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HOMOLOGY OPERATIONS ON INFINITE LOOP SPACES

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In the last few years, it has become clear that homology operations on infinite loop spaces play a very important role in algebraic topology. These operations are the fundamental tool in the study of the homology of such spaces as QX, BF, $B\,\mathrm{Top}$, etc. I recently observed that the language of topological PROP's developed by Boardman and Vogt in [2], allows extremely simple constructions of the operations and proofs of their properties. This approach to the operations was the subject of my talk at the summer institute, but is not yet ready for publication. Instead, I shall here summarize the basic algebraic results of the theory and shall briefly indicate the extent of our present information about the subject, including precise descriptions of the homology of F, F/O, and BF. The quoted results are due to various people. Only those results which are stated without historical references are due to the author; more complete statements and proofs will appear later. This paper is divided into three sections as follows:

(1) The Dyer-Lashof algebra and its dual,
(2) Allowable R-modules and $H_*(QX)$,
(3) The homology of F, F/O, BF, and B Top.

Much of this material has been previously circulated in preprint form, but some of the results in § 3 are new.

1. **The Dyer-Lashof algebra and its dual.** By an infinite loop sequence $B = \{B_i \mid i \geqq 0\}$, we understand a sequence of based spaces such that $B_i = \Omega B_{i+1}$; by a map $g: B \to C$ of infinite loop sequences, we understand a sequence of basepoint preserving maps $g_i: B_i \to C_i$ such that $g_i = \Omega g_{i+1}$, $i \geqq 0$. B_0 and g_0 are then called (perfect) infinite loop spaces and maps. These notions are equivalent

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for the purposes of homotopy theory to the more usual ones in which equalities are replaced by homotopies [8]. We define $H_*(B) = H_*(B_0;\ Z_p)$ for some fixed prime p and regard H_* as a functor from the category of infinite loop sequences to that of graded Z_p-modules. $H_*(B)$ admits homology operations which are analogous to the Steenrod operations in cohomology. For $p = 2$, these operations were first introduced by Araki and Kudo [1]; their work was later simplified by Browder [4]. Dyer and Lashof [5] introduced the operations for odd primes and developed many of their algebraic properties; we shall therefore refer to the operations as Dyer-Lashof operations. The following theorem summarizes their properties. Parts (1) through (5) were proven by Dyer and Lashof, and part (6) was first observed by Milgram. The Adem relations (7) were implicit in the work of Dyer and Lashof and the Nishida relations (8) were proven by Nishida in [14]. We state the results for an arbitrary prime p; the modifications needed in the case $p = 2$ are indicated in square brackets.

THEOREM 1.1. *There exist natural homomorphisms $Q^i : H_*(B) \to H_*(B)$, $i \geqq 0$, of degree $2i(p-1)$ [of degree i]. They satisfy the following properties:*

(1) *$Q^0(\phi) = \phi$ and $Q^i(\phi) = 0$ for $i > 0$, where $\phi \in H_0(B)$ is the identity element for the loop product in $H_*(B)$.*

(2) *$Q^i(x) = 0$ if $2i <$ degree (x) [if $i <$ degree (x)].*

(3) *$Q^i(x) = x^p$ if $2i =$ degree (x) [if $i =$ degree (x)].*

(4) *$\sigma_* Q^i = Q^i \sigma_*$, where $\sigma_* : IH_*(\Omega B) \to H_*(B)$ is the homology suspension.*

(5) *Cartan formula: $Q^s(xy) = \sum_{i=0}^{s} Q^i(x) Q^{s-i}(y)$ and, if $\psi(x) = \sum x' \otimes x''$, then $\psi Q^s(x) = \sum_{i=0}^{s} \sum Q^i(x') \otimes Q^{s-i}(x'')$.*

(6) *If $\chi : H_*(B) \to H_*(B)$ is the conjugation (induced from the map $\chi(l)(t) = l(1-t)$, $\chi : \Omega B_1 \to \Omega B_1$), then $\chi Q^i = Q^i \chi$.*

(7) *Adem relations: If $p \geqq 2$ and $a > pb$, then*

$$Q^a Q^b = \sum_i (-1)^{a+i} (pi - a, a - (p-1)b - i - 1) Q^{a+b-i} Q^i;$$

if $p > 2$, $a \geqq pb$, and β is the mod p Bockstein, then

$$Q^a \beta Q^b = \sum_i (-1)^{a+i} (pi - a, a - (p-1)b - i) \beta Q^{a+b-i} Q^i$$
$$- \sum (-1)^{a+i} (pi - a - 1, a - (p-1)b - i) Q^{a+b-i} \beta Q^i.$$

(8) *Nishida relations: Let $P^s_* : H_*(B) \to H_*(B)$, of degree $-2s(p-1)$, be dual to P^s (i.e. $P^s = \mathrm{Hom}_{Z_p}(P^s_*; 1)$ with $H^*(B) = \mathrm{Hom}_{Z_p}(H_*(B); Z_p)$) if $p > 2$, and let $P^s_* = Sq^s_*$, of degree $-s$, if $p = 2$; then*

$$P^s_* Q^r = \sum_i (-1)^{i+s} (s - pi, r(p-1) - ps + pi) Q^{r-s+i} P^i_*;$$

if $p > 2$,

$$P^s_* \beta Q^r = \sum_i (-1)^{i+s} (s - pi, r(p-1) - ps + pi - 1) \beta Q^{r-s+i} P^i_*$$
$$+ \sum_i (-1)^{i+s} (s - pi - 1, r(p-1) - ps + pi) Q^{r-s+i} P^i_* \beta.$$

In (7) and (8), $(i, j) = (i + j)!/i!\,j!$ if $i > 0$ and $j > 0$, $(i, 0) = 1 = (0, i)$ if $i \geqq 0$, and $(i, j) = 0$ if $i < 0$ or $j < 0$; the sums are over the integers.

Define the Dyer-Lashof algebra R to be the quotient F/J of the free associative algebra F generated by $\{Q^s, \beta Q^{s+1} \mid s \geqq 0\}$ (not β itself) [by $\{Q^s \mid s \geqq 0\}$] modulo the two-sided ideal J consisting of all elements which annihilate every homology class of every infinite loop space. We shall explicitly describe R, and shall then describe its dual R^*. We need the following definition:

DEFINITION 1.2. (a) $p > 2$. Consider sequences $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$, where $\epsilon_j = 0$ or 1 and $s_j \geqq \epsilon_j$. Define the degree, length, and excess of I by

$$d(I) = \sum_{j=1}^{k} [2s_j(p-1) - \epsilon_j], \qquad l(I) = k,$$

$$e(I) = 2s_k - \epsilon_1 - \sum_{j=2}^{k} [2ps_j - \epsilon_j - 2s_{j-1}] = 2s_1 - \epsilon_1 - \sum_{j=2}^{k} [2s_j(p-1) - \epsilon_j].$$

I is said to be admissible if $ps_j - \epsilon_j \geqq s_{j-1}$ for $2 \leqq j \leqq k$. Each I determines $Q^I = \beta^{\epsilon_1} Q^{s_1} \cdots \beta^{\epsilon_k} Q^{s_k} \in R$.

(b) $p = 2$. Consider sequences $I = (s_1, \ldots, s_k)$, $s_j \geqq 0$, and define $d(I) = \sum_{j=1}^k s_j$, $l(I) = k$, $e(I) = s_k - \sum_{j=2}^k (2s_j - s_{j-1}) = s_1 - \sum_{j=2}^k s_j$. I is said to be admissible if $2s_j \geqq s_{j-1}$ for $2 \leqq j \leqq k$. Each I determines the element $Q^I = Q^{s_1} \cdots Q^{s_k} \in R$.

(c) *Convention*. The empty sequence I is admissible and satisfies $d(I) = 0$, $l(I) = 0$, and $e(I) = \infty$ for all p; it determines $Q^I = 1 \in R$.

The structure of R is given by the following theorem:

THEOREM 1.3. *The ideal J is generated by the Adem relations* (7) (*and, if $p > 2$, the relations obtained by applying β to the Adem relations*) *and by the relations $Q^I = 0$ if $e(I) < 0$* (*see* (2)). *R has the Z_p-basis $\{Q^I \mid I$ is admissible and $e(I) \geqq 0\}$. R_0* (*degree zero*) *is the polynomial algebra generated by Q^0 and R is augmented via $\epsilon: R \to Z_p$ defined by $\epsilon((Q^0)^j) = 1$, $j \geqq 0$. R admits a structure of Hopf algebra with coproduct defined on generators by the formulas*

$$\psi(Q^s) = \sum_{i+j=s} Q^i \otimes Q^j; \qquad \psi(\beta Q^{s+1}) = \sum_{i+j=s} [\beta Q^{i+1} \otimes Q^j + Q^i \otimes \beta Q^{j+1}].$$

R admits a structure of left coalgebra over A^0, the opposite Hopf algebra of the Steenrod algebra; the operations P^i_* are determined by the Nishida relations (8) *and induction on the length of admissible monomials, starting with the formulas*

$$P^s_* Q^r = (-1)^s (s, r(p-1) - ps) Q^{r-s};$$
$$P^s_* \beta Q^r = (-1)^s (s, r(p-1) - ps - 1) \beta Q^{r-s}.$$

Here A^0 enters since we are writing Steenrod operations on the left in homology and $\mathrm{Hom}_{Z_p}(\ \ ; Z_p)$ is contravariant; for any space X, $H_*(X)$ is a left A^0-coalgebra. Observe that the inductive definition of the P^s_* on R could equally well be started with $P^s_*(1) = 0$ for $s > 0$ and $P^0_*(1) = 1$. The theorem is proven by showing that

R operates faithfully on a homology class of a certain infinite loop space [see Theorem 2.5].

If $i > 0$, then R_i is finite dimensional, although R_0 is not. Let $R[k] \subset R$ be the subspace spanned by $\{Q^I \mid I$ is admissible, $e(I) \geqq 0$, and $l(I) = k\}$ ($R[0]$ is spanned by 1). By (5), (7), and (8), each $R[k]$ is a sub A^0-coalgebra of R, and $R = \bigoplus_{k \geqq 0} R[k]$ as an A^0-coalgebra. $R[k]$ is connected and $R_0[k]$ is spanned by $(Q^0)^k$. The product takes $R[k] \otimes R[l]$ into $R[k+l]$ since, in marked contrast to the Steenrod algebra, the Adem relations for R are homogeneous with respect to length (because $Q^0 \neq 1$). As an A-algebra, $R^* = \prod_{k \geqq 0} R[k]^*$. The identity element of $R[k]^*$ is the dual ξ_{0k} of $(Q^0)^k$, and the identity of R^* is $\prod_{k \geqq 0} \xi_{0k}$. Strictly speaking, R^* is not a Hopf algebra, since R is not of finite type, but it is the inverse limit of its quotient A-algebras $\prod_{k=0}^n R[k]^*$, $n < \infty$, and each of these may be regarded as a sub Hopf algebra of R^* (dual to the quotient Hopf algebra $R/(\sum_{l>n} R[l])$ of R). $\prod_{k=0}^n R[k]^*$ is augmented by $\epsilon(\xi_{00}) = 1$ and $\epsilon(\xi_{0k}) = 0$, $k > 0$; clearly $\xi_{0k}^2 = \xi_{0k}$ and $\psi(\xi_{0k}) = \sum_{i=0}^k \xi_{0i} \otimes \xi_{0,k-i}$. We shall describe $R[k]^*$ as an A-algebra and shall give the coproduct on generators.

Define certain admissible sequences inductively as follows:

(a) I_{jk}, $1 \leqq j \leqq k$, $p \geqq 2$: $I_{11} = (0, 1)$; $I_{j,k+1} = (0, p^k - p^{k-j}, I_{jk})$ if $j \leqq k$; $I_{k+1,k+1} = (0, p^k, I_{kk})$ [if $p = 2$, omit the zeroes; if $p > 2$, the zeroes merely denote omission of Bockstein]. Then $d(I_{jk}) = 2(p^k - p^{k-j})$ if $p > 2$ and $d(I_{jk}) = 2^k - 2^{k-j}$ if $p = 2$.

(b) J_{jk}, $1 \leqq j \leqq k$, $p > 2$: $J_{1,1} = (1, 1)$; $J_{j,k+1} = (0, p^k - p^{k-j}, J_{jk})$ if $j \leqq k$; $J_{k+1,k+1} = (1, p^k, I_{kk})$. Then $d(J_{jk}) = 2(p^k - p^{k-j}) - 1$.

(c) K_{ijk}, $1 \leqq i < j \leqq k$, $p > 2$: $K_{ij,k+1} = (0, p^k - p^{k-i} - p^{k-j}, K_{ijk})$ if $j \leqq k$; $K_{i,k+1,k+1} = (1, p^k - p^{k-i}, J_{ik})$. Then $d(K_{ijk}) = 2(p^k - p^{k-i} - p^{k-j})$. Let $S_k = \{I_{jk}, J_{jk}, K_{ijk}\}$ if $p > 2$ and $S_k = \{I_{jk}\}$ if $p = 2$. Observe that the elements I_{jk} and K_{ijk} with $j < k$ all give pth powers when the corresponding Q^I is applied to a zero-dimensional class.

LEMMA 1.4. *$\{Q^I \mid I \in S_k\}$ is a basis for the primitive elements $PR[k]$ of $R[k]$. The Steenrod operations and Bockstein on $PR[k]$ are determined by the following formulas:*

(i) $P_*^{p^{k-j}} Q^{I_{jk}} = -Q^{I_{j-1,k}}$ *if* $2 \leqq j \leqq k$ ($P_*^{p^r} Q^{I_{jk}} = 0$ *otherwise*).

(ii) $P_*^{p^{k-j}} Q^{J_{jk}} = -Q^{J_{j-1,k}}$ *if* $2 \leqq j \leqq k$ ($P_*^{p^r} Q^{J_{jk}} = 0$ *otherwise*).

(iii) $P_*^{p^{k-j}} Q^{K_{ijk}} = -Q^{K_{i,j-1,k}}$ *if* $1 \leqq i < j-1 < k$;

$P_*^{p^{k-i}} Q^{K_{ijk}} = -Q^{K_{i-1,j,k}}$ *if* $2 \leqq i < j \leqq k$ ($P_*^{p^r} Q^{K_{ijk}} = 0$ *otherwise*).

(iv) $\beta Q^{I_{kk}} = Q^{J_{kk}}$; $\beta Q^{J_{jk}} = Q^{K_{jkk}}$ *if* $j < k$ ($\beta Q^I = 0$ *otherwise*, $I \in S_k$).

Let $\xi_{jk} = (Q^{I_{jk}})^*$, $\tau_{jk} = (Q^{J_{jk}})^*$, and $\sigma_{ijk} = (Q^{K_{ijk}})^*$ in $R[k]^*$ (in the dual basis to that of admissible monomials). Let $R^+[k] = R[k]$ if $p = 2$ and let $R^+[k]$ be the sub-coalgebra of $R[k]$ spanned by those Q^I such that $\epsilon_j = 0$ (that is, which do not involve β) if $p > 2$. Also, if $p > 2$, define elements $v_\rho \in R[k]^*$ for each set $\rho = \{r_1, \ldots, r_j\}$ such that $1 \leqq r_1 < \cdots < r_j \leqq k$ by the formulas:

(d) $v_\rho = \sigma_{r_1r_2k}\sigma_{r_3r_4k}\cdots\sigma_{r_{j-1}r_jk}$ if j is even; $v_\rho = \sigma_{r_1r_2k}\cdots\sigma_{r_{j-2}r_{j-1}}\tau_{r_jk}$ if j is odd.

Let $v_\rho = \xi_{0k}$ if ρ is the empty set, and let V_k be the subspace of $R[k]^*$ spanned by the v_ρ (the v_ρ are linearly independent). With these notations, the structure of R^* is determined by the following theorem.

THEOREM 1.5. *For $p \geqq 2$, $R^+[k]^*$ is the polynomial algebra generated by $\{\xi_{jk} \mid 1 \leqq j \leqq k\}$. If $p > 2$, the product defines an isomorphism $R^+[k]^* \otimes V_k \to R[k]^*$, and $R[k]^*$ is determined as an algebra by the relations ($\rho = \{r_1, \ldots, r_j\}$, $j = 2i - \epsilon$, $\epsilon = 0$ or 1):*

(i) *$v_\rho\sigma_{stk} = 0$ if $s \in \rho$ or $t \in \rho$; $v_\rho\tau_{sk} = 0$ if $s \in \rho$.*

(ii) *$v_\rho\sigma_{stk} = (-1)^\alpha v_{\rho\cup\{s,t\}}$ if $s \notin \rho$ and $t \notin \rho$, where α is the number of indices l such that $s < r_l < t$.*

(iii) *$v_\rho\tau_{sk} = (-1)^{\beta+\epsilon}\xi^\epsilon_{kk}v_{\rho\cup\{s\}}$ if $s \notin \rho$, where β is the number of indices l such that $r_l < s$.*

With sums taken over all integers which make sense ($\xi_{ij} = 0$ if $i < 0$ or $j \leqq i$, $\tau_{ij} = 0$ if $i < 1$ or $j \leqq i$, $\sigma_{hij} = 0$ if $h < 1$ or $i \leqq h$ or $j < i$), the coproduct is given on generators by the formulas:

$$\text{(iv)} \qquad \psi(\xi_{jk}) = \sum_{(h,i)} \xi^{p^i-p^{i-h}}_{k-i,k-i}\xi^{p^{i-h}}_{j-h,k-i} \otimes \xi_{hi},$$

$$\text{(v)} \qquad \psi(\tau_{jk}) = \sum_{(h,i)} \xi^{p^i-p^{i-h}}_{k-i,k-i}\xi^{p^{i-h}}_{j-h,k-i} \otimes \tau_{hi} + \sum_i \xi^{p^i-1}_{k-i,k-i}\tau_{j-i,k-i} \otimes \xi_{ii},$$

$$\text{(vi)} \qquad \begin{aligned}\psi(\sigma_{ijk}) = &\sum_{(f,g,h)}\left(\sum_{s=0}^{t} \xi^{p^h-p^{h-f}-p^{h-g}}_{k-h,k-h}\xi^{p^{h-g+i-f-s}}_{j-g+i-f-s,k-h}\xi^{p^{h-f}-p^{h-g+i-f-s}}_{i-f,k-h}\xi^{p^{h-g}}_{s,k-h}\right)\otimes \sigma_{fgh}\\ &-\sum_{(g,h)} \xi^{p^h-p^{h-g}-1}_{k-h,k-h}\xi^{p^{h-g}}_{i-g,k-h}\tau_{j-h,k-h} \otimes \tau_{gh}\\ &+\sum_h \xi^{p^h-1}_{k-h,k-h}\sigma_{i-h,j-h,k-h} \otimes \xi_{hh},\end{aligned}$$

where, in the first sum, $t =$ minimum$(i - f, j - g, k - h)$.

The proof is by direct dualization, using the Adem relations. In the case $p = 2$, the structure of R^* was first discovered by I. Madsen [7]. While complicated, the A-algebra structure of R^* is not unmanageable. The Steenrod operations on the indecomposable elements $QR[k]^*$ are computed by Lemma 1.3 which implies the following useful corollary.

COROLLARY 1.6. *If $p = 2$, $R[k]^*$ is generated as an A-algebra by ξ_{1k}. If $p > 2$, $R[1]^*$ is generated as an A-algebra by τ_{11} and $R[k]^*$, $k > 1$, is generated as an A-algebra by ξ_{1k} and σ_{12k}.*

In other words, $R[k]^*$ is a quotient A-algebra either of $H^*(K(Z_p, n))$ or of $H^*(K(Z_p, n) \otimes K(Z_p, m))$, for appropriate integers n and m.

REMARK 1.7. R is very closely related to the E_1-term of the Curtis-Kan et al. [3] version of the Adams spectral sequence. Precisely, R is a quotient algebra of the opposite algebra E_1^0. This relationship, which was first noticed by I. Madsen, deserves further study.

2. **Allowable R-modules and $H_*(QX)$.** Having introduced the Dyer-Lashof algebra, we can now give analogs of the notions of unstable A-modules and A-algebras in the cohomology of spaces. These notions will yield a concise description of $H_*(QX)$ as a functor of $H_*(X)$, where $QX = \text{inj lim } \Omega^n S^n X$. Of course, $QS^i X = \Omega QS^{i+1}X$ and QX is therefore a (perfect) infinite loop space. The QX are, in a precise sense, the free infinite loop spaces (see the proof of Corollary 2.7); they play a role in the theory of infinite loop spaces which is roughly analogous to that played by $K(\pi, n)$'s in the cohomology of spaces. Clearly $\pi_n(QX)$ is the nth stable homotopy group of X.

The following lemma is required in order to obtain a sensible analog to the notion of unstable A-module.

LEMMA 2.1. *Let K^q, $q \geqq 0$, denote the subspace of R spanned by $\{Q^I \mid I$ is admissible and $0 \leqq e(I) < q\}$. Then K^q is a two-sided ideal (and sub A^0-module) of R, and K^q is precisely the set of all elements of R which annihilate every homology class of degree $\geqq q$ of every infinite loop space. The quotient algebra $R^q = R/K^q$ has basis $\{Q^I \mid I$ admissible, $e(I) \geqq q\}$.*

In order to deal with nonconnected spaces, which are crucial to the applications, we need a preliminary definition.

DEFINITION 2.2. By a homology coalgebra C, we mean a cocommutative, unital $(\eta: Z_p \to C)$, augmented $(\epsilon: C \to Z_p)$ coalgebra C such that C is a direct sum of connected coalgebras. We then define $GC = \{g \mid g \in C, \psi(g) = g \otimes g, g \neq 0\}$. GC is a basis for C_0, $\epsilon(g) = 1$ for $g \in GC$, and each $g \in GC$ determines a component C_g of C whose positive degree elements are $\{c \mid \psi(c) = c \otimes g + \sum c' \otimes c'' + g \otimes c\}$; clearly C is the direct sum of its components C_g for $g \in GC$.

If X is a based space, then $H_*(X)$ is a homology coalgebra; its base-point determines the unit and its components determine the direct sum decomposition.

DEFINITION 2.3. An R-module D is said to be allowable if $K^q D_q = 0$ for all $q \geqq 0$. The category of allowable R-modules is the full subcategory of that of R-modules whose objects are allowable; it is an Abelian subcategory which is closed under the tensor product. An allowable R-algebra is an allowable R-module and a commutative algebra such that the product and unit are morphisms of R-modules and such that $Q^i(x) = x^p$ if $p > 2$ and $2i = \deg(x)$ or if p = 2 and $i = \deg(x)$. (Here R operates on Z_p through its augmentation.) An allowable R-coalgebra is an allowable R-module and homology coalgebra whose coproduct, unit, and augmentation are morphisms of R-modules. An allowable R-Hopf algebra is an allowable R-module and Hopf algebra which is both an allowable R-algebra and an allowable R-coalgebra and which admits a conjugation χ [**13**, Definition 8.4] such that $\chi^2 = 1$ and χ is a morphism of R-modules (χ is necessarily a morphism of Hopf algebras and, by definition, $\phi(1 \otimes \chi)\psi = \eta\epsilon$). For any of these structures, an allowable AR-structure is an allowable R-structure and an unstable A^0-structure of the same type (in the sense of homology: its dual, if of finite type, is an unstable A-structure of the dual type) such that the A^0 and R operations satisfy the Nishida relations.

With these definitions, the entire content of Theorem 1.1 is that the mod p homology of an infinite loop space carries a natural structure of allowable AR-Hopf algebra. It should be observed that if B is connected and satisfies the definition of an allowable R-Hopf algebra, except for the condition about χ, then this last condition is automatically satisfied. This is false in the nonconnected case, where we have

LEMMA 2.4. *Let B be an allowable R-Hopf algebra. Then each $g \in GB$ is invertible and $\chi(g) = g^{-1}$. If $x \in B$, $\deg(x) > 0$, and $\psi(x) = x \otimes g + \sum x' \otimes x'' + g \otimes x$, then $\chi(x) = -x \cdot g^{-2} - \sum x' \cdot \chi(x'') \cdot g^{-1}$.*

To take advantage of our definitions, we require free functors taking values in our various categories of allowable R and AR-structures. These are obtained as follows.

(a) *Z_p-modules to allowable R-modules.* If M is a Z_p-module, define $D(M) = \bigoplus_{q \geqq 0} R^q \otimes M_q (D_n(M) = \bigoplus_{q \geqq 0} R^q_{n-q} \otimes M_q$ gives the grading). R operates on the left of $D(M)$ via the maps $R \to R^q$.

(b) *Homology coalgebras to allowable R-coalgebras.* If C is a homology coalgebra with unit $\eta : Z_p \to C$, and $JC = \operatorname{Coker} \eta$, define $E(C) = Z_p \oplus D(JC)$ as an R-module; $C \subset E(C)$ and, by induction on the length of admissible monomials, the coproduct on C together with the Cartan formula define a structure of allowable R-coalgebra on $E(C)$.

(c) *Homology unstable A^0-coalgebras to allowable AR-coalgebras.* Given C, the Nishida relations and A^0 operations on C define an allowable AR-coalgebra structure on $E(C)$ by induction on the length of admissible monomials.

(d) *Allowable R-modules to allowable R-algebras.* Given D, define $V(D) = A(D)/I$, where $A(D)$ is the free commutative algebra generated by D and I is the ideal generated by $\{d^p - Q^i(d) \mid 2i = \deg(d)\}$ if $p > 2$ or by $\{d^2 - Q^i(d) \mid i = \deg(d)\}$ if $p = 2$; the Cartan formula and the requirement that the unit be a morphism of R-modules define a structure of allowable R-algebra on $V(D)$.

(e) *Allowable R-coalgebras to allowable R-Hopf algebras.* Given E, define $W(E) = V(JE)$, $JE = \operatorname{Coker} \eta$; $E \subset W(E)$ and the coproduct on E induces a coproduct on $W(E)$ such that $W(E)$ becomes a Hopf algebra over R, not necessarily allowable unless E is connected. $GW(E)$ is a commutative monoid, $W_0(E)$ is its monoid ring, and $W(E) = V(\bar{E}) \otimes W_0(E)$ as an algebra, where $\bar{E}$ is the set of positive degree elements of E. Let $\tilde{G}W(E)$ be the (commutative) group generated by $GW(E)$, let $\tilde{W}_0(E)$ be its group ring, and define $\tilde{W}(E) = V(\bar{E}) \otimes \tilde{W}_0(E)$ as an algebra. The coproduct on $W(E)$ is determined by those on $W(E) \subset \tilde{W}(E)$ and on $\tilde{W}_0(E)$, and (as in Lemma 2.4) $\tilde{W}(E)$ admits a conjugation extending that on $\tilde{W}_0(E)$. The R-operations on $\tilde{W}(E)$ are determined by those on $W(E)$ and by $Q^s\chi = \chi Q^s$. With these structures, $\tilde{W}(E)$ is an allowable AR-Hopf algebra. Of course, if E is connected, then $\tilde{W}(E) = W(E)$.

(f) *Allowable AR-coalgebras to allowable AR-Hopf algebras.* Given E, the

Cartan formula (for the Steenrod operations) defines a structure of allowable AR-Hopf algebra on $\tilde{W}(E)$.

In each case, verifications are required to prove that these functors are well defined and are adjoint to the forgetful functor going in the other direction. The functors V and W occur in other contexts in algebraic topology and are discussed in [**9**].

Let X be a space. $H_*(X)$ is a homology unstable A^0-coalgebra, hence $\tilde{W}EH_*(X)$ is defined and is the free allowable AR-Hopf algebra generated by $H_*(X)$. The natural inclusion $X \to QX$ induces a monomorphism on homology; by the freeness of $\tilde{W}EH_*(X)$, there results a morphism of AR-Hopf algebras f: $\tilde{W}EH_*(X) \to H_*(QX)$ which defines a natural transformation of functors on the category $\mathscr{T}$ of based spaces. Dyer and Lashof [**5**] proved that f is an isomorphism of algebras if X is connected and computed a component of $H_*(QS^0)$ as an algebra. Simple proofs of their results are obtainable by use of the Eilenberg-Moore spectral sequence. By a reinterpretation and generalization of their methods, we can prove the following theorem; observations of I. Madsen were instrumental in obtaining this result.

THEOREM 2.5. *$f: \tilde{W}EH_*(X) \to H_*(QX)$ is an isomorphism of AR-Hopf algebras for every space X.*

REMARK 2.6. The mod p Bockstein spectral sequence of QX is determined by that of X and the following formula [**10**, Proposition 6.8], which is valid for any three-fold loop space B.

(1) Let $y \in H_{2q}(B)$ and let $\beta_{r-1}(y)$ be defined; then, modulo indeterminacy, $\beta_r(y^p) = \beta_{r-1}(y)y^{p-1}$ unless $r = 2$ and $p = 2$, when $\beta_2(y^2) = \beta(y)y + Q^{2q}\beta(y)$. (Of course, if $p = 2$, Theorem 1.1 (8) implies that $\beta Q^s = (s-1)Q^{s-1}$.) Thus the Bockstein spectral sequences of QX are functors of those of X, and the (additive) integral homology of QX is explicitly known as a functor of the integral homology of X.

The following corollary of Theorem 2.5 gives an analog to the statement that the cohomology of any space is a quotient of a free unstable A-algebra.

COROLLARY 2.7. *If B is an infinite loop space, then $H_*(B)$ is a quotient of the free allowable AR-Hopf algebra $H_*(QB_0)$.*

PROOF. By [**8**, Proposition 1], there is an adjunction isomorphism

$$\mathrm{Hom}_{\mathscr{T}}(X, B_0) \to \mathrm{Hom}_{\mathscr{L}}(\tilde{Q}(X), B),$$

where $\mathscr{L}$ is the category of infinite loop sequences and $\tilde{Q}(X) = \{QS^iX \mid i \geq 0\}$. It follows that, for any infinite loop sequence $B = \{B_i\}$, there is a map $g: \tilde{Q}B_0 \to B$ in $\mathscr{L}$ such that the composite $B_0 \to QB_0 \xrightarrow{g_0} B_0$ is the identity in $\mathscr{T}$, the category of based spaces.

As an algebra, $H_*(QX) = VD(JH_*(X)) \otimes H_0(QX)$, where $VD(JH_*(X))$ is the free commutative algebra generated by the Z_p-module: $\{Q^I(x) \mid x \in JH_*(X)$, I admissible, $\deg(Q^I x) > 0$, $e(I) + \epsilon_1 > \deg(x)\}$. (If $p = 2$, $e(I) > \deg(x)$ is required; $Q^I = 1$ is allowed if $\deg(x) > 0$.) The $Q^I(x)$ with $e(I) = \deg(x) > 0$ and, if $p > 2$, $\epsilon_1 = 0$ precisely account for the pth powers of positive degree elements. Note that:

$$\pi_0(QX) = \pi_0^s(X) = \text{inj lim}\ \pi_i(S^iX) = \text{inj lim}\ H_i(S^iX) = JH_0(X).$$

Therefore $H_0(QX)$, as a Hopf algebra with conjugation, is the group ring of the free commutative group with one generator $x \in JH_0(X)$ for each component of X other than that of the base-point, in agreement with (e). The coproduct and Steenrod operations on $H_*(QX)$ are induced from those on $H_*(X)$ by the Cartan formulas and Nishida relations; their explicit evaluation requires use of the Adem relations for the Dyer-Lashof operations.

3. **The homology of F, F/O, BF, and B Top.** To illustrate the previous results and prepare for the discussion of F and BF, we consider QS^0 in detail. Let $\tilde{F}(n)$ denote the space of based maps $S^n \to S^n$, $S:\tilde{F}(n) \to \tilde{F}(n+1)$, and let $\tilde{F} = \text{inj lim}\ \tilde{F}(n)$. Then $\tilde{F}(n) = \Omega^nS^n$ and $\tilde{F} = QS^0$. Let $\tilde{F}_i$ denote the component of $\tilde{F}$ consisting of the maps of degree i, $i \in Z$. If $[i] \in H_0(\tilde{F})$ is represented by a map of degree i, then $\{[i] \mid i \in Z\}$ is a basis for $H_0(\tilde{F})$. Denote the loop product in $\tilde{F} = \Omega QS^1$ by $*$. Then $*:\tilde{F}_i \times \tilde{F}_j \to \tilde{F}_{i+j}$ and $[i] * [j] = [i+j]$. Each Q^s takes $H_*(\tilde{F}_i)$ to $H_*(\tilde{F}_{pi})$, and $Q^0[i] = [pi]$. $[0]$ is the identity for $*$ and $Q^s[0] = 0$ for $s > 0$. Let $\bar{R}$ denote the set of positive degree elements of R and let $H_*(S^0)$ have basis $[0]$ and $[1]$. Then $EH_*(S^0) = Z_p[0] \oplus R \cdot [1]$,

$$WEH_*(S^0) = V(R[1]) = V(\bar{R}[1]) \otimes P\{[1]\},$$

and

$$H_*(\tilde{F}) = \tilde{W}EH_*(S^0) = V(\bar{R}[1]) \otimes H_0(\tilde{F}).$$

Let $A(X)$ denote the free commutative algebra generated by the set X, then

$$V(\bar{R}[1]) = A\{Q^I[1] \mid I \text{ admissible}, d(I) > 0, e(I) + \epsilon_1 > 0\}.$$

$R \to H_*(\tilde{F})$ defined by $Q^I \to Q^I[1]$ is a monomorphism of A^0-coalgebras; by the Cartan formulas and Nishida and Adem relations, R determines the Steenrod operations and coproduct in $H_*(\tilde{F})$ and the Dyer-Lashof operations on $V(R[1])$; for $i > 0$, $Q^s[-i] = \mathcal{X}Q^s[i]$.

Of course, $\tilde{F}$ is a topological monoid under the product c defined by composition of maps, $c:\tilde{F}_i \times \tilde{F}_j \to \tilde{F}_{ij}$. This product is homotopic to that obtained from the smash product of maps $\tilde{F}(m) \times \tilde{F}(n) \to \tilde{F}(m+n)$ by passage to limits, and is therefore homotopy commutative.

With the product c, $\tilde{F}$ plays a special role in the theory of infinite loop spaces. Thus let $B = \{B_i \mid i \geqq 0\}$ be any infinite loop sequence. Then $B_0 = \Omega^nB_n$ is homeomorphic to the spaces of based maps $S^n \to B_n$, and composition of maps defines an operation $c_n:B_0 \times \tilde{F}(n) \to B_0$. Since $c_{n+1}(1 \times S) = c_n$, we obtain $c:B_0 \times \tilde{F} \to B_0$ by passage to limits. Clearly $c:QS^0 \times \tilde{F} \to QS^0$ coincides with the composition product on $\tilde{F}$. The basic properties of $c_*:H_*(B) \otimes H_*(\tilde{F}) \to H_*(B)$ are given in the following theorem.

THEOREM 3.1. *c_* gives $H_*(B)$ a structure of Hopf algebra over the Hopf algebra $H_*(\tilde{F})$ and, with $c_*(b \otimes f) = bf$,*

(i) *$\phi f = \epsilon(f)\phi$, $\epsilon: H_*(B) \to Z_p$, where $\phi \in H_0(B)$ is the identity.*

(ii) *$P^k_*(bf) = \sum P^i_*(b)P^{k-i}_*(f)$ and $\beta(bf) = \beta(b)f + (-1)^{\deg b} b\beta(f)$.*

(iii) *$Q^k(b) \cdot f = \sum_i Q^{k+i}(bP^i_*(f))$ and, if $p > 2$,*

$$\beta Q^k(b) \cdot f = \sum_i \beta Q^{k+i}(bP^i_*(f)) - \sum_j (-1)^{\deg b} Q^{k+j}(b \cdot P^j_* \beta(f)).$$

(iv) *$\sigma_*(b)f = \sigma_*(bf)$, where σ_* is the homology suspension.*

Thus $H_*(B)$ is a Hopf algebra over each of R, A^0, and $H_*(\tilde{F})$, all of these homology operations are stable, and we have precise commutation formulas relating these three types of operations. Applied to $B = \{QS^i \mid i \geqq 0\}$, the theorem completely determines the structure of $H_*(\tilde{F})$ as an algebra under c_*. Explicitly, we have the following corollary.

COROLLARY 3.2. *If $x, y, z \in H_*(\tilde{F})$ and $\psi(z) = \sum z' \otimes z''$, then*

(a) *$(x * y)z = \sum (-1)^{\deg y \deg z'} xz' * yz''$, and*

(b) *$(x * [i])(y * [j]) = \sum (-1)^{\deg x'' \deg y'} x'y' * x''[j] * y''[i] * [ij]$.*

Thus c_* is determined by the products between the generators of $H_*(\tilde{F})$ under $*$ namely $[\pm 1]$ and the $Q^I[1]$. The products $Q^I[1] \cdot Q^J[1]$ are determined by (iii) of the theorem and induction on $l(I)$; in particular, all $Q^I[1]$ with $l(I) > 1$ are decomposable under c_* in terms of the $\beta^\epsilon Q^s[1]$. Finally, $x \cdot [1] = x$, $x \cdot [-1] = \mathscr{X}(x)$, and $x \cdot [0] = \epsilon(x)[0]$ for $x \in H_*(\tilde{F})$.

Now recall that $F = \tilde{F}_1 \cup \tilde{F}_{-1}$ is the space of based homotopy equivalences of spheres and that $SF = \tilde{F}_1$. It is easy to compute $H_*(F)$ as an algebra from the corollary. The following result was first proved by Milgram [**11**] in the case $p = 2$ and later by the author and Tsuchiya [**17**], independently, in the case $p > 2$. The proofs of Milgram and Tsuchiya rely on similar, but less general, results than Theorem 3.1. It will be convenient to first fix notations for various elements and sets of elements in $H_*(SF)$. Thus define

(a) $x_s = Q^s[1] * [1 - p]$; $\deg x_s = s$ if $p = 2$, and $\deg x_s = 2s(p - 1)$, if $p > 2$;

(b) $y_s = Q^{s(p-1)}Q^s[1] * [1 - p^2]$; $\deg y_s = 2s$ if $p = 2$ and $\deg y_s = 2sp(p - 1)$ if $p > 2$;

(c) $z_s = Q^{s+1}Q^s[1] * [-3]$ if $p = 2$; $\deg z_s = 2s + 1$;

(d) $z_s = Q^{(p-1)s}\beta Q^s[1] * [1 - p^2]$ if $p > 2$; $\deg z_s = 2sp(p - 1) - 1$;

(e) $I = Q^I[1] * [1 - p^{l(I)}]$, where I is admissible, $l(I) \geqq 2$, and $e(I) + \epsilon_1 > 0$; let X denote the set of all such $I \in H_*(SF)$; observe that $z_s \in X$ but $y_s \notin X$ and, if $p > 2$, $\beta z_s \notin X$. Define

(f) $Y = \{y_s\} \cup X$ if $p = 2$ and $Y = \{\beta z_s\} \cup X$ if $p > 2$ (here $y_s \notin Y$).

COROLLARY 3.3. *As an algebra under c_*, $H_*(SF) = E\{x_s\} \otimes P(Y)$ if $p = 2$ and $H_*(SF) = E\{\beta x_s\} \otimes P\{x_s\} \otimes A(Y)$ if $p > 2$.*

Let $J: SO \to SF$ be the natural inclusion. We next give precise information on J_* and describe $H_*(F/O)$.

Let $p = 2$. Then $H_*(SO) = E\{\alpha_s \mid s \geqq 1\}$, where $\deg a_s = s$, $\psi(a_s) = \sum_{i=0}^{s} a_i \otimes a_{s-i}$, and $\langle w_{s+1}, \sigma_*(a_s)\rangle = 1$. We may define Stiefel-Whitney classes $W_s = \phi^{-1}Sq^s\phi(1) \in H^*(BSF)$, where ϕ is the Thom isomorphism, and then $(BJ)^*(w_s) = w_s$. Thus $(BJ)^*: H^*(BSF) \to H^*(BSO)$ is an epimorphism, hence so is J^*.

THEOREM 3.4. *Let $p = 2$. Then $J_*(a_s) = x_s$, hence $\operatorname{Im} J_* = E\{x_s\}$. $H_*(F/O) = H_*(SF)//J_* \cong P(Y)$, $H_*(SF) \to H_*(F/O)$ is the natural epimorphism, and $H_*(F/O) \to H_*(BO)$ is trivial.*

Now let $p > 2$. Then $H_*(SO) = E\{a_s \mid s \geqq 1\}$, where $\deg a_s = 4s - 1$, a_s is primitive, and $\langle P_s, \sigma_*(a_s)\rangle = 1$, with P_s the Pontrjagin class reduced mod p. We may define Wu classes $q_s = \phi^{-1}P^s\phi(1)$ in both $H^*(BF)$ and $H^*(BO)$. Then $(BJ)^*(q_s) = q_s$ and, in $H^*(BO)$, $q_s - k_sP_{ms}$ is decomposable, where $m = (1/2)(p-1)$ and $0 \neq k_s \in Z_p$ [12, p. 120]. By $\sigma^*(BJ)^* = J^*\sigma^*$ and dualization, $J_*(a_{ms}) \neq 0$ in $H_*(F)$.

We need the following lemma.

LEMMA 3.5. *Let $p > 2$. Then the sub Hopf algebra $E\{\beta x_s\} \otimes P\{x_s\}$ of $H_*(SF)$ contains unique primitive elements b_s such that $b_s - \beta x_s$ is decomposable, and $E\{\beta x_s\} \otimes P\{x_s\} = E\{b_s\} \otimes P\{x_s\}$.*

THEOREM 3.6. *Let $p > 2$. Then $J_*(a_{ms}) = lb_s$, $0 \neq l \in Z_p$, and $J_*(a_s) = 0$ for $s \not\equiv 0 \bmod m$, hence $\operatorname{Im} J_* = E\{b_s\}$. Further, $H_*(F/O) = [H^*(BO)/\!/P\{qs\}]^* \otimes H_*(SF)/\!/J_*$, where $H_*(SF)/\!/J_* \cong P\{x_s\} \otimes A(Y)$; $H_*(SF) \to H_*(F/O)$ is the natural epimorphism onto $H_*(SF)/\!/J_*$; and $H_*(F/O) \to H_*(BO)$ is the inclusion on $[H^*(BO)/\!/P\{q_s\}]^*$ and is trivial on $H_*(SF)/\!/J_*$.*

We next discuss the classifying space BF. If $p = 2$, $H_*(BF)$ is already determined, as a coalgebra, by $H_*(F)$ since $E^2 = E^\infty$ for dimensional reasons in the Eilenberg-Moore spectral sequence converging from $\operatorname{Tor}_{H_*(SF)}(Z_2, Z_2)$ to $H_*(BSF)$. Thus Milgram [11] first computed $H_*(BF; Z_2)$ as a coalgebra. For $p > 2$, and to compute $H_*(BF; Z_2)$ as an algebra, more information is needed. This information for $p > 2$ was obtained first by Tsuchiya [17] and then by the author and for $p = 2$ by Madsen [7]; it will be discussed below.

THEOREM 3.7. *If $p = 2$, $H_*(BF) = H_*(BO) \otimes BC$ as a Hopf algebra, where BC is the primitively generated Hopf algebra*

$$E\{\sigma_*(y_s)\} \otimes P\{\sigma_*(z_s)\} \otimes P\{\sigma_*(I) \mid I \in X, e(I) > 1\}.$$

If $p > 2$, $H_*(BF) = [P\{q_s\} \otimes E\{\beta q_s\}]^* \otimes BC$ as a Hopf algebra, where BC is the primitively generated Hopf algebra

$$E\{\sigma_*(\beta z_s)\} \otimes P\{\sigma_*(z_s)\} \otimes A\{\sigma_*(I) \mid I \in X, e(I) + \epsilon_1 > 1\}.$$

In both cases, BC is closed under the Steenrod operations in $H_*(BF)$.

Of course, the pth powers in $(BC)^*$ are all zero. There are spaces $B \operatorname{Im} J$ (one for each p), constructed by Stasheff [16], such that $H^*(B \operatorname{Im} J) = P\{q_s\} \otimes E\{\beta q_s\}$ if $p > 2$. Peterson and Toda [15] first proved that $H^*(BF) \cong H^*(B \operatorname{Im} J) \otimes (BC)^*$, as a Hopf algebra over the Steenrod algebra, but without computing $(BC)^*$. In principle, the theorem determines the Steenrod operations in $H_*(BF)$ since the given generators of BC are suspensions of elements with known Steenrod operations in $H_*(SF)$. There is one practical difficulty, however. In applying the Nishida relations to the $Q^I[1] * [1 - p^{l(I)}]$, one sometimes reaches terms $Q^J[1] * [1 - p^{l(I)}]$, where $Q^J[1]$ is a pth power in the loop product $*$. It is not known that all such elements (other than the y_s if $p = 2$) are decomposable under c_*. Thus such elements could conceivably suspend nontrivially to $H_*(BF)$.

It is instructive to compare $H_*(BSF)$ to $H_*(QS^1)$. The latter is the free commutative primitively generated Hopf algebra

$$A\{Q^I i \mid I \text{ is admissible and } e(I) + \epsilon_1 > 1\},$$

where i is the fundamental class of $H_*(S^1)$. In the case $p = 2$, the presence of $H_*(BSO)$ in $H_*(BSF)$ forced the appearance of the additional generators $\sigma_*(y_s)$ and $\sigma_*(z_s)$. In the case $p > 2$, the presence of the Wu classes and their Bocksteins in $H^*(BF)$ forced the appearance of the additional generators $\sigma_*(\beta^\epsilon z_s)$.

The description of $H_*(BF)$ just given is clearly inappropriate for most applications. The interest in BF lies mainly in its relationship to BO, B Top, BPL, F/pl, etc. By the work of Boardman and Vogt [2], all of these spaces are infinite loop spaces and the maps between them are infinite loop maps. To study these maps and to compute $H_*(BBF)$, etc., one must describe $H_*(BF)$ in terms of its own Dyer-Lashof operations rather than in terms of the suspensions of the Q^s for the $*$ product in $H_*(\tilde{F})$.

To study this problem, it is again convenient to use all of $\tilde{F}$. Although $\tilde{F}$ is not an infinite loop space under c, since $\pi_0(\tilde{F})$ is not a group under c_*, it is easy to prove by use of topological PROP's that $H_*(\tilde{F})$ admits Dyer-Lashof operations $\tilde{Q}^s: H_*(\tilde{F}_l) \rightarrow H_*(\tilde{F}_{l^p})$ which coincide with the infinite loop operations for the composition product on $H_*(F)$ and which satisfy all of the properties stated in Theorem 1.1 except (6). The information required for the proof of Theorem 3.7 was, in the case $p = 2$, the determination of the first operation above the square, namely $\tilde{Q}^s(x)$ for $x \in H_{s-1}(SF)$ and, in the case $p > 2$, the determination of $\beta^\epsilon \tilde{Q}^s(x)$ for $x \in H_{2s-1}(SF)$. If $p > 2$, these $\beta\tilde{Q}^s(x)$ produced nontrivial differentials (d_{p-1}) in the Eilenberg-Moore spectral sequence and, for all p, these $\tilde{Q}^s(x)$ determined the algebra extensions from E_∞ to $H_*(BF)$. An earlier preprint of mine claimed a complete algebraic determination of the higher $\tilde{Q}^s$; the methods used did give considerable information, but less than was claimed. Tsuchiya [17] initiated a direct geometric study of the $\tilde{Q}^s$, but his results did not give an explicit hold on the higher operations. Recently Madsen [7] obtained precise formulas, in the case $p = 2$, for the evaluation of the $\tilde{Q}^s$ on $H_*(\tilde{F})$ in terms of the Q^s and the loop and composition products. Modulo one ambiguity, I have since obtained such an evaluation for all p. The key result is the following theorem, which evaluates the $\tilde{Q}^s$ on elements of $H_*(\tilde{F})$ which are $*$-decomposable.

THEOREM 3.8. *There exist operations* $\tilde{Q}_i^s: H_*(\tilde{F}) \otimes H_*(\tilde{F}) \to H_*(\tilde{F})$ *for* $0 \leq i \leq p$ *such that, for all* $x, y \in H_*(\tilde{F})$,

(1) $\tilde{Q}^s(x * y) = \sum\sum \tilde{Q}_0^{s_0}(x^{(0)} \otimes y^{(0)}) * \cdots * \tilde{Q}_p^{s_p}(x^{(p)} \otimes y^{(p)})$, $\sum s_i = s$, *where* $\psi(x \otimes y) = \sum x^{(0)} \otimes y^{(0)} \otimes \cdots \otimes x^{(p)} \otimes y^{(p)}$ *gives the iterated coproduct in* $H_*(\tilde{F}) \otimes H_*(\tilde{F})$.

The $\tilde{Q}_0^s$ *and* $\tilde{Q}_p^s$ *are determined from the* $\tilde{Q}^s$ *by*

(2) $\tilde{Q}_0^s(x \otimes y) = \tilde{Q}^s(\epsilon(y)x)$ *and* $\tilde{Q}_p^s(x \otimes y) = \tilde{Q}^s(\epsilon(x)y)$, $\epsilon: H_*(\tilde{F}) \to Z_p$.

The $\tilde{Q}_i^s$, $0 < i < p$, *are determined from the* Q^s *by the formulas*

(3) $\sum_{k \geq 0} \tilde{Q}_i^{s+k} P_*^k(x \otimes y) = \sum \tilde{Q}_i^s([1] \otimes [1]) x^{(1)} \cdots x^{(p-i)} y^{(1)} \cdots y^{(i)}$, *where* $\psi(x) = \sum x^{(1)} \otimes \cdots \otimes x^{(p-i)}$ *and* $\psi(y) = \sum y^{(1)} \otimes \cdots \otimes y^{(i)}$ *give the iterated coproducts: this formula is inductively solvable for* $\tilde{Q}_i^s(x \otimes y)$ *in terms of the Steenrod operations, the composition product, and the elements* $\tilde{Q}_i^t([1] \otimes [1])$;

(4) $\tilde{Q}_i^s([1] \otimes [1]) = \sum \tilde{Q}_1^{s_1}([1] \times [1]) * \cdots * \tilde{Q}_1^{s_{r_i}}([1] \otimes [1])$, $\sum s_j = s$, *for* $1 < i < p$, *where* $r_i = (1/p)(i, p - i)$ *and*

(5) $\tilde{Q}_1^s([1] \otimes [1]) = Q^s[1]$.

In particular, formulas (3) *and* (5) *imply*

(6) $\tilde{Q}_1^s([1] \otimes y) = Q^s(y)$ *for all* $y \in H_*(\tilde{F})$.

The crucial formula (5) is obtainable algebraically, from knowledge of $H_*(O)$, if $p = 2$, but requires a very explicit geometric hold on the operations if $p > 2$. The theorem reduces the problem of calculating the $\tilde{Q}^s$ on $H_*(\tilde{F})$ to their evaluation on elements which are indecomposable under both the loop and composition products. We have the following lemma.

LEMMA 3.9. *If* $s > 0$, *then* $\tilde{Q}^s[0] = 0$, $\tilde{Q}^s[1] = 0$, *and, if* $p > 2$, $\tilde{Q}^s[-1] = 0$; *if* $p = 2$, *then* $\tilde{Q}^s[-1] = x_s = Q^s[1] * [-1]$.

Thus, by Corollary 3.2, we are reduced to the evaluation of the $\tilde{Q}^s\beta^\epsilon Q^r[1]$, and, by Theorem 3.8, it suffices to evaluate the $\tilde{Q}^s\beta^\epsilon x_r$ instead. Now Kochman [6] has succeeded in completely determining all Dyer-Lashof operations in the homology of all spaces involved in Bott periodicity. By Theorems 3.4 and 3.6, his results on $H_*(SO)$ compute the $\tilde{Q}^s(x_r)$ if $p = 2$ and the $\tilde{Q}^s(b_r)$ if $p > 2$. With $H_*(SO) = E\{a_r\}$, as above, Kochman has proven the following theorem.

THEOREM 3.10. *If* $p = 2$, *then, with* $a_0 = 1$,

$$Q^s(a_r) = (r, s - r - 1)a_{r+s} + \sum_{0 \leq i < j < k,\, i+j+k = r+s} [(r - i, i + j - 2r - 1) + (r - i, s - j - r - 1) + (r - j, s - i - r - 1)]a_i a_j a_k$$

If $p > 2$, then $Q^s(a_r) = (-1)^s(2r - 1, s - 2r)a_{r+ms}$, $m = (1/2)(p - 1)$. If $p = 2$, these formulas completely determine the $\tilde{Q}^s$; if $p > 2$, the $\tilde{Q}^s(x_r)$ are not yet determined, but this problem appears to be solvable geometrically. Of course, the algebraic complexity of these results is enormous. They evaluate the $\tilde{Q}^s$ in terms of our basis for $H_*(\tilde{F})$ defined by means of the loop product and its operations Q^s, but their purpose is to enable us to prove that $H_*(F)$ admits a reasonable basis described in terms of the $\tilde{Q}^s$ themselves. The important result is therefore the following, which we state provisionally as a conjecture, although a complete proof should be available shortly.

CONJECTURE 3.11. Theorem 3.7 remains true with $\sigma_*(I)$ replaced by $\tilde{Q}^J\sigma_*(K)$ for those $I = (J, K) \in X$ such that $l(K) = 2$ and $l(J) > 0$ (with $e(I) + \epsilon_1 > 1$).

The conjecture implies the weaker statement that BC is generated as an R-algebra by the elements $\sigma_*(K)$ with $l(K) = 2$ (and the $\sigma_*(y_s)$ and $\sigma_*(\beta^{\epsilon}z_s)$), and so reduces the number of R-algebra generators of $H_*(BF)$ to manageable proportions. For $p = 2$, this weaker statement has been proven by Madsen [7].

Finally, we shall very briefly discuss $H_*(B\,\mathrm{Top})$. Let $p > 2$. Sullivan (unpublished) has shown that BPL, which is mod p homotopy equivalent to $B\,\mathrm{Top}$ by the triangulation theorem, splits as a space, after localization at p, into a product $BO \times B\,\mathrm{Coker}\,J$. (Here $BO \overset{(p)}{\cong} Y \times Y'$, where $H^*(Y) = P\{q_s\}$, and the relevant map $BO \to BPL$ is the natural inclusion on $Y \subset BO$ and is the composite $Y' \to BO \overset{(p)}{\cong} F/PL \to BPL$ on Y'.) By the Adams conjecture, F is mod p homotopy equivalent to $\mathrm{Im}\,J \times \mathrm{Coker}\,J$ (the analogous statement for BF is an exceedingly difficult open question). Using these facts and Theorem 3.8, I have proven the following result.

THEOREM 3.12. *If $p > 2$, then $H_*(BPL)$ is isomorphic as a Hopf algebra to $H_*(BO) \otimes BC$, and the following composite is an isomorphism of A^0-coalgebras*:

$$H_*(B\,\mathrm{Coker}\,J) \to H_*(BPL) \to H_*(BF) \to H_*(BF)/\!/(E\{\beta q_s\} \otimes P\{q_s\})^* \cong BC.$$

I do not claim that $H_*(B\,\mathrm{Coker}\,J)$ maps into $BC \subset H_*(BF)$, and I thus do not have complete information on the map $H_*(BPL) \to H_*(BF)$. The previous result had been conjectured, and proven in low dimensions, by Peterson.

For $p = 2$, the problem is considerably more difficult Madsen [7] has obtained very useful information on the Dyer-Lashof operations in $H_*(F/\mathrm{Top}; Z_2)$ and his work may well lead to a computation of $H_*(B\,\mathrm{Top}; Z_2)$.

ADDED IN PROOF. A number of changes have occurred, since the summer conference, in the state of our knowledge.

(1) For $p = 2$, Madsen has proven Conjecture 3.11 and has shown that BF does not split as $BJ \times B\,\mathrm{Coker}\,J$; however, such a splitting has always appeared far less likely for $p = 2$ than for $p > 2$, and the question for $p > 2$ is still open.

(2) Madsen, Brumfiel, and Milgram have succeeded in computing $H_*(B\,\mathrm{Top}; Z_2)$, and Tsuchiya has given an independent proof of Theorem 3.12, in "Characteristic classes for PL-microbundles" (mimeographed notes).

(3) I have obtained very simple proofs of the results of Boardman and Vogt, and this work has greatly streamlined the construction of the operations and the proofs of the results of this paper.

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