E_∞ SPACES, GROUP COMPLETIONS, AND PERMUTATIVE CATEGORIES

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In the last few years, a number of authors have developed competing theories of iterated loop spaces. Among the desirable properties of such a theory are:

(1) A recognition principle for n-fold loop spaces, $1 \leq n \leq \infty$, which applies when $n = \infty$ to such spaces at Top, BF, PL/O, etc. and to the classifying spaces of categories with appropriate structure.

(2) An approximation theorem which describes the homotopy type of $\Omega^n\Sigma^n X$, $1 \leq n \leq \infty$, in terms of iterated smash products of X and canonical spaces.

(3) A theory of homology operations on n-fold loop spaces, $1 \leq n \leq \infty$, at least sufficient to describe $H_*\Omega^n\Sigma^n X$, with all structure in sight, as a functor of H_*X.

(4) Computations and applications of the homology operations on interesting spaces to which (1) applies.

In addition, rigor and aesthetics dictate (5) and suggest (6).

(5) Complete proofs of all non-trivial technical details are to be given.

(6) Only simple and easily visualized topological constructions are to be used.

Point (5) is particularly important since several quite plausible sketched proofs of recognition principles have foundered on seemingly minor technical details. At the moment, the author's theory [17], which shall be referred to as [G], provides the only published solution to (1) and (2) which makes any claim to satisfy (5). For this reason, and because it includes the deeper cases $1 < n < \infty$ of (1) and (2) as well as machinery designed for use in (3) and (4), [G] is quite lengthy. The main purpose of the present paper is to outline and to generalize to non-connected cases the solution of (1) and (2) in the case $n = \infty$. The present proofs will not use the solution to (1) and (2) for $n < \infty$. The recognition

principle will be proven in §2, after the notion of a group completion is discussed in §1. In §3 we obtain some consistency statements concerning loop spaces and classifying spaces of E_∞ spaces and discuss various approximation theorems. Finally, we demonstrate in §4 that our recognition principle applies in particular to the classifying spaces of permutative categories; the competing theories of Segal [26], Anderson [1, 2], and Tornehave [28] are designed primarily for application to such spaces. It follows that our theory can be used to construct algebraic K-theory.

In an appendix, we sharpen some of the results of [G]; these improvements are required in order to handle non-connected spaces.

I would like to emphasize that the present theory is a synthesis which incorporates many ideas borrowed from or inspired by the papers of Dyer and Lashof [11], Milgram [20], Boardman and Vogt [6, 7], Beck [5], and (in the non-connected case) Barratt, Priddy, and Quillen [3, 4, 22, 24]. My aim has been to mold these disparate lines of thought into a single coherent theory, geared towards explicit calculations in homology and homotopy and based on an absolute minimum of categorical and simplicial machinery.

The key topological applications and homological calculations based on this theory will appear in [18]. That paper will also contain a multiplicative elaboration of the theory which studies E_∞ ring spaces and applies in particular to the spaces $B\mathcal{F}$ used in §4 to define the K-theory of a commutative topological ring.

§1. Group completions

The notion of a group completion will be central to our recognition principle for non-connected spaces. The following development is motivated by Quillen's generalization [24] of the work of Barratt and Priddy [4].

Definition 1.1. Let G be a monoid. Define the <u>translation category</u> $\tilde{G}$ of G to be the category with objects the elements of G and with morphisms from g' to g'' those elements $g \in G$ such that $g'g = g''$.

Lemma 1.2. Let G be a central submonoid of a ring R and let $i : R \to R[G^{-1}]$ denote the localization of R at G; thus $i(g)$ is invertible for $g \in G$ and i is universal with this property. Then $R[G^{-1}]$ is isomorphic as a (left and right) R-module to the limit of that functor from $\tilde{G}$ to the category of R-modules which sends each object to R and each morphism $g : g' \to g''$ to multiplication by g.

Definition 1.3. An H-space X will be said to be admissible if X is homotopy associative and if left translation by any given element of X is homotopic to right translation by the same element. A group completion $g : X \to Y$ of X is an H-map between admissible H-spaces such that Y is grouplike ($\pi_0 Y$ is a group) and the unique morphism of k-algebras

$$\bar{g}_* : H_*(X; k)[\pi_0^{-1} X] \to H_*(Y; k)$$

which extends g_* is an isomorphism for all commutative coefficient rings k.

Remark 1.4. By the following argument of Quillen [24], the condition on $\bar{g}_*$ will be satisfied if it is satisfied for $k = Z_p$ (the integers mod p) for all primes p and for $k = Q$ (the rationals). For any Abelian group A, $\pi_0 X$ acts on $H_*(X; A)$ and $H_*(X; A)[\pi_0^{-1} X]$ can be defined as the evident limit (and is a homological functor of A). Clearly, given the condition on $\bar{g}_*$ for the cited k,

$$\bar{g}_* : H_*(X; A)[\pi_0^{-1} X] \to H_*(Y; A)$$

will be an isomorphism of Abelian groups for any Q-module A, for any Z_p-module A, for $A = Z_{p^n}$ (by induction on n), and for A any torsion group (by passage to limits). Now the conclusion follows by use of the exact sequence

$$0 \to tk \to k \to k \otimes_Z Q \to t'k \to 0,$$

where tk denotes the torsion subgroup of k.

Remark 1.5. If X is a grouplike admissible H-space, then X is H-equivalent to $\pi_0 X \times X_0$, where X_0 denotes the component of the identity element [see, e.g., 18, 4.6]. Thus, by the Whitehead theorem for connected H-spaces, a group completion of a grouplike admissible H-space is a weak homotopy equivalence.

The following theorem is a special case of a result stated by Quillen [24, §9].

Theorem 1.6. Let G be a topological monoid such that G and ΩBG are admissible H-spaces. Then the natural inclusion $\zeta : G \to \Omega BG$ is a group completion.

Here B denote the standard classifying space functor (described in §3). According to Quillen, the admissibility of ΩBG need not be assumed, but I do not know how to prove the more general result. This hypothesis is clearly satisfied if G is strongly homotopy commutative, since BG is then an H-space [27, p. 251 and 269] and thus ΩBG is homotopy commutative. As I shall show in [19], Quillen's proof can be simplified under the present hypotheses.

§2. The recognition principle for E_∞ spaces

Recall from [G, 1.1 and 3.5] that an E_∞ operad $\mathcal{C}$ is a suitably compatible collection of contractible spaces $\mathcal{C}(j)$ on which the symmetric group Σ_j acts freely. Thus the orbit spaces $\mathcal{C}(j)/\Sigma_j$ are $K(\Sigma_j, 1)$'s. An action θ of $\mathcal{C}$ on a space X is a suitably compatible collection of Σ_j-equivariant maps $\theta_j : \mathcal{C}(j) \times X^j \to X$ [G, 1.2 and 1.4]. $\mathcal{C}[\mathcal{J}]$ denotes the category of $\mathcal{C}$-spaces (X, θ). An E_∞ space is a $\mathcal{C}$-space over some E_∞ operad $\mathcal{C}$. Given an E_∞ space (X, θ) and a prime p, the map $\theta_{p*} : H_*(\mathcal{C}(p) \times_{\Sigma_p} X^p) \to H_*X$ determines homology operations on H_*X. These operations will be studied and calculated on some of the spaces of primary geometric interest in [18]; see [16] for references and a partial summary.

Recall from [G, §2] that an operad $\mathcal{C}$ determines a monad (C, μ, η) such that an action θ of $\mathcal{C}$ on X is equivalent to an action

$\theta : CX \to X$ of C on X. As a space, $CX = \amalg\, \mathcal{C}(j) \times_{\Sigma_j} X^j/(\approx)$, where the equivalence relation uses base-point identifications to glue the $\mathcal{C}(j) \times_{\Sigma_j} X^j$ together (as in the proof of A. 2 below). The natural transformations $\mu : CCX \to CX$ and $\eta : X \to CX$ are given by the compatibility conditions in the definition of an operad.

A monad C can act from the right on a functor F as well as from the left on an object X [G, 9. 4]. Given such a triple (F, C, X) in the category $\mathcal{J}$ (of nice [G, p. 1] based spaces), we can construct a space $B(F, C, X)$ by forming the geometric realization ([G, 11. 1], or see the proof of A. 4 below) of the simplicial space $B_*(F, C, X)$ defined in [G, 9. 6]. The space $B_q(F, C, X)$ of q-simplices is FC^qX (where C^q is the q-fold composite); the faces are given by $FC \to F$, by $CC \to C$ applied in successive positions, and by $CX \to X$; the degeneracies are given by $1 \to C$ applied in successive positions. In an obvious sense, $B(F, C, X)$ is a functor of all three variables [G, 9. 6]. $B(F, C, X) = \amalg\, FC^qX \times \Delta_q/(\approx)$, where $\approx$ gives the appropriate face and degeneracy identifications. Thus (see [G, 9. 2 and 11. 8]) any map $\rho : Y \to FX$ determines a map $\tau(\rho) = |\tau_*(\rho)| : Y \to B(F, C, X)$ and any map $\lambda : FX \to Y$ such that $\lambda\partial_0 = \lambda\partial_1 : FCX \to Y$ determines a map $\varepsilon(\lambda) = |\varepsilon_*(\lambda)| : B(F, C, X) \to Y$. This two-sided bar construction provides all of the spaces and maps required in our theory.

We shall need to know that E_∞ spaces have group completions in order to prove that they have group completions which are infinite loop spaces.

Lemma 2. 1. Let $\mathcal{C}$ be an E_∞ operad. Then there is a functor $G : \mathcal{C}[\mathcal{J}] \to \mathcal{J}$ and a natural transformation $g : 1 \to G$ such that GX is an admissible H-space and $g : X \to GX$ is a group completion for all $\mathcal{C}$-spaces (X, θ).

Proof. As explained in [G, 3. 11], the E_∞ space (X, θ) determines an A_∞ space $(X, \theta\pi_1)$, $\pi_1 : \mathcal{C} \times \mathcal{M} \to \mathcal{C}$. Let $C \times M$ denote the monad associated to $\mathcal{C} \times \mathcal{M}$. As explained in [G, 13. 5], the A_∞ space $(X, \theta\pi_1)$ determines a topological monoid $B(M, C \times M, X)$ and maps of $\mathcal{C} \times \mathcal{M}$-spaces (in particular of H-spaces)

$$X \xleftarrow{\varepsilon(\theta\pi_1)} B(C \times M,\ C \times M,\ X) \xrightarrow{B(\pi_2,\ 1,\ 1)} B(M,\ C \times M,\ X)$$

such that $\varepsilon(\theta\pi_1)$ is a strong deformation retraction with right inverse $\tau(\eta)$. Since $\pi_2 : \mathcal{C} \times \mathfrak{M} \to \mathfrak{M}$ is a local Σ-equivalence (by [G, 3.7 or 3.11]), $B(\pi_2, 1, 1)$ is a homotopy equivalence by A.2 (ii) and A.4 (ii) of the appendix (which show that the connectivity hypothesis of [G, 13.5 (ii)] is unnecessary). Define

$$GX = \Omega BB(M,\ C \times M,\ X)$$

and define $g : X \to GX$ to be the composite

$$X \xrightarrow{\tau(\eta)} B(C \times M,\ C \times M,\ X) \xrightarrow{B(\pi_2,\ 1,\ 1)} B(M,\ C \times M,\ X) \xrightarrow{\zeta} GX.$$

Since X is an E_∞ space, it is certainly strongly homotopy commutative. The result follows immediately from Theorem 1.6.

Next, recall the definition, [G, 4.1], of the little cubes operads $\mathcal{C}_n$. The points of $\mathcal{C}_n(j)$ are to be thought of as j-tuples of disjoint patches on the n-sphere, and $\mathcal{C}_n(j)$ is Σ_j-equivariantly homotopy equivalent to the configuration space $F(R^n; j)$ of j-tuples of distinct points of R^n via the map which sends patches to their center points [G, 4.8]. There is an obvious natural action θ_n of $\mathcal{C}_n$ on n-fold loop spaces [G, 5.1]. (These results are due to Boardman and Vogt [7].) By use of Cohen's calculations of $H_*(F(R^n; j)/\Sigma_p)$ [10], the maps

$$(\theta_{np})_* : H_*(\mathcal{C}_n(p) \times_{\Sigma_p} (\Omega^n X)^p) \to H_*\Omega^n X$$

can be used to obtain a complete theory of homology operations on n-fold loop spaces; we recall that, for finite n and odd p, the operations of Dyer and Lashof [11] were insufficient to allow computation of $H_*\Omega^n\Sigma^n X$ and Milgram's calculation [20] of $H_*\Omega^n\Sigma^n X$ (for connected X) did not yield convenient operations.

We should point out that we are using Σ to denote the reduced suspension, whereas S was used in [G]; the purpose of this change is to emphasize by the notation that Σ^n is a functor and not a sphere. Define $\alpha_n : C_n X \to \Omega^n\Sigma^n X$ to be the composite

$$C_nX \xrightarrow{C_n\eta} C_n\Omega^n\Sigma^nX \xrightarrow{\theta_n} \Omega^n\Sigma^nX,$$

where $\eta : X \to \Omega^n\Sigma^nX$ is the standard inclusion. Our approximation theorem, proven in [G, §6 and §7], asserts that α_n is a weak homotopy equivalence if (and, by [G, 8.14], only if) X is connected. By the Whitehead theorem for connected H-spaces, this assertion in the case $n = \infty$ is also a consequence of the following result.

Theorem 2.2. $\alpha_\infty : C_\infty X \to QX = \varinjlim \Omega^n\Sigma^nX$ <u>is a group completion for every space</u> $X \in \mathcal{J}$.

Here α_∞ is a map of $\mathcal{C}_\infty$-spaces [G, 5.2] and in particular of admissible H spaces [G, p. 4]. The homologies of $C_\infty X$ and QX with Z_p coefficients are computed in [18, 4.1 and 4.2], and it follows immediately from these calculations that the map $\overline{\alpha}_{\infty *}$ of Definition 1.3 is an isomorphism; the truth of this assertion for rational coefficients can be verified by parallel (but much simpler) computations or by appeal to the form of the E^∞-terms of the Bockstein spectral sequences of $C_\infty X$ and QX [18, 4.13]. (For connected X, H_*QX was computed by Dyer and Lashof [11].)

I am reasonably certain that $\alpha_n : C_nX \to \Omega^n\Sigma^nX$ is a group completion for all X and all n, $1 < n < \infty$, but a rigorous calculation of H_*C_nX is not yet available.

By [G, 5.2], $\alpha_n : C_n \to \Omega^n\Sigma^n$ is a morphism of monads. By [G, 9.5], it follows that the adjoint $\lambda_n : \Sigma^nC_n \to \Sigma^n$ is a C_n-functor. Therefore, if X is a $\mathcal{C}_n$-space, then $B(\Sigma^n, C_n, X)$ is defined. This space should be thought of as an n-fold de-looping of X. In particular, by [G, 13.1], $B(\Sigma^n, C_n, \Omega^nY)$ is weakly homotopy equivalent to Y if Y is n-connected and $B(\Sigma^n, C_n, C_nY)$ is homotopy equivalent to Σ^nY for any Y.

Now suppose given an arbitrary E_∞ operad $\mathcal{C}$. We wish to use the α_n to study $\mathcal{C}$-spaces. To this end, let $\mathcal{D}_n$ denote the product operad $\mathcal{C} \times \mathcal{C}_n$ [G, 3.8] and let $\psi_n : \mathcal{D}_n \to \mathcal{C}$ and $\pi_n : \mathcal{D}_n \to \mathcal{C}_n$ denote the projections. $\mathcal{D}_\infty$ is again an E_∞ operad and, if (X, θ) is a $\mathcal{C}$-space, then $(X, \theta\psi_\infty)$ is a $\mathcal{D}_\infty$-space. Thus we lose no information

by studying $\mathcal{D}_\infty$-spaces instead of $\mathcal{C}$-spaces. If (X, ξ) is a $\mathcal{D}_\infty$-space, then the monad D_n acts on X via the restriction ξ_n of ξ to $D_nX \subset D_\infty X$. D_n acts on Σ^n via the composite $\lambda_n \circ \Sigma^n\pi_n$. Thus $B(\Sigma^n, D_n, X)$ is defined. We have maps of $\mathcal{D}_n$-spaces

$$X \xleftarrow{\varepsilon(\xi_n)} B(D_n, D_n, X) \xrightarrow{B(\alpha_n\pi_n, 1, 1)} B(\Omega^n\Sigma^n, D_n, X) \xrightarrow{\gamma^n} \Omega^n B(\Sigma^n, D_n, X),$$

where γ^n is defined by iteration of the obvious natural comparison $\gamma : |\Omega_* Y| \to \Omega|Y|$ for simplicial spaces Y [G, p. 115]. We are interested in the limit case, and we can define (see [G, p. 143] for the details)

$$B_iX = \lim_{\to} \Omega^j B(\Sigma^{i+j}, D_{i+j}, X).$$

Visibly $B_iX = \Omega B_{i+1}X$, and we thus obtain a functor B_∞, written $B_\infty X = \{B_iX\}$, from $\mathcal{D}_\infty[\mathcal{J}]$ to the category $\mathcal{L}_\infty$ of infinite loop sequences. Let $W : \mathcal{L}_\infty \to \mathcal{D}_\infty[\mathcal{J}]$ denote the functor given on objects $Y = \{Y_i | i \geq 0\} \in \mathcal{L}_\infty$ by $WY = (Y_0, \theta_\infty\pi_\infty)$, where θ_∞ is the action of $\mathcal{C}_\infty$ given in [G, 5.1]; W is to be thought of as an 'underlying E_∞ space' functor. The following recognition theorem compares the categories $\mathcal{D}_\infty[\mathcal{J}]$ and $\mathcal{L}_\infty$ by comparing WB_∞ and $B_\infty W$ to the respective identity functors. In particular, the categories of grouplike $\mathcal{D}_\infty$-spaces and of connective (Y_i is (i-1)-connected) infinite loop sequences are essentially equivalent.

Theorem 2.3. *Let (X, ξ) be a $\mathcal{D}_\infty$-space, where $\mathcal{D}_\infty = \mathcal{C} \times \mathcal{C}_\infty$ for some E_∞ operad $\mathcal{C}$. Let $\pi_\infty : \mathcal{D}_\infty \to \mathcal{C}_\infty$ be the projection. Consider the following maps of $\mathcal{D}_\infty$-spaces:*

$$X \xleftarrow{\varepsilon(\xi)} B(D_\infty, D_\infty, X) \xrightarrow{B(\alpha_\infty\pi_\infty, 1, 1)} B(Q, D_\infty, X) \xrightarrow{\gamma^\infty} WB_\infty X = B_0X .$$

(i) *$\varepsilon(\xi)$ is a strong deformation retraction with right inverse $\tau(\eta)$, where $\eta : X \to D_\infty X$ is given by the unit of D_∞.*

(ii) *$B(\alpha_\infty\pi_\infty, 1, 1)$ is a group completion and is therefore a weak homotopy equivalence if X is grouplike.*

(iii) *$\gamma^\infty : B(Q, D_\infty, X) \to B_0X$ is a weak homotopy equivalence.*

(iv) *The map $\iota = \gamma^\infty B(\alpha_\infty\pi_\infty, 1, 1)\tau(\eta) : X \to B_0X$ is a group completion.*

(v) B_iX *is* $(m+i)$*-connected if* X *is* m*-connected.*

(vi) *Let* $Y = \{Y_i\} \in \mathcal{L}_\infty$; *there is a natural map* $\omega : B_\infty WY \to Y$ *in* $\mathcal{L}_\infty$ *such that* $\omega_0 \iota = 1$ *and the following diagram commutes:*

$$\begin{tikzcd}
B(D_\infty, D_\infty, Y_0) \arrow[r, "B(\alpha_\infty \pi_\infty, 1, 1)"] \arrow[d, "\varepsilon(\theta_\infty \pi_\infty)"'] & B(Q, D_\infty, Y_0) \arrow[d, "\gamma^\infty"] \\
Y_0 & B_0 Y_0 \arrow[l, "\omega_0"']
\end{tikzcd}$$

; $\omega_i : B_i Y_0 \to Y_i$ *is a weak homotopy equivalence if* Y *is connective.*

(vii) *Let* $Z \in \mathcal{J}$; *then* $(D_\infty Z, \mu)$ *is a* $\mathfrak{D}_\infty$*-space, where* $\mu : D_\infty D_\infty Z \to D_\infty Z$ *is given by the product of* D_∞, *and the composite*

$$B_\infty D_\infty Z \xrightarrow{B_\infty \alpha_\infty \pi_\infty} B_\infty QZ \xrightarrow{\omega} Q_\infty Z = \{Q\Sigma^i Z\}$$

is a strong deformation retraction of infinite loop sequences with right inverse the adjoint $\phi_\infty(\iota\eta) : Q_\infty Z \to B_\infty D_\infty Z$ *of the inclusion* $\iota\eta : Z \to B_0 D_\infty Z$.

Proof. $\varepsilon(\xi)$ and $B(\alpha_\infty \pi_\infty, 1, 1)$ are realizations of maps of simplicial $\mathfrak{D}_\infty$-spaces and are therefore maps of $\mathfrak{D}_\infty$-spaces by [G, 12.2]. Part (i) holds before realization by [G, 9.8], hence after realization by [G, 11.10]. To prove (ii), consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[d, "g"'] & B(D_\infty, D_\infty, X) \arrow[l, "\varepsilon(\xi)"'] \arrow[r, "B(\alpha_\infty \pi_\infty, 1, 1)"] \arrow[d, "B(gD_\infty, 1, 1)"] & B(Q, D_\infty, X) \arrow[d, "B(gQ, 1, 1)"] \\
GX & B(GD_\infty, D_\infty, X) \arrow[l, "|G_* \varepsilon_*(\xi)|"] \arrow[r, "B(G\alpha_\infty \pi_\infty, 1, 1)"] & B(GQ, D_\infty, X)
\end{tikzcd}$$

Here G_* denotes the simplicial functor obtained by applying G in each degree. $B_*(GD_\infty, D_\infty, X) = G_* B_*(D_\infty, D_\infty, X)$ by [G, 9.7], hence $|G_* \varepsilon_*(\xi)|$ makes sense; this map is a homotopy equivalence since we can apply G_* to the simplicial homotopy of [G, 9.8] and then apply [G, 11.10]. By the left-hand square, $B(gD_\infty, 1, 1)$ is a group completion

since g is. By 2.2 and the fact that, by A.2 (i), $\pi_\infty : D_\infty Z \to C_\infty Z$ induces an isomorphism on homology, $\alpha_\infty \pi_\infty : D_\infty Z \to QZ$ is a group completion for any space Z. Therefore, by the very definition of a group completion,

$$G\alpha_\infty\pi_\infty : GD_\infty Z \to GQZ \quad \text{and} \quad gQ : QZ \to GQZ$$

induce isomorphisms on homology for any Z. Thus

$$B(G\alpha_\infty\pi_\infty, 1, 1) \quad \text{and} \quad B(gQ, 1, 1)$$

induce isomorphisms on homology by A.4 (i). Now (ii) follows from the right-hand square. The map γ^∞ of (iii) is obtained by passage to limits from the γ^n (see [G, p. 143]); it is a map of $\mathcal{D}_\infty$-spaces by [G, 12.4] and a weak homotopy equivalence by [G, 12.3]. Now (iv) follows from (i), (ii), and (iii), and (v) follows from [G, 11.12] (see also A.5). The ω_i in (vi) are defined by passage to loops and limits from the maps $\varepsilon\phi^n(1) : B(\Sigma^n, D_n, \Omega^n Y_n) \to Y_n$, where $\phi^n(1)$ is the evaluation; the diagram follows formally [G, p. 146 and p. 130]. Finally, (vii) follows from [G, 9.9 and 11.10], which give that $\Sigma^n Z$ is a strong deformation retract of $B(\Sigma^n, D_n, D_n Z)$, by passage to loops and limits; see [G, p. 42-43] for the adjunction ϕ_∞.

The theorem implies a uniqueness statement for the infinite loop sequence constructed from an E_∞ space.

Corollary 2.4. Suppose given maps of $\mathcal{D}_\infty$-spaces

$$(X, \xi) \xleftarrow{f} (X', \xi') \xrightarrow{g} (Y_0, \theta_\infty\pi_\infty)$$

such that f is a weak homotopy equivalence, g is a group completion and $Y = \{Y_i\} \in \mathcal{L}_\infty$ is connective. Then the maps

$$B_\infty X \xleftarrow{B_\infty X} B_\infty X' \xrightarrow{B_\infty g} B_\infty Y_0 \xrightarrow{\omega} Y$$

display a weak homotopy equivalence in $\mathcal{L}_\infty$ between $B_\infty X$ and Y.

Observe that there are obvious functors $\Omega^j : \mathcal{L}_\infty \to \mathcal{L}_\infty$ such that $\Omega_0^j Y = \Omega^j Y_0$ and $\Omega_0^{-j} Y = Y_j$ for $j \geq 0$ [G, p. 147]. The following con-

sequence of 2.3 (vi) shows in particular that the i-th de-looping functor B_i is weakly equivalent to the i-fold iterate of B_1.

Corollary 2.5. Let (X, ξ) be a $\mathfrak{D}_\infty$-space. Then the map

$$\omega_i : B_iB_jX = B_iW\Omega^{-j}B_\infty X \to \Omega_i^{-j}B_\infty X = B_{i+j}X$$

is a weak homotopy equivalence for all $i \geq 0$.

In [G, 14.4], the following further information about the homotopy type of the de-looping B_iX is obtained.

Theorem 2.6. Let (X, ξ) be a $\mathfrak{D}_\infty$-space. For $i > 0$, the following maps of $\mathfrak{D}_\infty$-spaces are weak homotopy equivalences:

$$B(D_\infty\Sigma^i, D_\infty, X) \xrightarrow{B(\alpha_\infty\pi_\infty\Sigma^i, 1, 1)} B(Q\Sigma^i, D_\infty, X) \xrightarrow{\gamma^\infty} W\Omega^{-i}B_\infty X = B_iX.$$

As discussed in [G, p. 155], the Segal spectral sequence of $B(D_\infty\Sigma^i, D_\infty, X)$ converges to H_*B_iX and has an E^2-term which, at least in principle, is a computable functor of H_*X.

§3. Loop spaces and classifying spaces

We shall obtain some useful consistency statements here. As an application, we shall rederive the Barratt-Quillen [3, 26] homotopy approximation to QX by relating it to the approximation we have already given in 2.3 (vii). Finally, we shall obtain a homology variant of the recognition theorem which includes Priddy's theorem [22] relating $K(\Sigma_\infty, 1)$ to QS^0.

Recall from [G, 1.5] that if $\mathcal{C}$ is any operad and if (X, θ) is a $\mathcal{C}$-space, then ΩX is again a $\mathcal{C}$-space with action defined pointwise and denoted by $\Omega\theta$. By iteration, we have functors $\Omega^i : \mathcal{C}[\mathcal{J}] \to \mathcal{C}[\mathcal{J}]$ for $i > 0$.

Theorem 3.1. Let $\mathcal{C}$ be an E_∞ operad and let $\mathfrak{D}_\infty = \mathcal{C} \times \mathcal{C}_\infty$ with projections $\psi_\infty : \mathfrak{D}_\infty \to \mathcal{C}$ and $\pi_\infty : \mathfrak{D}_\infty \to \mathcal{C}$. Let (X, θ) be a $\mathcal{C}$-space. Then, for $i > 0$, there is a $\mathfrak{D}_\infty$-space Y_iX and there are maps of $\mathfrak{D}_\infty$-spaces

$$(X,\ \theta\psi_\infty) \xleftarrow{\varepsilon} Y_iX \xrightarrow{\delta} (B_i\Omega^iX,\ \theta_\infty\pi_\infty) = W\Omega^{-i}B_\infty\Omega^iX$$

such that $\Omega^i\varepsilon$ and δ are weak homotopy equivalences. Therefore, if X is (i-1)-connected, then the infinite loop sequences $B_\infty X$ and $\Omega^{-i}B_\infty\Omega^iX$ are weakly homotopy equivalent.

Proof. The second statement will follow from the first by use of 2.3 (vi). The basic point is that when taking limits of loop spaces (via $\Omega^nX \to \Omega^{n+1}\Sigma X$) the new coordinate is the last coordinate, whereas when forming loop spaces the new coordinate is the first coordinate. Since X is a $\mathcal{D}_\infty$-space via $\theta\psi_\infty$ and Ω^iX is a $\mathcal{D}_\infty$-space via $\Omega^i(\theta\psi_\infty) = (\Omega^i\theta)\psi_\infty$ and since the suspensions $\mathcal{D}_n \to \mathcal{D}_{n+1}$ involve only the little cubes coordinates, it is plausible that our categorical constructions can be suspended so as to free the first i coordinates. The detailed constructions occupy much of [G, §14] and Y_iX, ε, and δ are specified in the statement of [G, 14.9]. We need only indicate here how the connectivity hypothesis of [G, 14.9] can be improved. In the proof of [G, 14.7], we must show that the map $B(\delta_{i\infty}\tau'_{i\infty},\ 1,\ 1)$ in the bottom diagram of [G, p. 150] is a weak homotopy equivalence (which will imply that $\Omega^i\varepsilon$ of the present statement is a weak homotopy equivalence). To prove this, consider the following commutative diagram $(\xi_i = \Omega^i\theta\cdot\psi_\infty)$:

$$\begin{tikzcd}
\Omega^iX \arrow[d, "g"'] & B(D_\infty, D_\infty, \Omega^iX) \arrow[l, "\varepsilon(\xi_i)"'] \arrow[r, "B(\delta_{i\infty}\tau'_{i\infty},\ 1,\ 1)"] \arrow[d, "B(gD_\infty,\ 1,\ 1)"] & B(\Omega^iD'_\infty\Sigma^i, D_\infty, \Omega^iX) \arrow[d, "B(g\Omega^iD'_\infty\Sigma^i,\ 1,\ 1)"] \\
G\Omega^iX & B(GD_\infty, D_\infty, \Omega^iX) \arrow[l, "|G_*\varepsilon_*(\xi_i)|"'] \arrow[r, "B(G\delta_{i\infty}\tau'_{i\infty},\ 1,\ 1)"] & B(G\Omega^iD'_\infty\Sigma^i, D_\infty, \Omega^iX)
\end{tikzcd}$$

Here $\delta_{i\infty}\tau'_{i\infty} : D_\infty Z \to \Omega^iD'_\infty\Sigma^iZ$ is a group completion for any Z by [G, 14.2], 2.2, and A.2 (applied to the local Σ-equivalence $\tau'_{i\infty}$). By arguments precisely analogous to those used to prove 2.3 (ii), $B(\delta_{i\infty}\tau'_{i\infty},\ 1,\ 1)$ is a group completion and thus, since Ω^iX is grouplike, a weak homotopy equivalence. Again, connectivity was assumed in [G, 14.8] because of references to [G, 3.4 and 11.13], and we can now refer to A.2 and A.4 instead. With these changes, the result follows as in the proof of [G, 14.9].

Remark 3.3. If $Y \in \mathcal{L}_\infty$ and $i > 0$, then $\Omega^i WY$ and $W\Omega^i Y$ are $\Omega^i Y_0$ together with two different actions $D_\infty \Omega^i Y_0 \to \Omega^i Y_0$. By [G, 14.10] and A.2, these action maps are homotopic.

To relate our de-loopings to the standard classifying space functor, we require some recollections from [G, §10]. If $\mathcal{U}$ is any category with finite products, then the notions of a monoid G in $\mathcal{U}$ and of right and left G-objects Y and X in $\mathcal{U}$ are defined. There is an obvious category $\mathcal{A}\mathcal{U}$ with objects such triples (Y, G, X) and there is a two-sided simplicial bar construction $B_* : \mathcal{A}\mathcal{U} \to \mathcal{S}\mathcal{U}$. When $\mathcal{U}$ is the category of (nice, unbased) spaces, we can compose B_* with geometric realization to obtain a functor $B : \mathcal{A}\mathcal{U} \to \mathcal{U}$. Indeed, this construction is just another application of that used in the previous section. We shall use $B(Y, G, X)$ to study the classification of various types of fibrations in [19]. Let $\delta : G \to *$ be the unique map onto the one-point G-space $*$ and define

$$p = B(1, 1, \delta) : EG = B(*, G, G) \to B(*, G, *) = BG.$$

BG is the standard classifying space of the monoid G, EG is a contractible right G-space, and p is a principal quasi G-fibration if G is grouplike (or G-bundle if G is a group).

Now, until otherwise specified, let $\mathcal{C}$ be any operad. By [G, 1.6 and 1.7], $\mathcal{C}[\mathcal{T}]$ has finite products. By [G, 12.2], the geometric realization of a simplicial $\mathcal{C}$-space is a $\mathcal{C}$-space. Thus $B? = |B_*?|$ defines a functor $\mathcal{A}\mathcal{C}[\mathcal{T}] \to \mathcal{C}[\mathcal{T}]$. An object (Y, G, X) of $\mathcal{A}\mathcal{C}[\mathcal{T}]$ consists of a topological monoid G and right and left G-spaces Y and X such that Y, G, and X are $\mathcal{C}$-spaces and the product and unit of G and the actions of G on Y and on X are maps of $\mathcal{C}$-spaces. For clarity, write (G, θ, ϕ) for a monoid in $\mathcal{C}[\mathcal{T}]$, where θ is the action of $\mathcal{C}$ and ϕ is the monoid product. The unit condition ensures that the base-point (for θ) coincides with the identity element e (for ϕ). The product ϕ need not be (and in practice is not) $\theta_2(c)$ for any element $c \in \mathcal{C}(2)$, but we have the following observation.

Lemma 3.4. *Let (G, θ, ϕ) be a monoid in $\mathcal{C}[\mathcal{T}]$. If $\mathcal{C}(1)$ is connected, then ϕ is homotopic to $\theta_2(c)$ for any $c \in \mathcal{C}(2)$.*

Proof. Write $\phi(g, g') = gg'$ and $\theta_2(c)(g, g') = g \wedge g'$. Since ϕ is a map of $\mathcal{C}$-spaces, $(g_1g_2) \wedge (g_1'g_2') = (g_1 \wedge g_1')(g_2 \wedge g_2')$ and therefore $g_1 \wedge g_2' = (g_1 \wedge e)(e \wedge g_2')$. Since e is a two-sided homotopy identity for $\wedge$, by [G, p. 4], the conclusion follows.

The following result asserts the existence and essential uniqueness of classifying spaces in $\mathcal{C}[\mathcal{T}]$ for monoids in $\mathcal{C}[\mathcal{T}]$.

Proposition 3.5. Let (G, θ, ϕ) be a monoid in $\mathcal{C}[\mathcal{T}]$. Then BG and EG admit actions $B\theta$ and $E\theta$ of $\mathcal{C}$ such that EG is a right G-space in $\mathcal{C}[\mathcal{T}]$ and $p : EG \to BG$ is a map of $\mathcal{C}$-spaces. If G is grouplike and if $p' : E' \to B'$ is a map of $\mathcal{C}$-spaces and a principal quasi G-fibration such that E' is contractible and is a right G-space in $\mathcal{C}[\mathcal{T}]$, then B' is weakly homotopy equivalent as a $\mathcal{C}$-space to BG.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc}
E' & \xleftarrow{\varepsilon(\lambda)} & B(E', G, G) & \xrightarrow{q} & B(*, G, G) = EG \\
\downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
B' & \xleftarrow{\varepsilon(p')} & B(E', G, *) & \xrightarrow{q} & B(*, G, *) = BG
\end{array}$$

Here $\lambda : E' \times G \to E'$ is the given right action, $\varepsilon(?) = |\varepsilon_*(?)|$ with ε_* as in [G, 9.2], $p = B(1, 1, \delta)$, and $q = B(\delta, 1, 1)$. All maps are realizations of maps of simplicial $\mathcal{C}$-spaces and $\varepsilon(\lambda)$ and q are realizations of maps of simplicial right G-spaces in $\mathcal{C}[\mathcal{T}]$. Since p' and (see [19]) both maps p are principal quasi G-fibrations with contractible total spaces, $\varepsilon(p')$ and the bottom map q are weak homotopy equivalences.

The comparison of G to ΩBG can also be carried out in $\mathcal{C}[\mathcal{T}]$.

Proposition 3.6. Let (G, θ, ϕ) be a monoid in $\mathcal{C}[\mathcal{T}]$. Then the natural inclusion $\zeta : G \to \Omega BG$ is a map of $\mathcal{C}$-spaces.

Proof. $\zeta(g)(t) = |[g], (t, 1-t)|$ for $g \in G$ and $t \in I$, where $[g]$ is g regarded as a 1-simplex of B_*G and $(t, 1-t) \in \Delta_1$. $B\theta$ is the

composite

$$CBG = C|B_*G| \xrightarrow{\nu^{-1}} |C_*B_*G| \xrightarrow{|B_*\theta|} |B_*G| = BG,$$

where $B_*\theta$ is the simplicial action specified on q-simplices by $B_q\theta = \theta^q$ and where ν is the homeomorphism of [G, 12.2]. Therefore, if $c \in \mathcal{C}(j)$, $g_i \in g$, and $t \in I$, then

$$\begin{aligned}(\Omega B\theta)_j(c, \zeta(g_1), \ldots, \zeta(g_j))(t) &= (B\theta)_j(c, \zeta(g_1)(t), \ldots, \zeta(g_j)(t)) \\ &= (B\theta)_j(c, |[g_1], (t, 1\text{-}t)|, \ldots, |[g_j], (t, 1\text{-}t)|) \\ &= |B_*\theta|\,|[c; [g_1], \ldots, [g_j]], (t, 1\text{-}t)| \\ &= |[\theta_j(c, g_1, \ldots, g_j)], (t, 1\text{-}t)| \\ &= \zeta\theta_j(c, g_1, \ldots, g_j)(t)\end{aligned}$$

(Here $[c; [g_1], \ldots, [g_j]]$ is an element of $CB_1G = CG$.)

Returning to the context of E_∞ spaces, we can now compare our de-looping B_1 to the classifying space functor B.

Theorem 3.7. Let $\mathcal{C}$ be an E_∞ operad and let $\mathcal{D}_\infty = \mathcal{C} \times \mathcal{C}_\infty$ with projection $\psi_\infty : \mathcal{D}_\infty \to \mathcal{C}$. Let (G, θ, ϕ) be a monoid in $\mathcal{C}[\mathcal{T}]$. Then $(\Omega BG, \Omega B\theta \circ \psi_\infty)$ is weakly homotopy equivalent as a $\mathcal{D}_\infty$-space to $WB_\infty G$ and $(BG, B\theta \circ \psi_\infty)$ is weakly homotopy equivalent as a $\mathcal{D}_\infty$-space to $W\Omega^{-1}B_\infty G$ (which has underlying space B_1G). Therefore the infinite loop sequences $B_\infty\Omega BG$, $B_\infty G$, and $\Omega B_\infty BG$ are all weakly homotopy equivalent.

Proof. Since $\zeta : G \to \Omega BG$ is a map of $\mathcal{D}_\infty$-spaces and a group completion, $B_\infty\zeta : B_\infty G \to B_\infty\Omega BG$ is defined and is a weak homotopy equivalence of infinite loop sequences. This obviously implies that $B_i\zeta : W\Omega^{-i}B_\infty G \to W\Omega^{-i}B_\infty\Omega BG$ is a weak homotopy equivalence of $\mathcal{D}_\infty$-spaces for $i \geq 0$. When $i = 0$, $WB_\infty\Omega BG$ is weakly homotopy equivalent as a $\mathcal{D}_\infty$-space to ΩBG by 2.3. When $i = 1$, $W\Omega^{-1}B_\infty\Omega BG$ is weakly homotopy equivalent as a $\mathcal{D}_\infty$-space to BG by 3.2. The last statement follows by use of 2.3 (vi).

Barratt [3] and Quillen [unpublished; see Segal [26]] have given

approximations to QX. The key fact behind any such result is 2. 2: any natural group completion of CX, where $\mathcal{C}$ is an E_∞ operad, should approximate QX. In this sense, 2. 3 (vii) is another such approximation theorem and so is [G, 6. 4], which asserts that $\Omega C \Sigma X$ approximates QX. Barratt and Quillen focus attention (implicitly, see [G, 6. 5]) on one particular E_∞ operad, namely $\mathcal{D} = \mathcal{D}\mathfrak{M}$, as defined in [G, p. 161]. This is a 'minimal' E_∞ operad in that $\mathcal{D}(1)$ is a point; $\mathcal{D}(j)$ is the normalized version of Milnor's universal bundle for Σ_j (see 4. 7 and 4. 8). Let (D, μ, η) be the monad associated to $\mathcal{D}$. As shown in [G, p. 161], DX is a topological monoid; denote its product by $\oplus$.

Theorem 3. 7. <u>For all</u> $X \in \mathcal{T}$, $(DX, \mu, \oplus)$ <u>is a monoid in</u> $\mathcal{D}[\mathcal{T}]$ <u>and</u> ΩBDX <u>is weakly homotopy equivalent as a</u> $\mathcal{D}_\infty$<u>-space to</u> QX (<u>that is</u>, <u>to</u> $WQ_\infty X$), <u>where</u> $\mathcal{D}_\infty = \mathcal{D} \times \mathcal{C}_\infty$.

Proof. Let $c \in \mathcal{D}(j)$ and let $[d_i; y_i], [e_i, z_i] \in DX$, $1 \le i \le j$. By [G; 1. 7, 2. 4 (iii), and p. 161], in order to show that $\oplus$ is a map of $\mathcal{D}$-spaces we must verify the formula

$$\begin{aligned}[\gamma(c; d_1, \ldots, d_j) \oplus \gamma(c; e_1, \ldots, e_j); y_1, \ldots, y_j, z_1, \ldots, z_j] \\ = [\gamma(c; d_1 \oplus e_1, \ldots, d_j \oplus e_j); y_1, z_1, \ldots, y_j, z_j].\end{aligned}$$

By [G, 15. 1], γ is obtained by applying the product-preserving functor $|D_* ?|$ (see [G, 10. 2]) to γ for the operad $\mathfrak{M}$ of [G, 3. 1]. On the level of symmetric groups (that is, in $\mathfrak{M}$), if $\tau \oplus \tau'$ denotes the permutation of successive blocks of letters determined by τ and τ', then

$$\gamma(\sigma; \tau_1, \ldots, \tau_j) = \tau_{\sigma^{-1}(1)} \oplus \ldots \oplus \tau_{\sigma^{-1}(j)},$$

hence

$$\gamma(\sigma; \tau_1, \ldots, \tau_j) \oplus \gamma(\sigma; \mu_1, \ldots, \mu_j) = \gamma(\sigma; \tau_1 \oplus \mu_1, \ldots, \tau_j \oplus \mu_j)\nu$$

for a certain permutation ν (which shuffles blocks of letters). It follows (by a diagram chase) that the same relation holds in $\mathcal{D}$, and the desired equality results in view of the equivariance identifications used to define DX. By the previous theorem, ΩBDX is weakly homotopy

equivalent as a $\mathcal{D}_\infty$-space to $WB_\infty DX$. The projection $\psi_\infty : D_\infty X \to DX$ is a map of $\mathcal{D}_\infty$-spaces and induces an isomorphism on homology by A. 2 (i). By 2. 3, $B_0\psi_\infty : WB_\infty D_\infty X \to WB_\infty DX$ induces an isomorphism on homology and is therefore a weak homotopy equivalence of $\mathcal{D}_\infty$-spaces. Now the conclusion follows from 2. 3 (vii).

The most striking instance of the theorem is the case $X = S^0$, when, by [G, 8. 11], $DS^0 = \amalg\, \mathcal{D}(j)/\Sigma_j$ and the assertion is that QS^0 is homotopy equivalent to the group completion $\Omega B \amalg K(\Sigma_j, 1)$. We shall obtain a related homology approximation to QX in [18, §5]; when $X = S^0$, this result will reduce to Priddy's theorem [22] which states that $K(\Sigma_\infty, 1) \times Z$ is homologically equivalent to QS^0. We here give another instance of the same type of homology approximation which also contains Priddy's theorem.

Construction 3. 8. Let G be the free monoid on one generator g and let $\mathcal{C}$ be any operad. Consider the category an object of which is a $\mathcal{C}$-space (X, θ) together with an inclusion of monoids $G \subset \pi_0 X$ and a chosen base-point $a \in g$; morphisms are to preserve all of these data. Construct a functor from this category to spaces as follows. Fix an element $c \in \mathcal{C}(2)$. Define $\rho(a) : X \to X$ by $\rho(a)(x) = \theta_2(x)(x, a)$; thus, $\rho(a)$ is right translation by a. Assume (for simplicity) or arrange (by use of mapping cylinders) that $\rho(a) : X_n \to X_{n+1}$ is a cofibration, where X_n denotes the component g^n, $n \geq 0$. Then define $\overline{X}$ to be the limit of the X_n under the maps $\rho(a)$; clearly morphisms f in our domain category determine maps $\overline{f}$ by passage to limits.

Proposition 3. 9. Let $\mathcal{D}_\infty = \mathcal{C} \times \mathcal{C}_\infty$ where $\mathcal{C}$ is an E_∞ operad such that $\mathcal{C}(j)$, $j \geq 1$, has the Σ_j-equivariant homotopy type of a CW free Σ_j-complex. Let (X, ξ) be a $\mathcal{D}_\infty$-space such that X has the homotopy type of a CW-complex and $\pi_0 X$ is a free monoid on one generator g. Let $(B_0X)_0$ denote the component of the base-point of B_0X. Then there is a map $\overline{\iota} : \overline{X} \to (B_0X)_0$ such that $\overline{\iota}_* : H_*(\overline{X}; k) \to H_*((B_0X)_0; k)$ is an isomorphism of algebras for all commutative rings k.

Proof. By 3.8 applied to the operad $\mathcal{D}_\infty$, we have spaces and maps

$$\overline{X} \xleftarrow{\overline{\varepsilon}(\xi)} \overline{B}(D_\infty, D_\infty, X) \xrightarrow{\overline{B}(\alpha_\infty \pi_\infty, 1, 1)} \overline{B}(Q, D_\infty, X) \xrightarrow{\overline{\gamma}^\infty} \overline{B}_0 X ;$$

the relevant inclusions of $G = \pi_0 X$ are evident. By A.3, A.6, and Milnor's theorem [21], all spaces in sight have the homotopy type of CW-complexes. $\overline{\varepsilon}(\xi)$ and the natural inclusion of $(B_0X)_0$ in $\overline{B}_0X$ are homotopy equivalences (since $\varepsilon(\xi)$ and each $\rho(a) : (B_0X)_n \to (B_0X)_{n+1}$ are homotopy equivalences); choose homotopy inverses $\overline{\tau}(\eta)$ and λ and define

$$\overline{\iota} = \lambda \circ \overline{\gamma}^\infty \circ \overline{B}(\alpha_\infty \pi_\infty, 1, 1) \circ \overline{\tau}(\eta) .$$

$H_*(\overline{X}; k)$ is a well-defined algebra since $\pi_0 X$ is central in the associative algebra $H_*(X; k)$, and $\overline{\iota}_*(x) = x \circ g^{-n}$ for $x \in H_*(X_n; k)$ since the restriction of λ to $(B_0X)_n$ is homotopic to right translation by any point in the component $g^{-n} \in \pi_0 B_0 X$. For any $\mathcal{D}_\infty$-space X, 1.2, 1.5, and 2.3 imply that

$$H_*(B_0X;k) = k\pi_0B_0X \otimes H_*((B_0X)_0;k) \quad \text{and} \quad H_*((B_0X)_0;k) \cong \varinjlim H_*(X_a; k)$$

where $k\pi_0B_0X$ is the group ring and the limit is taken over the translation functor from $\widetilde{\pi_0 X}$ to k-modules which sends the object $a \in \widetilde{\pi_0 X}$ to $H_*(X_a; k)$ and the morphism $b : a \to a'$ to multiplication by b. In our case, the isomorphism is realized as $\lambda_*^{-1} : H_*((B_0X)_0; k) \to H_*(\overline{B}_0X;k)$, and the desired conclusion follows from the definition of $\overline{X}$.

Priddy's theorem is obtained by taking $X = D_\infty S^0$ (or $X = CS^0$), since then $\overline{X}$ is a $K(\Sigma_\infty, 1)$ and B_0X is homotopy equivalent to QS^0.

§4. Symmetric monoidal and permutative categories

Our goal here is to demonstrate that our theory associates infinite loop spaces with good properties to categories of the specified types by observing that the classifying space of a permutative category is naturally an E_∞ space. This observation (and it is no more than that: the proof is a triviality) has been known to Stasheff and myself for some time; it only acquires usefulness with the present extension of my theory to non-

connected spaces. We shall also indicate how to use our infinite loop spaces to define algebraic K-theory and shall compute our K^0R, $K^{-1}R$, and $K^{-2}R$ for a topological ring R.

Recall that a topological category $\mathcal{A}$ is a small category in which the set $\mathcal{O}\mathcal{A}$ of objects of $\mathcal{A}$ and the set $\mathcal{M}\mathcal{A}$ of morphisms of $\mathcal{A}$ are (compactly generated Hausdorff) spaces and the four structural functions

$$\text{Source } S\colon\ \mathcal{M}\mathcal{A} \to \mathcal{O}\mathcal{A}, \qquad \text{Target } T\colon\ \mathcal{M}\mathcal{A} \to \mathcal{O}\mathcal{A}$$
$$\text{Identity } I\colon\ \mathcal{O}\mathcal{A} \to \mathcal{M}\mathcal{A}, \qquad \text{Composition } C\colon \mathcal{M}\mathcal{A} \times_{\mathcal{O}\mathcal{A}} \mathcal{M}\mathcal{A} \to \mathcal{M}\mathcal{A}$$

are continuous, where $\mathcal{M}\mathcal{A} \times_{\mathcal{O}\mathcal{A}} \mathcal{M}\mathcal{A} = \{(g, f) \mid f, g \in \mathcal{M}\mathcal{A},\ Sg = Tf\}$. Henceforward, in all definitions, theorems, etc., all given categories are tacitly assumed to be topological and all given functors and natural transformations are tacitly assumed to be continuous; all constructed gadgets must be proven to be consistent with the topology. Of course, if no topology is in sight, we can always impose the discrete topology.

Monoidal, strict monoidal, and symmetric monoidal categories are defined in [14, VII §1 and §7]. Provided that the collection of isomorphism classes of objects forms a set, we can replace a given large, hence non-topological, (symmetric) monoidal category by an equivalent small (symmetric) monoidal category simply by choosing any skeleton [14, p. 91].

Definition 4.1. A permutative category $(\mathcal{A}, \square, *, c)$ is a symmetric strict monoidal category. In detail, $\square : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is an associative bifunctor, $* \in \mathcal{O}\mathcal{A}$ is a two-sided identity for $\square$, and $c : \square \to \square\tau$ is a natural transformation (where $\tau : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ is the transposition) such that $c^2 = 1$, $c(A \square *) = IA$ for $A \in \mathcal{O}\mathcal{A}$, and the following diagram is commutative for $A, B, C \in \mathcal{O}\mathcal{A}$:

$$\begin{tikzcd}
A \square B \square C \arrow[rr, "c"] \arrow[dr, "1 \square c"'] & & C \square A \square B \\
& A \square C \square B \arrow[ur, "c \square 1"'] &
\end{tikzcd}$$

By a trivial special case of MacLane's coherence theorem [13, 5.1], all diagrams built up from $\square$, $*$, and c are commutative.

Following MacLane [14], we are using the neutral symbol $\square$ rather than $\otimes$ or $\oplus$, which are distractingly suggestive of special cases. Symmetric monoidal categories are ubiquitous but permutative categories are seldom found in nature. This is no limitation in view of the following result, which is due to Isbell [12].

Proposition 4.2. Let $\mathcal{A}$ be a monoidal category. Then there is a naturally equivalent strict monoidal category $\mathcal{B}$; if $\mathcal{A}$ is symmetric, then $\mathcal{B}$ is permutative.

Proof. Define $\mathcal{O}\mathcal{B} = M\mathcal{O}\mathcal{A}$ as a topological monoid where $M\mathcal{O}\mathcal{A}$, the James construction on $\mathcal{O}\mathcal{A}$, is the free topological monoid generated by $\mathcal{O}\mathcal{A}$ subject to the single relation $* = e$ [G, 3.2]. Let $\eta : \mathcal{O}\mathcal{A} \to \mathcal{O}\mathcal{B}$ be the standard inclusion. Write objects of $\mathcal{B}$ by juxtaposition of objects of $\mathcal{A}$ and define $\pi : \mathcal{O}\mathcal{B} \to \mathcal{O}\mathcal{A}$ by

$$\pi(A_1 \dots A_n) = A_1 \square (A_2 \square (A_3 \square \dots (A_{n-1} \square A_n) \dots)), \quad A_i \in \mathcal{O}\mathcal{A}.$$

Define $\mathcal{M}\mathcal{B}$ by $\mathcal{B}(B, B') = \{B\} \times \mathcal{A}(\pi B, \pi B') \times \{B'\}$. The singleton sets are required for disjointness of hom sets and they determine S and T for $\mathcal{B}$; I and C for $\mathcal{B}$ are induced from I and C for $\mathcal{A}$. $\mathcal{M}\mathcal{B}$ is topologized as a subspace of $\mathcal{O}\mathcal{B} \times \mathcal{M}\mathcal{A} \times \mathcal{O}\mathcal{B}$, and the four functions are clearly continuous. $\square : \mathcal{M}\mathcal{B} \times \mathcal{M}\mathcal{B} \to \mathcal{M}\mathcal{B}$ and, if $\mathcal{A}$ is symmetric, the symmetry c of $\mathcal{B}$ are determined by the following morphisms of $\mathcal{A}$:

$$\pi(B \square C) \xrightarrow{\cong} \pi B \square \pi C \xrightarrow{f \square g} \pi B' \square \pi C' \xrightarrow{\cong} \pi(B' \square C')$$

for morphisms (B, f, B') and (C, g, C') of $\mathcal{B}$ and

$$\pi(B \square C) \xrightarrow{\cong} \pi B \square \pi C \xrightarrow{c} \pi C \square \pi B \xrightarrow{\cong} \pi(C \square B)$$

for objects B and C of $\mathcal{B}$; the unlabelled isomorphisms are uniquely determined by the monoidal structure of $\mathcal{A}$. Define $\eta : \mathcal{M}\mathcal{A} \to \mathcal{M}\mathcal{B}$ by $\eta(f) = (A, f, A')$ for $f : A \to A'$ and define $\pi : \mathcal{M}\mathcal{B} \to \mathcal{M}\mathcal{A}$ by

$\pi(B, g, B') = g$ for $g : \pi B \to \pi B'$. Then η and π are functors such that $\pi\eta = 1$, and the morphisms $(B, I\pi B, \eta\pi B)$ of $\mathcal{B}$ define a natural isomorphism between 1 and $\eta\pi$.

Let $(\mathcal{A}, \square, *, c)$ be a fixed permutative category. For $\sigma \in \Sigma_j$, c determines a natural transformation $c_\sigma : \square_j \to \square_j\sigma$, where Σ_j acts on the left of $\mathcal{A}^j$ by permutation of coordinates and $\square_j : \mathcal{A}^j \to \mathcal{A}$ denotes the iterate of $\square$; by coherence, $c_\sigma c_\tau = c_{\sigma\tau}$. We have the following three lemmas, the first of which is an observation due to Anderson [2]; the other two are direct consequences of coherence: we need only observe that the assertions make sense on objects. Recall the definition, 1. 1, of the translation category $\tilde{G}$ of a monoid G. Observe that G acts on the right of $\tilde{G}$ via the product of G. Observe too that if G is a group, then there is a unique morphism $g \to g'$ for $g, g' \in G$ and a functor with range $\tilde{G}$ is therefore uniquely determined by its object function.

Lemma 4. 3. For $j \geq 0$, there is a Σ_j-equivariant functor

$$c_j : \tilde{\Sigma}_j \times \mathcal{A}^j \to \mathcal{A}$$

defined, on objects and morphisms respectively, by

$$c_j(\sigma, A_1, \ldots, A_j) = A_{\sigma^{-1}(1)} \square \ldots \square A_{\sigma^{-1}(j)}$$

and

$$c_j(\sigma \to \tau, f_1, \ldots, f_j) = c_{\tau\sigma^{-1}} \circ (f_{\sigma^{-1}(1)} \square \ldots \square f_{\sigma^{-1}(j)}).$$

(If $j = 0$, $\tilde{\Sigma}_0 \times \mathcal{A}^0$ is the unit category, with one object and one morphism, and c_0 is the functor determined by $* \in \mathcal{O}\mathcal{A}$.)

Lemma 4. 4. The following diagram is commutative for all $j \geq 0$, $k \geq 0$, and $j_i \geq 0$ with $j_1 + \ldots + j_k = j$:

$$\begin{tikzcd}
\tilde{\Sigma}_k \times \tilde{\Sigma}_{j_1} \times \dots \times \tilde{\Sigma}_{j_k} \times \mathcal{A} \arrow[rr, "\tilde{\gamma} \times 1"] \arrow[d, "1 \times \mu"'] & & \tilde{\Sigma}_j \times \mathcal{A}^j \arrow[dr, "c_j"] & \\
& & & \mathcal{A} \\
\tilde{\Sigma}_k \times \tilde{\Sigma}_{j_1} \times \mathcal{A}^{j_1} \times \dots \times \tilde{\Sigma}_{j_k} \times \mathcal{A}^{j_k} \arrow[rr, "1 \times c_{j_1} \times \dots \times c_{j_k}"] & & \tilde{\Sigma}_k \times \mathcal{A}^k \arrow[ur, "c_k"'] &
\end{tikzcd}$$

where μ is the evident shuffle isomorphism and the functors $\tilde{\gamma} : \tilde{\Sigma}_k \times \tilde{\Sigma}_{j_1} \times \dots \times \tilde{\Sigma}_{j_k} \to \tilde{\Sigma}_j$ are defined on objects by

$$\tilde{\gamma}(\sigma; \tau_1, \dots, \tau_k) = \tau_{\sigma^{-1}(1)} \oplus \dots \oplus \tau_{\sigma^{-1}(k)} .$$

Lemma 4.5. For $j \geq 0$, c_ν determines a natural isomorphism between the two composites in the following diagram:

$$\begin{tikzcd}
\tilde{\Sigma}_j \times (\mathcal{A} \times \mathcal{A})^j \arrow[r, "\Delta \times \nu"] \arrow[d, "1 \times \square^j"'] & \tilde{\Sigma}_j \times \tilde{\Sigma}_j \times \mathcal{A}^j \times \mathcal{A}^j \arrow[r, "1 \times \tau \times 1"] & \tilde{\Sigma}_j \times \mathcal{A}^j \times \tilde{\Sigma}_j \times \mathcal{A}^j \arrow[d, "c_j \times c_j"] \\
\tilde{\Sigma}_j \times \mathcal{A}^j \arrow[r, "c_j"] & \mathcal{A} & \mathcal{A} \times \mathcal{A} \arrow[l, "\square"']
\end{tikzcd}$$

Here $\nu \in \Sigma_{2j}$ determines the evident shuffle isomorphism, τ is the transposition, and Δ is the diagonal functor.

Now recall the following definition, due to Segal [25].

Definition 4.6. Let $\mathcal{A}$ be a category. The nerve, or morphism complex, $B_*\mathcal{A}$ of $\mathcal{A}$ is the simplicial space specified as follows:

$B_0\mathcal{A} = \mathcal{O}\mathcal{A}$, $B_1\mathcal{A} = \mathcal{M}\mathcal{A}$, and if $q > 1$

$B_q\mathcal{A} = \mathcal{M}\mathcal{A} \times_{\mathcal{O}\mathcal{A}} \dots \times_{\mathcal{O}\mathcal{A}} \mathcal{M}\mathcal{A}$, q factors $\mathcal{M}\mathcal{A}$;

$\partial_0 = S$ and $\partial_1 = T$ on $B_1\mathcal{A}$; $s_0 = I$ on $B_0\mathcal{A}$;

$$\partial_i[f_1, \ldots, f_q] = \begin{cases} [f_2, \ldots, f_q] & \text{if } i = 0 \\ [f_1, \ldots, f_{i-1}, f_i f_{i+1}, \ldots, f_q] & \text{if } 0 < i < q \text{ for } q > 1; \\ [f_1, \ldots, f_{q-1}] & \text{if } i = q \end{cases}$$

$$s_i[f_1, \ldots, f_q] = [f_1, \ldots, f_i, ISf_i = ITf_{i+1}, f_{i+1}, \ldots, f_q] \text{ for } q > 0.$$

Define the classifying space $B\mathfrak{a}$ of $\mathfrak{a}$ to be the geometric realization of $B_*\mathfrak{a}$. Then B is a functor from the category of (topological) categories and functors to the category of spaces and maps; B preserves products by [G, 11.5]. Let $\mathcal{J}$ denote the category with objects 0 and 1 and a single non-identity morphism $0 \to 1$; $B\mathcal{J}$ is homeomorphic to the unit interval I. A natural transformation $\lambda : F \to G$ between functors $\mathfrak{a} \to \mathfrak{a}'$ determines a functor $\lambda : \mathfrak{a} \times \mathcal{J} \to \mathfrak{a}'$ and thus a homotopy $B\lambda : B\mathfrak{a} \times I \to B\mathfrak{a}'$ between BF and BG.

In [G, §9 and §10], the bar construction was set up in sufficient generality to ensure that anything which looks like a bar construction is indeed a bar construction. The above construction is no exception. A category with object space $\mathcal{O}$ is a monoid in the monoidal category of (topological) $\mathcal{O}$-Graphs [14, p. 49 and 167], and $B_*\mathfrak{a} = B_*(\mathcal{O}, \mathfrak{a}, \mathcal{O})$ where $\mathcal{O}$ is the identity object of $\mathcal{O}$-Graph. When $\mathcal{O}$ is a single point $*$, $\mathcal{O}$-Graph is the category $\mathcal{U}$ of spaces and the present functor B reduces to the classifying space functor for topological monoids. We also have the following familiar special case, the normalized version of Milnor's universal bundle for topological groups.

Lemma 4.7. For a topological group G, $B\tilde{G} = |D_*G|$ where D_*G is as defined in [G, 10.2]. Therefore $B\tilde{G}$ and EG are homeomorphic as right G-spaces.

Proof. $D_qG = G^{q+1}$ with faces and degeneracies given by projections and diagonals. A simplicial homeomorphism $B_*\tilde{G} \leftrightarrow D_*G$ is given by $g \leftrightarrow g$ on 0-simplices and by

$$[g_1 \leftarrow g_2, \ldots, g_q \leftarrow g_{q+1}] \longleftrightarrow (g_1, \ldots, g_{q+1})$$

on q-simplices for $q > 0$. The second statement follows from [G, 10.3].

An equally trivial comparison of definitions gives the following

addendum (compare 4. 4 to the proof of 3. 7).

Lemma 4. 8. *The* E_∞ *operad* $\mathcal{D}$ *of* [G, p. 161] *satisfies* $\mathcal{D}(j) = B\tilde{\Sigma}_j$ *as a right* Σ_j*-space and its structural maps* γ *coincide with the maps* $B\tilde{\gamma} : B\tilde{\Sigma}_k \times B\tilde{\Sigma}_{j_1} \times \ldots \times B\tilde{\Sigma}_{j_k} \to B\tilde{\Sigma}_j$, $j_1 + \ldots + j_k = j$.

Now, by the very definition of an action by an operad [G, 1. 4], 4. 3 and 4. 4 immediately imply the following result.

Theorem 4. 9. *If* $(\mathcal{A}, \square, *, c)$ *is a permutative category, then the maps*

$$\Gamma_j = Bc_j : \mathcal{D}(j) \times (B\mathcal{A})^j \to B\mathcal{A}$$

define a natural action Γ *of the* E_∞ *operad* $\mathcal{D}$ *on* $B\mathcal{A}$.

Similarly, 4. 5 implies the following consistency statement.

Theorem 4. 10. *If* $(\mathcal{A}, \square, *)$ *is a strict monoidal category, then* $B\mathcal{A}$ *is a topological monoid with product* $B\square$. *If* $\mathcal{A}$ *is permutative, then* $(B\mathcal{A}, \Gamma, B\square)$ *is a homotopy monoid in* $\mathcal{D}[\mathcal{T}]$ *in the sense that the following diagrams are* Σ_j*-equivariantly homotopy commutative,* $j \geq 0$:

$$\begin{tikzcd}
\mathcal{D}(j) \times (B\mathcal{A} \times B\mathcal{A})^j \arrow[r, "(\Gamma \times \Gamma)_j"] \arrow[d, "1 \times (B\square)^j"'] & B\mathcal{A} \times B\mathcal{A} \arrow[d, "B\square"] \\
\mathcal{D}(j) \times (B\mathcal{A})^j \arrow[r, "\Gamma_j"] & B\mathcal{A}
\end{tikzcd}$$

In particular, $B\square$ *is homotopic to* $\theta_2(d)$ *for any* $d \in \mathcal{D}(2)$.

Proof. For the last statement, merely replace equalities by homotopies in the proof of 3. 4.

In the spirit of Quillen's work [23, 24], we suggest the following as a reasonable construction of algebraic K-theory.

Definition 4. 11. For a space X, a permutative category $\mathcal{A}$, and an integer n, define $K^n(X; \mathcal{A}) = [X, B_n B\mathcal{A}]$, where B_n is the n-th de-

looping functor if $n \geq 0$ and $B_{-n} = \Omega^n B_0$ if $n > 0$. In particular, define the K-groups of $\mathcal{A}$ by $K^n\mathcal{A} = \pi_0 B_n B\mathcal{A}$.

For $n \geq 0$, $K^{-n}\mathcal{A} = \pi_n B_0 B\mathcal{A}$. We have that $K^0\mathcal{A}$ is the group completion of $\pi_0 B\mathcal{A}$ where, by [G, 11.11], $\pi_0 B\mathcal{A}$ is the quotient monoid obtained from $\pi_0 \mathcal{O}\mathcal{A}$ by identifying two components [A] and [A'] whenever there is a morphism between A and A'. Thus, if all morphisms of $\mathcal{A}$ are isomorphisms and $\mathcal{O}\mathcal{A}$ is discrete, then $\pi_0 B\mathcal{A}$ is the monoid of isomorphism classes of objects of $\mathcal{A}$ and $K^0\mathcal{A}$ is the obvious Grothendieck group.

Now let R be a (topological) ring with unit. By an abuse of terminology justified by 4.2, let $\mathcal{F} \subset \mathcal{P}$ denote permutative categories under $\oplus$ obtained by choosing skeletons of the categories of finitely generated free and finitely generated projective left R-modules and their isomorphisms. Here $\mathcal{O}\mathcal{P}$ is given the discrete topology, but Aut P for $P \in \mathcal{O}\mathcal{B}$ and therefore $\mathcal{M}\mathcal{P} = \coprod_{P \in \mathcal{O}\mathcal{P}} \operatorname{Aut} P$ are assumed to be appropriately topologized. (For instance, we could topologize free, hence also projective, modules in the evident way and then use the compact-open topology.) We assume that each Aut P has the homotopy type of a CW-complex. Clearly

$$H_*B\mathcal{F} = \coprod_{n \geq 0} H_*BGL(R, n) \quad \text{and} \quad H_*B\mathcal{P} = \coprod_{P \in \mathcal{O}\mathcal{P}} H_*B \operatorname{Aut} P.$$

For example, if R is the real numbers (resp., complex numbers) and if GL(R, n) is topologized as usual, then $K^n(X; \mathcal{F})$ is real (resp., complex) connective K-theory. (For $n > 0$, this requires an easy consistency argument based on the fact that the iterated Bott maps define morphisms of permutative categories; the details are similar to those in [18, §6] where a different, more geometric, construction of these K-theories within the context of E_∞ spaces is given.)

We use $\mathcal{P}$ to define the K-theory of R; that is, we set $K^nR = K^n\mathcal{P}$. For $n < 0$, we could just as well use $\mathcal{F}$. Indeed, the translation functor (defined in the proof of 3.9) for $B\mathcal{F}$ is clearly cofinal with that for $B\mathcal{P}$ and the map $(B_0B\mathcal{F})_0 \to (B_0B\mathcal{P})_0$ of base-point components is therefore a homotopy equivalence since it is a map of connected H-spaces of the homotopy type of CW-complexes (by A.3,

A. 6, and [21]) which induces an isomorphism on homology.

Recall that Bass defines $K_1R = GLR/ER$ and Milnor defines $K_2R = \mathrm{Ker}(STR \to GLR)$ where ER denotes the commutator subgroup of GLR and STR denotes the Steinberg group. Provided that GLR and its topological subgroup ER are discrete, it follows that $K_1R \cong H_1BGLR$ and $K_2R \cong H_2BER$. In our general topological situation, we have the following result.

Proposition 4.12. K^0R <u>is isomorphic to the projective class group of</u> R, $K^{-1}R$ <u>is isomorphic to</u> H_1BGLR, <u>and, if the homogeneous space</u> GLR/ER <u>is discrete,</u> $K^{-2}R$ <u>is isomorphic to</u> H_2BER.

Proof. We have already verified the first statement and, by the following argument, essentially due to Anderson (see Quillen [24]), the other two statements are consequences of 3.9. By 3.8 and 4.10, $\overline{B\mathfrak{F}}$ has the homotopy type of BGLR and therefore, by 3.9, the evident isomorphism from H_*BGLR to $H_*(B_0B\mathfrak{F})_0$ is induced by $\bar{\iota} : \overline{B\mathfrak{F}} \to (B_0B\mathfrak{F})_0$. Since $\pi_1(B_0B\mathfrak{F})_0$ is Abelian,

$$K^{-1}R \cong \pi_1(B_0B\mathfrak{F})_0 \cong H_1(B_0B\mathfrak{F})_0 \cong H_1\overline{B\mathfrak{F}} \cong H_1BGLR.$$

Let E(R, n) be the commutator subgroup of GL(R, n). Regard $\mathfrak{F}$ as the category with objects $N = \{n \mid n \geq 0\}$ whose only morphisms are $\mathfrak{F}(n, n) = GL(R, n)$. Let $\mathscr{E}$ and $\mathfrak{F}/\mathscr{E}$ be the categories with objects N whose only morphisms are

$$\mathscr{E}(n, n) = E(R, n) \quad \text{and} \quad (\mathfrak{F}/\mathscr{E})(n, n) = GL(R, n)/E(R, n).$$

Let $i : \mathscr{E} \to \mathfrak{F}$ and $\pi : \mathfrak{F} \to \mathfrak{F}/\mathscr{E}$ denote the evident morphisms of permutative categories under $\oplus$ (and cognate maps). We then have the following homotopy commutative diagram:

$$\begin{tikzcd}
BER \arrow[r, phantom, "\simeq"] \arrow[d, "Bi"'] & \overline{B\mathcal{E}} \arrow[r, "\bar{\iota}"] \arrow[d, "\overline{Bi}"] & (B_0B\mathcal{E})_0 \arrow[r, "\subset"] \arrow[d, "B_0Bi"] & (B_0B\mathcal{E})'_0 \arrow[d] \\
BGLR \arrow[r, phantom, "\simeq"] \arrow[d, "B\pi"'] & \overline{B\mathfrak{F}} \arrow[r, "\bar{\iota}"] \arrow[d, "\overline{B\pi}"] & (B_0B\mathfrak{F})_0 \arrow[r, "\subset"] \arrow[d, "B_0B\pi"] & (B_0B\mathfrak{F})'_0 \arrow[d, "(B_0B\pi)'"] \\
B(GLR/ER) \arrow[r, phantom, "\simeq"] & \overline{B\mathfrak{F}/\mathcal{E}} \arrow[r, "\bar{\iota}"] & (B_0B\mathfrak{F}/\mathcal{E})_0 \arrow[r, equal] & (B_0B\mathfrak{F}/\mathcal{E})_0
\end{tikzcd}$$

On the right, $(B_0B\pi)'$ if the fibration with fibre $(B_0B\mathcal{E})'_0$ obtained by replacing $B_0B\pi$ by a fibration in the standard fashion; $(B_0B\pi)'$ is an H-map and therefore has trivial local coefficients in homology [see 9, p. 16.08, 16.09]. As noted by Quillen [24, p. 18], $B\pi$ on the left is a fibration with trivial local coefficients because if $x = \pi(y) \in GLR/ER$ and $z \in H_*BER$, then there is a conjugate subgroup G of $E(R, n)$ in ER for some n such that z is in the image of H_*BG and y commutes with the elements of G, hence x acts trivially on z. Observe that, by suitably expanding the diagram, following the proof of 3.9, we could arrange to have all squares commute. We may thus regard the diagram as a map of fibrations. It follows by the comparison theorem that $H_*BER \to H_*(B_0B\mathcal{E})'_0$ is an isomorphism. Since GLR/ER is discrete and Abelian, $B(GLR/ER)$ is a $K(GLR/ER, 1)$ and an Abelian H-space. It is easily verified by consideration of the relevant limits that $\bar{\iota} : \overline{B\mathfrak{F}/\mathcal{E}} \to (B_0\mathfrak{F}/\mathcal{E})_0$ is an H-map and therefore (since it induces an isomorphism on homology) a homotopy equivalence. Thus

$$K^{-2}R \cong \pi_2(B_0B\mathfrak{F})_0 \cong \pi_2(B_0B\mathcal{E})'_0 \cong H_2(B_0B\mathcal{E})'_0 \cong H_2BER.$$

APPENDIX

In two results of [G], namely [G, 3.4 and 11.13], connectivity was assumed only because I was unaware of the following fundamental 'glueing theorem' of R. Brown [8; 7.5.7].

Theorem A.1. Suppose given a commutative diagram

$$\begin{tikzcd}
& A' \arrow[rr, "f'"] \arrow[dd, "i'" near start] & & C' \arrow[dd, "\tilde{i}'"] \\
A \arrow[ur, "\alpha"] \arrow[rr, "f"'] \arrow[dd, "i"'] & & C \arrow[ur, "\gamma"] \arrow[dd, "\tilde{i}" near start] & \\
& B' \arrow[rr, "\tilde{f}'"] & & D' \\
B \arrow[ur, "\beta"] \arrow[rr, "\tilde{f}"'] & & D \arrow[ur, "\delta"'] &
\end{tikzcd}$$

of spaces and maps such that i *and* i' *are cofibrations and* $\tilde{i}$ *and* $\tilde{i}'$ *are the cofibrations induced by* f *and* f'. *(Thus the front and back squares are pushouts.) Assume that* α, β, *and* γ *are homotopy equivalences. Then* δ *is also a homotopy equivalence.*

The second part of the following sharpening of [G, 3.4] was known to Beck; it is closely related to his result [5, Theorem 8] on topological theories.

Proposition A.2. *Let* $\psi : \mathcal{C} \to \mathcal{C}'$ *be a morphism of operads and let* $X \in \mathcal{T}$.

(i) *If* ψ *is a local equivalence and* $\mathcal{C}$ *and* $\mathcal{C}'$ *are* Σ-*free, then* $\psi : CX \to C'X$ *induces an isomorphism on (integral) homology.*

(ii) *If* ψ *is a local* Σ-*equivalence, then* $\psi : CX \to C'X$ *is a homotopy equivalence.*

Proof. The hypotheses are explained in [G, p. 21] and (i) is proven (but not stated) in [G, p. 22]. CX is constructed by means of successive pushouts

$$\begin{tikzcd}
\mathcal{C}(j+1) \times_{\Sigma_{j+1}} sX^j \arrow[rr, "f"] \arrow[d, "\cap"'] & & F_jCX \arrow[d] \\
\mathcal{C}(j+1) \times_{\Sigma_{j+1}} X^{j+1} \arrow[rr] & & F_{j+1}CX
\end{tikzcd}$$

where $sX^j = \bigcup_{0 \le i \le j} s_i X^j$, $s_i(x_1, \ldots, x_j) = (x_1, \ldots, x_{i-1}, *, x_i, \ldots, x_j)$,

and $f(c, s_i y) = [\sigma_i c, y]$ for $c \in \mathcal{C}(j+1)$ and $y \in X^j$ (see [G, 2.3 and 2.4] for the notations). By induction on j, the hypothesis of (ii), the non-degeneracy of the base-point of X, and the glueing theorem imply that $\psi : F_j CX \to F_j C'X$ is a homotopy equivalence for all j. The result follows.

Corollary A. 3. Let $\mathcal{C}$ be a Σ-free operad such that $\mathcal{C}(j)$, $j \geq 1$, has the Σ_j-equivariant homotopy type of a CW-complex. Then CX has the homotopy type of a CW-complex.

Proof. Let S and T denote the total singular complex and geometric realization functors between spaces and simplicial sets and let $\Phi : TS \to 1$ be the standard natural transformation. Recall that Φ is a homotopy equivalence on spaces of the homotopy type of CW-complexes [15, 16.6]. Let $\mathcal{C}'$ be the operad $TS\mathcal{C}$ ($\mathcal{C}'(j) = TS\mathcal{C}(j)$, etc.) and let $X' = TSX$. By the freeness of the Σ_j actions, $\Phi : \mathcal{C}' \to \mathcal{C}$ is a local Σ-equivalence, hence $\Phi : C'X' \to CX'$ is a homotopy equivalence. By an argument just like the previous proof, $C\Phi : CX' \to CX$ is also a homotopy equivalence. $\mathcal{C}'(j)$ is a CW free Σ_j-complex, and it follows by induction and the glueing diagrams that $C'X'$ is a CW-complex.

The second part of the following sharpening of [G, 11.13] is due to Tornehave [28, A. 3]; we give his proof for completeness. Zisman (private communication) has an alternative proof based on a homotopy analog of the Segal [25; G, 11.14] spectral sequence in homology.

Theorem A. 4. Let $f : X \to X'$ be a map of proper simplicial spaces.

(i) If each $f_q : X_q \to X'_q$ induces an isomorphism on homology, then $|f| : |X| \to |X'|$ induces an isomorphism on homology.

(ii) If each $f_q : X_q \to X'_q$ is a homotopy equivalence, then $|f| : |X| \to |X'|$ is a homotopy equivalence.

Proof. (i) follows either from the Segal spectral sequence or from a slight refinement (see A. 5) of the proof of [G, 11.13]. We prove (ii). Since X is proper, the inclusion $sX_q \to X_{q+1}$ is a cofibration, where $sX_q = \bigcup_{0 \leq j \leq q} s_j X_q$. We have successive pushouts

$$\begin{tikzcd}
sX_q \times \partial\Delta_{q+1} \arrow[r] \arrow[d, "\cap"'] & X_{q+1} \times \partial\Delta_{q+1} \arrow[d] & \\
sX_q \times \Delta_{q+1} \arrow[r] & (sX_q \times \Delta_{q+1}) \cup (X_{q+1} \times \partial\Delta_{q+1}) \arrow[r, "g"] \arrow[d, "\cap"'] & F_q|X| \arrow[d] \\
& X_{q+1} \times \partial\Delta_{q+1} \arrow[r] & F_{q+1}|X|
\end{tikzcd}$$

where $q(s_i x, u) = |x, \sigma_i u|$ and $q(x, \delta_i v) = |\partial_i x, v|$ (see [G, 11.1] for the notation). By induction on q and the glueing theorem, it suffices to show that $f_{q+1} : sX_q \to sX'_q$ is a homotopy equivalence for all q. Let $s^kX_q = \bigcup_{0\le j\le k} s_jX_q$ for $0 \le k \le q$. In the following diagram, the right square is a pushout and the maps s_k are homeomorphisms:

$$\begin{tikzcd}
s^{k-1}X_{q-1} \arrow[r, "s_k"] \arrow[d] & s^{k-1}X_q \cap s_kX_q \arrow[r] \arrow[d] & s^{k-1}X_q \arrow[d] \\
X_q \arrow[r, "s_k"] & s_kX_q \arrow[r] & s^kX_q
\end{tikzcd}$$

If we assume inductively that $s^{k-1}X_{q-1} \to s^kX_{q-1}$ is a cofibration for $0 < k < q$ (a vacuous assumption if $q = 0$ or 1), then we conclude (by [G, A.5] and propriety) that $s^{k-1}X_q \to s^kX_q$ is a cofibration, and similarly for X'. Since $s_0 : X_q \to s_0X_q$ is a homeomorphism, $f_{q+1} : s^0X_q \to s^0X'_q$ is a homotopy equivalence. By induction on q and for fixed q by induction on k, the diagram above and the glueing theorem imply that $f_{q+1} : s^kX_q \to s^kX'_q$ is a homotopy equivalence for all k and q. The result follows.

Remarks A.5. In the proof of [G, 11.12], which asserts that $|X|$ is n-connected if each X_q is $(n - q)$-connected, we can use the Mayer-Vietoris sequence of the excision

$$(s_kX_{q-1}, s_kX_{q-1} \cap s^{k-1}X_{q-1}) \to (s^kX_q, s^{k-1}X_q)$$

instead of that used in [G, p. 108-109] (in view of the proof above). It

follows that 'strict' propriety (see [G, 11.2]) was an unnecessary hypothesis in [G, 11.12]. Since strictness was not required elsewhere, it is an unnecessary notion and references to it should be deleted throughout [G].

The following corollary is also due to Tornehave [28, A.5].

Corollary A.6. If X is a proper simplicial space such that each X_q has the homotopy type of a CW-complex, then $|X|$ has the homotopy type of a CW-complex.

Proof. With notations as in the proof of A.3, $|\Phi_*| : |T_*S_*X| \to |X|$ is a homotopy equivalence. Since T_*S_*X is a cellular simplicial space (each ∂_i and s_i is cellular), $|T_*S_*X|$ is a CW-complex by [G, 11.4].

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