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H^*Spin(n) AS A HOPF ALGEBRA

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This paper contains a calculation of H^*Spin(n) as a Hopf algebra over the Steenrod algebra A (where all homology and cohomology is to be taken with Z_2 coefficients). The calculation was carried out independently by the authors some eight or nine years ago. Since then, a great deal of progress has been made in determining which A-algebras can possibly be realized as the cohomology of finite H-spaces. While all such results depend vitally on the existence of a Hopf algebra structure, they shed little light on the coalgebra structure.

Our reason for publishing the calculation at this time is to advertise the efficacy of the methods involved. Briefly, they amount to this. Suppose given a finite H-space G which admits a classifying space BG and suppose one can calculate H^*BG as an algebra (in an appropriate range of dimensions). Then one can calculate the E_2-term of the Eilenberg–Moore spectral sequence converging to H^*G. With luck, the differentials and extensions will be manageable, and complete information will be obtained. The essential point is that the Eilenberg–Moore spectral sequence can well be an efficient tool for the calculation of the coalgebra structure in situations where the Serre spectral sequence is virtually useless. A technical point of some interest is that the program can succeed even if H^*BG is known only in a range of dimensions well below the actual dimension of G.

Since the Spin(n) are classical the reader may find it hard to believe that their Pontryagin rings are not in the literature. To the best of our knowledge, the previously published calculations are those of Borel [3], who computed H^*Spin (n) as an algebra for all n and as a Hopf algebra for $n \leq 10$; of Araki [2], who obtained a particular simple system of generators for H_*Spin (n); and of Kojima [6], who computed the coproduct when $n = 2^s + 1$ or $n = 2^s + 2$ and verified that H^*Spin(n) is not cocommutative when $n \geq 10$ unless n is of the form $2^s + 1$.

We begin by recalling some standard facts about H^*BSO(n) and establishing

notation. Of course, $H^*\mathrm{BSO}(n) = P\{w_i \mid 2 \leq i \leq n\}$, where the w_i are the Stiefel–Whitney classes. Write $w_0 = 1$, $w_1 = 0$, and $w_i = 0$ for $i > n$. We then have the Wu formula

$$\text{(1)} \qquad \mathrm{Sq}^i w_j = \sum_{k=0}^{i} (k, j-i-1) w_{i-k} w_{j+k} \text{ if } i < j, \text{ where } (a,b) = (a+b)!/a!\,b!.$$

In particular we have

$$\text{(2)} \qquad \mathrm{Sq}^{j-1} w_j = w_{2j-1} + \sum_{k=j}^{2j-3} w_k w_{2j-1-k}.$$

Define elements $v_i \in H^i\mathrm{BSO}(n)$ by the equations

$$\text{(3)} \qquad v_i = w_i \text{ if } i \neq 2^j + 1,\ v_2 = w_2, \text{ and } v_{2^{j+1}+1} = \mathrm{Sq}^{2^j}(v_{2^j+1}) \text{ if } j \geq 0.$$

By (2), $H^*\mathrm{BSO}(n) = P\{v_i \mid 2 \leq i \leq n\}$, and we shall always work with the v_i rather than with the w_i. We agree to write n in the form

$$\text{(4)} \qquad n = 2^s + t \text{ with } s \geq 1 \text{ and } 0 < t \leq 2^s.$$

With these notations, our main theorem reads as follows. Recall that the cohomology suspension $\sigma^*: H^q BG \to \tilde{H}^{q-1}G$ has image contained in the primitive elements.

Theorem 1. *$H^*\mathrm{Spin}(n)$ has a simple system of generators*

$$\{u_i \mid 3 \leq i < n,\ i \neq 2^j\} \cup \{u \mid \deg u = 2^{s+1} - 1\},$$

where $u_i = \sigma^*(v_{i+1})$ satisfies $\mathrm{Sq}^r(u_i) = (r, i-r)u_{i+r}$ (with $u_j = 0$ for $j = 2^k$ or $j \geq n$). If $n \leq 9$, then $u = \sigma^* x$ for some x, and $\mathrm{Sq}^r u = 0$ for $r > 0$. If $n \geq 10$, then u is not in the image of σ^*, and the coproduct and Steenrod operations on u are given by the following formulas:

(i) *$\psi(u) = u \otimes 1 + \sum_{(i,j)} u_i \otimes u_j + 1 \otimes u$, where the sum is taken over all pairs (i,j) such that $n > i > j \geq 3$, $i + j = 2^{s+1} - 1$, and neither i nor j is a power of two.*

(ii) *$\mathrm{Sq}^r u = \sum_{(p,q)} a_{pq} u_p u_q$ if $r > 0$, where the sum is taken over all pairs (p,q) such that $n > p > q \geq 3$, $p + q = 2^{s+1} + r - 1$, and neither p nor q is a power of two and where $a_{pq} = \sum_{(i,j)} (p-i, 2i-p)(q-j, 2j-q)$ summed over all pairs (i,j) as in (i). In particular, $\mathrm{Sq}^1 u = 0$ and $\mathrm{Sq}^r u = 0$ for $r > 2(t-1)$, hence $u^2 = 0$. If $n = 2^s + 1$, then u is primitive and $\mathrm{Sq}^r u = 0$ for $r > 0$.*

Remark 2. The set $\{1, u_i, u\}$ is a Z_2-basis for a sub restricted coalgebra of $H^*\mathrm{Spin}(n)$, and $H^*\mathrm{Spin}(n)$ is characterized as the universal enveloping Hopf algebra $W(C)$ of C. The definitions behind this observation are given in [8].

The following corollaries will be obvious from the proof.

Corollary 3. *If $p: \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the covering projection, then $p^*\sigma^* w_{i+1} = u_i$ for $i \neq 2^j$ and $p^*\sigma^* w_{2^j+1} = 0$.*

Corollary 4. *If i_n: Spin$(n) \to$ Spin$(n+1)$ is the inclusion, then $i_n^*(u_j) = u_j$, $i_n^*(u) = u$ if $2^s < n < 2^{s+1}$, and $i_n^*(u) = 0$ if $n = 2^s$. Therefore i_n^* is an epimorphism for $2^s < n < 2^{s+1}$.*

The fact that $\beta u = 0$ leads to a simple proof of the following result of Borel [3].

Corollary 5. Spin(n) *has 2-torsion if and only if $n \geq 7$, and then all of its torsion is precisely of order 2.*

Proof. Let $\{E_r\}$ denote the mod 2 cohomology Bockstein spectral sequence of Spin(n). $E_1 = H^*$Spin(n) with $\beta u_{2j-1} = u_{2j}$, $\beta u_{2j} = 0$, and $\beta u = 0$. Therefore $E_1 = E_2$ if and only if $n \leq 6$ and, for all n, E_2 is the exterior algebra on the following set of generators:

$$\{x_{4j-1} \mid 3 \leq j < n/2 \text{ and } j \neq 2^k\} \cup \{u_{2^j-1} \mid 2 \leq j \leq s\} \cup \{u\} \cup \{u_{n-1} \mid n \text{ even}\},$$

where $x_{4j-1} = u_{2j-1}u_{2j} + u_{4j-1}$. Modulo torsion, the integral cohomology of Spin(n) is the exterior algebra generated by

$$\{p^*\sigma^*P_i \mid 1 \leq i < n/2\} \cup \{p^*\sigma^*\chi \mid n \text{ even}\},$$

where P_i and χ are the Pontryagin and Euler classes of H^*BSO(n). $E_2 = E_\infty$ by a comparison of dimensions and the conclusion follows.

To begin the proof of the theorem, define

(5) $$v = v_{2^{s+1}+1} = \mathrm{Sq}^{2^s} v_{2^s+1} \text{ in } H^{2^{s+1}+1}\mathrm{BSO}(n).$$

The element v is decomposable, and how it decomposes is crucial to the properties of the exceptional generator u of H^*Spin(n). Write IR for the augmentation ideal of a connected algebra R. Define ideals $J(k)$ of H^*BSO(n) by

(6) $$J(k), k \geq 0, \text{ is the ideal generated by } \{v_{2^j+1} \mid 0 \leq j \leq k\}.$$

Lemma 6. *If $n \leq 9$, then $v \in J(s)$. If $n \geq 10$, then $v \notin J(s)$ and*

$$v \equiv \sum_{(i,j)} v_i v_j \bmod J(s) + (IH^*\mathrm{BSO})^3,$$

where (i,j) runs over the set specified in (i) *of Theorem* 1.

Proof. Observe first that if X is a space and if $a_i \in H^{j_i}X$ for $1 \leq i \leq q$, then

(7) $$\mathrm{Sq}^{j-1}(a_1 \cdots a_q) = \sum_{i=1}^{q} a_1^2 \cdots a_{i-1}^2 \mathrm{Sq}^{j_i-1}(a_i) a_{i+1}^2 \cdots a_q^2, \; j = \sum_{i=1}^{q} j_i.$$

Clearly this implies the assertion

(8) If $b \in J(k)$ with $\deg b = j$, then $\mathrm{Sq}^{j-1}(b) \in J(k+1)$.

Now $v_2 = w_2$, $v_3 = w_3$, and thus $v = \mathrm{Sq}^2 v_3 \in J(1)$ if $n = 3$ or $n = 4$ by (2). For $n \geq 5$,

$v_5 = w_5 + v_2v_3$, hence $w_5 \in J(2)$, and thus $v = Sq^4v_5 \in J(2)$ if $5 \leq n \leq 8$ by (2) and (8). For $n \geq 9$, $v_9 \equiv w_9 \bmod J(2)$, hence $w_9 \in J(3)$, and thus $v = Sq^8v_9 \in J(3)$ if $n = 9$ by (2) and (8). Assume that $n \geq 10$ and abbreviate $I = IH^*BSO(n)$. By (7), if $x \in I^q$ has degree j, then $Sq^{j-1}(x) \in I^{2q-1}$. Since $v_{2^j+1} \equiv w_{2^j+1} \bmod I^2$ for $0 \leq j \leq s$, $v \equiv Sq^{2^s}w_{2^s+1} \bmod I^3$. The congruence asserted in the lemma follows immediately from (2). When $n \neq 2^s + 1$, $v \notin J(s)$ follows. When $n = 2^s + 1$, $v \equiv 0 \bmod J(s) + I^3$. However, by use of (2), (3), and (7), it is easy to verify that

$$v \equiv \sum_{j=2^{s-1}+2}^{n-2} (v_j^2 v_{2n-2j-1} + v_j v_{n-j} v_{n-1}) \bmod J(s) + I^4$$

when $n = 2^s + 1$ with $s \geq 4$, and $v \notin J(s)$ follows.

We turn to the calculation of $H^*BSpin(n)$ in the degrees needed. Actually, Quillen [9] has computed $H^*BSpin(n)$ in all degrees, but we shall only use the following trivial lemma. Define quotient algebras $R(n)$ and $S(n)$ of $H^*BSO(n)$ by

(9) $R(n) = H^*BSO(n)/J(s)$; thus $R(n) \cong P\{v_i \mid 2 \leq i \leq n, i \neq 2^j + 1\}$

(10) $S(n) = H^*BSO(n)/J(s+1) = R(n)/(\bar{v})$, where $\bar{v}$ denotes the image of v in $R(n)$.

Lemma 7. *If $n \leq 9$, then $H^*BSpin(n) = R(n) \otimes P\{x\}$, $\deg x = 2^{s+1}$. If $n \geq 10$, then $H^*BSpin(n) = S(n)$ in degrees less than 2^{s+2}.*

Proof. $BSpin(n)$ is the fibre of w_2: $BSO \to K(Z_2, 1)$. We therefore have a fibration π: $BSpin(n) \to BSO(n)$ with fibre $K(Z_2, 1)$ in which the fundamental class $i \in H^1K(Z_2, 1)$ transgresses to w_2 in the Serre spectral sequence $\{E_r\pi\}$. For $k \geq 0$, i^{2^k} transgresses to v_{2^k+1} since $sq^{2^k}\tau = \tau Sq^{2^k}$. Therefore

$$E_{2^k+2}\pi = H^*BSO(n)/J(k) \otimes P\{i^{2^{k+1}}\}$$

for $0 \leq k \leq s$. If $n \leq 9$, $i^{2^{s+1}}$ transgresses to zero, since $v \in J(s)$, and thus $E_{2^s+2}\pi = E_\infty\pi$. The first statement follows, with x projecting to $i^{2^{s+1}} \in E_\infty\pi$. If $n \geq 10$, $\tau(i^{2^{s+1}}) \neq 0$ since $v \notin J(s)$. Since $R(n)$ is a polynomial algebra, v is not a zero divisor in $E_{2^{s+1}+1}\pi$ and therefore $E_{2^{s+2}+1}\pi = S(n) \otimes P\{i^{2^{s+2}}\}$. The second statement follows.

The crux of the proof of Theorem 1 is the following homological result.

Proposition 8. $Tor_{S(n)}(Z_2, Z_2) = E\{\sigma v_i \mid 4 \leq i \leq n, i \neq 2^j + 1\} \otimes \Gamma\{u\}$ *as an algebra, where* bideg $u = (-2, 2^{s+1} + 1)$ *and where, with $u_i = \sigma(v_{i+1})$, the coproduct on u is given by formula* (i) *of Theorem* 1.

Here torsion products are bigraded, the first or homological degree being given by non-positive superscripts and the second degree being the internal degree. For

any connected Z_2-algebra R, σ is the composite of the projection of IR onto the indecomposable elements QB and the natural isomorphism between QB and $\mathrm{Tor}_R^{-1,*}(Z_2, Z_2)$. Elements in the image of σ are primitive.

Before proceeding to its proof, we show how Proposition 8 implies Theorem 1. Let $\{E_r\}$ denote the Eilenberg–Moore spectral sequence of the universal bundle of $\mathrm{Spin}(n)$ [1, 4, 5, 10, 11]. This is a spectral sequence of differential Hopf algebras over the Steenrod algebra which converges from $E_2 = \mathrm{Tor}_{H^*\mathrm{BSpin}(n)}(Z_2, Z_2)$ to $H^*\mathrm{Spin}(n)$. It satisfies

$$E_r^{pq} = 0 \text{ if } p > 0, q < 0, \text{ or } -4p > q \text{ and } d_r\colon E_r^{pq} \to E_r^{p+r,q-r+1}.$$

Here $E_r^{pq} = 0$ for $-4p > q$ because $H^q\mathrm{BSpin}(n) = 0$ for $0 < q < 4$. The suspension σ^* factors as the composite

$$H^q\mathrm{BSpin}(n) \xrightarrow{\sigma} E_2^{-1,q} \xrightarrow{\pi} E_\infty^{-1,q} \subset \tilde{H}^q\,\mathrm{Spin}(n),$$

where π is the natural epimorphism.

We first dispose of the case $n \leq 9$ of Theorem 1. Since $\mathrm{Tor}_{P\{y\}}(Z_2, Z_2) = E\{\sigma y\}$ for any y, we have

$$E_2 = E\{\sigma v_i \mid 4 \leq i \leq n,\ i \neq 2^j + 1\} \otimes E\{\sigma x\}$$

$E_2 = E_\infty$ since $E_2^{-1,*}$ consists of permanent cycles. The Steenrod operations on $u_{i-1} = \sigma^* v_i$ are obviously determined by $v_i = \pi^* w_i$ and (1). $\mathrm{Sq}^r u$ is primitive since $u = \sigma^* x$ is primitive, and no non-zero primitive elements have degree greater than $2^{s+1} - 1 = \deg u$. Thus $\mathrm{Sq}^r u = 0$ for $r > 0$.

Henceforward, assume that $n \geq 10$. We have a morphism of algebras $j\colon S(n) \to H^*\mathrm{BSpin}(n)$ since $\pi^*\colon H^*\mathrm{BSO}(n) \to H^*\mathrm{BSpin}(n)$ factors through $S(n)$. We therefore have a morphism of Hopf algebras $j_*\colon \mathrm{Tor}_{S(n)}(Z_2, Z_2) \to E_2$, and j_* is an isomorphism in internal degrees $q < 2^{s+2}$ by Lemma 7. As an algebra, the divided polynomial algebra $\Gamma\{u\}$ of Proposition 8 may be written $E\{\gamma_{2^j}(u) \mid j \geq 0\}$. Let $D = E\{\sigma v_i\} \otimes E\{u\}$. Then D is a sub Hopf algebra of $\mathrm{Tor}_{S(n)}(Z_2, Z_2)$. All primitive elements of D have internal degree $q < 2^{s+2}$, hence j_* is a monomorphism when restricted to D. Identify D with $j_* D \subset E_2$. All elements of D are permanent cycles since its generators have homological degrees -1 or -2. We claim that no element of D bounds. Indeed, assume the contrary, let r be minimal such that some non-zero $d_r x$ is in D and let x be of minimal total degree with this property. As usual, since $\psi d_r x = (d_r \otimes 1 + 1 \otimes d_r)\psi(x)$, $d_r x$ must be primitive. Thus either $x \in E_r^{-1-r,i+r-1}$ and $d_r x = u_i$ or $n = 2^s + 1$, $x \in E_r^{2-r,2^{s+1}+r}$, and $d_r x = u$. Since $E_r^{pq} = 0$ for $-4p > q$, it is trivial to check that x has internal degree $q < 2^{s+2}$ and therefore lies in D. This contradicts $d_r x \neq 0$ and proves our claim.

We now have that D is a sub Hopf algebra of E_∞, and it follows that $E_\infty = D \otimes D'$ as a vector space. Of course, $\mathrm{Spin}(n)$ is a compact manifold of dimension $\frac{1}{2}n(n-1)$. Since

$$\deg\left(\left(\underset{i}{\times}\ u_i\right) u\right) = \sum_{i=1}^{n-1} i - \sum_{j=0}^{s} 2^j + 2^{s+1} - 1 = \tfrac{1}{2}n(n-1),$$

D' can have no elements of positive total degree and $E_\infty = D$. The fact that the u_i and u lift to a simple system of generators for $H^*\mathrm{Spin}(n)$ is standard. Since $\pi^*(w_i) = v_i$ and $u_{i-1} = \sigma^* v_i$, it only remains to verify (i) and (ii) of Theorem 1. Observe first that the lifting of $u \in E_\infty^{-2,*}$ to $F^{-2}H^*\mathrm{Spin}(n)$ is unique if $n \neq 2^{s+1}$ and admits two choices which differ by u_{n-1} if $n = 2^{s+1}$ since the lifting is well-defined modulo $\mathrm{Im}\,\sigma^*$. Since u_{n-1} is primitive and $\mathrm{Sq}^r u_{n-1} = 0$ for $r > 0$, the choice of lifting can have no effect on the validity of (i) and (ii).

Consider $\psi(u)$. If $x' \otimes x''$ is a non-zero summand of $\psi(u)$, its filtration must be at least -2 and therefore precisely -2 if x' and x'' both have positive degree. Thus the coproduct on $u \in H^*\mathrm{Spin}(n)$ must be the same as on $u \in E_\infty$, which proves (i). Similarly, summands of $\mathrm{Sq}^r u$ have filtration at least -2 and therefore precisely -2 if $r > 0$ since $\deg u_i \leq \deg u$. We can thus write $\mathrm{Sq}^r u$ in the form $\sum_{(p,q)} a_{pq} u_p u_q$ for some constants $a_{pq} \in Z_2$ and the range of summation stated in (ii). To evaluate the a_{pq}, we use $\psi \mathrm{Sq}^r u = \mathrm{Sq}\, \psi u$. Clearly

$$\psi \mathrm{Sq}^r u = \mathrm{Sq}^r u \otimes 1 + \sum_{(p,q)} a_{pq} (u_p \otimes u_q + u_q \otimes u_p) + 1 \otimes \mathrm{Sq}^r u.$$

On the other hand, by (i) and the Wu and Cartan formulas,

$$\mathrm{Sq}^r \psi u = \mathrm{Sq}^r u \otimes 1 + \sum_{(b,c)} \left(\sum_{(i,j)} (b-i, 2i-b)(c-j, 2j-c) \right) u_b \otimes u_c + 1 \otimes \mathrm{Sq}^r u$$

summed over all pairs (i,j) as in (i) and all pairs (b,c) such that $b + c = 2^{s+1} + r - 1$. Formula (ii) of Theorem 1 follows upon equating coefficients.

We must still prove Proposition 8. To simplify the notation and clarify the nature of the argument, we generalize the context. Thus let R be a polynomial algebra on positive degree generators $x_1, \ldots, x_k$ and let $S = R/(w)$, where w is a non-zero decomposable element of R. Write $w = \sum_{j=1}^k y_j x_j$ with $y_j \in IR$ and write $y_j = z_j + d_j$ with z_j a linear combination of the x_i and $d_j \in (IR)^2$. Thus $w \equiv \sum_{j=1}^k z_j x_j \bmod (IR)^3$. Proposition 8 is a special case of the following result.

Proposition 9. $\mathrm{Tor}_S(Z_2, Z_2) = E\{\sigma x_i \mid 1 \leq i \leq k\} \otimes \Gamma\{u\}$ *as an algebra, where* $\mathrm{bideg}\, u = (-2, \deg w)$ *and where*

$$\psi u = u \otimes 1 + \sum_{j=1}^k \sigma z_j \otimes \sigma x_j + 1 \otimes u.$$

Proof. Define a differential S-algebra $X = S \otimes \bar{X}$ by letting $\bar{X} = E\{\sigma x_i\} \otimes \Gamma\{u\}$ and specifying the differential on generators by $d(\sigma x_i) = x_i$ and $d\gamma_r(u) = \sum_{j=1}^k y_j (\sigma x_j) \gamma_{r-1}(u)$. We claim that X is acyclic and is thus a minimal S-free resolution of Z_2, minimal meaning that the induced differential on $Z_2 \otimes_S X$ is zero. The description of $\mathrm{Tor}_S(Z_2, Z_2)$ as an algebra will follow.

To prove the claim, write $X = Y \otimes \Gamma\{u\}$, where Y is the subdifferential S-algebra $S \otimes E\{x_i\}$. As a complex, Y is a quotient of the differential R-algebra

$Z = R \otimes E\{\sigma x_i\}$ with $d(\sigma x_i) = x_i$, and we have the short exact sequence of chain complexes

$$(*) \qquad 0 \to Z \xrightarrow{w} Z \to Y \to 0.$$

Since R is a polynomial algebra, Z is acyclic; that is, $H^{pq}Z = 0$ unless $p = q = 0$, when $H^{0,0}Z = Z_2$. Let $\deg w = m$. Multiplication by w has bidegree $(0, m)$, hence $w_* = 0$ on homology, and the long exact homology sequence of $(*)$ yields $H^{p,q}Y = 0$ except in the cases $H^{0,0}Y = Z_2$ and $H^{-1,m}Y = Z_2$; the latter group maps to $H^{0,0}Z$ under the connecting homomorphism and is generated by the class α of the cycle $\sum y_j(\sigma x_j)$.

Now filter X by the internal degree of Y, $F^qX = \sum_{i \le q} Y^{*,i} \otimes \Gamma\{u\}$. This is a decreasing filtration by subcomplexes, and the induced differential on the associated graded complex E_0X is just $d \otimes 1$. Thus $E_1X = HY \otimes \Gamma\{u\}$. (This is actually trigraded, but we can ignore such pedantry.) In view of the original differential on the $\gamma_r(u)$, we have $d_m\gamma_r(u) = \alpha \otimes \gamma_{r-1}(u)$. Therefore $E_{m+1}X = Z_2$ (in tridegree $(0,0,0)$) and our claim is proven.

It remains to compute ψu, and for this we use the bar construction (e.g. [5 or 7]). Define a comparison of resolutions $\mu : X \to BS$ by the inductive formula $\mu = s\mu d$ on $\bar{X}$, where s is the standard contracting homotopy of BS [7, (3.1) and Proposition 13]. *Then* $\mu(\sigma x_i) = [x_i]$ and $\mu(u) \approx \sum_j [y_j \mid x_j]$. Passing to $\bar{B}S = Z_2 \otimes_S BS$, we find by [7, (3.7)] that

$$\Psi\mu(u) = \mu(u) \otimes 1 + \sum [y_j] \otimes [x_j] + 1 \otimes \mu(u).$$

Here $y_j = z_j + d_j$ and $[d_j]$ obviously bounds in $\bar{B}S$, by [7, (3.3)]. The claimed formula for ψu follows on passage to homology.

The construction of X from Y in the proof above is a special case of a general procedure for the construction of resolutions in commutative algebra by successively attaching exterior or divided polynomial algebras to kill off homology classes. In the ungraded case, the method is due to Tate [12].

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