on which G acts. Since any two elements of G in a p-torus A of $\mathscr{A}_p(G)$ commute, they have the same orbit R if they are not the identity. Therefore each A gives a unique partition of S into two A-invariant subsets, each with p elements. The set of such partitions of S into two subsets with p elements gives a corresponding decomposition of $\mathscr{A}_p(G)$ into disjoint subposets, each consisting of those A which partition S in the prescribed way.

Under the action of G, these partitions are permuted transitively, meaning that, given two partitions, there is an element of G that permutes one into the other. Consider for definiteness the partition into the first p and last p elements of S. Let H be the subgroup of those elements of G that fix this partition. The corresponding subposet of $\mathscr{A}_p(G)$ is $\mathscr{A}_p(H)$. Here H is the *wreath product* $\Sigma_2 \int \Sigma_p$, which is the semi-direct product of Σ_2 with $\Sigma_p \times \Sigma_p$ determined by the permutation action of Σ_2 on $\Sigma_p \times \Sigma_p$.

Since p is odd, $\mathscr{A}_p(H) = \mathscr{A}_p(\Sigma_p \times \Sigma_p)$, which, after passage to realizations of simplicial complexes, is the join $\mathscr{A}_p(\Sigma_p) * \mathscr{A}_p(\Sigma_p)$. Since Σ_p has $(p-2)!$ Sylow subgroups, each of order p, $\mathscr{A}_p(\Sigma_p)$ is the disjoint union of $(p-2)!$ points. After counting the number of partitions and inspecting the join of our two discrete spaces $\mathscr{A}_p(\Sigma_p)$, Quillen informs us, and we can work out for ourselves, that $|\mathscr{A}_p(\Sigma_{2p})|$ is a disconnected graph with $(2p)!/2(p!)^2$ components, each of which is homotopy equivalent to a one-point union of $((p-2)!-1)^2$ circles. For example, for $p = 5$, there are 25 circles. The same analysis applies to the alternating groups A_n for $n \leq 2p$ since $\mathscr{A}_p(A_n) = \mathscr{A}_p(\Sigma_n)$. Of course, these $\mathscr{A}_p(G)$ are not weakly contractible.

5. The components of $\mathscr{S}_p(G)$

Let p be a prime which divides the order of G. We describe the set of components $\pi_0(\mathscr{S}_p(G))$, which of course is the same as $\pi_0(\mathscr{A}_p(G))$. Recall that two elements of a poset are in the same component if they can be connected by a chain of elements, each either $\leq$ or $\geq$ the next. In the poset $\pi_0(\mathscr{S}_p(G))$, each element is a p-group and is contained in a Sylow subgroup. Therefore there is at least one Sylow subgroup in each component. Since any one Sylow subgroup P generates all the others by conjugation by elements of G, G acts transitively on $\pi_0(\mathscr{S}_p(G))$, in the sense that there is a single orbit. If $N = N_P$ denotes the subgroup of G that fixes the component $[P]$ of P, then G/N is isomorphic to the G-set $\pi_0(\mathscr{S}_p(G))$ via $gN \mapsto [P^g]$. We want to determine the subgroup N. Let $\mathrm{Syl}_p(G)$ denote the set of p-Sylow subgroups of G and let $N_G H$ denote the normalizer in G of a subgroup H. Recall that $H^g = gHg^{-1}$.

Proposition 5.1. *The following conditions on a subgroup M of G are equivalent.*

(i) For some $P \in Syl_p(G)$, $M \supset N_P$.

(ii) For some $P \in Syl_p(G)$, $M \supset N_G H$ for all $H \in \mathscr{S}_p(P)$.

(iii) $M \supset N_G P$ for some $P \in Syl_p(G)$ and $K \subset M$ if K is a p-subgroup of G that intersects M non-trivially.

(iv) p divides the order of M and $M \cap M^g$ is of order prime to p for all $g \notin M$.

Moreover, $\mathscr{S}_p(G)$ is connected if and only if there is no proper subgroup M which satisfies these equivalent conditions.

Proof. For the last statement, G is connected if and only if $G = N_P$ for all $P \in \mathrm{Syl}_p(G)$. (i) $\Longrightarrow$ (ii): If $g \in N_G H$ with $H \subset P$, then $H^g = H$ is contained in both P and P^g, so that $[P] = [P^g] = g[P]$. This means that $g \in N_P \subset M$.
(ii) $\Longrightarrow$ (iii): Obviously $M \supset N_G P$. Since P is a p-Sylow subgroup of G, it is also a p-Sylow subgroup of M. Thus if H is a non-trivial p-subgroup of M, then H is conjugate in M to a subgroup, H^m say, of P. Since $M \supset N_G(H^m)$ and $(N_G H)^m = N_G(H^m)$, $M \supset N_G H$. Let K be a p-subgroup of G such that $K \cap M$ is non-trivial. We have

$$K \cap M \subset N_K(K \cap M) = K \cap N_G(K \cap M) \subset K \cap M.$$

Since K is a p-group, the first inclusion is proper if $K \cap M$ is a proper subgroup of K. Since this is a contradiction, $K \cap M = K$ and $K \subset M$.
(iii) $\Longrightarrow$ (iv): Since $M \supset P$, p divides the order of M. Assume that p divides the order of $M \cap M^g$ for some $g \in G$. Then there is a non-trivial p-subgroup $H \subset M \cap M^g$. Let $H \subset Q$ for $Q \in \mathrm{Syl}_p(G)$. Since $Q \cap M$ is non-trivial, we have $Q \subset M$. Since $Q^{g^{-1}} \supset H^{g^{-1}}$ and $H^{g^{-1}} \subset M$, we also have $Q^{g^{-1}} \subset M$. Since P, Q, and $Q^{g^{-1}}$ are p-Sylow subgroups of M, they are conjugate in M, say $Q^m = P$ and $Q^{g^{-1}} = P^n$ for $m, n \in M$. Then a quick check shows that $mgn \in N_G P \subset M$ and therefore $g \in M$, proving (iv).
(iv) $\Longrightarrow$ (i): Writing G as the disjoint union of double cosets MgM, one calculates that the index of M in G is the sum over double coset representatives g of the indices of $M \cap M^g$ in M. Since p divides the order of M and does not divide the order of $M \cap M^g$ if $g \notin M$, these indices are divisible by p except for the double coset represented by e. Thus the index of M in G is congruent to 1 mod p, hence M must contain some p-sylow subgroup P. Let $N = N_P$. For $n \in N$, P and P^n are in the same component. Considering p-Sylow subgroups containing groups in a chain connecting them, we see that there is a sequence of p-Sylow subgroups $P = P_0, P_1, \ldots, P_q = P^n$ such that $P_i \cap P_{i+1} \neq \{e\}$. There are elements g_i such that $P_{i-1}^{g_i} = P_i$, and we can choose g_q so that $g_q \cdots g_1 = n$. We have $P \subset M$, and we assume inductively that $P_{i-1} \subset M$. Then $P_{i-1} \cap P_i \subset M \cap M^{g_i}$, so this intersection contains a p-group and, by (iv), $g_i \in M$. This implies that $P_i \subset M$ and, inductively, we conclude that $n \in M$, so that $N \subset M$. □

Corollary 5.2. *N_P is generated by the groups $N_G H$ for $H \in \mathscr{S}_p(P)$.*

Proof. N_P contains all of these $N_G H$, so it contains the subgroup they generate, and it is the smallest such subgroup. □

By the contrapositive, G is *not* connected if and only if there is a *proper* subgroup M of G that satisfies the equivalent properties of the proposition. For example, if $r_p(G) = 1$ and G has no non-trivial normal p-subgroup, then $\mathscr{A}_p(G)$ is discrete and not contractible, and is therefore not connected. Quillen gives a condition on G under which these are the only examples.

Proposition 5.3. *Let H $(= O_{p'}(G))$ be the largest normal subgroup of G of order prime to p and let K $(= O_{p',p}(G))$ be specified by requiring K/H to be the largest normal p-subgroup of the quotient group G/H. If K/H is non-trivial and $\mathscr{S}_p(G)$ is not connected, then $r_p(G) = 1$.*

Proof. If Q is a p-Sylow subgroup of K, then $K = QH$ since H is a p'-group and K/H is a p-group. This implies that H acts transitively on $\pi_0(\mathscr{S}_p(K))$ since it implies that any two p-Sylow subgroups are conjugate by the action of some $h \in H$. The intersection with K of a p-Sylow subgroup P of G is a p-Sylow subgroup of K. A p-subgroup of K is a p-subgroup of G, and the induced map $\pi_0(\mathscr{S}_p(K)) \longrightarrow \pi_0(\mathscr{S}_p(G))$ is surjective since $P \cap K \subset P$ implies that $[P]$ is the image of $[P \cap K]$. Therefore H also acts transitively on $\pi_0(\mathscr{S}_p(G))$. Let A be a maximal p-torus of G. The map $\pi_0(\mathscr{S}_p(AH)) \longrightarrow \pi_0(\mathscr{S}_p(G))$ is also surjective since H acts transitively on the target and the map is H-equivariant. Therefore $\mathscr{S}_p(AH)$ is not connected. The component $[A]$ is fixed by the centralizers $C_H(B)$ for all non-trivial subgroups B of A since $B^h = B \subset A$ for $h \in C_H(B)$. A result that I have not looked up, [1, 6.2.4], shows that if A is not cyclic (= rank one), then H is generated by these centralizers, which contradicts the fact that $\mathscr{S}_p(AH)$ is not connected. Thus A is cyclic. □

References

[1] D. Gorenstein. Finite Groups. Harper and Row. 1968.

[2] M.C. McCord. Singular homology groups and homotopy groups of finite topological spaces. Duke Mathematical Journal 33(1966), 465–474.

[3] D. Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. Advances in Mathematics 28(1978), 101–128.

[4] R.E. Stong. Finite topological spaces. Trans. Amer. Math. Soc. 123(1966), 325–340.

[5] R.E. Stong. Group actions on finite spaces. Discrete Mathematics 49(1984), 95–100.