OPERADIC CATEGORIES, A_∞-CATEGORIES AND n-CATEGORIES

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When I offered to talk here, I asked myself if I had something suitable for a nice short 20 minute talk, and sent in an abstract about "cores of spaces and spectra". However, since I have been given 50 minutes, I have decided to scrap that for something a little more substantial. It is brand new. I assume that I am among friends and that you won't mind if some of what I say turns out to be nonsense.

On April 1, a paper of mine on triangulated categories was posted on the algebraic K-theory web site. The next day, Maxim Kontsevich called me from Paris to talk about a possible use of "h-categories", which are homotopical analogues of "A_∞-categories", as an alternative way of thinking about triangulated categories. At the time, I had no idea what an A_∞-category was. Now I do, and the notion fits together with some ideas I have long had about parametrizing the composition in a category by use of operads. I wrote up some notes for Kontsevich on this and sent them off on April 15. Two days later, Carlos Simpson gave an Adrian Albert talk at Chicago on n-categories. It was immediately apparent to me that the ideas in my notes for Kontsevich would give a considerably simpler and probably equivalent alternative way of thinking about n-categories. This is based on a model category of enriched operadic A_∞-categories. The model category point of view also gives an easy construction of the triangulated derived category of complexes of modules over an A_∞-category. I shall try to explain the ideas in the simplest possible terms. There are no real applications yet, but the concepts just feel right to me. I will give the idea by presenting an intuitive example that is naturally related to topological conformal field theory.

You have all seen the pair of pants picture of a composite of cobordisms, with say two inputs (around the ankles) and one output (around the waist). Now pants hide more structure, legs with knee joints. Instead of glueing along matching outputs and inputs, one might more naturally and flexibly sew in a cobordism with one input and one output at each joint. For symmetry, we don't want different looking knees, so we sew in the same cobordism at joints at the same level. Formally, the idea looks like this. Consider the moduli space $\mathscr{M}(j,k)$ whose points are (possibly disconnected) Riemann surfaces Σ, say of arbitrary genus or more modestly just of genus zero, together with $j+k$ biholomorphic maps from the disc D into Σ, with disjoint interiors. We think of the first j disks as inputs and the last k as outputs. Let $\mathscr{C}(j)$ be the Cartesian product of $j-1$ copies of $\mathscr{M}(1,1)$. We get maps

$$\theta : \mathscr{C}(j) \times \mathscr{M}(i_{j-1}, i_j) \times \cdots \times \mathscr{M}(i_0, i_1) \longrightarrow \mathscr{M}(i_0, i_j)$$

as follows. For $(c_1, \ldots, c_{j-1}) \in \mathscr{C}(j)$, we sew the input disc of c_r to each of the i_r output discs of a point of $\mathscr{M}(i_{r-1}, i_r)$ and we sew the output disc of c_r to each of the

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input discs of a point of $\mathscr{M}(i_r, i_{r+1})$. These maps are $\Sigma_{i_1} \times \cdots \times \Sigma_{i_{j-1}}$-equivariant, but that is not a feature I want to emphasize. We have maps

$$\gamma : \mathscr{C}(k) \times \mathscr{C}(j_1) \times \cdots \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j)$$

where $j = j_1 + \cdots + j_k$. These just order the $j-1$ points of $\mathscr{M}(1,1)$ given by a point in the domain by shuffling so as to put the rth entry of a point of $\mathscr{C}(k)$ between the entries given by points of $\mathscr{C}(j_{r-1})$ and $\mathscr{C}(j_r)$. With these structure maps, the $\mathscr{C}(j)$ give an operad without permutations, and the maps θ give that the $\mathscr{M}(j,k)$ are the morphism spaces of a "$\mathscr{C}$-category". This is exactly like the notion of a $\mathscr{C}$-space, or $\mathscr{C}$-algebra, X given by maps $\mathscr{C}(j) \times X^j \longrightarrow X$ that I introduced in 1970. That notion is the special case of a $\mathscr{C}$-category X with a single object.

Now I want to take a different starting point. Even earlier, Stasheff introduced the notion of an A_∞-space X. Simplifying still earlier work of Sugawara, his notion of an A_∞-space gave an intrinsic characterization of what structure on a connected H-space X ensures that it admits a delooping or classifying BX. The structure is given by higher associativity homotopies encoded in maps $\mathscr{K}(j) \times X^j \longrightarrow X$ for certain polytopes $K(j)$ that are homeomorphic to I^{j-2} when $j \geq 2$. Here $\mathscr{K}(0)$, $\mathscr{K}(1)$, and $\mathscr{K}(2)$ are points, giving the basepoint of X, the identity map $X \longrightarrow X$, and the unital product $X \times X \longrightarrow X$; $K(3) = I$, giving homotopy associativity, and $K(4)$ is a pentagon, encoding an evident homotopy between homotopies relating the five ways that one can associate products of four elements. In retrospect, the $\mathscr{K}(j)$ comprise an operad (again, without permutations), and an A_∞-space is exactly a $\mathscr{K}$-space in the operad action sense.

Theorem 1. *A connected A_∞-space X admits a classifying space BX such that X is equivalent as an A_∞-space to ΩBX.*

I will briefly indicate three different proofs. The details won't matter to us, but the method given in the second proof will generalize to show how to replace A_∞-categories by equivalent genuine categories.

First proof. Mimic the standard inductive construction of the classifying space of a topological group or monoid. □

Second proof. Show that the A_∞-space X is equivalent to the topological monoid $M = B(J, K, X)$, where J is the James construction or free topological monoid functor and K is an analogous functor constructed from the operad $\mathscr{K}$. The categorical bar construction used here was also constructed in 1970. We can take $BX = BM$. □

Third proof. Change operads to the little 1-cubes operad $\mathscr{C}_1$, which we can do functorially, and show that $X \simeq \Omega B(\Sigma, C_1, X)$; that is, use the special case $n = 1$ of my n-fold delooping machine. □

There are two notions of a map of A_∞-spaces $f : X \longrightarrow Y$. For a strict map, the following diagrams commute:

$$\begin{array}{ccc} \mathscr{K}(j) \times X^j & \xrightarrow{\mathrm{id} \times f^j} & \mathscr{K}(j) \times Y^j \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y. \end{array}$$

For a weak, or "strong homotopy map", there are suitably compatible maps

$$\mathscr{K}(j) \times I \times X^j \longrightarrow Y \quad \text{or} \quad I^{j-1} \times X^j \longrightarrow Y$$

starting from f. This notion is due to Sugawara and gives just enough information to get an induced map $Bf : BX \longrightarrow BY$, using the first proof above. In the rest of this talk, I will only consider strict maps, but the complete story will have to deal with weak maps. This can be done by the methods of Boardman and Vogt or Lada, as modernized by Batanin, but perhaps a more model theoretic approach might allow a simplified repackaging of the combinatorics. One way to handle this is to show that a weak map induces a strict map with respect to a suitably enlarged operad weakly equivalent to $\mathscr{K}$. One could then change operads, as in the third proof above, to get back to a strict map between $\mathscr{K}$-spaces that are equivalent to the original $\mathscr{K}$-spaces.

We now turn to algebra. We work over a fixed commutative ground field k. A differential graded algebra A has a bar construction BA that has a spectral sequence

$$E^2_{p,q} = Tor^{H_*(A)}(k,k) \Longrightarrow H_*(BA).$$

Stasheff formulated a notion of an A_∞-algebra A such that

$$C_*(BX) \cong B(C_*X)$$

for an A_∞-space X, thus obtaining a spectral sequence

$$E^2_{p,q} = Tor^{H_*(X)}(k,k) \Longrightarrow H_*(BX).$$

The notion of an A_∞-algebra can be described as an action of an algebraic operad $\mathscr{K}^{alg}$ of chain complexes on a chain complex A that is obtained from a simplicial structure on the operad $\mathscr{K}$. Instead, staring at the $\mathscr{K}(n)$, Stasheff came up with the following elegant but perhaps mysterious looking equivalent definition. Kontsevich has an even more elegant and mysterious looking reformulation.

Definition 2. An A_∞-algebra A is a graded unital k-module together with maps $m_j : A^j \longrightarrow A$ of degree $j-2$, $j \geq 1$, such that

$$\sum_{1\leq i\leq j,\, p+q=j+1} (-1)^{\varepsilon(i,q,a)} m_p(a_1 \otimes \cdots a_{i-1} \otimes \\ m_q(a_i \otimes \cdots \otimes a_{i+q-1}) \otimes a_{i+q} \otimes \cdots \otimes a_j) = 0.$$

for $j \geq 1$, where $\varepsilon(i,q,a) = i + q(i + j + \sum_{h=1}^{i-1} \deg a_h)$. The conditions required of the unit are that

$$m_2(a \otimes 1) = a = m_2(1 \otimes a) \quad \text{and} \quad m_j(a_1 \otimes \cdots \otimes a_j) = 0$$

if any $a_i = 1$.

Let $d = m_1$ and write $m_2(a \otimes b) = ab$. For $j = 1$, the equation gives $d \circ d = 0$. For $j = 2$ it gives that the product is a map of chain complexes. For $j = 3$ it gives that the product is chain homotopy associative. And so on.

Theorem 3. *An A_∞-algebra A admits a bar construction BA with a spectral sequence exactly as above.*

First proof. Mimic the standard description of the algebraic bar construction. □

Second proof. Replace A by the equivalent DGA

$$B(J^{alg}, K^{alg}, A)$$

and take its bar construction; here J^{alg} is the free algebra functor. $\square$

Again, there are two notions of a map, strict and weak, with the weak notion being just sufficient to induce a map of bar constructions.

We define A_∞-categories in a way that should now be obvious.

Definition 4. An A_∞-category A with object set $\mathscr{O}$ consists of graded k-modules $A(S,T)$ for each pair of objects (S,T), unit elements $1 \in A(S,S)$, and maps

$$m_j : A(S_{j-1}, S_j) \otimes \cdots A(S_0, S_1) \longrightarrow A(S_0, S_j)$$

of degree $j-2$, $j \geq 1$, such that the same conditions as in the definition of an A_∞-algebra are satisfied.

Of course, A_∞-algebras are A_∞-categories with a single object. These structures arose in work of Fukaya, who formulated the definition. His examples, now called Fukaya categories, play a central role in Kontsevich's conjectures on mirror symmetry. There are serious technical issues (unit condition problems, for example) that prevent direct interpretation of his geometry in these algebra terms, but the concept and examples seems to be of fundamental importance. From an algebraic point of view, especially in view of my work with Kriz on operadic algebra, it seems sensible to go to an operadic view of these structures, using $\mathscr{K}^{alg}$ instead of the m_n. This allows use of a monadic reformulation that is better suited to homotopical work. However, it unfortunately means that, on a technical level, we are moving away from precise concepts currently in use in connection with mirror symmetry.

Contrasting our moduli space example with our A_∞-category example, it seems reasonable to define the basic terms in full generality. This will lead us quite directly to the promised theory of n-categories and to the derived category of an A_∞-category. We fix once and for all a preferred base category $\mathscr{B}$. Pick your favorite. It might be compactly generated unbased topological spaces, simplicial sets, or $\mathbb{Z}$-graded chain complexes of k-modules, for example. It could also be $\mathscr{C}at$, the category of small categories, or it could be the category of simplicial sheaves in the Nisnevich topology. We assume that $\mathscr{B}$ is closed symmetric monoidal, meaning that it has a product $\otimes_{\mathscr{B}}$ with unit object $u_{\mathscr{B}}$ and an internal hom functor $\mathscr{B}$ satisfying the usual adjunction

$$\mathscr{B}(A \otimes_{\mathscr{B}} B, C) \cong \mathscr{B}(A, \mathscr{B}(B, C)).$$

We want to set up the foundations to do serious homotopy theory in categories of operadic categories, so we assume that $\mathscr{B}$ is a Quillen model category. For the experts, we want $\mathscr{B}$ to be cofibrantly generated and proper.

For full generality, we fix a ground category $\mathscr{G}$ that has all of the same structure that we required of $\mathscr{B}$, except that we do not require internal hom objects, which may or may not be present. We assume instead that $\mathscr{G}$ is enriched over $\mathscr{B}$. This means that $\mathscr{G}$ has hom objects $\mathscr{G}(X,Y)$ in $\mathscr{B}$, unit maps $u_{\mathscr{B}} \longrightarrow \mathscr{G}(X,X)$ in $\mathscr{B}$, and a composition law

$$\mathscr{G}(Y,Z) \otimes_{\mathscr{B}} \mathscr{G}(X,Y) \longrightarrow \mathscr{G}(X,Z)$$

that is associative and unital in the evident sense. The underlying category of $\mathscr{G}$ has as its morphism sets the set of morphisms $u_{\mathscr{B}} \longrightarrow \mathscr{G}(X,Y)$ in $\mathscr{B}$, sometimes denoted $\mathscr{G}_0(X,Y)$. Examples show that this always give the underlying category that you thought you were looking at in the first place.

You can think of the case $\mathscr{G}=\mathscr{B}$, but then you will miss the punch line. Algebraically, you can think of $\mathscr{B}$ as the category of chain complexes of k-modules and $\mathscr{G}$ as the category of chain complexes of A-modules for a commutative DGA A over k. Topologically, you can think of $\mathscr{B}$ as the category of simplicial sets or topological spaces and $\mathscr{G}$ as one of the modern categories of spectra, such as symmetric spectra, orthogonal spectra, or S-modules, that are closed symmetric monoidal Quillen model categories. Or you can think of $\mathscr{B}$ as simplicial sheaves and $\mathscr{G}$ as one of the model categories of spectra in that category. We will construct some surprising new examples.

We assume that $\mathscr{G}$ is complete and cocomplete in the enriched sense that it has all indexed limits and colimits. This is equivalent to requiring that it have ordinary colimits and limits in its underlying category and that it be tensored and cotensored over $\mathscr{B}$. That means that there must be functorial "tensors" $X\odot B$ and "cotensors" $F(B,X)$ in $\mathscr{G}$ between objects $B\in\mathscr{B}$ and $X\in\mathscr{G}$ with natural isomorphisms in $\mathscr{B}$

$$\mathscr{G}(X\odot B,Y)\cong\mathscr{B}(B,\mathscr{G}(X,Y))\cong\mathscr{G}(X,F(B,Y)).$$

We emphasize that the construction of tensors is often by no means obvious, and they need not be given by some familiar kind of tensor product. For example, it is entirely obvious how to tensor a k-module with an A-module, but it is not at all obvious how to tensor a k-module with an A-algebra. Similarly, it is obvious how to tensor a space with an S-module, but it is not obvious how to tensor a space with an S-algebra. In both cases, the required tensors exist, but they are not standard and simple constructions. This sounds awfully technical and categorical, but it was absolutely central to the deeper parts of the work of Elmendorf, Kriz, Mandell, and myself (EKMM) in stable homotopy theory, and it is even more central to the new theory of n-categories. Fortunately, the work done in EKMM makes it perfectly clear how to proceed in this new theory.

Since $\mathscr{B}$ is symmetric monoidal, we can define operads in $\mathscr{B}$. For the present purposes, "operad" means an operad without permutations. An A_∞-operad is one each of whose objects is weak equivalent to $u_{\mathscr{B}}$. There are many examples, for example $\mathscr{K}$ and $\mathscr{C}_1$ in spaces, $\mathscr{K}^{alg}$ and $C_*(\mathscr{C}_1)$ in chain complexes. There is always the trivial example $\mathscr{A}ss$ with $\mathscr{A}ss(j)=u_{\mathscr{B}}$; its algebras are just the monoids in the symmetric monoidal category $\mathscr{B}$. We fix any operad $\mathscr{C}$ in $\mathscr{B}$. We indicate later how to change operads in favorable circumstances. We can define actions of $\mathscr{C}$ on objects of $\mathscr{G}$ in terms of maps $\mathscr{C}(j)\odot X^j\longrightarrow X$, where X^j is the j-fold power of X under $\otimes_{\mathscr{G}}$. Such mixing of categories has often arisen in topological contexts. We follow Mac Lane in misusing the word "graph" for the following obvious concept.

Definition 5. For a set $\mathscr{O}$, the elements of which are called objects, an $\mathscr{O}$-graph $\mathscr{M}$ in $\mathscr{G}$ is a set of objects $\mathscr{M}(S,T)$ of $\mathscr{G}$, one for each pair (S,T) of objects in $\mathscr{O}$. We call $\mathscr{M}(S,T)$ the object of morphisms with source S and target T. A morphism $F:\mathscr{M}\longrightarrow\mathscr{N}$ of $\mathscr{O}$-graphs is a set of morphisms $F:\mathscr{M}(S,T)\longrightarrow\mathscr{N}(S,T)$ in $\mathscr{G}$. More precisely, these are the morphisms in the underlying category of the enriched category of $\mathscr{O}$-graphs in $\mathscr{G}$, which has morphism objects in $\mathscr{B}$, namely

$$\prod_{(S,T)}\mathscr{G}(\mathscr{M}(S,T),\mathscr{N}(S,T))$$

Ignoring the enrichment, this is just a silly but convenient way of describing the category $\mathscr{G}^{\mathscr{O}^2}$ of functors from the discrete category (identity morphisms only) $\mathscr{O} \times \mathscr{O}$ to $\mathscr{G}$. As a diagram category, $\mathscr{G}^{\mathscr{O}^2}$ inherits a model structure from $\mathscr{G}$.

Definition 6. Define a product $\otimes$ on $\mathscr{G}^{\mathscr{O}^2}$ by setting

$$(\mathscr{M} \otimes \mathscr{N})(S,V) = \coprod_T \mathscr{M}(T,V) \otimes_{\mathscr{G}} \mathscr{N}(S,T).$$

Define the unit $\mathscr{O}$-graph $\mathscr{U} = \mathscr{U}_{\mathscr{O}}$ by letting $\mathscr{U}(S,S) = u$ and $\mathscr{U}(S,T) = \emptyset$, the initial object of $\mathscr{G}$, if $S \neq T$. Since $X \amalg \emptyset = X$ and $X \otimes \emptyset = \emptyset$, $\mathscr{G}^{\mathscr{O}^2}$ is monoidal (but of course not symmetric monoidal) with unit object $\mathscr{U}$. Also, define the tensor of an object A of $\mathscr{B}$ with an $\mathscr{O}$-graph $\mathscr{M}$ in the obvious way:

$$(A \odot \mathscr{M})(S,T) = A \odot \mathscr{M}(S,T).$$

Definition 7. Define the category of graphs to have objects $(\mathscr{M}, \mathscr{O})$, where $\mathscr{M}$ is an $\mathscr{O}$-graph, and to have morphisms $(F,f) : (\mathscr{M}, \mathscr{O}) \longrightarrow (\mathscr{N}, \mathscr{P})$, where $f : \mathscr{O} \longrightarrow \mathscr{P}$ is a function and F consists of a set of maps $\mathscr{M}(S,T) \longrightarrow \mathscr{N}(f(S), f(T))$. More conceptually, f determines a functor f^* from $\mathscr{P}$-graphs to $\mathscr{O}$-graphs by setting $f^*\mathscr{N}(S,T) = \mathscr{N}(f(S), f(T))$, and then F is just a map $\mathscr{M} \longrightarrow f^*\mathscr{N}$ of $\mathscr{O}$-graphs. Again, this describes the underlying category of an enriched category with morphism objects in $\mathscr{B}$, namely

$$\prod_{(S,T)} \mathscr{G}(\mathscr{M}(S,T), \mathscr{N}(f(S), f(T))).$$

We usually abbreviate $(\mathscr{M}, \mathscr{O})$ to $\mathscr{M}$, leaving $\mathscr{O}$ understood.

Clearly, a category enriched over $\mathscr{G}$ with object set $\mathscr{O}$ is exactly a monoid in the monoidal category of $\mathscr{O}$-graphs in $\mathscr{G}$. We call these "categories in $\mathscr{G}$ over $\mathscr{O}$". The notion of a $\mathscr{C}$-category in $\mathscr{G}$ over $\mathscr{O}$ is now nearly obvious, as is the corresponding notion of a $\mathscr{C}$-functor between such categories. Write $\mathscr{M}^j$ for the j-fold $\otimes$-power of an $\mathscr{O}$-graph $\mathscr{M}$, with $\mathscr{M}^0 = \mathscr{U}_{\mathscr{O}}$.

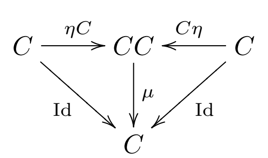
Definition 8. A $\mathscr{C}$-category in $\mathscr{G}$ over $\mathscr{O}$ is a $\mathscr{C}$-object $\mathscr{M}$ in the category of $\mathscr{O}$-graphs in $\mathscr{G}$. That is, there must be maps $\mathscr{C}(j) \odot \mathscr{M}^j \longrightarrow \mathscr{M}$ of $\mathscr{O}$-graphs such that the usual unit and associativity diagrams commute.

Unravelling, we have maps in the underlying category of $\mathscr{G}$

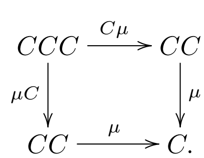
$$\mathscr{C}(j) \odot \mathscr{M}(S_{j-1}, S_j) \otimes_{\mathscr{G}} \cdots \otimes_{\mathscr{G}} \mathscr{M}(S_0, S_1) \longrightarrow \mathscr{M}(S_0, S_j)$$

and can write the relevant diagrams accordingly. When $\mathscr{C} = \mathscr{A}ss$, this is just a category in $\mathscr{G}$ over $\mathscr{O}$. As usual, we have a monadic reinterpretation. We recall some standard categorical definitions.

Definition 9. Let $\mathscr{S}$ be any category. A monad in $\mathscr{S}$ is a functor $C : \mathscr{S} \to \mathscr{S}$ together with natural transformations $\mu : CC \to C$ and $\eta : \mathrm{Id} \to C$ such that the following diagrams commute:



and



A C-algebra is an object A of $\mathscr{S}$ together with a map $\xi : CA \to A$ such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta} & CA \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \xi} \\ & & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} CCA & \xrightarrow{C\xi} & CA \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \xi} \\ CA & \xrightarrow{\xi} & A. \end{array}$$

Taking $\xi = \mu$, we see that CX is a C-algebra for any $X \in \mathscr{S}$. It is the free C-algebra generated by X. That is, for C-algebras A, restriction along $\eta : X \to CX$ gives an adjunction isomorphism

$$C[\mathscr{S}](CX, A) \cong \mathscr{S}(X, A),$$

where $C[\mathscr{S}]$ is the category of C-algebras.

Construction 10. The operad $\mathscr{C}$ in $\mathscr{B}$ determines a monad $C = C_{\mathscr{O}}$ on the category of $\mathscr{O}$-graphs for any set $\mathscr{O}$. For an $\mathscr{O}$-graph $\mathscr{M}$,

$$C\mathscr{M} = \coprod_{j \geq 0} \mathscr{C}(j) \odot \mathscr{M}^j.$$

The unit $\eta : \mathrm{Id} \longrightarrow C$ and product $\mu : CC \longrightarrow C$ are defined from the structure maps of $\mathscr{C}$.

Proposition 11. *The category of $\mathscr{C}$-categories in $\mathscr{G}$ over $\mathscr{O}$ is isomorphic to the category of algebras over the monad $C_{\mathscr{O}}$ in the category of $\mathscr{O}$-graphs.*

For fixed $\mathscr{O}$, we have implicitly defined a morphism between $\mathscr{C}$-categories with the same object set $\mathscr{O}$ to be a map of $\mathscr{C}$-algebras (or equivalently $C_{\mathscr{O}}$-algebras). This defines the $\mathscr{C}$-functors whose object functions are identity maps. We define $\mathscr{C}$-functors between $\mathscr{C}$-categories with varying maps of object sets by reducing to this special case. This requires some easy categorical observations.

Proposition 12. *Let $f : \mathscr{O} \longrightarrow \mathscr{P}$ be a function. The functor f^* from $\mathscr{P}$-graphs to $\mathscr{O}$-graphs is lax monoidal and commutes with tensors. As $\mathscr{O}$ varies, the monads $C_{\mathscr{O}}$ specify a monad C in the category of graphs.*

Definition 13. Define a $\mathscr{C}$-functor F with object function f from a $\mathscr{C}$-category $\mathscr{M}$ to a $\mathscr{C}$-category $\mathscr{N}$ in $\mathscr{G}$ to be a $\mathscr{C}$-functor $F : \mathscr{M} \longrightarrow f^*\mathscr{N}$. Equivalently, $F : \mathscr{M} \longrightarrow \mathscr{N}$ is a morphism of C-algebras in the category of graphs.

The definition just keeps track of the additional structure that we require on the maps of morphism objects $F : \mathscr{M}(S,T) \longrightarrow \mathscr{N}(f(S), f(T))$. We have now defined the (honest) category of $\mathscr{C}$-categories in $\mathscr{G}$ and $\mathscr{C}$-functors. The standard notion of an enriched natural transformation carries over amusingly to this context, but I am not sure this is the best definition.

Definition 14. Let F and G be $\mathscr{C}$-functors from $\mathscr{M}$ to $\mathscr{N}$ with object functions f and g. A $\mathscr{C}$-natural transformation $\alpha : F \longrightarrow G$ is a set of morphisms

$$\alpha : u_{\mathscr{G}} \longrightarrow \mathscr{C}(2) \odot \mathscr{N}(f(S), g(S))$$

in $\mathscr{G}$ such that the following diagrams commute in $\mathscr{G}$:

$$\begin{array}{ccc} \mathscr{M}(S,T) & \xrightarrow{F} & \mathscr{N}(f(S),f(T)) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ \mathscr{N}(g(S),g(T)) & \xrightarrow[\alpha^*]{} & \mathscr{N}(f(S),g(T)). \end{array}$$

Here, if ξ is the action of $\mathscr{C}$ on $\mathscr{N}$, then α_* is the composite map in $\mathscr{G}$ displayed in the diagram

$$\begin{array}{ccc} \mathscr{N}(f(S),f(T)) & \xrightarrow{\alpha_*} & \mathscr{N}(f(S),g(T)) \\ {\scriptstyle \cong}\downarrow & & \uparrow{\scriptstyle \xi} \\ u\otimes\mathscr{N}(f(S),f(T)) & \xrightarrow{\alpha\otimes\mathrm{id}} & \mathscr{C}(2)\odot\mathscr{N}(f(T),g(T))\otimes\mathscr{N}(f(S),f(T)). \end{array}$$

The map α^* in $\mathscr{G}$ is defined similarly, using $\mathscr{N}(g(S),g(T))\otimes u$.

Now change notations and fix an object set $\mathscr{O}$ and a $\mathscr{C}$-category $\mathscr{A}$ in $\mathscr{G}$ over $\mathscr{O}$. We think of $\mathscr{A}$ as some kind of algebra, and we want to define modules over it. When $\mathscr{C}=\mathscr{A}ss$, we can specialize the usual notion of a left action of a monoid on an object: an $\mathscr{A}$-module should then be an $\mathscr{O}$-graph $\mathscr{M}$ together with an associative and unital action map $\mathscr{A}\otimes\mathscr{M}\longrightarrow\mathscr{M}$ of $\mathscr{O}$-graphs. In operad theory, there is a standard notion of a module over a $\mathscr{C}$-algebra that specializes to give the appropriate generalization of this notion.

Definition 15. A left $\mathscr{A}$-module $\mathscr{M}$ is an $\mathscr{O}$-graph $\mathscr{M}$ together with maps

$$\mu:\mathscr{C}(j)\odot\mathscr{A}^{j-1}\otimes\mathscr{M}\longrightarrow\mathscr{M}$$

of $\mathscr{O}$-graphs such that the appropriate associativity and unit diagrams commute.

When $\mathscr{O}$ has only one object, the model category theory of module categories like this is well understood, and the generalization to modules with many objects presents no difficulty. We easily obtain the following result.

Theorem 16. *The category of $\mathscr{A}$-modules is complete and cocomplete, and it is enriched, tensored, and cotensored over $\mathscr{B}$. It is a cofibrantly generated proper model category in which the weak equivalences and fibrations $\mathscr{M}\longrightarrow\mathscr{N}$ are the levelwise weak equivalences and fibrations: each map $\mathscr{M}(S,T)\longrightarrow\mathscr{N}(S,T)$ must be a weak equivalence or fibration in $\mathscr{G}$.*

When $\mathscr{B}=\mathscr{G}$ is the category of chain complexes over k, its homotopy category with respect to the standard model structure is just the derived category $\mathscr{D}(k)$, and of course it is a triangulated category. The homotopy category of $\mathscr{O}$-graphs inherits a triangulation from $\mathscr{D}(k)$, and the homotopy category of $\mathscr{A}$-modules inherits a triangulation from the category of $\mathscr{O}$-graphs.

Corollary 17. *Let $\mathscr{B}=\mathscr{G}$ be the category of chain complexes over k and let $\mathscr{D}(\mathscr{A})$ be the homotopy category associated to the model category of $\mathscr{A}$-modules for a $\mathscr{C}$-category $\mathscr{A}$ in $\mathscr{B}$ over $\mathscr{O}$. Then $\mathscr{D}(\mathscr{A})$ is a triangulated category with a faithful and exact forgetful functor to the triangulated homotopy category of $\mathscr{O}$-graphs.*

When $\mathscr{C}=\mathscr{K}^{alg}$, this gives the promised triangulated derived category $\mathscr{D}(\mathscr{A})$ of an A_∞-category $\mathscr{A}$.

Returning to the main theme, we revert to the notation $\mathscr{M}$ and $\mathscr{N}$ for $\mathscr{C}$-categories in $\mathscr{G}$. We have three theorems on the structure of the category of $\mathscr{C}$-categories in $\mathscr{G}$.

Theorem 18. *The category of $\mathscr{C}$-categories in $\mathscr{G}$ is complete and cocomplete, and it is enriched, tensored, and cotensored over $\mathscr{B}$.*

I have not written down a complete proof of the following deeper result, but I have thought through the proof and I have no doubt that it is correct. The main problem concerns set theory, and that should not be a major stumbling block.

Theorem 19. *The category of $\mathscr{C}$-categories in $\mathscr{G}$ is a cofibrantly generated proper model category. The weak equivalences are the essentially surjective levelwise weak equivalences and the fibrations are the levelwise fibrations.*

Here "levelwise" refers to the maps $F : \mathscr{M}(S,T) \longrightarrow \mathscr{N}(f(S), f(T))$ in $\mathscr{G}$ of a morphism (F, f) of $\mathscr{C}$-categories in $\mathscr{G}$. To explain the term "essentially surjective", I will assume for simplicity that $\mathscr{C}$ is an A_∞-operad. Then, on passage to the homotopy category associated to $\mathscr{G}$, a $\mathscr{C}$-category $\mathscr{M}$ gives rise to a category enriched over $Ho\mathscr{G}$ with the same object set. We require F to be surjective on isomorphism classes of objects in the underlying categories.

The next theorem requires that the operad $\mathscr{C}$ be a Hopf operad: there must be a coassociative and cocommutative map of operads $\mathscr{C} \longrightarrow \mathscr{C} \otimes \mathscr{C}$, where the operad $\mathscr{C} \otimes \mathscr{C}$ has jth object $\mathscr{C}(j) \otimes_{\mathscr{B}} \mathscr{C}(j)$ and the evident structure maps. When $\mathscr{B}$ is Cartesian closed, $\otimes_{\mathscr{B}} = \times$, diagonal maps give the required coaction. There are interesting examples when $\mathscr{B}$ is the category of k-chain complexes.

Theorem 20. *If $\mathscr{C}$ is a Hopf operad, then the category of $\mathscr{C}$-categories in $\mathscr{G}$ is a symmetric monoidal category.*

Of course, we cannot expect internal hom objects in the precise sense of the usual $\otimes$-adjunction, since we do not have them in categories of algebras.

Here is the punch line: when $\mathscr{C}$ is a Hopf operad, the category of $\mathscr{C}$-algebras in $\mathscr{G}$ has all of the structure that we assumed on $\mathscr{G}$. Therefore, we can iterate the construction. This gives a new theory of n-categories.

Definition 21. We define the category $\mathscr{B}(n;\mathscr{C})$ of n-$\mathscr{C}$-categories enriched over $\mathscr{B}$. We take $\mathscr{B}(0;\mathscr{C}) = \mathscr{B}$, so that a 0-category is just an object of $\mathscr{B}$. Inductively, we define $\mathscr{B}(n;\mathscr{C})$ to be the category of $\mathscr{C}$-categories in $\mathscr{B}(n-1;\mathscr{C})$.

One can even start with $\mathscr{B} = \mathscr{C}at$, in which case it would be sensible to start the inductive definition with $\mathscr{B}(1;\mathscr{C}) = \mathscr{C}at$, letting $\mathscr{B}(0;\mathscr{C})$ just be the category of sets. This makes the construction conceptually interior to category theory and is in conformity with past practice in n-category theory.

For the cases when $\mathscr{B}$ is spaces or simplicial sets and $\mathscr{G}$ is a good category of spectra, these categories give a starting point for a higher categorical homotopy theory of stable homotopy theory.

When $\mathscr{C}$ is an A_∞-operad, the objects of $\mathscr{B}(n;\mathscr{C})$ deserve to be called (weak) n-categories. There is a vast literature of n-categories with a chaotic confusion of competing definitions. In all of the earlier approaches, 0-categories are understood to be sets, whereas we prefer a context in which 0-categories come with their own homotopy theory. Otherwise our approach is similar in spirit to an approach of Simpson and Tamsamani, although they make no use of operads. Our approach is very much simpler in detail. Operads do appear in the two quite different alternative treatments due to Baez and Dolan and to Batanin. Our approach is also similar in spirit to Batanin's, but again it is very much simpler in detail. Our definition should form a bridge that allows comparison of the definitions of Simpson-Tamsamani and Batanin to our new definition and therefore to each other.

Incidentally, I believe that everything I have said works for braided monoidal rather than symmetric monoidal base and ground categories.

As promised, we can sometimes change operad. Given operads $\mathscr{C}$ and $\mathscr{D}$ in $\mathscr{B}$, we write $C \otimes D$ for the monad associated to the product operad $\mathscr{C} \otimes \mathscr{D}$, by abuse. We assume, as holds automatically when $\mathscr{B}$ is Cartesian monoidal, that we have projections $\mathscr{C} \longleftarrow \mathscr{C} \otimes \mathscr{D} \longrightarrow \mathscr{D}$. For a $\mathscr{C}$-category $\mathscr{M}$, we then have a two-sided bar construction $B(D, C \otimes D, \mathscr{M})$, which is a $\mathscr{D}$-category, and we have a pair of natural maps of $(\mathscr{C} \otimes \mathscr{D})$-categories in $\mathscr{G}$

$$\mathscr{M} \longleftarrow B(C \otimes D, C \otimes D, \mathscr{M}) \longrightarrow B(D, C \times D, \mathscr{M}).$$

The first map is a weak equivalence and the second map is a weak equivalence if $\mathscr{C}$ is an A_∞-operad. In particular, taking $\mathscr{D} = \mathscr{A}ss$ and $\mathscr{C}$ to be any A_∞-operad, this shows how to replace A_∞ categories enriched over $\mathscr{G}$ with honest categories enriched over $\mathscr{G}$.

Strict n-categories are defined using $\mathscr{C} = \mathscr{A}ss$. It might seem at first glance that the equivalence that I just gave should lead inductively to an equivalence between our homotopy categories of n-categories over an A_∞ operad $\mathscr{C}$ and of strict n-categories enriched over $\mathscr{B}$, but that is not the case. The extra generality in the choice of operad appears naturally and gives flexibility. It is expected to be essential to the potential applications in algebraic geometry and topological quantum field theory. As I said at the start, these ideas and definitions just feel right to me. Time alone will tell how valuable they really are.