PROJECT DESCRIPTION: HIGHER CATEGORICAL STRUCTURES AND THEIR APPLICATIONS

PROPOSED RESEARCH

1. Background, context, and perspective

Eilenberg and Mac Lane introduced categories, functors, and natural transformations in their 1945 paper [54]. The language they introduced transformed modern mathematics. Their focus was not on categories and functors, but on natural transformations, which are maps between functors. Implicitly, they were introducing the 2-category "Cat" of categories, functors, and natural transformations.

Higher category theory concerns higher level notions of naturality, which can be expressed as maps between natural transformations, maps between such maps, and so on, that is, maps between maps between maps. Just as the original definitions of Eilenberg and Mac Lane gave a way of thinking about categorical structures and analogies between such structures in different fields, higher category theory promises to allow serious thinking about and study of higher categorical structures that appear in a variety of specific fields. The need for such a language has become apparent, almost simultaneously, in mathematical physics, algebraic geometry, computer science, logic, and, of course, category theory. As we shall explain, such a language and a relevant body of results is already implicit throughout algebraic topology. In all of these areas, higher categorical structures are there in nature, and one needs a coherent way of thinking about them.

To give some feel for how such structures appear, we briefly consider topological quantum field theory (TQFT). A TQFT is a structure-preserving functor from a suitable cobordism category of manifolds to an analogously structured algebraic category. With the usual definitions, the objects of the domain category are closed manifolds of a certain dimension and the maps between them are equivalence classes of cobordisms between them, which are manifolds with boundary in the next higher dimension. However, it is in many respects far more natural to deal with an n-cobordism "category" constructed from points, edges, surfaces, and so on through n-manifolds that have boundaries with corners. The structure encodes cobordisms between cobordisms between cobordisms. This is an n-category with additional structure, and one needs analogously structured linear categories as targets for the appropriate "functors" that define the relevant TQFT's. One could equally well introduce the basic idea in terms of formulations of programming languages that describe processes between processes between processes. A closely analogous idea has long been used in the study of homotopies between homotopies between homotopies in algebraic topology. Analogous structures appear throughout mathematics.

In contrast to the original Eilenberg-Mac Lane definitions, there are many possible definitions of n-categories for larger n. This is already visible when $n = 2$, where there are strict 2-categories and weak 2-categories. However, the two notions are suitably equivalent, whereas this is false for $n \geq 3$. It is *not* to be expected that a single all embracing definition that is equally suited for all purposes will emerge. It is not a question as to whether or not a good definition exists. Not one, but many, good definitions already do exist, although they have been worked out to varying degrees. There is growing general agreement on the basic desiderata of a good definition of n-category, but there does not yet exist an axiomatization, and there are grounds for believing that only a partial axiomatization may be in the cards. This is a little like the original axiomatization of a cohomology theory in algebraic topology, where the core axioms give a uniqueness theorem, but full generality allows variants. Thus, for example, singular and Cech cohomology are different, although they agree on finite complexes. There are deeper and more relevant analogies with algebraic topology that we will explain below.

In the 1930's, there were many definitions of cohomology theories and no clear idea of the relationship between them. Definitions of algebraic K-theory (in the late 1960's), of spectra in algebraic topology (in the early 1960's on a crude level and in the 1990's on a more highly structured level), and of mixed motives in algebraic geometry (ongoing) give other instances where a very major problem was, or is, the foundational one of obtaining good definitions and a clear understanding of how different good definitions are related. That is the nature of the main focus of the project we are proposing.

We are also working on applications that are intertwined with these foundational issues, and we propose related work in several directions. Our technical proposal gives a brief and inadequate overview of the state of the art in these areas. In fact, only one of the six co-Directors of this project has his mathematical roots in category theory. The others have roots in algebraic topology, mathematical physics, algebraic geometry, logic, and computer science. We intend to continue our work in these areas. Nevertheless, like all others interested in n-categories, we agree that the fundamental problem of understanding the foundations must be addressed satisfactorily for this subject to reach maturity. We believe that, despite their intrinsic complexity, when developed coherently, higher categorical structures such as n-categories, will eventually become part of the standard mathematical culture.

In the United States, the last sentence may well read as hyperbole, a gross overstatement. That is far less true elsewhere. Category theory was invented in the United States, and many of its pioneers worked here. However, the category theory community has burgeoned elsewhere while it has contracted here. For example, the category theory discussion list on the web has more subscribers than the algebraic topology discussion list, but a far smaller proportion of Americans, and a still smaller proportion of the active mathematicians on the list are American. It is hard to think of another important field of mathematics in which the United States is so woefully weak. As far as we know, no category theorist is supported by the NSF. It is not even clear to which program such a person could apply.

This might not matter if the field were peripheral, but it is not. Categorical ideas are implicit in many major branches of modern mathematics, and it is painfully apparent to a categorically knowledgable reader that much recent work contains reinventions in specific areas of material that has been familiar to category theorists for decades. There are common categorical structures that appear in a variety of mathematical contexts, and there is a lot to be said about them in full generality. It is a beautiful part of the essential nature and structure of mathematics that the best way to solve a problem is often to generalize it. In particular, it is often very much harder to understand a categorical structure in a specialized context than it is to understand it in its natural level of generality.

We have no doubt that this applies to higher category theory and its applications. While the theory must be developed with a close eye to the applications, the internal logic and fundamental nature of the mathematical structures at hand, their aesthetics if you will, must be the main guide to the development of the foundations. Anything less would be fundamentally shortsighted. In the technical part of the proposal to follow, we shall give our current vision of how the project of unification seems likely to proceed, outlining a precise technical program. We shall also try to explain the relevance of foundational work in higher category theory to applications in specific fields.

The U.S. PI's on this proposal are not themselves expert in modern category theory. They have ideas that they wish to see implemented, but expertise that is unavailable in the United States will be needed to carry out these ideas in detail. There are strong and active groups of mathematicians working in category theory in other countries, and they are training a new generation of excellent young people. There are no such groups in this country. Some of these young people are directly interested in n-category theory, and we are already working in collaboration with a few of them. We hope to use the resources of an FRG to bring such young people to this country for sustained collaboration. We hope that this, and other facets of our project, will help initiate a rejuvenation of category theory in the United States.

We wish to emphasize the communal nature of this proposal. Around fifty "informal participants" have actively aided us in its preparation. A list of their names and affiliations is given below. The list includes most of the people who have worked on the definition and analysis of n-categories and most of the people who are directly involved in their applications. These people come from a wide range of backgrounds, interests, and locations. They are all in agreement that the time is right to make a concerted effort to establish firm and coherent foundations for higher category theory. The lack of such foundations is a major impediment to further progress in a number of important areas of application. They join us in hoping to develop modes and habits of collaboration that will allow us to give this entire field of mathematics a more coherent and unified structure.

2. The relationship with higher homotopies

Higher category theory is related both mathematically and by analogy to the higher homotopies that have been used since the late 1940's in algebraic topology and homological algebra. The analogy is obvious and precise: a natural transformation between functors $\mathcal{C} \to \mathcal{D}$ can be viewed as a functor $\mathcal{C} \times \mathcal{I} \to \mathcal{D}$, where $\mathcal{I}$ is the evident category with two objects and one non-identity morphism. Category theory is intrinsically more general, in a transparently obvious way: homotopies have inverses, but natural transformations do not.

Steenrod's 1947 paper [137] already used a sequence of higher commutativity homotopies, called $\cup_i$-products, on cochains to define the Steenrod squares in mod 2 cohomology. The $\cup_i$ were quickly made combinatorially explicit, but that feature disappeared from memory, returning in the 1990's in one proof of Deligne's conjecture on the cochain algebra of Hochschild cohomology [116, 117]. The reason for the amnesia is that, to define Steenrod operations in mod p cohomology for odd primes p, the higher homotopies were encapsulated within the language of resolutions and the cohomology of groups [138].

A similar passage from explicit higher homotopies to abstract encapsulations reappeared on the space level in the theory of A_∞-spaces of Sugawara and Stasheff [135, 136, 144, 145, 146] and its later codifications in terms of operad actions (E_1-spaces) [113] or diagrams of spaces (Δ-spaces) [126]. Here the notion of a topological monoid is relaxed to a notion where strict associativity is replaced by a sequence of higher associativity homotopies. The combinatorics of higher homotopies are encapsulated in a more conceptual framework, and it is found that any A_∞-space is equivalent to a topological monoid.

A deeper manifestation of similar ideas appeared around 1970 in the theory of iterated loop spaces. Here Steenrod's two approaches to cohomology operations were recapitulated homotopically in the development of homology operations, mod 2 by Kudo and Araki [2] and mod p by Dyer and Lashof [51]. In parallel, the theory of A_∞-spaces was generalized to a topological theory of iterated loop spaces that aimed at an understanding of what internal structure on a space ensures that it is equivalent to an n-fold or infinite loop space. Here work of Boardman and Vogt, May, and Segal [21, 113, 126] gave conceptual encapsulations that hid the implicit higher homotopies, whose combinatorial structure is still somewhat obscure. These encapsulations are given in terms of PROP or operad actions (E_n and E_∞-spaces) or diagrams of spaces (Γ-spaces).

3. Higher categorical structures

It is gradually becoming more and more apparent that the categorical analogues of higher homotopies appear throughout mathematics and its applications, at least implicitly. Explicit use of 2-categories is by now commonplace. To give just one example, the theory of algebraic stacks [91] makes pervasive use of this language. Quite generally, homotopy theory of the classical sort, both on spaces and on chain complexes, is intrinsically 2-categorical: homotopies are maps between maps.

Like categories themselves, 2-categories are easy to understand. However, there is a glimpse of difficulties to come. There are strict 2-categories, defined by Ehresmann and Eilenberg and Kelly [52, 53] and there are weak 2-categories, defined by Benabou [18] under the name of bicategories. Intuitively, in strict 2-categories, coherence diagrams are required to commute; in weak ones, they are required to commute only up to natural isomorphism. Here the difference is only technical, since there is a theorem that any weak 2-category is equivalent in a suitable sense to a strict one [89].

The difficulties in understanding n-categories become fully apparent only for $n = 3$, where the corresponding result does *not* hold: not every weak 3-category is equivalent to a strict one. Indeed, while strict n-categories for all n were defined by Eilenberg and Kelly in the same 1965 paper in which they defined strict 2-categories, even for $n = 3$ the more general weak ones were only defined by Gordon, Power and Street thirty years later [66], under the name of tricategories.

Despite their greater subtlety, it is the weak n-categories that are most important in applications. This becomes clear already for $n = 3$. For example, knot theory and the quantum invariants of 3-manifolds make extensive use of braided monoidal categories, which are a special kind of weak 3-category [81]. Also, Joyal and Tierney [83] have shown that weak 3-categories are a suitable framework for studying homotopy 3-types, while the strict ones are not sufficiently general. For this reason, current research is concentrated on understanding weak n-categories. As is becoming standard, we call these simply 'n-categories' from now on.

There are several other kinds of higher categorical structures that occur naturally and are directly relevant to the study of n-categories. For example, there are n-fold categories and multicategories which have many compositions, but not arranged hierarchically as in n-categories. There are also weakenings of the notion of a category. For example, we may view a category as a monoid with many objects, and there is a notion of an A_∞-category analogous to the notion of an A_∞-space (e.g. [11]). Both notions have cochain level algebraic versions, and the Fukaya categories relevant to mirror symmetry are naturally occurring examples of such algebraic A_∞-categories [61, 62].

4. The focus of this proposal

Our central focus is the understanding of n-categories and their applications, where $n \geq 3$. We allow n to go to infinity and include the study of ω-categories (or ∞-categories). The theory of 3-categories worked out in [66] is explicit and combinatorial. While not every 3-category is equivalent to a strict 3-category, there is an intermediate notion of a Gray category [68], or semi-strict 3-category, such that every 3-category is equivalent to a Gray category. The idea is that some but not all coherence diagrams can be arranged to commute strictly without essential loss of information. Trimble [152] has defined 4-categories in a similarly concrete vein, but it is apparent that more conceptual encapsulations are essential to a coherent theory of n-categories that can be useful in applications.

The problem of understanding n-categories is so very natural that a dozen different definitions have been proposed. Ten of them are sketched in Leinster [99], with no attempt at comparisons. Many are direct descendents of ideas introduced in the study of iterated loop space theory, involving operad actions or categories of presheaves (diagram categories in the topological literature). While the plethora is indicative of the broad interest in the subject, the situation is untenable for as potentially important an area of mathematics as this. People working on these definitions are spread around the world, with the largest groups in Australia, France, and the British Isles. Only a very few people in the United States are currently involved. Moreover, the definitions have been proposed by people with backgrounds in mathematical physics, algebraic geometry, and algebraic topology, as well as by category theorists with backgrounds in logic and/or computer science. There is a need for both specific theories of n-categories and for a metatheory, ideally an axiomatization, that describes what features the specific theories should have in common. Further, there is an evident need for precise comparisons among theories. Once the theory is better understood, there is a need to explain it in intuitive non-technical terms that are accessible to those without categorical expertise.

5. Programs for comparisons of definitions

May has initiated one program for comparison. It is based on the use of techniques introduced in algebraic topology, especially Quillen model category theory, which have recently been used to obtain precise comparisons among very different definitions of highly structured spectra in stable homotopy theory [111, 112, 125]. The much earlier axiomatizations of infinite loop space machines and of one-fold loop space machines given in [56, 115, 149] are also directly relevant. The analogies are close enough that at least a portion of the theory of n-categories works in an analogous fashion.

However, the project here is far larger than the cited comparison projects in algebraic topology. The definitions of n-categories are very different one from another, having arisen from such widely different perspectives, and they involve awesomely complicated combinatorics. The hope and expectation is that this proposed method of comparison will cut through the combinatorics, just as the cited comparisons in topology turned out to depend on general features rather than on the distinctive intricacies of the various definitions being compared.

Nevertheless, the combinatorics must be retained, perhaps in encapsulated form, because it is the central feature for many areas of application. An alternative approach to unification, one which confronts rather than hides the combinatorics, has been proposed by Makkai [107], thinking from a logical perspective that seems worlds apart from the perspective arising from algebraic topology. There is an another program for comparison that again involves ideas from algebraic topology but retains more of the combinatorics. It is based on a rethinking of homotopical algebra, still in a somewhat preliminary stage, in terms of variant kinds of simplicial analogues of A_∞-categories, which are variously called Segal categories (Dwyer, Kan, and Smith [50]), weak Kan complexes or quasi-categories (Joyal [80]), or h-categories (Kontsevich [90]), and which first appeared implicitly in the work of Segal [126] and explicitly in the work of Boardman and Vogt [21]. We shall later refer ambiguously to any such notion as a "weak category". Boardman and Vogt were concerned with homotopy invariant structures in algebraic topology, and the issues they confronted are closely analogous to the issues involved in retaining the combinatorics in a comparison of n-categories. We need to understand not just how to compare definitions but how to compare different kinds of comparisons.

The tension and interplay between the encapsulated global perspective and the unravelled combinatorial perspective is a major source of richness and difficulty in this entire subject. The definition of strict n-categories illustrates the point. There is a well understood notion of a category $\mathcal{C}$ enriched over a category $\mathcal{B}$. For each pair of objects X, Y of $\mathcal{C}$, there must be an internal hom object $\mathcal{C}(X,Y)$ in $\mathcal{B}$. The category $\mathcal{B}$ must have a product, say $\otimes$, and there must be composition maps in $\mathcal{B}$

$$\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y)\longrightarrow\mathcal{C}(X,Z).$$

Starting from (small) categories as 1-categories and using $\times$ as $\otimes$, we define strict $(n+1)$-categories inductively as categories enriched over the category of strict n-categories. This is clear, concise, and conceptual. In this case, it is not hard to unravel the encoded combinatorics and write down in explicit non-recursive terms exactly what a strict n-category or strict ω-category is. Ignoring well understood set-theoretic issues, as we shall do throughout this proposal, we find that the category of strict n-categories is itself a strict $(n+1)$-category. A major expected feature of any good definition of (weak) n-categories is that the category of n-categories should itself be an $(n+1)$-category. We shall return to a more detailed discussion of the foundational issues after describing some special cases, some related higher categorical structures, and some areas of application. The foundational issues we intend to address are intertwined with the structures needed in the applications.

6. Strict ω-categories and concurrency problems

The unravelled combinatorical structure of strict ω-categories turns out to be directly relevant to the study of concurrency problems in computer science. An example of such a problem is the management of simultaneous transactions by a centralized data bank so as to avoid deadlock. More precisely, it is relevant to understanding the deformations of higher dimensional automata that leave invariant the quantities that are relevant to such applications. This theory is related both to the higher category theory that is our central focus and to the ongoing efforts to understand homotopy theory in terms of higher categorical structures. It leads to an interesting new variant of homotopy theory that feels natural both categorically and topologically, called directed homotopy, or "dihomotopy". The point is that in concurrency problems one is dealing with time directed flows, with no time reversal. Dihomotopy theory is concerned with spaces with a local partial ordering that represents the time flow and with directed, or non-decreasing, maps. Each map h_t in a directed homotopy h between directed maps h_0 and h_1 must be directed. There is an obvious forgetful functor mapping dihomotopy theory to homotopy theory. Using the natural ordering of the interval, the unreduced suspension provides a functor "Glob" relating homotopy theory to directed homotopy theory; each value of the suspension coordinate gives a copy of X in Glob$X$ as an "achronal cut", a subspace on which the partial order reduces to equality. The relationships among homotopy theory, dihomotopy theory, and higher category theory are not well understood. An idea of the state of the subject is given in the papers [63, 64, 65, 67] of Gaucher and Goubault. Work in this area currently use strict n-categories, but there are interesting theoretical and computational interactions with the weak theory.

7. Unravelling of definitions and k-tuply monoidal n-categories

Unravelling some of the recursive definitions of n-categories appears to be quite difficult. While the specialization of most to 1-categories and 2-categories is reasonably well understood, relating the various proposed definitions to 3-categories as described in [66] is already a difficult undertaking. Thus detailed explorations in low dimensions are essential. A related and fascinating specialization is to $(n+k)$-categories with only one object, one 1-morphism, and so on up to one $(k-1)$-morphism for some k. There is a reindexing process that allows one to transform such an $(n+k)$-category to an n-category with additional structure, called a k-tuply monoidal n-category. Baez and Dolan [8, 9] have formulated very interesting conjectures as to the structure of such special kinds of n-categories.

As yet unknown higher dimensional analogues of Gray categories are relevant here since these speculations are best understood in semi-strict situations. For example, as a starting point, McCrudden [118] has given a fully explicit combinatorial definition of a symmetric monoidal 2-category.

In low dimensions, the Baez and Dolan conjectures work as follows. When $n = 0$, we obtain sets for $k = 0$, monoids for $k = 1$, commutative monoids for $k = 2$ and, conjecturally, commutative monoids for all $k \geq 2$. When $n = 1$, we obtain categories for $k = 0$, monoidal categories for $k = 1$, braided monoidal categories for $k = 2$, and, conjecturally, symmetric monoidal categories for all $k \geq 3$. When $n = 2$, one starts at $k = 0$ with 2-categories and conjectures that the sequence stabilizes at strongly involutory monoidal 2-categories for $k \geq 4$. This point of view gives a clear conceptual reason for the appearance and importance of braided monoidal categories and higher analogues that appear naturally in such areas as knot theory and mathematical physics. For example, categories of representations of quantum groups are braided monoidal categories [81].

The general stabilization conjecture, analogous to the Freudenthal suspension theorem for homotopy groups, is that the sequence stabilizes when $k \geq n+2$. This is a test property for a good definition of n-categories, and Simpson [132] has proven it for one of the proposed definitions [129, 147]. Batanin's recent work [15] relating his higher operads (without permutations) to classical operads is an important step towards proving it for his entirely different proposed definition [12].

8. Categorical modelling of homotopy types

The comparison with homotopy theory is not frivolous. The reason for the stabilization when $n = 0$ is a categorical version of the argument used to show that the homotopy groups of spaces are Abelian for $k \geq 2$. It is a basic fact that this Abelianization fails for groupoids and still more so for the higher order versions of fundamental groups defined in terms of paths, paths between paths, and so on. In fact, Grothendieck [69] conjectured that homotopy theory itself could be entirely algebraicized in terms of n-groupoids, which are n-categories in which all morphisms are isomorphisms, and he seemed to have strict n-groupoids in mind. However, it is now understood that the only spaces that can be modelled by strict n-groupoids have trivial Whitehead products and indeed trivial k-invariants. There are correct versions of Grothendieck's program that use n-fold groupoids or "n-categorical groups", and these objects have been shown to be equivalent to more calculationally accessible algebraic objects, such as "crossed n-cubes"; see for example [24, 25, 38, 55, 100, 122]. There is a generalized Van Kampen theorem [30] that allows the computation of some homotopy types, with explicit computations of homotopy groups, compositions and Whitehead products in some cases.

The relationship of these methods in homotopy theory with other methods, such as stabilisation, localisation, and completion, needs much further study. More explicitly, it would be of considerable interest to find a precise relationship between this categorical algebraization of homotopy theory and the traditional algebraizations of rational, and, recently (Mandell [110]), p-adic homotopy theory in terms of cochain algebras, but that is totally unknown territory at present. A current impediment to such comparisons is that the use of higher dimensional analogues of groupoids in algebraic topology is little known in the U.S, although it has been pursued to good effect in Europe since the late 1960's. For example, the survey [24] cites 110 references, including only a few background works written by Americans.

The higher groupoid approach to homotopy theory gives an example of the kind of technical interaction that this project should foster. In formulating and proving a generalized Van Kampen theorem in the form of colimit theorems for relative homotopy modules or crossed modules, Brown and Higgins [26, 27] were led to extend cubical techniques (which have generally been discarded in the U.S.). They introduced extra cubical degeneracies, called connections because of an analogy with connections in differential topology, that obey an appropriate transport law. Now, much later, this insight is playing a significant role in a homological approach to concurrency problems, where cubical sets with connections and compositions, and their relationship with strict ω-categories, play a major role.

This is related to a fundamental test property for definitions of n-categories. It is known for some definitions (e.g. [83, 131, 148]) and must hold for any good definition that homotopy theory can be modelled on (weak) n-groupoids, which are the n-categories in which every morphism is an equivalence. More precisely,

n-groupoids model spaces with trivial homotopy groups for $i > n$, and ω-groupoids model all spaces. The relationship between this categorical modelling of homotopy types and that in terms of n-fold groupoids requires elucidation. A closely related problem is to understand n-categories in cubical terms rather than the traditional simplicial terms. On the strict level, relationships have been worked out in [1, 28, 29], but none of the current definitions of n-categories are expressed in the cubical terms that appear naturally in the n-fold groupoid approach to homotopy theory.

9. Categorical modelling of iterated loop spaces

There is another, quite different, categorical modelling of homotopy theory that is conceptually related to the ideas just discussed and which well illustrates the depth of the study of coherence maps and their properties. Balteanu, Fiedorowicz, Schwänzl, and Vogt [10] introduced a theory of n-fold monoidal categories that mimics and models categorically the tautology that an $(n+1)$-fold loop space is a loop space in the category of n-fold loop spaces. A 1-fold monoidal category is just a monoidal category and, inductively, an $(n+1)$-fold monoidal category is a monoidal category in the category of n-fold monoidal categories. To make sense of this, the category of n-fold monoidal categories must itself be monoidal, and here the stabilization phenomenon is circumvented by relaxing the notion of a morphism of n-fold monoidal categories. For example, a map $F : \mathcal{A} \longrightarrow \mathcal{B}$ of monoidal categories requires a coherence natural transformation $\eta : F(A) \otimes F(B) \longrightarrow F(A \otimes B)$. If such transformations are required to be isomorphisms, then stabilization occurs and n-fold monoidal categories model infinite loop spaces for any $n \geq 3$ [81]. In contrast, with general transformations allowed, the theories of n-fold loop spaces and of n-fold monoidal categories are essentially equivalent.

The relationship between the categorical modelling of spaces discussed in the previous section and this categorical modelling of n-fold loop spaces is illuminated by recent work of Batanin [15], who reinterprets the theory of [10] as a particular case of a theory of suspensions of n-categories. The classical operads used in [10] are actually constructed out of higher "n-operads". Recent work of Day and Street [40, 41] suggests a q-fold version of Batanin's theory.

10. The semantics of programming languages

To return to our low dimensional examples, another example of categorical modelling comes from the study of the structure of programming languages. A rigorous mathematical programming semantics gives rise to categories with various kinds of algebraic structure. This uses the theory of T-algebras for a 2-monad T, as in [20]: already there are strict and weak notions. Much relevant structure on the resulting 2-categories is weak and, as explained by Power [123], the study of this structure and the study of the natural (weak) functors between the relevant 2-categories involves 3-categories. By the coherence theorem mentioned in Section 3, the relevant combinatorics can then be handled using Gray categories (semi-strict 3-categories). In extending this programme, Hyland and Power in [77] have studied 'pseudo-commutative' 2-monads. With this additional structure, the 2-category of T-algebras reflects a general higher dimensional linear algebra [78]; in particular it seems that these give examples of symmetric monoidal bicategories, but the relevant combinatorics are not yet understood. Similar considerations to these appear to arise in the study of concurrent processes, the theory of higher dimensional automata, and probably also in proof theory. The higher dimensional algebra being developed in connection with programming semantics is also relevant to a rigorous formulation of TQFT (see below). But for that one will also need a good theory of higher dimensional duality. At the level of monoidal bicategories this is treated in [42], and the related theory of 2-Hilbert spaces is developed in [6].

11. Applications to TQFT

There are many applications of n-categories to mathematical physics; let us mention just two. The most well-established application is to topological quantum field theory. The connections between 3-dimensional TQFTs and category theory have been intensively studied since the early 1980s by researchers including Jones [79], Drinfel'd [44], Witten [155], Joyal–Street [81], Freyd–Yetter [60], Reshetikhin–Turaev [154], and many others. Indeed, our current understanding of braided monoidal categories, and of the necessity for "weakening" in n-category theory, is to a large extent a spinoff of this body of work. By now it is known that 3-dimensional TQFTs can be constructed either from Hopf algebras or monoidal categories. These constructions are related by the fact that the representations of a Hopf algebra form a monoidal category. Many interesting examples are known, mainly coming from the Hopf algebras known as quantum groups. Though research in this area is still active, by now one can say that quantum groups and their relation to quantum field theory—specifically, Chern–Simons theory—are well-understood.

It is widely believed that this whole story generalizes to higher dimensions. However, even in dimension 4, work is just beginning. On the bright side, we know how to construct 4-dimensional TQFTs from certain Hopf categories [37] and monoidal 2-categories [101], and we also know that representations of a Hopf category form a monoidal 2-category [120]. However, it has not yet been shown that the TQFT coming from a given Hopf category is the same as that coming from its monoidal 2-category of representations. Still worse, the only concrete examples of these constructions so far come from the relation between n-categories and homotopy theory, and thus give only homotopy invariants of 4-manifolds [102]. We really want examples that give explicit combinatorial formulas for Donaldson invariants, Seiberg–Witten invariants, and related invariants of smooth 4-manifolds. For this, we need to translate the quantum–field theoretic approach to these invariants into the language of algebra, as was successfully done for Chern–Simons theory. One obstacle is that, at least as currently formulated, Donaldson and Seiberg-Witten theory are not TQFTs in the strict Atiyah sense [3], since they apply only to simply–connected 4-manifolds. The implications of this fact for the algebra are puzzling.

However, the current proposal will mainly have impact on another aspect of this problem, namely developing the necessary n-categorical infrastructure for studying TQFTs in higher dimensions. One crucial task is to clarify the concept of "extended TQFT" [58, 92]. For this, we should define an n-category nCob whose objects are collections of points, whose morphisms are cobordisms between these, whose 2-morphisms are "cobordisms between cobordisms", and so on up to the nth dimension. An n-dimensional extended TQFT will then be a well-behaved n-functor from nCob to the n-category nVect of "n-vector spaces". Baez and Dolan have conjectured a purely algebraic description of nCob in terms of a universal property [4], which if verified will greatly assist in constructing examples. At present we expect to focus on developing the theory of n-categories to the point where nCob and nVect have been given rigorous definitions for all n; currently this has only been done for low values of n [6, 84].

12. Applications to string theory

A second, newer, application of n-categories to mathematical physics arises from string theory. In traditional quantum field theory particles are treated as pointlike, and the parallel transport of a particle along a path is described using gauge fields, that is, connections on bundles. In string theory, however, point particles are just the bottom of a hierarchy which includes 1-dimensional strings, 2-dimensional "2-branes", and so on. It has gradually become clear that a consistent theory of these higher-dimensional objects involves an n-categorical generalization of the concepts of bundle and connection. So far this has most successfully been formalized using the concept of n-gerbe, rigorously defined so far only for $n \leq 2$ [22]. Intuitively, a 0-gerbe is just a sheaf, which assigns to any open set in the manifold representing spacetime a set of "sections", for example the sections of some bundle. Similarly, a 1-gerbe assigns to any open set a groupoid of sections, and a 2-gerbe assigns to any open set a 2-groupoid of sections. Using a more precise and restrictive definition of an n-gerbe, the set of equivalence classes of n-gerbes associated to a group G and living over a space X can be viewed as the G-valued degree n non-Abelian cohomology of X. For $n = 1$, this is the familiar description of equivalence classes of principal G-bundles over X as the degree one G-valued cohomology of X. The notion of connection and higher curvature data can be generalized to these contexts, both when the coefficient group G is abelian [31, 103] and when it is non-Abelian [23]. It has been suggested that the transport, and the interactions, of n-branes are described by connections on n-gerbes [43, 59, 157], with the higher curvature forms as the associated field strengths. This gives some urgency to the task of developing the theory of n-gerbes and their differential geometry for higher values of n.

13. Applications to algebraic geometry

A classical application of algebraic topology to another area of mathematics is the study of the topology of the complex points of algebraic varieties. In the middle of the last century, this branched off into a closely related area, that of defining analogues of algebraic topology (such as étale cohomology) for algebraic varieties defined over general ground fields. It is natural that a close study of higher homotopies is useful for the study of this aspect of algebraic geometry. In a parallel development, at about the same time, category theory came into heavy use in the formalization of algebraic geometry, especially with the concept of "sheaf". This again naturally leads to consideration of a form of higher homotopies, and from there algebraic (and differential) geometers were led to a sort of homotopical notion of sheaf called "stack". This appears for example in the notion of how to glue together sheaves which are defined on open subsets of a variety.

In more recent years these two directions have converged, with the development of notions of "higher stacks" that encode sheaf-like data that takes into account the existence of many different levels of higher homotopies. Intuitively, the n-gerbes mentioned in the previous section are to n-stacks as n-groupoids are to n-categories.

To give an idea of the definition of an n-stack, suppose that we have a good theory of n-categories on hand. Workers in this area have so far been using the

Tamsamani-Simpson definition [129, 147]. We can regard a 1-category as an $(n+1)$-category, and in the cited theory it makes sense to form the $(n+1)$-category $n\mathcal{C}at$ of n-categories and the internal hom $(n+1)$-category $HOM(C^{op}, n\mathcal{C}at)$. This is the $(n+1)$-category of n-prestacks on C. When C is a Grothendieck site, we can say what it means for an n-prestack to satisfy the descent condition for hypercoverings, and the resulting objects are the n-stacks for the site C. One can go on to define ω-stacks. In particular, for a scheme X, one can define an ω-stack of perfect complexes of $\mathcal{O}_X$-modules.

These and related definitions only make sense once one has sufficiently developed foundations of n-category theory, and they are at the heart of the development of non-Abelian Hodge theory of Simpson and his collaborators. We mention, for examples, the construction of the non-abelian Hodge filtration, the Gauss-Manin connection and the proof of its regularity (see [127, 133]), the higher Kodaira-Spencer deformation classes (see [128]), the non-abelian (p,p)-classes theorem (see [85]), a non-abelian analogue of the density of the monodromy (see [86]), and the notion of non-abelian mixed Hodge structure (see [87]). Recently Toen has made progress on Grothendieck's program of defining "schematic homotopy types" (see [151]) with a view toward the development of Hodge theory for these objects; this in turn has given new restrictions on homotopy types of projective manifolds (see [88]). All of these results were guessed and proved using higher stack and/or higher category theory.

The Tamsamani-Simpson definitions of n-categories and related structures used in this work have already allowed the generalization of a number of basic techniques from category theory and topology [129, 130, 134]. On the other hand, in this formalism, the composition operations giving the structure of an n-category are implicit rather than explicit, which means that in some basic sense we lack calculability. The paper [57] of Fiedorowicz and May, which compares definitions of homology operations in terms of operads and in terms of Segal machinery, illustrates an analogous situation in infinite loop space theory. A comparison between different theories of n-categories could potentially add a dimension of calculability to the current applications of n-category theory to non-Abelian cohomology.

A closely related reason for alternative definitions is that the Tamsamani-Simpson definition appears to be ill-adapted to more general contexts that deal with linear higher categories, or more exotic enrichment. This problem has already proven to be an obstacle to completing Toen's program of Tannaka duality for n-categories [150], an idea which is philosophically behind his work on schematization but which has not yet been fully worked out. Again, a precisely analogous situation already appeared in infinite loop space theory. Because Segal machinery depends on the projections present on Cartesian products, it does not work for general kinds of products, which usually lack projections. In contrast, operads are defined in any symmetric monoidal category. The Batanin [12] and May [114] definitions of n-categories are operadic, and the comparison, now nearing completion, between the May definition and the Tamsamani-Simpson definition based on Segal machinery should allow effective simultaneous use of these definitions in just the same way that the corresponding two equivalent infinite loop space machines have long been used in algebraic topology. These ideas are illustrative of the gains that we can expect from a successful comparison program in n-category theory.

14. The existing definitions of n-categories

Returning to the basic foundational questions, such as the comparison between different definitions of n-categories, that are our main theme, we put forward a tentative and provisional picture of some ways the theory might evolve.

The starting point must of course be the present state of the field: we have a dozen definitions of n-categories, and a few comparisons. The various definitions are at uneven stages of internal development. The earliest, due to Street, was presented briefly and tentatively in [139, 140], 1987–88 papers that focused primarily on strict n-categories. The explosion of interest in the area came almost a decade later. As already noted, the basic monograph [66] on 3-categories did not appear until 1995. A wave of definitions of n-categories, some still available only in brief summaries, began appearing soon after. The list of references for this proposal includes around fifty items dealing primarily with n-categories, all more recent than 1995.

We give a brief and inadequate overview of the various definitions, suppressing the most important details, which concern, for example, what shapes of diagrams are allowed for the various compositions of morphisms (which may or may not actually be directly present, or which may be suitably parametrized when present). One can think in terms of diagrams based on iteration of the idea that a relation $h = g \circ f$ is given as a triangle with edges f, g, and h. This gives simplicial shapes. Alternatively, one can think in terms of iteration of the idea that a morphism of maps is an arrow between parallel arrows. It is standard to write the "parallel" arrows as upper and lower curves connecting source and target, hence the natural shape of the resulting diagrams is "globular". Much more general shapes are obtained if one thinks in terms of arrows between composites of arrows, and the combinatorics used to describe this idea in conceptual form is new and interesting.

There are several ways of classifying the various definitions "taxonomically". Although there are persuasive reasons for enlarging the perspective to an enriched context, as mentioned above, it is generally accepted that 0-categories should be sets, 1-categories should be ordinary categories, and 2-categories should be bicategories, at least up to suitable equivalence. In recursive definitions, $(n+1)$-categories are defined as n-categories with suitable additional structure. In some cases, the recursive definition is not the original one, but rather is obtained by examining a more direct definition or by suitably specializing a definition of ω-categories.

Street's original definition [139], which he has since developed much further, is motivated by the classical simplicial approach to algebraic topology. A complete analysis in low dimensions has been given by Duskin [45]. The starting point is that any category can be viewed as a simplicial set via the nerve functor, and that one can characterize combinatorially exactly which simplicial sets arise from categories in this fashion. The definition identifies ω-categories as simplicial sets with appropriate higher level combinatorial restrictions, and it then identifies n-categories by specialization. This is conceptually the simplest of a number of definitions that can be thought of as given by presheaves with suitable combinatorial restrictions.

There are several definitions that are motivated by the operadic approach to iterated loop spaces, and a direct comparison among them may be possible. In the definitions of Batanin [12, 17, 141] and their later variants due to Leinster [95, 96, 97, 98], n-categories are specializations of operadically defined ω-categories. The shapes of diagrams in these definitions are globular, and the relevant operads live in a category of globular sets that specifies the underlying diagrams of sets of

ω-categories. A related definition has been given by Penon [121], and Batanin has already compared that definition with his own [13]. In the same paper, Batanin claimed that certain key combinatorial structures relevant to his definition, called computads, were given as a category of presheaves. Makkai and Zawadowski noticed that this fails in general, but Batanin [16] recently determined for which of his higher operads the associated computads do form a presheaf category. The required condition is expressed in terms of a sequence of classical operads determined by his higher operad. There seems to be a close connection between the coherence problem for weak n-categories and the behaviour of these operads, and Batanin has formulated precise conjectures [16]. These points may sound technical, but their understanding is an important first step towards further comparisons.

The definitions of Trimble [153] and May [114] are recursive, with $(n+1)$-categories defined to be algebras over some operad that defines a monad on the category of n-categories. Trimble's approach is geared towards relations to homotopy theory, and his operads are operads of n-categories. They are obtained by applying a recursively defined functor from spaces to n-categories to a specific operad of spaces that is defined in terms of paths on intervals. In contrast, May works in an enriched context in which one can start with any operad $\mathcal{C}$ in $\mathcal{C}at$ and make sense recursively of an action of $\mathcal{C}$ on an n-category defined with respect to $\mathcal{C}$; these algebras are the $(n+1)$-categories defined with respect to $\mathcal{C}$.

Work in progress by Batanin and Weber promises to give a recursive version of Batanin's definition, and that should open the way to a comparison between his definition and May's. May's definition allows many as yet unexplored generalizations of the notion of an n-category. In particular, there is a notion of a q-fold monoidal n-category that builds a direct analogue of iterated loop space theory into higher category theory. This should admit comparison with the more highly developed work of Batanin [15] relating iterated loop space theory to his suspension theory on n-categories.

The definitions due to Tamsamani and Simpson [129, 147] are motivated by the Γ-space approach to infinite loop spaces or, more precisely, the Δ-space approach to 1-fold loop spaces. They were developed with a view towards the theory of n-stacks and its application to the development of non-Abelian Hodge theory that we have already discussed. Simpson's definition can be expressed recursively and it can be compared to May's definition by recursive use of arguments analogous to those that Thomason [149] and Fiedorowicz [56] used to compare the operadic and Δ-space approaches to 1-fold loop space theory. With considerable input from Simpson and Toen, May has recently made progress on this comparison, although further model theoretic work is necessary to pin down all of the details.

Baez and Dolan [5, 8, 9] gave a remarkable definition designed to allow very general diagram shapes, with what they call "opetopic sets" replacing globular sets. With n-opetopic sets replacing simplicial sets, their definition is conceptually analogous to Street's definition, except that it works one n at a time. Their n-opetopic sets are given by a presheaf category, and their n-categories are suitably restricted n-opetopic sets. Variants of their definition are given and studied by Hermida, Makkai, and Power [70, 71, 72, 73, 73, 75], Makkai and Zawadowski [106, 108, 156], Leinster [94, 95] and Cheng, who also gives comparisons among definitions within this family [32, 33, 34, 35]. Another conceptually similar definition, using an alternative diagram scheme, has recently been given by Higuchi, Miyada and

Tsujishita [76, 119]. Part of the point of these alternative definitions is to make the scheme of allowable shapes more combinatorially complete and more computable.

Finally, there is a definition due to Joyal [80]. It is part of a general program for redoing all of category theory in terms of the weak categories mentioned above and then viewing weak categories as the case $n = 1$ of (weak) n-categories. For this purpose, a category Θ built up out of categorically defined "discs" is used to specify the diagram shapes of a presheaf category of Θ-sets. Then ω-categories are defined as suitably restricted Θ-sets. The category Θ is a colimit of categories $\Theta(n)$ that are used similarly to define n-categories. Berger [19] has taken initial steps towards a comparison of the definitions of Batanin and Joyal.

15. The issues involved in comparing definitions

These definitions and their internal development, which ranges from just starting to quite fully developed, together with the few comparisons already mentioned, give a rough idea of the present state of the foundations. While some of the definitions seem complicated, each has a compelling internal logic and structure. We wish to go from here to a more coherent subject, and there are many paths that we must explore. To make comparisons, we must specify precisely what it is that we are comparing. The sketch above discusses only the objects of our various n-categories. We must go on to define morphisms of n-categories and morphisms between morphisms, and so forth. As stated before, we expect in the end to have well-defined $(n+1)$-categories of n-categories. The intuition is, first, that there should be an n-category $\mathrm{HOM}(X, Y)$ between any two n-categories X and Y and, second, that there should be suitable composition operations

$$HOM(Y, Z) \times HOM(X, Y) \longrightarrow HOM(X, Z).$$

The first intuition is clear and precise, and it is known to hold in several cases. The second intuition is less obvious; it means that there should be the kind of "composition" that defines $(n+1)$-categories in terms of n-categories for a given (recursive) definition of $(n+1)$-categories. It is known to hold with Simpson's definition [129]. We then should make sense of comparisons via a suitable notion of equivalence of $(n+1)$-categories. More modest goals may serve as stepping stones. More ambitious goals must be explored as well.

We discuss some of the issues involved and then come back to possible approaches and to other central problems. In every case, there is an evident notion of a strict map of n-categories and thus an evident (ordinary) category of n-categories and strict maps. Strict maps are not what we want combinatorially: weaker notions will be essential to applications that involve the combinatorics. The analogue in infinite loop space theory is well understood, and as a first approximation we might hope that something similar happens in n-category theory. If so, we will arrive at the following picture, which is part of our program of exploration.

The first problem is to define what it means for a strict map of n-categories to be a "weak equivalence". The notion of an equivalence of categories is generally agreed to be the right starting point, at $n = 1$, and there is a well understood notion of a biequivalence of bicategories that gives the next stage. With recursive definitions of n-category, there is an essentially obvious recursive definition of a weak equivalence. Thus assume that we agree on a definition of a strict weak equivalence of n-categories. In analogous topological situations that already appeared in the work of Stasheff and Sugawara [135, 136, 144] on A_∞ spaces and were first studied systematically in Boardman and Vogt's fundamental work on homotopy invariant algebraic structures [21], there are weak maps as well as strict maps, but there is a sense in which they are equivalent notions.

In crude form, with the same structured objects, one obtains the same morphism sets $[X, Y]$ on passage to homotopy categories when we define homotopy categories by passing to homotopy classes of weak maps or by starting with strict maps and formally inverting the weak equivalences. This miracle happens because there is an object, WX say, such that a strict map $WX \longrightarrow Y$ is the same thing as a weak map $X \longrightarrow Y$, and there is a strict weak equivalence $WX \longrightarrow X$. A general topological version was given by Boardman and Vogt [21], and a more recent categorical version was given by Batanin [11] in the context of A_∞ categories. His paper works out the explicit combinatorics of composition of weak maps of A_∞ categories and works out its relationship to the construction W. This provides one prototype for the study of weak maps of n-categories.

This equivalence of homotopy categories is not good enough. We want a sense in which the relevant categories are equivalent before passage to homotopy, since passage to homotopy throws out the combinatorial structure. The immediate problem is that there are no relevant categories in sight to compare. It is a fact of life, already apparent in topological contexts in the 1950's, that weak maps do not compose to give a category, but only a weak category or A_∞ category, and we have wrong way arrows if we think in terms of strict weak equivalences. There is a long understood solution if we take the second approach. Dwyer and Kan [46, 47, 48, 49] defined and studied a "simplicial localization" $L\mathcal{C}$ of a category $\mathcal{C}$ with a given subcategory $\mathcal{W}$ of weak equivalences. This construction retains all of the combinatorial structure, and passage to components from this category gives the homotopy category, $\mathcal{C}[\mathcal{W}^{-1}]$, that is obtained from $\mathcal{C}$ by formally inverting the maps in $\mathcal{W}$. The solution in the first approach is to expect, not a category, but rather a weak category (or Segal category, or A_∞-category) $W\mathcal{C}$ of weak maps. This is part of the motivation for Joyal's program to redevelop category theory in terms of weak categories. Kontsevich has also embarked on such a program [90]. Several others have ideas towards such a program, and the extensive "coherent homotopy theory" of Cordier and Porter [36] is directly relevant. The work of Rezk [124] is also relevant. He obtains a model category with certain generalized Segal categories as the fibrant objects, thus obtaining one model for a "homotopy theory of homotopy theories".

16. The homotopical comparison program

This leads to two (modest!) programs of comparison. One might try to compare simplicial localizations or one might try to compare weak categories of n-categories. One advantage of the first program is that there is a very well established methodology for making comparisons of this sort. It originated in the topological literature, but it applies in general. A second advantage is that it does not require any definition or study of weak maps. That is also its disadvantage: eventually, we do want to understand the weak maps. However, based on experience in algebraic topology, it seems likely that weak maps are entirely irrelevant to some kinds of applications. For example, they play no role in the work on n-stacks and non-Abelian Hodge theory described above. Thus the first program is of intrinsic interest in its own right. Moreover, as we shall explain in a moment, it leads to an alternative to a direct attack on the second program. Such a direct attack may be unrealistic at this time. The definition of weak maps in a given theory of n-categories will surely depend heavily on the internal combinatorics of that theory, hence direct comparison of the weak categories associated to two definitions is likely to be quite difficult. In any case, we do not yet have a fully worked out theory of weak maps in any one of our categories of n-categories, let alone in two of them.

Optimistically, once the theory of weak maps is worked out, we should be able to prove that the miracle cited above works. Indeed, we expect the definition of the construction WX to go hand in hand with the definition and analysis of weak maps. It should be a general principle of homotopical algebra that, when the miracle occurs, the simplicial localization $L\mathcal{C}$ and the weak category $W\mathcal{C}$ are suitably equivalent. To the best of our knowledge, no precise theorem of this form has yet been written down, but there is little doubt that such a result must hold. We expect this part of the general theory to be understood within the time span of our proposal, and we take it for granted now. The point to emphasize is that the problems of defining $W\mathcal{C}$ and proving that it is equivalent to $L\mathcal{C}$ are part of the internal development of $\mathcal{C}$. It is a major part of our program to work out the internal structure of certain of our categories $\mathcal{C}$ of n-categories, at least to the point that we can either prove such an equivalence or, as it might turn out, see that it is unrealistic to expect such a result.

Suppose that we have such a result for two of our categories $\mathcal{C}$ and $\mathcal{D}$. Then $L\mathcal{C}$ is equivalent to $W\mathcal{C}$ and $L\mathcal{D}$ is equivalent to $W\mathcal{D}$. Thus, to prove that $W\mathcal{C}$ is equivalent to $W\mathcal{D}$, we need only prove that $L\mathcal{C}$ is equivalent to $L\mathcal{D}$. Conceptually, this reduces the comparison of weak categories of weak maps, when we finally define and understand them, to the comparison of categories defined entirely in terms of strict maps, a problem that may already be accessible. In fact, in definitions that are global and encapsulated, it may be unrealistic to attempt a direct definition of weak maps. If $\mathcal{D}$ is of that sort and $L\mathcal{C}$ is equivalent to $L\mathcal{D}$, then the theory of weak maps in $\mathcal{C}$ can be viewed as implicitly giving a theory of weak maps in $\mathcal{D}$. Precisely this philosophy applies in stable homotopy theory. Although the theory there is almost as recent as the theory here, it is already routine to obtain information coming from the infrastructure of one highly structured category of spectra and to interpret it as giving information in another such category.

The standard method for comparing simplicial localizations of categories $\mathcal{C}$ and $\mathcal{D}$ with prescribed subcategories of weak equivalences is to define classes of cofibrations and fibrations such that $\mathcal{C}$ and $\mathcal{D}$ are Quillen model categories, and then to prove that $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent. The requisite definitions are simple and conceptual, and the relevant theory is very highly developed, providing a simple and convenient framework for homotopical algebra where it applies. The notion of Quillen equivalence, which requires a well-behaved adjunction, may be too restrictive, but Mandell [109] has worked out general necessary and sufficient conditions for two model categories to have equivalent simplicial localizations. Some categories $\mathcal{C}$ of n-categories are known to be model categories, and it is hoped that all of them are. In the cases of categories of suitably restricted presheaves, an optimistic guess is that one can give the entire category of presheaves a model structure whose fibrant objects are the ω-categories, with suitable specializations to n-categories. Since the restrictions are sophisticated analogues of the Kan extension condition that defines fibrant simplicial sets (Kan complexes), this does not seem unreasonable.

Baez, together with Leinster and Cheng, has made progress in this direction in the case of the Baez-Dolan definition of n-categories. If there are such model structures, then methods of comparison that were used in [111, 112, 125] to compare model categories of presheaves of spaces (which give versions of spectra) should adapt to give comparisons here. The first step must be to define comparison functors relating the domain categories of the relevant presheaf categories, and several people have made progress in this direction.

17. An alternative perspective

A different conceptual perspective explains why it seems plausible to expect equivalences between categories of presheaves based on different diagram shapes (globular and opetopic, say). No matter how expressed in terms of specific definitions, the intuition is that we are weakening the well understood notion of a strict n-category, replacing equations by equivalences throughout in a systematic way. It makes sense to seek a general theory of this kind of "weakening" that applies not only to n-categories but to more general mathematical structures and which might lead to conceptual comparisons. Cogent insights in this direction come not only from homotopical algebra but also from work of Makkai [104, 105, 107], which leads to alternative perspectives on the nature of the definitions of n-categories and the comparisons that we seek. It is based on an interesting alternative perspective on the conceptual foundations of category theory.

To give some idea of this perspective, let us return to the "suitable restrictions" required of n-categories in the various presheaf definitions. These conditions are existential: they require the existence of "horn fillers" satisfying certain conditions, but the fillers are not specified and are not unique. They encode "virtual" composition operations, "virtual" meaning that composition is many-valued, but with values varying within suitable equivalence classes. In this framework, we expect to be able to to find exact equivalences between notions of n-category based on different diagram shapes. Makkai [107] is proposing an exact general scheme of formal specifications of concepts of n-category of this sort, together with a means of comparing these specifications. The ideas have been shown to work on low dimensional fragments of the concepts, and there is some computational evidence that they work in higher dimensions.

18. The problem of coherence

One of the thorniest issues in the emerging subject of n-categories is that of coherence. This can be viewed as the issue of satisfying two potentially conflicting desiderata of a good notion of an n-category. On the one hand, one certainly requires a compact and explicit specification of the concept. On the other hand, one wants the definition to encode implicitely precise coherence conditions of the sort familiar from traditional applications of categorical concepts. From the perspective of computer science, for example, one surely wants all of the structure present to be computable in a reasonable sense. We would like the definitions to be more determinate than many of them seem to be.

One interpretation of our coherence desiderata is to ask for a description of n-categories, up to suitable equivalence, as algebraic theories in the sense of Lawvere [93]. For example, even in the definition of a tricategory in [66], certain transformations are required to be equivalences, but inverse equivalences are not specified.

For that reason, tricategories are not algebraic in Lawvere's sense. That could be rectified by specifying inverses and adding further axioms to an already unwieldy definition. A much more satisfying solution is given by the pair of statements that semistrict Gray categories are algebraic and every tricategory is triequivalent to a Gray category. It is this pair of statements that entered into the results behind our allusion to the importance of Gray categories in the study of the logic and semantics of programming languages, and it is this kind of coherence result that is needed to interpret the various kinds of k-tuply monoidal n-categories in a fashion similar to the standard definition of a braided monoidal category. A perhaps overoptimistic hope is that there is a notion of a semi-strict n-category such that semi-strict n-categories are algebraic and every n-category is suitably equivalent to a semi-strict n-category, just as in the case $n = 3$. It might be that there is more than one such semi-strict notion. However, coherence of n-categories for $n > 3$ may require and certainly is illuminated by rethinking the very meaning of coherence.

In fact, the problem of comparing definitions is intimately related to the meaning of coherence. For example, the equivalence of a definition based on globular shapes with a definition based on opetopic shapes would be a kind of coherence theorem saying intuitively that, despite the intrinsic relevance of more general shapes coming from the non-strict associativity of the various compositions, any n-category defined in terms of such general shapes is suitably equivalent to one defined only in terms of the familiar globular shapes specified iteratively in terms of source and target. However, it may well be that greater clarity can be obtained by focusing instead on the most general shapes possible. In some sense, that is analogous to the problem of defining "all" in the classical idea that coherence theory seeks to make precise the notion that "all" coherence diagrams (in terms of specified coherence maps) that can reasonably be expected to commute do in fact commute.

The last point of view appears naturally in the comparison scheme discussed in the previous section. The horn fillers that appear in some of the presheaf definitions of n-categories are specified by virtual compositions that encode universal properties analogous to the standard universal properties used to define such categorical concepts as tensor products, colimits, and limits. There is an intuition that these definitions give a notion of n-category that is "coherent", in the sense that they are defined in terms of conditions that must be preserved by any reasonable kind of comparison map. Moreover, any of the presheaf definitions should be exactly comparable to a definition in which "all" possible shapes are accounted for.

19. Conclusion

This reinterpretation in terms of coherence of the filler conditions that single out the n-categories in a suitable category of presheaves seems very far away from the idea that precisely the same conditions specify the fibrant objects of a model category. We must make precise comparisons between such disparate ideas. There is an enormous amount of work to be done in this field and an enormous number of natural directions to pursue. The exploration of categorical structures like these is in its infancy, and our project can have a major impact on the shape this emerging subject takes. The natural appearance of these categorical structures in algebraic topology, differential geometry, algebraic geometry, mathematical physics, and computer science makes it abundantly clear that this subject is likely to be an important part of twenty-first century mathematics.

20. Participants

The people on the following list made active contributions to the preparation of this proposal. Those starred are the co-Directors.

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