**H*Spin(n) AS A HOPF ALGEBRA**

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This paper contains a calculation of $H^*\text{Spin}(n)$ as a Hopf algebra over the Steenrod algebra $A$ (where all homology and cohomology is to be taken with $\mathbb{Z}_2$ coefficients). The calculation was carried out independently by the authors some eight or nine years ago. Since then, a great deal of progress has been made in determining which $A$-algebras can possibly be realized as the cohomology of finite $H$-spaces. While all such results depend vitally on the existence of a Hopf algebra structure, they shed little light on the coalgebra structure.

Our reason for publishing the calculation at this time is to advertise the efficacy of the methods involved. Briefly, they amount to this. Suppose given a finite $H$-space $G$ which admits a classifying space $BG$ and suppose one can calculate $H^*BG$ as an algebra (in an appropriate range of dimensions). Then one can calculate the $E_2$-term of the Eilenberg–Moore spectral sequence converging to $H^*G$. With luck, the differentials and extensions will be manageable, and complete information will be obtained. The essential point is that the Eilenberg–Moore spectral sequence can well be an efficient tool for the calculation of the coalgebra structure in situations where the Serre spectral sequence is virtually useless. A technical point of some interest is that the program can succeed even if $H^*BG$ is known only in a range of dimensions well below the actual dimension of $G$.

Since the $\text{Spin}(n)$ are classical the reader may find it hard to believe that their Pontryagin rings are not in the literature. To the best of our knowledge, the previously published calculations are those of Borel [3], who computed $H^*\text{Spin}(n)$ as an algebra for all $n$ and as a Hopf algebra for $n \leq 10$; of Araki [2], who obtained a particular simple system of generators for $H_*\text{Spin}(n)$; and of Kojima [6], who computed the coproduct when $n = 2^a + 1$ or $n = 2^a + 2$ and verified that $H^*\text{Spin}(n)$ is not cocommutative when $n > 10$ unless $n$ is of the form $2^a + 1$.

We begin by recalling some standard facts about $H^*\text{BSO}(n)$ and establishing
notation. Of course, $H^*BSO(n) = P\{w_i \mid 2 \leq i \leq n\}$, where the $w_i$ are the Stiefel-Whitney classes. Write $w_0 = 1$, $w_1 = 0$, and $w_i = 0$ for $i > n$. We then have the Wu formula

$$\text{Sq'} w_j = \sum_{k=0}^j (k, j - i - 1) w_{i+k} w_{j-k} \text{ if } i < j, \text{ where } (a, b) = (a + b)!/a!b!.$$ 

In particular we have

$$\text{Sq}' w_j = w_{2j-1} + \sum_{k=1}^{2j-3} w_k w_{2j-1-k}.$$ 

Define elements $v_i \in H^*BSO(n)$ by the equations

$$v_i = w_i \text{ if } i \neq 2j + 1, v_2 = w_2, \text{ and } v_{2j+1} = \text{Sq}^{j'}(v_{2j+1}) \text{ if } j \geq 0.$$ 

By (2), $H^*BSO(n) = \mathcal{P}\{v_i \mid 2 \leq i \leq n\}$, and we shall always work with the $v_i$ rather than with the $w_i$. We agree to write $n$ in the form

$$n = 2^s + t \text{ with } s \geq 1 \text{ and } 0 < t \leq 2^s.$$ 

With these notations, our main theorem reads as follows. Recall that the cohomology suspension $\sigma^*: H^*BG \to \tilde{H}^{*+1}G$ has image contained in the primitive elements.

**Theorem 1.** $H^*Spin(n)$ has a simple system of generators

$$\{u_i \mid 3 \leq i < n, \ i \neq 2^s - 1\} \cup \{u \mid \deg u = 2^{s+1} - 1\},$$

where $u_i = \sigma^*(v_{i+1})$ satisfies $\text{Sq}'(u_i) = (r, i - r) u_{i+r}$ (with $u_i = 0$ for $j = 2^k$ or $j \geq n$). If $n \leq 9$, then $u = \sigma^*x$ for some $x$, and $\text{Sq}' u = 0$ for $r > 0$. If $n \geq 10$, then $u$ is not in the image of $\sigma^*$, and the coproduct and Steenrod operations on $u$ are given by the following formulas:

(i) $\psi(u) = u \otimes 1 + \sum_{(i, j)} u_i \otimes u_j + 1 \otimes u$, where the sum is taken over all pairs $(i, j)$ such that $n > i > j \geq 3$, $i + j = 2^{s+1} - 1$, and neither $i$ nor $j$ is a power of two.

(ii) $\text{Sq}' u = \sum_{(p, q)} a_{pq} u_p u_q$ if $r > 0$, where the sum is taken over all pairs $(p, q)$ such that $n > p > q \geq 3$, $p + q = 2^{s+1} + r - 1$, and neither $p$ nor $q$ is a power of two and where $a_{pq} = \sum_{(i, j)} (p - i, 2i - p)(q - j, 2j - q)$ summed over all pairs $(i, j)$ as in (i). In particular, $\text{Sq}' u = 0$ and $\text{Sq}' u = 0$ for $r > 2(t - 1)$, hence $u^2 = 0$. If $n = 2^s + 1$, then $u$ is primitive and $\text{Sq}' u = 0$ for $r > 0$.

**Remark 2.** The set $\{1, u, u\}$ is a $\mathbb{Z}_2$-basis for a sub restricted coalgebra of $H^*Spin(n)$, and $H^*Spin(n)$ is characterized as the universal enveloping Hopf algebra $W(C)$ of $C$. The definitions behind this observation are given in [8].

The following corollaries will be obvious from the proof.

**Corollary 3.** If $p: Spin(n) \to SO(n)$ is the covering projection, then $p^*\sigma^* w_{i+1} = u_i$ for $i \neq 2^s$ and $p^*\sigma^* w_{2^s+1} = 0$. 
Corollary 4. If \( i_n : \text{Spin}(n) \to \text{Spin}(n + 1) \) is the inclusion, then \( i_n^*(u_i) = u_i \), \( i_n^*(u) = u \) if \( 2^i < n < 2^{i+1} \), and \( i_n^*(u) = 0 \) if \( n = 2^i \). Therefore \( i_n^* \) is an epimorphism for \( 2^i < n < 2^{i+1} \).

The fact that \( \beta u = 0 \) leads to a simple proof of the following result of Borel [3].

Corollary 5. \( \text{Spin}(n) \) has 2-torsion if and only if \( n \geq 7 \), and then all of its torsion is precisely of order 2.

Proof. Let \( \{ E_r \} \) denote the mod 2 cohomology Bockstein spectral sequence of \( \text{Spin}(n) \). \( E_1 = H^*\text{Spin}(n) \) with \( \beta u_{2j} = u_{2j} \), \( \beta u_{2j} = 0 \), and \( \beta u = 0 \). Therefore \( E_1 = E_2 \) if and only if \( n \leq 6 \) and, for all \( n, E_2 \) is the exterior algebra on the following set of generators:

\[
\{ x_{4j-1} | 3 \leq j < n/2 \text{ and } j \neq 2k \} \cup \{ u_{2j-1} | 2 \leq j \leq s \} \cup \{ u \} \cup \{ u_{n-1} | n \text{ even} \},
\]

where \( x_{4j-1} = u_{2j-1}u_{2j} + u_{4j-1} \). Modulo torsion, the integral cohomology of \( \text{Spin}(n) \) is the exterior algebra generated by

\[
\{ p^* \sigma^*P_i | 1 \leq i < n/2 \} \cup \{ p^* \sigma^* \chi | n \text{ even} \},
\]

where \( P \) and \( \chi \) are the Pontryagin and Euler classes of \( H^*\text{BSO}(n) \). \( E_2 = E_\infty \) by a comparison of dimensions and the conclusion follows.

To begin the proof of the theorem, define

\[
(5) \quad v = v_{2^{r+1}+1} = \text{Sq}^2v_{2^{r+1}} \text{ in } H^{2^{r+1}+1}\text{BSO}(n).
\]

The element \( v \) is decomposable, and how it decomposes is crucial to the properties of the exceptional generator \( u \) of \( H^*\text{Spin}(n) \). Write \( IR \) for the augmentation ideal of a connected algebra \( R \). Define ideals \( J(k) \) of \( H^*\text{BSO}(n) \) by

\[
(6) \quad J(k), k \geq 0, \text{ is the ideal generated by } \{ v_{2^i} | 0 < j < k \}.
\]

Lemma 6. If \( n \leq 9 \), then \( v \in J(s) \). If \( n \geq 10 \), then \( v \notin J(s) \) and

\[
\quad v = \sum_{(i,j)} v_{ij} \mod J(s) + (IH^*\text{BSO})^q,
\]

where \( (i,j) \) runs over the set specified in (i) of Theorem 1.

Proof. Observe first that if \( X \) is a space and if \( a_i \in H^iX \) for \( 1 \leq i \leq q \), then

\[
(7) \quad \text{Sq}^{q-i}(a_1 \cdots a_q) = \sum_{(i,j)} a_1^2 \cdots a_{i-1}^2 \text{Sq}^{q-i}(a_i)a_{i+1}^2 \cdots a_q^2, j = \sum_{i \leq j} j.
\]

Clearly this implies the assertion

\[
(8) \quad \text{If } b \in J(k) \text{ with deg } b = j, \text{ then } \text{Sq}^{q-i}(b) \in J(k+1).
\]

Now \( v_2 = w_2, v_3 = w_3 \), and thus \( v = \text{Sq}^2v_3 \in J(1) \) if \( n = 3 \) or \( n = 4 \) by (2). For \( n \geq 5 \),
\[ v_2 = w_2 + v_2 v_3, \] hence \[ w_2 \in J(2), \] and thus \[ v = Sq^4 v_5 \in J(2) \] if \( 5 \leq n \leq 8 \) by (2) and (8). For \( n \geq 9 \), \( v_9 = w_9 \) mod \( J(2) \), hence \( w_9 \in J(3) \), and thus \( v = Sq^3 v_9 \in J(3) \) if \( n = 9 \) by (2) and (8). Assume that \( n \geq 10 \) and abbreviate \( I = IH^*BSO(n) \). By (7), if \( x \in I^s \) has degree \( j \), then \( Sq^{-1}(x) \in I^{q-1} \). Since \( v_{2j+1} = w_{2j+1} \) mod \( I^2 \) for \( 0 \leq j \leq s \), \( v = Sq^2 w_{2s+1} \) mod \( I^3 \). The congruence asserted in the lemma follows immediately from (2). When \( n \neq 2^s + 1 \), \( v \not\in J(s) \) follows. When \( n = 2^s + 1 \), \( v = 0 \) mod \( J(s) + I^3 \). However, by use of (2), (3), and (7), it is easy to verify that

\[ v = \sum_{j=s}^{n-2} (v_j^j v_{2^{n-j} - 1} + v_j v_{j-1} v_{n-1}) \mod J(s) + I^4 \]

when \( n = 2^s + 1 \) with \( s \geq 4 \), and \( v \not\in J(s) \) follows.

We turn to the calculation of \( H^*BSpin(n) \) in the degrees needed. Actually, Quillen [9] has computed \( H^*BSpin(n) \) in all degrees, but we shall only use the following trivial lemma. Define quotient algebras \( R(n) \) and \( S(n) \) of \( H^*BSO(n) \) by

\[ R(n) = H^*BSO(n)/J(s); \text{ thus } R(n) = P\{v_i \mid 2 \leq i \leq n, i \neq 2^s + 1\} \]

\[ S(n) = H^*BSO(n)/J(s+1) = R(n)/\langle \bar{v} \rangle, \]

where \( \bar{v} \) denotes the image of \( v \) in \( R(n) \).

**Lemma 7.** If \( n \leq 9 \), then \( H^*BSpin(n) = R(n) \otimes P\{x\} \), deg \( x = 2^{s+1} \). If \( n \geq 10 \), then \( H^*BSpin(n) = S(n) \) in degrees less than \( 2^{s+2} \).

**Proof.** \( BSpin(n) \) is the fibre of \( w_2: BSO \to K(Z_2, 1) \). We therefore have a fibration \( \pi: BSpin(n) \to BSO(n) \) with fibre \( K(Z_2, 1) \) in which the fundamental class \( i \in H^1 K(Z_2, 1) \) transgresses to \( w_2 \) in the Serre spectral sequence \( \{E, \pi\} \). For \( k \geq 0 \), \( i^{2k} \) transgresses to \( v_{2k+1} \) since \( sq^{2k} \pi = \tau Sq^{2k} \). Therefore

\[ E_2^{s+2} \pi = H^*BSO(n)/J(k) \otimes P\{i^{2^{s+1}}\} \]

for \( 0 \leq k \leq s \). If \( n \leq 9 \), \( i^{2^{s+1}} \) transgresses to zero, since \( v \in J(s) \), and thus \( E_2^{s+2} \pi = E_{2^{s+1}} \). The first statement follows, with \( x \) projecting to \( i^{2^{s+1}} \in E_{2^{s+1}} \). If \( n \geq 10 \), \( \tau(i^{2^{s+1}}) \neq 0 \) since \( v \not\in J(s) \). Since \( R(n) \) is a polynomial algebra, \( v \) is not a zero divisor in \( E_2^{s+2} \pi \) and therefore \( E_2^{s+2} \pi = S(n) \otimes P\{i^{2^{s+1}}\} \). The second statement follows.

The crux of the proof of Theorem 1 is the following homological result.

**Proposition 8.** \( Tors_S(n)(Z_2, Z_2) = E\{v_i \mid 4 \leq i \leq n, i \neq 2^j + 1\} \otimes \Gamma\{u\} \) as an algebra, where bideg \( u = (-2, 2^{s+1} + 1) \) and where, with \( u_i = \alpha(v_{i+1}) \), the coproduct on \( u \) is given by formula (i) of Theorem 1.

Here torsion products are bigraded, the first or homological degree being given by non-positive superscripts and the second degree being the internal degree. For
any connected \( Z_2 \)-algebra \( R \), \( \sigma \) is the composite of the projection of \( IR \) onto the indecomposable elements \( QB \) and the natural isomorphism between \( QB \) and \( \text{Tor}_R^{-1,\ast}(Z_2, Z_2) \). Elements in the image of \( \sigma \) are primitive.

Before proceeding to its proof, we show how Proposition 8 implies Theorem 1. Let \( \{E_r\} \) denote the Eilenberg–Moore spectral sequence of the universal bundle of \( \text{Spin}(n) \) [1, 4, 5, 10, 11]. This is a spectral sequence of differential Hopf algebras over the Steenrod algebra which converges from \( E_2 = \text{Tor}_{H^*\text{BSpin}(n)}(Z_2, Z_2) \) to \( H^*\text{Spin}(n) \). It satisfies

\[
E_\infty = 0 \text{ if } p > 0, \ q < 0, \text{ or } -4p > q \text{ and } d_r: E_\infty \to E_\infty^{r,q+r+1}.
\]

Here \( E_\infty = 0 \) for \(-4p > q \) because \( H^*\text{BSpin}(n) = 0 \) for \( 0 < q < 4 \). The suspension \( \sigma^* \) factors as the composite

\[
H^*\text{BSpin}(n) \xrightarrow{\sigma} E_2^{-1,q} \twoheadrightarrow E_2^{-1,q} \subset \tilde{H}^q \text{Spin}(n),
\]

where \( \pi \) is the natural epimorphism.

We first dispose of the case \( n < 9 \) of Theorem 1. Since \( \text{Tor}_{P(n)}(Z_2, Z_2) = E \{\sigma y\} \) for any \( y \), we have

\[
E_2 = E \{\sigma v_i \mid 4 \leq i \leq n, i \neq 2^j + 1\} \otimes E \{\sigma x\}
\]

\( E_2 = E_\ast \) since \( E_2^{-1,\ast} \) consists of permanent cycles. The Steenrod operations on \( u_i = \sigma^* v_i \) are obviously determined by \( n_i = \pi^* w_i \) and (1). \( \text{Sq}^q u \) is primitive since \( u = \sigma^* x \) is primitive, and no non-zero primitive elements have degree greater than \( 2^{r+1} - 1 = \deg u \). Thus \( \text{Sq}^q u = 0 \) for \( r > 0 \).

Henceforward, assume that \( n \geq 10 \). We have a morphism of algebras \( j: S(n) \to H^*\text{BSpin}(n) \) since \( \pi^*: H^*\text{BSO}(n) \to H^*\text{BSpin}(n) \) factors through \( S(n) \). We therefore have a morphism of Hopf algebras \( j_\ast: \text{Tor}_{S(n)}(Z_2, Z_2) \to E_2 \), and \( j_\ast \) is an isomorphism in internal degrees \( q < 2^{r+2} \) by Lemma 7. As an algebra, the divided polynomial algebra \( \Gamma(u) \) of Proposition 8 may be written \( E \{\gamma_{2^r}(u) \mid j \geq 0\} \). Let \( D = E \{\sigma v_i\} \otimes E \{u\} \). Then \( D \) is a sub Hopf algebra of \( \text{Tor}_{S(n)}(Z_2, Z_2) \). All primitive elements of \( D \) have internal degree \( q < 2^{r+2} \), hence \( j_\ast \) is a monomorphism when restricted to \( D \). Identify \( D \) with \( j \ast D \subseteq E_2 \). All elements of \( D \) are permanent cycles since its generators have homological degrees \(-1 \) or \(-2 \). We claim that no element of \( D \) bounds. Indeed, assume the contrary, let \( r \) be minimal such that some non-zero \( d_x \) is in \( D \) and let \( x \) be of minimal total degree with this property. As usual, since \( \psi dx = (d \otimes 1 + 1 \otimes d_0) \psi(x) \), \( dx \) must be primitive. Thus either \( x \in E_\ast^{-1,\ast-1} \) and \( dx = x \), or \( n = 2^r + 1 \), \( x \in E_\ast^{-r,2^{r+1}} \), and \( dx = u \). Since \( E\infty = 0 \) for \(-4p > q \), it is trivial to check that \( x \) has internal degree \( q < 2^{r+2} \) and therefore lies in \( D \). This contradicts \( dx \neq 0 \) and proves our claim.

We now have that \( D \) is a sub Hopf algebra of \( E_\ast \), and it follows that \( E_\ast = D \otimes D' \) as a vector space. Of course, \( \text{Spin}(n) \) is a compact manifold of dimension \( \frac{1}{2} n(n - 1) \). Since

\[
\deg \left( \left( \prod_{i=1}^{n-1} u_i \right) u \right) = \sum_{i=1}^{n-1} i - \sum_{j=0}^{k} 2^j + 2^{r+1} - 1 = \frac{1}{2} n(n - 1),
\]
$D'$ can have no elements of positive total degree and $E_\omega = D$. The fact that the $u_i$ and $u$ lift to a simple system of generators for $H^*\text{Spin}(n)$ is standard. Since $\pi^*(w_1) = v_1$ and $u_{-1} = \sigma^*v_0$, it only remains to verify (i) and (ii) of Theorem 1. Observe first that the lifting of $u \in E_{-2}^*$ to $F^{-2}H^*\text{Spin}(n)$ is unique if $n \neq 2^{*+1}$ and admits two choices which differ by $u_{-1}$ if $n = 2^{*+1}$ since the lifting is well-defined modulo $\text{Im} \sigma^*$. Since $u_{-1}$ is primitive and $\text{Sq}^r u_{-1} = 0$ for $r > 0$, the choice of lifting can have no effect on the validity of (i) and (ii).

Consider $\psi(u)$. If $x' \otimes x''$ is a non-zero summand of $\psi(u)$, its filtration must be at least $-2$ and therefore precisely $-2$ if $x'$ and $x''$ both have positive degree. Thus the coproduct on $u \in H^*\text{Spin}(n)$ must be the same as on $u \in E_\omega$, which proves (i). Similarly, summands of $\text{Sq}^r u$ have filtration at least $-2$ and therefore precisely $-2$ if $r > 0$ since $\deg u \leq \deg u$. We can thus write $\text{Sq}^r u$ in the form $\mathbf{\Sigma}_{(p,q)} a_{pq} u_p u_q$ for some constants $a_{pq} \in Z_2$ and the range of summation stated in (ii). To evaluate the $a_{pq}$, we use $\psi \text{Sq}^r u = \text{Sq} \psi^r u$. Clearly

$$\psi \text{Sq}^r u = \text{Sq}^r u \otimes 1 + \sum_{(p,q)} a_{pq} (u_p \otimes u_q + u_q \otimes u_p) + 1 \otimes \text{Sq}^r u.$$  

On the other hand, by (i) and the Wu and Cartan formulas,

$$\text{Sq}^r \psi u = \text{Sq}^r u \otimes 1 + \sum_{(b,c)} \left( \sum_{(i,j)} (b - i, 2i - b)(c - j, 2j - c) \right) u_b \otimes u_c + 1 \otimes \text{Sq}^r u$$

summed over all pairs $(i, j)$ as in (i) and all pairs $(b, c)$ such that $b + c = 2^{*+1} + r - 1$. Formula (ii) of Theorem 1 follows upon equating coefficients.

We must still prove Proposition 8. To simplify the notation and clarify the nature of the argument, we generalize the context. Thus let $R$ be a polynomial algebra on positive degree generators $x_1, \ldots, x_k$ and let $S = R/(w)$, where $w$ is a non-zero decomposable element of $R$. Write $w = x_{k+1} \cdots x_k$ with $y_j \in IR$ and write $y_i = z_i + d_i$ with $z_i$ a linear combination of the $x_i$ and $d_i \in (IR)^3$. Thus $w = \mathbf{\Sigma}_{i=1}^k z_i x_i \mod (IR)^3$. Proposition 8 is a special case of the following result.

**Proposition 9.** $\text{Tors}_S(Z_2, Z_2) = E\{\sigma x_i \mid 1 \leq i \leq k\} \otimes \Gamma'\{u\}$ as an algebra, where $\text{bideg} u = (-2, \deg w)$ and where

$$\psi u = u \otimes 1 + \sum_{i=1}^k \sigma z_i \otimes \sigma x_i + 1 \otimes u.$$  

**Proof.** Define a differential $S$-algebra $X = S \otimes \tilde{X}$ by letting $\tilde{X} = E\{\sigma x_i \} \otimes \Gamma'\{u\}$ and specifying the differential on generators by $d(\sigma x_i) = x_i$ and $d\gamma, (u) = \mathbf{\Sigma}_{j=1}^k y_j (x_j)^{y_j - 1}(u)$. We claim that $X$ is acyclic and is thus a minimal $S$-free resolution of $Z_2$, minimal meaning that the induced differential on $Z_2 \otimes_S X$ is zero. The description of $\text{Tors}_S(Z_2, Z_2)$ as an algebra will follow.

To prove the claim, write $X = Y \otimes \Gamma'\{u\}$, where $Y$ is the subdifferential $S$-algebra $S \otimes \mathcal{E}\{x_i\}$. As a complex, $Y$ is a quotient of the differential $R$-algebra
\[ Z = R \otimes E \{ \sigma \_i \} \text{ with } d(\sigma \_i) = x_\_i, \text{ and we have the short exact sequence of chain complexes} \]

\[ (*) \quad 0 \rightarrow Z \xrightarrow{d} Z \rightarrow Y \rightarrow 0. \]

Since \( R \) is a polynomial algebra, \( Z \) is cyclic; that is, \( H^mZ = 0 \) unless \( p = q = 0 \), when \( H^{0,0}Z = Z_2 \). Let \( \deg w = m \). Multiplication by \( w \) has bidegree \((0, m)\), hence \( w_x = 0 \) on homology, and the long exact homology sequence of \((*)\) yields \( H^{0,0}Y = 0 \) except in the cases \( H^{0,0}Y = Z_2 \) and \( H^{-1,0}Y = Z_2 \); the latter group maps to \( H^{0,0}Z \) under the connecting homomorphism and is generated by the class \( \alpha \) of the cycle \( \sum \gamma_i(\sigma \_i) \).

Now filter \( X \) by the internal degree of \( Y \), \( F^*X = \sum_{i < q} Y^{*,i} \otimes \Gamma \{ u \} \). This is a decreasing filtration by subcomplexes, and the induced differential on the associated graded complex \( E_\circ X \) is just \( d \otimes 1 \). Thus \( E_1X = HY \otimes \Gamma \{ u \} \). (This is actually trigraded, but we can ignore such pedantry.) In view of the original differential on the \( \gamma_i(u) \), we have \( d \circ \gamma_i(\alpha) = \alpha \otimes \gamma_{-1}(u) \). Therefore \( E_{m+1}X = Z_2 \) (in tridegree \((0,0,0)\)) and our claim is proven.

It remains to compute \( \psi u \), and for this we use the bar construction \((e.g. [5 or 7])\). Define a comparison of resolutions \( \mu : X \rightarrow BS \) by the inductive formula \( \mu = s \mu d \) on \( \tilde{X} \), where \( s \) is the standard contracting homotopy of \( BS \) \([7, (3.1) \text{ and Proposition 13}] \). Then \( \mu (\sigma \_i) = [x_\_i] \) and \( \mu (u) = \sum [y_i | x_i] \). Passing to \( \tilde{BS} = Z_2 \otimes_s BS \), we find by \([7, (3.7)]\) that

\[ \Psi \mu (u) = \mu (u) \otimes 1 + \sum [y_i] \otimes [x_i] + 1 \otimes \mu (u). \]

Here \( y_i = z_i + d_i \) and \([d_i] \) obviously bounds in \( \tilde{BS} \), by \([7, (3.3)]\). The claimed formula for \( \psi u \) follows on passage to homology.

The construction of \( X \) from \( Y \) in the proof above is a special case of a general procedure for the construction of resolutions in commutative algebra by successively attaching exterior or divided polynomial algebras to kill off homology classes. In the ungraded case, the method is due to Tate \([12]\).

References