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THE UNIQUENESS OF INFINITE LOOP SPACE MACHINES

J. P. MAY and R. THOMASON

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AN INFINITE loop space machine is a functor which constructs spectra out of simpler space level data. There are many such machines known[1–4, 8, 15]. They differ somewhat in the data they accept. Worse, they are given by such widely disparate topological constructions that it is far from obvious that they turn our equivalent spectra when fed the same data. The purpose of this paper is to prove that all machines which satisfy certain reasonable properties do in fact turn out equivalent spectra. The properties are satisfied by Segal's machine[15], but require use of somewhat more general input data than the other machines in the literature are geared to accept. We generalize May's machine[8, 9] so that it acts in the requisite generality and satisfies the requisite properties. Thus the May and Segal machines are equivalent. This proof will illustrate what would be involved in the corresponding generalization of other machines, and we are quite confident that an exhaustive case-by-case verification would lead to the conclusion that there is really only one infinite loop space machine.

To avoid leaving a wrong impression, we hasten to add that this does not mean we can now discard all but one of the explicit constructions. The purpose of the constructions is to prove theorems and make calculations, of the sort sketched in [11], and such applications may only be accessible to one or another of the machines. For example, the passage from E_∞ ring spaces to E_∞ ring spectra, the construction of classifying spectra for bundle and fibration theories oriented with respect to an E_∞ ring spectrum, and the passage from E_∞ ring spectra to H_∞ ring spectra[12, 13] are part of a calculationally powerful circle of ideas which depends on use of the particular geometry of May's machine. The point here is that while there is now a uniqueness theorem for infinite loop space machines, there is no uniqueness theorem for the assembly lines of multiplicative infinite loop space factories. On the other hand, Segal's machine has the distinct advantage of being very much simpler to construct than the others. Moreover, it will play a canonical role in our theory. Rather than compare two machines directly, we compare each of them to Segal's machine.

We give a general discussion of the input data of infinite loop space machines in §1 and give a way to construct examples in §4. We prove the uniqueness theorem in §§2 and 3, except that we relegate the proof of a key result about spectra to the first appendix. We give the promised generalization of May's machine in §§5 and 6.

As is traditional in this subject, there is also an appendix about cofibrations.

In the course of proving our new results, we have had to redevelop and systematize the foundations of infinite loop space theory, and it is our hope that the present paper can serve as a readable source for its main ideas and techniques.

The first author wishes to acknowledge that the key new idea is entirely due to the second author and the latter wants to thank Waldhausen for a very helpful conversation. Both authors wish to acknowledge that the basic insight comes from Fiedorowicz' paper[6].

§1. CATEGORIES OF OPERATORS AND THEIR ACTIONS

We here describe a general framework that seems to encompass appropriate domain data for infinite loop space machines and other such theories of algebraic structure up to homotopy. We first define the notion of a $\mathscr{G}$-space for a category of

operators $\mathcal{G}$ and then compare the categories of $\mathcal{G}$-spaces as $\mathcal{G}$ varies. It will be apparent that Segal's Γ-spaces fit into this framework, and we shall see in §4 that Boardman and Vogt's homotopy everything H-spaces and May's E_∞ spaces also fit into this framework. This section should be regarded as an elaboration of the first half and alternative to the second half of Segal[15, App. B]. No originality is claimed.

Let $\mathcal{F}$ denote the category of finite based sets $\mathbf{n} = \{0, 1, \ldots, n\}$ with basepoint 0; its morphisms are the based functions. Let Π denote the subcategory of $\mathcal{F}$ consisting of all morphisms ϕ: $\mathbf{m} \to \mathbf{n}$ such that $\phi^{-1}(j)$ has at most one element for $1 \le j \le n$ ($\phi^{-1}(0)$ may have more than one element).

Regard the set N of objects of $\mathcal{F}$ as a discrete space.

Definition 1.1. A category of operators is a topological category $\mathcal{G}$ with object space N such that $\mathcal{G}$ contains Π and is augmented over $\mathcal{F}$ by a functor ϵ: $\mathcal{G} \to \mathcal{F}$ which restricts to the inclusion on Π. A map of categories of operators is a continuous functor ν: $\mathcal{G} \to \mathcal{H}$ such that $\nu(\mathbf{n}) = \mathbf{n}$ and the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{G} \arrow[dd, "\nu"] \arrow[dr] & \\
\Pi \arrow[ur] \arrow[dr] & & \mathcal{F} \\
& \mathcal{H} \arrow[ur] &
\end{tikzcd}$$

The map ν is said to be an equivalence if each map ν: $\mathcal{G}(\mathbf{m}, \mathbf{n}) \to \mathcal{H}(\mathbf{m}, \mathbf{n})$ is an equivalence.

We shall add a minor technical condition to the definition in Addendum 1.7 below. That $\mathcal{G}$ is topological means that its set of morphisms is a space and its structural functions are continuous but, as should be but is not standard, we require in addition that the identity function from objects to morphisms be a cofibration, so that $1 \in \mathcal{G}(\mathbf{n}, \mathbf{n})$ is a non-degenerate basepoint for each n.

Intuitively, we think of $\mathcal{G}(\mathbf{m}, \mathbf{1})$ as a space of m-ary operations and we think of $\mathcal{G}(\mathbf{m}, \mathbf{n})$ as a space of operations with m inputs and n outputs. Π consists of the elementary operations in $\mathcal{F}$, namely those which do not combine distinct variables.

Let $\mathcal{T}$ denote the category of nondegenerately based compactly generated weak Hausdorff spaces. We do not insist that spaces have the homotopy type of CW-complexes, and by an equivalence we agree to mean a weak homotopy equivalence; we emphasize that this convention is to remain in force throughout the paper. Recall the notion of an equivariant cofibration from[4, App. §2].

Definition 1.2. Let $\mathcal{G}$ be a category of operators. A $\mathcal{G}$-space is a functor X: $\mathcal{G} \to \mathcal{T}$, written $\mathbf{n} \to X_n$ on objects, such that the adjoints $\mathcal{G}(\mathbf{m}, \mathbf{n}) \times X_m \to X_n$ are continuous and the following properties hold (where we use the same name for maps in $\mathcal{G}$ and for their images under X).

(1) X_0 is aspherical (that is, equivalent to a point).

(2) For $n > 1$, the map $X_n \to X_1{}^n$ with coordinates δ_i is an equivalence, where δ_i: $\mathbf{n} \to \mathbf{1}$ is the map in Π given by $\delta_i(j) = 1$ if $i = j$ and $\delta_i(j) = 0$ otherwise.

(3) If $\phi: \mathbf{m} \to \mathbf{n}$ is an injection in Π and Σ_ϕ is the group of permutations $\sigma: \mathbf{n} \to \mathbf{n}$ such that $\sigma\phi = \phi$, then ϕ: $X_m \to X_n$ is a Σ_ϕ-equivariant cofibration.

Let $\mathcal{G}[\mathcal{T}]$ denote the category of $\mathcal{G}$-spaces, its morphisms being the natural transformations under $\mathcal{G}$. A map $X \to X'$ of $\mathcal{G}$-spaces is said to be an equivalence if each $X_n \to X'_n$ is an equivalence.

Restriction gives a forgetful functor $\mathcal{G}[\mathcal{T}] \to \Pi[\mathcal{T}]$, and a $\mathcal{G}$-space is to be thought of as an underlying Π-space with additional structure. In turn, a Π-space X is to be thought of as a sequence of spaces X_n with all the formal and homotopical properties that would be present if X_n were the n-fold product Y^n for some based space Y. The following definitions make this more precise.

Definition 1.3. Let L: $\Pi[\mathcal{T}]\to\mathcal{T}$ be the functor which sends a Π-space X to the space X_1 and let R: $\mathcal{T}\to\Pi[\mathcal{T}]$ be the functor which sends a space Y to the Π-space with nth space Y^n; for ϕ: $\mathbf{m}\to\mathbf{n}$ in Π, ϕ: $Y^m\to Y^n$ is specified by $\phi(y)=z$ where $z_j=y_i$ if $\phi(i)=j$ and $z_j=*$ if $j\not\in \mathrm{Im}\,\phi$, $1\le j\le n$. Observe that L and R are left and right adjoints,

$$\mathcal{T}(LX, Y)\cong\Pi[\mathcal{T}](X, RY),$$

since a map f: $X_1\to Y$ extends uniquely to a map $\tilde{f}$: $X\to RY$, $\tilde{f}_n$: $X_n\to Y^n$ having ith coordinate $f\delta_i$ for $1\le i\le n$ (and $\tilde{f}_0$ mapping X_0 to the point Y^0).

As the following remarks make precise, $\mathcal{F}$-spaces are essentially the same thing as Γ-spaces.

Remarks 1.4. As was first observed by Anderson, the category $\mathcal{F}$ is isomorphic to the opposite of the category Γ introduced by Segal[15]. The only differences between our notion of $\mathcal{F}$-space and Segal's notion of Γ-space are that we have chosen to introduce basepoints and impose a cofibration condition. The former change is reasonable since basepoints are always present in practice and in any case must be introduced as soon as loop spaces are considered. The latter change has the usual technical convenience.

The reader unhappy about the cofibration condition in our definition of a $\mathcal{G}$-space should be reassured by the following definition and proposition, which show that this condition results in no loss of generality.

Definition 1.5. An improper $\mathcal{G}$-space is a functor from $\mathcal{G}$ to based spaces which satisfies conditions (1) and (2), but not necessarily condition (3), in the definition of a $\mathcal{G}$-space.

Thus an improper $\mathcal{F}$-space is exactly a Γ-space with basepoints. The proof of the following "whiskering proposition" is deferred until Appendix B. This result will play a technical role in obtaining the full generality of our uniqueness theorem.

PROPOSITION 1.6. *For appropriate categories of operators $\mathcal{G}$, such as $\mathcal{F}$, there is a functor W from improper $\mathcal{G}$-spaces to $\mathcal{G}$-spaces and a natural equivalence π: $WX\to X$ of improper $\mathcal{G}$-spaces.*

To compare categories of $\mathcal{G}$-spaces, we assume given a map ν: $\mathcal{G}\to\mathcal{H}$ of categories of operators. For an $\mathcal{H}$-space Y, pullback along ν gives a $\mathcal{G}$-space ν^*Y. We want a converse construction which assigns an $\mathcal{H}$-space ν_*X to a $\mathcal{G}$-space X. We essentially follow Segal[15, App. B], but we rephrase his argument in terms of the generalization of the two-sided bar construction presented in [10, §12] in hopes that this may provide some clarification.

Let $\mathcal{O}$ be a fixed space, thought of as a space of objects. Recall from [10, §12] that a "right graph" is a space $\mathcal{Y}$ together with a source map $\mathcal{Y}\to\mathcal{O}$, a "left graph" is a space $\mathcal{X}$ together with a target map $\mathcal{X}\to\mathcal{O}$, and a "graph" is a space with both a source and target map to $\mathcal{O}$. There is an evident product (composable pairs) on the category of graphs such that a category $\mathcal{G}$ with object space $\mathcal{O}$ is precisely a monoid in the category of graphs, composition giving the product and identity giving the unit. There are evident notions of right and left graphs over $\mathcal{G}$. Given such structures $\mathcal{Y}$ and $\mathcal{X}$, we can construct a two-sided bar construction $B(\mathcal{Y},\mathcal{G},\mathcal{X})$ which enjoys most of the good properties familiar from the clasical situation in which $\mathcal{O}$ is replaced by a single point. Our interest is in the case $\mathcal{O}=\mathbf{N}$.

We refer the reader to [10, §12] for details and return to our map ν: $\mathcal{G}\to\mathcal{H}$. For each fixed $n\ge 0$, the space $\mathcal{H}_n=\amalg\mathcal{H}(\mathbf{m},\mathbf{n})$ is a right graph over $\mathcal{G}$, the requisite maps $\mathcal{H}(\mathbf{m},\mathbf{n})\times\mathcal{G}(\mathbf{q},\mathbf{m})\to\mathcal{H}(\mathbf{q},\mathbf{n})$ being given by $1\times\nu$ followed by composition. For a $\mathcal{G}$-space X, let X also denote the space $\amalg X_q$ regarded as a left graph over $\mathcal{G}$ via the maps $\mathcal{G}(\mathbf{q},\mathbf{m})\times X_q\to X_m$. We thus obtain $B(\mathcal{H}_n,\mathcal{G},X)$. We have a trivial $\mathcal{G}$-space $*$, with nth space a point, and a natural map $*\to X$. The nondegeneracy of basepoints implies that the induced map

$$B(\mathcal{H}_n, \mathcal{G}, *) \to B(\mathcal{H}_n, \mathcal{G}, X)$$

is a cofibration. Define

$$(\nu_* X)_n = B(\mathcal{H}_n, \mathcal{G}, X)/B(\mathcal{H}_n, \mathcal{G}, *).$$

A map $\mathbf{m} \to \mathbf{n}$ in $\mathcal{H}$ induces a map $\mathcal{H}_m \to \mathcal{H}_n$ of right graphs over $\mathcal{G}$, by composition, and there results a map $(\nu_* X)_m \to (\nu_* X)_n$. Thus $\nu_* X$ is a functor $\mathcal{H} \to \mathcal{T}$, and the adjoints $\mathcal{H}(\mathbf{m}, \mathbf{n}) \times (\nu_* X)_m \to (\nu_* X)_n$ are continuous by the continuity of the functor B. We impose the following addition to the definition of a category of operators in order to ensure that ϕ: $(\nu_* X)_m \to (\nu_* X)_n$ is a Σ_ϕ-equivariant cofibration if ϕ: $\mathbf{m} \to \mathbf{n}$ is an injection in Π.

Addendum 1.7. We require of a category of operators $\mathcal{G}$ that left composition $\mathcal{G}_m \to \mathcal{G}_n$ by an injection ϕ: $\mathbf{m} \to \mathbf{n}$ in Π be a Σ_ϕ-equivariant cofibration.

This holds trivially for $\mathcal{G} = \mathcal{F}$ and for our examples in §4. The following result is due to Segal[15, B.1].

THEOREM 1.8. *Let ν: $\mathcal{G} \to \mathcal{H}$ be an equivalence of categories of operators. For a $\mathcal{G}$-space X, $\nu_* X$ is an $\mathcal{H}$-space and there are natural equivalences of $\mathcal{G}$-spaces*

$$\nu^* \nu_* X \longleftarrow 1_* X \longrightarrow X,$$

where 1_* is induced by the identity functor of $\mathcal{G}$. For an $\mathcal{H}$-space Y, there is a natural equivalence of $\mathcal{H}$-spaces $\nu_* \nu^* Y \to Y$.

Proof. Consider the following equivalences:

$$B(\mathcal{H}_n, \mathcal{G}, X) \xleftarrow{\nu_n} B(\mathcal{G}_n, \mathcal{G}, X) \xrightarrow{\epsilon_n} X_n.$$

Here ν_n is short for $B(\nu, 1, 1)$ and is an equivalence by[9, A.4] and ϵ_n is obtained by restriction of nth components from the canonical equivalence ϵ: $B(\mathcal{G}, \mathcal{G}, X) \to X$ of [8, 9.8 and 11.10]. When $X = *$, these show that each $B(\mathcal{H}_n, \mathcal{G}, *)$ is contractible, and it follows that the projections

$$B(\mathcal{H}_n, \mathcal{G}, X) \longrightarrow (\nu_* X)_n \quad \text{and} \quad B(\mathcal{G}_n, \mathcal{G}, X) \longrightarrow (1_* X)_n$$

are equivalences. Thus the equivalences above induce equivalences

$$(\nu^* \nu_* X)_n \longleftarrow (1_* X)_n \longrightarrow X_n.$$

These define natural transformations of functors $\mathcal{H} \to \mathcal{T}$ and show that $(\nu_* X)_0$ is aspherical and $(\nu_* X)_n \to (\nu_* X)_1{}^n$ is an equivalence. Thus $\nu_* X$ is an $\mathcal{H}$-space and the first part is proven. For an $\mathcal{H}$-space Y, the same references as above show that the maps

$$B(\mathcal{H}_n, \mathcal{G}, \nu^* Y) \xrightarrow{\nu_n} B(\mathcal{H}_n, \mathcal{H}, Y) \xrightarrow{\epsilon_n} Y_n,$$

$\nu_n = B(1, \nu, 1)$, are equivalences. By passage to quotients, they induce equivalences

$$(\nu_* \nu^* Y)_n \xrightarrow{\nu_n} (1_* Y)_n \xrightarrow{\epsilon_n} Y_n,$$

where 1_* is induced from the identity functor of $\mathcal{H}$. These composites give the required equivalence $\nu_* \nu^* Y \to Y$.

§2. THE UNIQUENESS THEOREM

We contend that an appropriate domain of definition for an infinite loop space machine is the category of $\mathcal{G}$-spaces for any category of operators $\mathcal{G}$ such that ϵ: $\mathcal{G} \to \mathcal{F}$ is an equivalence. We assume given such a $\mathcal{G}$ throughout this section. Given two such $\mathcal{G}$, there will usually not be an equivalence between them. It is therefore unreasonable to attempt to compare directly two machines based on different $\mathcal{G}$. However, Theorem 1.8 shows that the categories of $\mathcal{G}$-spaces and of $\mathcal{F}$-spaces are

essentially equivalent. We thus find it sensible to compare a machine based on $\mathscr{G}$ to a canonical machine based on $\mathscr{F}$, and we take Segal's machine for the latter. To effect such a comparison, we must first specify precisely what we mean by an infinite loop space machine.

We require conventions about spectra. By a spectrum, we shall here understand a sequence of based spaces E_i and equivalences $\sigma_i\colon E_i \to \Omega E_{i+1}$. A map $f\colon E \to E'$ will be a sequence of maps $f_i\colon E_i \to E_i'$ such that the diagram

$$(*) \qquad \begin{tikzcd} E_i \arrow[r, "f_i"] \arrow[d, "\sigma_i"'] & E_i' \arrow[d, "\sigma_i'"] \\ \Omega E_{i+1} \arrow[r, "\Omega f_{i+1}"] & \Omega E_{i+1}' \end{tikzcd}$$

commutes (on the nose, not up to homotopy). A homotopy $h\colon f \simeq f'$ will be a parametrized family of maps $h_t\colon E \to E'$ of spectra such that the $h_{t,i}$ specify a homotopy $f_i \simeq f_i'$ for each i. A map f will be called an equivalence if each f_i is an equivalence. Such maps need not have inverses, and we shall also use the term equivalence for chains of equivalences with arrows going either forwards or backwards. We defer further discussion of this category of spectra until after the statements of our main results.

All of our spectra will be connective, in the sense that each E_i is $(i-1)$-connected. Note that a map $f\colon E \to E'$ between connective spectra is an equivalence if and only if $f_0\colon E_0 \to E_0'$ is an equivalence. In turn, since E_0 is equivalent to the product of its identity component and the discrete group $\pi_0(E_0)$, f_0 is an equivalence if and only if it induces an isomorphism on π_0 and on integral (or field coefficient) homology.

Definition 2.1. An infinite loop space machine defined on $\mathscr{G}$-spaces is a functor E from $\mathscr{G}$-spaces to connective spectra, written $EX = \{E_iX, \sigma_i\}$, together with a natural group completion $\iota\colon X_1 \to E_0X$.

That ι is a group completion means that $\pi_0(E_0X)$ is the universal group associated to the monoid π_0X_1 and that $H_*(E_0X)$ is the localization of the Pontryagin ring H_*X_1 at its submonoid π_0X_1 for every commutative ring of coefficients or, equivalently by [9, 1.4], for every field of coefficients. It is by now well understood that this group completion property is an essential feature of any worthwhile machine. We require several direct consequences of the definition.

LEMMA 2.2. *If π_0X_1 is a group, then $\iota\colon X_1 \to E_0X$ is an equivalence.*

LEMMA 2.3. *If $f\colon X \to X'$ is a map of $\mathscr{G}$-spaces such that $f_1\colon X_1 \to X_1'$ is either an equivalence or a group completion, then $Ef\colon EX \to EX'$ is an equivalence.*

Proof. E_0f induces an isomorphism on π_0 and on homology under either hypothesis.

The categories of $\mathscr{G}$-spaces and of spectra have evident products, and we have the following commutation relation.

LEMMA 2.4. *For $\mathscr{G}$-spaces X and X', the projections specify an equivalence $E(X \times X') \to EX \times EX'$.*

Proof. For commutative algebras R and S over a field k with multiplicative submonoids M and N, the tensor product over k of the localization of R at M and the localization of S at N is the localization of $R \otimes S$ at the image of $M \times N$, this being a formal consequence of the defining universal property of localization and the fact that tensor product is the coproduct in the category of algebras. Since we can restrict

attention to field coefficients, it follows by the Kunneth theorem that the map $\iota \times \iota$ in the following commutative diagram is a group completion:

$$\begin{tikzcd} & X_1 \times X_1' \arrow[dl, "\iota"'] \arrow[dr, "\iota\times\iota"] & \\ E_0(X \times X') \arrow[rr] & & E_0X_1 \times E_0X_1'. \end{tikzcd}$$

Therefore the bottom map induces an isomorphism on π_0 and on homology.

Segal[15] has constructed an infinite loop space machine S defined on $\mathcal{F}$-spaces, and the following uniqueness theorem is our main result.

THEOREM 2.5. *For any infinite loop space machine E defined on $\mathcal{G}$-spaces, there is a natural equivalence of spectra between $E(\epsilon^* Y)$ and SY for $\mathcal{F}$-spaces Y.*

The proof will be given in the next section, and the definition of S will be recalled there. It should be observed that this is really a statement about the infinite loop space machine $E\epsilon^*$ defined on $\mathcal{F}$-spaces. Thus, for purposes of proof, we may assume without loss of generality that E itself is defined on $\mathcal{F}$-spaces. This has technical advantages in that $\mathcal{F}$ admits certain constructions that would not be available for general $\mathcal{G}$. However, the theorem has the following consequence for $\mathcal{G}$-spaces.

COROLLARY 2.6. *For any infinite loop space machine E defined on $\mathcal{G}$-spaces, there is a natural equivalence of spectra between EX and $S(\epsilon_* X)$ for $\mathcal{G}$-spaces X.*

Proof. $S(\epsilon_* X)$ is equivalent to $E(\epsilon^*\epsilon_* X)$, and the latter is equivalent to EX by Theorem 1.8 and Lemma 2.3.

Thus the machines E and S are completely equivalent.

Although all known machines take values in our category of spectra, it may be worth remarking that there is an alternative version of our results valid for infinite loop space machines which take values in the category of spectra and weak maps, by which we understand sequences f_i: $E_i \to E_i'$ for which the diagrams (*) above are only required to commute up to homotopy; here maps f and f' are homotopic if $f_i \simeq f_i'$ for each i, with no compatibility between homotopies as i varies. The proofs proceed along the same lines as those below and will be omitted. The interest is that the homotopy category of spectra and weak maps is essentially the same (see [12, p. 40]) as the category of cohomology theories on spaces.

We regard the category of spectra in which we have chosen to work here as merely a convenient first approximation to the stable homotopy category. In [12, p. 40], spectra as defined here were called weak Ω-prespectra, the structural maps σ_i: $E_i \to \Omega E_{i+1}$ of Ω-prespectra being required to be inclusions as well as equivalences and the structural maps of spectra being required to be homeomorphisms. With this hierarchy of terms, and with maps, homotopies, and equivalences defined as for our present category of weak Ω-prespectra, the iterated mapping cylinder construction of [7, Thm 4] yields a functor T which replaces weak Ω-prespectra by naturally equivalent Ω-prespectra and then the direct limit construction of [12, II.1.4] yields a functor Ω^∞ which replaces Ω-prespectra by naturally equivalent spectra. These functors preserve homotopies and equivalences. The point is that, as was summarized in [12, II] and will be explained in detail in [14], the stable category can and should be defined as that category obtained from the homotopy category of spectra (with structural maps homeomorphisms) by formally adjoining inverses to equivalences. Therefore Theorem 2.5 and Corollary 2.6 yield isomorphisms

$$\Omega^\infty TE(\epsilon^* Y) \cong \Omega^\infty TSY \quad \text{and} \quad \Omega^\infty TEX \cong \Omega^\infty TS(\epsilon_* X)$$

in the stable category. Actually, S takes values in Ω-prespectra, hence TS is

equivalent to S. If E happens to take values in spectra, as holds for May's machine for example, then $\Omega^\infty TE$ is equivalent to E and we conclude that the spectra-valued machines E and $\Omega^\infty S$ are equivalent. We have chosen to work with weak Ω-prespectra, and to call them spectra, in order to avoid excess verbiage involving Ω^∞ and T, but it is the derived conclusions in the stable category that we are really after.

§3. $\mathcal{FG}$-SPACES, SEGAL'S MACHINE, AND BISPECTRA

The proof of the uniqueness theorem is based on the use of "$\mathcal{FF}$-spaces" on the input side and of "bispectra" on the output side. We begin by defining the former notion but, for clarity and for use in a later paper, we proceed in greater generality than needed and assume given a category of operators $\mathcal{G}$.

Definition 3.1. An $\mathcal{FG}$-space is a functor Z: $\mathcal{F} \to \mathcal{G}[\mathcal{T}]$, written $\mathbf{n} \to Z_n$ on objects, such that the following properties hold.

(1) The component spaces of the $\mathcal{G}$-space Z_0 are aspherical.

(2) For $n > 1$, the map $Z_n \to Z_1{}^n$ with coordinates δ_i is an equivalence of $\mathcal{G}$-spaces. Let $\mathcal{FG}[\mathcal{T}]$ denote the category of $\mathcal{FG}$-spaces, its morphisms being the natural transformations under $\mathcal{F}$.

Thus an $\mathcal{FG}$-space consists of spaces Z_{nq}, maps ϕ_q: $Z_{mq} \to Z_{nq}$ for ϕ: $\mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$, and maps ψ_n: $Z_{np} \to Z_{nq}$ for ψ: $\mathbf{p} \to \mathbf{q}$ in $\mathcal{G}$ such that the following diagrams commute (because ϕ is required to be a map of $\mathcal{G}$-spaces):

$$\begin{tikzcd} Z_{mp} \arrow[r, "\phi_p"] \arrow[d, "\psi_m"'] & Z_{np} \arrow[d, "\psi_n"] \\ Z_{mq} \arrow[r, "\phi_q"] & Z_{nq} \end{tikzcd}$$

We shall sometimes write Z_{n*} for the $\mathcal{G}$-space Z_n. Symmetrically, the maps ϕ_p on Z_{*p} for fixed p give a functor $\mathcal{F} \to \mathcal{T}$, and conditions (1) and (2) imply that this functor satisfies the corresponding conditions of Definition 1.2. Thus Z_{*p} is an improper $\mathcal{F}$-space (as defined in Definition 1.5).

We have the following lemma on the condensation of $\mathcal{FG}$-spaces to improper $\mathcal{F}$-spaces via appropriate functors.

LEMMA 3.2. *Let D be any functor from $\mathcal{G}$-spaces to based spaces which satisfies the following properties.*

(i) *If X_1 is aspherical, then DX is aspherical.*

(ii) *If f: $X \to X'$ is an equivalence of $\mathcal{G}$-spaces, then Df: $DX \to DX'$ is an equivalence.*

(iii) *The map $D(X \times X') \to DX \times DX'$ given by the projections is an equivalence. Then for any $\mathcal{FG}$-space Z, the spaces DZ_n and maps $D\phi$ for ϕ: $\mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$ specify an improper $\mathcal{F}$-space DZ.*

Proof. The map $DZ_n \to (DZ_1)^n$ with coordinates $D\delta_i$ factors as the composite of equivalences

$$DZ_n \xrightarrow{D(\times \delta_i)} D(Z_1{}^n) \longrightarrow (DZ_1)^n.$$

For an infinite loop space machine E defined on $\mathcal{G}$-spaces, Lemmas 2.2–2.4 show that each functor E_i, $i \geq 0$, satisfies the properties specified in the lemma. The following facts about the resulting improper $\mathcal{F}$-spaces E_iZ are immediate by naturality.

LEMMA 3.3. *The group completions ι: $Z_{n1} \to E_0Z_n$ and equivalences σ_i: $E_iZ_n \to \Omega E_{i+1}Z_n$ specify maps ι: $Z_{*1} \to E_0Z$* and *σ_i: $E_iZ \to \Omega E_{i+1}Z$ of improper $\mathcal{F}$-spaces.*

For the particular E and Z of interest to us, one can verify directly that all improper $\mathcal{F}$-spaces in sight do in fact satisfy the cofibration condition of Definition 1.2. However, rather than complicating our axioms by insisting on conditions which ensure this, we shall make use of Proposition 1.6 to replace improper $\mathcal{F}$-spaces by $\mathcal{F}$-spaces.

We shall only need $\mathcal{F}\mathcal{F}$-spaces, and in fact we shall only need those $\mathcal{F}\mathcal{F}$-spaces which arise via the following functor from $\mathcal{F}$-spaces to $\mathcal{F}\mathcal{F}$-spaces; its definition is abstracted from ideas of Segal [15].

Construction 3.4. Construct a functor $\mathcal{F}[\mathcal{T}] \to \mathcal{F}\mathcal{F}[\mathcal{T}]$, written $Y \to \bar{Y}$ on objects, as follows. Let $\wedge : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be the functor "smash product of finite based sets", with $\mathbf{n} \wedge \mathbf{q} = \mathbf{nq}$ on objects and

$$(\phi \wedge \psi)((i-1)p + j) = \begin{cases} (\phi(i) - 1)q + \psi(j) & \text{if } \phi(i) > 0 \text{ and } \psi(j) > 0 \\ 0 & \text{if } \phi(i) = 0 \text{ or } \psi(j) = 0 \end{cases}$$

on morphisms ϕ: $\mathbf{m} \to \mathbf{n}$ and ψ: $\mathbf{p} \to \mathbf{q}$, where $1 \le i \le m$ and $1 \le j \le p$. Let ν_n: $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ be the functor specified by $\mathbf{q} \to (\mathbf{n}, \mathbf{q})$ on objects and $\psi \to (1, \psi)$ on morphisms. Define $\bar{Y}_n$ to be the $\mathcal{F}$-space given by the composite functor

$$\mathcal{F} \xrightarrow{\nu_n} \mathcal{F} \times \mathcal{F} \xrightarrow{\wedge} \mathcal{F} \xrightarrow{Y} \mathcal{T}.$$

$\bar{Y}_n$ is to be thought of as a pseudo n-fold product of Y; it has qth space Y_{nq}. For a map ϕ: $\mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$, the maps $\phi_q = \phi \wedge 1$: $Y_{mq} \to Y_{nq}$ specify a morphism ϕ: $\bar{Y}_m \to \bar{Y}_n$ of $\mathcal{F}$-spaces, and these give $\bar{Y}$ a structure of $\mathcal{F}\mathcal{F}$-space. Observe that $\bar{Y}_0 = \bar{Y}_{0*}$ and $\bar{Y}_{*0}$ are both constant at Y_0 and that $\bar{Y}_1 = \bar{Y}_{1*}$ and $\bar{Y}_{*1}$ are both copies of the given $\mathcal{F}$-space Y. The functoriality of this passage from $\mathcal{F}$-spaces to $\mathcal{F}\mathcal{F}$-spaces is evident.

This construction will play two distinct and independent roles in our work, one being that it is the starting point for the definition of Segal's machine S. Since we need some elementary facts about S not recorded in [15] and since we wish to modify Segal's definitions slightly, we review his constructions.

LEMMA 3.5. *The category Δ such that a simplicial object is a covariant functor defined on Δ maps to $\mathcal{F}$.*

Proof. A map f: $\mathbf{m} \to \mathbf{n}$ in Δ is a non-decreasing function $\{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$, and the corresponding map ϕ: $\mathbf{m} \to \mathbf{n}$ in $\mathcal{F}$ is specified by

$$\phi^{-1}(j) = \{i \mid f(j-1) < i \le f(j)\} \quad \text{for} \quad 1 \le j \le n$$

and

$$\phi^{-1}(0) = \mathbf{m} - \bigcup_{j=1}^{n} \phi^{-1}(j).$$

(Note that this functor $\Delta \to \mathcal{F}$ is not an embedding.)

Therefore an $\mathcal{F}$-space Y has an underlying simplicial based space and thus a geometric realization $|Y|$ in $\mathcal{T}$. We take realization in the classical sense, with degeneracies. The cofibration condition in Definition 1.2 ensures that Y is a proper simplicial space in the sense of [8, §11 and 9, App.], or a good simplicial space in the sense of [15, App. A]; by the latter, $|Y|$ has the same homotopy type as Segal's thick realization.

We have a cofibration $Y_0 \to |Y|$ and we define $DY = |Y|/Y_0$; of course, the quotient map $|Y| \to DY$ is an equivalence. In practice, Y_0 is usually a point (and this could easily be arranged functorially if desired), but the construction used to prove Theorem 1.8 leads to $\mathcal{F}$-spaces with large contractible zeroth spaces. With its standard filtration, the first filtration of $|Y|$ is the quotient of $Y_0 \amalg (Y_1 \times I)$ by an equivalence relation which identifies (y, t) to y for $y \in Y_0$, where Y_0 is embedded in Y_1 via the cofibration s_0, and identifies points of $Y_1 \times \partial I$ to points of Y_0. There results a natural map $\Sigma Y_1 \to DY$ with adjoint ι: $Y_1 \to \Omega DY$, which is an inclusion, and the main theorems of Segal's paper [15] assert that ι is a group completion.

By [9, A.5], DY is i-connected if Y_1 is $(i-1)$-connected. Since geometric realization preserves products [8, 11.4] and equivalences of proper simplicial spaces [9, A.4], the functor D from $\mathscr{F}$-spaces to based spaces satisfies the conditions of Lemma 3.2. For an $\mathscr{F}\mathscr{F}$-space of the form $\bar{Y}$, it is straightforward to verify that the resulting improper $\mathscr{F}$-space $D\bar{Y}$ inherits the cofibration condition of Definition 1.2 from Y and is therefore an $\mathscr{F}$-space. We can now define Segal's machine.

Definition 3.6. Define the classifying $\mathscr{F}$-space BY of an $\mathscr{F}$-space Y to be $D\bar{Y}$ and observe that

$$(BY)_1 = D\bar{Y}_1 = DY.$$

Inductively, define $B^0 Y = Y$ and $B^i Y = B(B^{i-1}Y)$ for $i > 0$. Define the Segal spectrum $SY = \{S_i Y, \sigma_i\}$ by

$$S_0 Y = \Omega DY \quad \text{and} \quad S_{i+1} Y = DB^i Y \quad \text{for} \quad i \geq 0,$$

$$\sigma_0 = 1\colon S_0 Y \to \Omega S_1 Y \quad \text{and} \quad \sigma_i = \iota\colon S_i Y \to \Omega S_{i+1} Y \quad \text{for} \quad i > 0.$$

The σ_i are equivalences since $S_i Y$ is $(i-1)$-connected, and $\iota\colon Y_1 \to \Omega DY = S_0 Y$ is a natural group completion.

Thus S is an infinite loop space machine defined on $\mathscr{F}$-spaces. Its relationship with loops will be vital to the proof of the uniqueness theorem.

Observe that, for any category of operators $\mathscr{G}$, composition with the loop functor on based spaces yields a loop functor on improper $\mathscr{G}$-spaces. By use of [8, A.7], the looping of a $\mathscr{G}$-space satisfies the cofibration condition requisite to again be a $\mathscr{G}$-space.

The loop functor on spectra is specified by $(\Omega E)_i = \Omega(E_i)$, the structural equivalences being the composites

$$\Omega E_i \xrightarrow{\Omega\sigma_i} \Omega\Omega E_{i+1} \xrightarrow{\tau} \Omega\Omega E_{i+1},$$

where τ is given by twisting coordinates, $(\tau g)(s)(t) = g(t)(s)$. The twist is correct geometrically and is vital to the following result.

PROPOSITION 3.7. *For $\mathscr{F}$-spaces Y, there is a natural map $\xi\colon S\Omega Y \to \Omega SY$ such that the following diagram commutes*:

$$\begin{tikzcd} & \Omega Y_1 \arrow[dl, "\iota"'] \arrow[dr, "\Omega\iota"] & \\ S_0\Omega Y \arrow[rr, "\xi_0"] & & \Omega S_0 Y. \end{tikzcd}$$

If $\pi_0 Y_1$ is a group, then ξ is an equivalence.

Proof. The last statement will be immediate from Lemma 2.2 and the diagram. There is a natural transformation $\gamma\colon |\Omega Y| \to \Omega|Y|$ for simplicial spaces Y (studied in [8, §12]), namely

$$\gamma(|f, u|)(t) = |f(t), u|$$

for $f \in \Omega Y_q$, $u \in \Delta_q$, and $t \in I$. Clearly this induces $\gamma\colon D\Omega Y \to \Omega DY$ for $\mathscr{F}$-spaces Y. A trivial calculation shows that the following diagram commutes.

$$\begin{tikzcd} & \Omega Y_1 \arrow[dl, "\iota"'] \arrow[dr, "\Omega\iota"] & \\ \Omega D\Omega Y \arrow[r, "\Omega\gamma"] & \Omega\Omega DY \arrow[r, "\tau"] & \Omega\Omega DY. \end{tikzcd}$$

We therefore define $\xi_0 = \tau \circ \Omega\gamma$: $S_0\Omega Y \to \Omega S_0 Y$, and we then define $\xi_1 = \gamma$: $S_1\Omega Y \to \Omega S_1 Y$. These make sense since $S_0 Y = \Omega DY$ and $S_1 Y = DY$. Because the twist τ in the diagram above is the zeroth structural map of the spectrum ΩSY, the equality $\Omega\gamma = \tau \circ \tau \circ \Omega\gamma$ is the compatibility relation $\Omega\zeta_1 \circ \sigma_0 = \sigma_0 \circ \zeta_0$. To proceed further, observe that composition with the loop functor on $\mathcal{F}[\mathcal{T}]$ defines a loop functor on $\mathcal{F}\mathcal{F}[\mathcal{T}]$ such that $\overline{\Omega Y} = \Omega \bar{Y}$. By naturality, the maps

$$\gamma\colon (B\Omega Y)_n = D\Omega \bar{Y}_n \to \Omega D\bar{Y}_n = \Omega(BY)_n$$

specify a map γ_*: $B\Omega Y \to \Omega BY$ of $\mathcal{F}$-spaces. Inductively, let $\gamma_*{}^0$ be the identity map of ΩY and define $\gamma_*{}^i$ for $i > 0$ to be the composite

$$B^i\Omega Y \xrightarrow{B\gamma_*{}^{i-1}} B\Omega B^{i-1} Y \xrightarrow{\gamma_*} \Omega B^i Y.$$

By the naturality of ι and the commutativity of the diagram above we have the following commutative diagram for $i > 0$.

$$\begin{tikzcd}
DB^{i-1}\Omega Y \arrow[r, "D\gamma_*{}^{i-1}"] \arrow[d, equal] & D\Omega B^{i-1}Y \arrow[r, "\gamma"] \arrow[d, equal] & \Omega DB^{i-1}Y \arrow[d, equal] \\
(B^i\Omega Y)_1 \arrow[r, "(B\gamma_*{}^{i-1})_1"] \arrow[dd, "\iota"'] & (B\Omega B^{i-1}Y)_1 \arrow[r, "(\gamma_*)_1"] \arrow[d] & \Omega(B^iY)_1 \arrow[d, "\Omega\iota"] \arrow[ddl, "\iota"'] \\
 & \Omega DB\Omega B^{i-1}Y \arrow[d, "\Omega D\gamma_*"] & \Omega\Omega DB^iY \arrow[d, "\tau"] \\
\Omega DB^i\Omega Y \arrow[r, "\Omega D\gamma_*{}^i"] \arrow[ur, "\Omega DB\gamma_*{}^{i-1}"] & \Omega D\Omega B^iY \arrow[r, "\Omega\gamma"] & \Omega\Omega DB^iY.
\end{tikzcd}$$

We define $\xi_i = \gamma \circ D\gamma_*{}^{i-1}$: $S_i\Omega Y \to \Omega S_i Y$ for $i > 1$ and conclude from the diagram that ξ: $S\Omega Y \to \Omega SY$ is then a map of prespectra such that $\xi_0 \circ \iota = \Omega\iota$, as desired.

The key to the proof of the uniqueness theorem is the following definition and theorem. These are inspired by and closely analogous to, but nevertheless different from, the corresponding parts of Fiedorowicz' paper[6]. We defer a precise comparison until Appendix A.

Definition 3.8. A bispectrum is a sequence of connective spectra F_i and equivalences τ_i: $F_i \to \Omega F_{i+1}$.

In more detail, a bispectrum consists of spaces F_{ij} and equivalences σ_{ij}: $F_{ij} \to \Omega F_{i,j+1}$ and τ_{ij}: $F_{ij} \to \Omega F_{i+1,j}$ such that the following diagrams commute:

$$\begin{tikzcd}
F_{ij} \arrow[rr, "\sigma_{ij}"] \arrow[d, "\tau_{ij}"'] & & \Omega F_{i,j+1} \arrow[d, "\Omega\tau_{i,j+1}"] \\
\Omega F_{i+1,j} \arrow[r, "\Omega\sigma_{i+1,j}"] & \Omega\Omega F_{i+1,j+1} \arrow[r, "\tau"] & \Omega\Omega F_{i+1,j+1}
\end{tikzcd}$$

We shall sometimes write F_{i*} for the spectrum F_i. Symmetrically, the spaces F_{ij} and maps τ_{ij} for fixed j form a spectrum F_{*j}. The following "up and across theorem" will be proven in Appendix A.

THEOREM 3.9. *The spectra $F_0 = F_{0*}$ and F_{*0} associated to a bispectrum F are naturally equivalent.*

We now have all the ingredients required to prove that EY is naturally equivalent to SY if E is an infinite loop space machine defined on $\mathcal{F}$-spaces. By Lemma 3.2 and Construction 3.4, we have improper $\mathcal{F}$-spaces $E_i\bar{Y}$. For clarity of exposition, we assume here that the $E_i\bar{Y}$ are actual $\mathcal{F}$-spaces, relegating the technical elaboration of the argument necessary when this assumption fails to Appendix B. Under this assumption, we have Segal spectra $SE_i\bar{Y}$, and Lemmas 2.3 and 3.3 and Proposition 3.7 yield a composite equivalence

$$\tau_i\colon SE_i\bar{Y} \xrightarrow{S\sigma_i} S\Omega E_{i+1}\bar{Y} \xrightarrow{\varepsilon} \Omega SE_{i+1}\bar{Y}.$$

We have a bispectrum and are entitled to conclude that $SE_0\bar{Y}$ and $S_0E\bar{Y} = \{S_0E_i\bar{Y}, \tau_{i,0}\}$ are equivalent. By Lemmas 2.3 and 3.3 again, $S\iota\colon SY \to SE_0\bar{Y}$ is an equivalence of spectra. By the naturality of ι for S and the diagram in Proposition 3.7, the maps ι: $E_iY = (E_i\bar{Y})_1 \to S_0E_i\bar{Y}$ specify an equivalence of spectra $EY \to S_0E\bar{Y}$. The required natural equivalence between SY and EY follows.

§4. OPERADS AND CATEGORIES OF OPERATORS

We now know how to compare different machines and are faced with the problem of showing that any particular given machine acts in the requisite generality. We begin by relating the data of other machines to the general context of §1. It will be apparent that most existing machines can be viewed as defined only on those $\mathcal{G}$-spaces with underlying Π-space of the form RY for a space Y, hence the problem is to allow for more general underlying Π-spaces. We shall solve this problem in the case of May's machine and leave the remaining cases to the interested reader.

The domain of definition of May's machine is the category of $\mathcal{C}$-spaces for a suitable "operad" $\mathcal{C}$. Aside from their working in the world of simplicial sets and restricting attention to one particular operad, that is also the domain of definition of the theory of Barratt and Eccles[2]. The original theory of Boardman and Vogt[4] was based on the notion of an action by a PROP on a space, and a PROP has an operad as part of its structure. Beck[3] also used PROP-actions. Thus to analyze the domain of definition of any of these theories it suffices to consider $\mathcal{C}$-spaces.

Recall from [8, §1] that an operad $\mathcal{C}$ consists of a sequence of (unbased) spaces $\mathcal{C}(j)$ such that $\mathcal{C}(0)$ is a point $*$, there is a unit element $1 \in \mathcal{C}(1)$, $\mathcal{C}(j)$ has a right action by the symmetric group Σ_j, and there are maps

$$\gamma\colon \mathcal{C}(k)\times\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_k)\to\mathcal{C}(j_1+\cdots+j_k),$$

all subject to appropriate axioms. By [8, App.], we may as well assume that 1 is a nondegenerate basepoint in $\mathcal{C}(1)$. Points of $\mathcal{C}(j)$ are to be thought of as j-ary operations. From these we will construct a space $\hat{\mathcal{C}}(\mathbf{m},\mathbf{n})$ of operations accepting m inputs and yielding n outputs.

Construction 4.1. Let $\mathcal{C}$ be an operad. Construct the associated category of operators $\hat{\mathcal{C}}$ as follows. Let

$$\hat{\mathcal{C}}(\mathbf{m},\mathbf{n}) = \coprod_{\phi\in\mathcal{F}(\mathbf{m},\mathbf{n})}\ \prod_{1\le j\le n} \mathcal{C}(|\phi^{-1}(j)|),$$

where $|S|$ denotes the cardinality of a set S (and $\hat{\mathcal{C}}(\mathbf{m},\mathbf{0})$ is to be interpreted as a point $*$ indexed on the unique map $\mathbf{m}\to\mathbf{0}$ in $\mathcal{F}$). Write elements in the form $(\phi; c_1,\ldots,c_n)$, or $(\phi; c)$ for brevity. For $(\phi; c)\in\hat{\mathcal{C}}(\mathbf{m},\mathbf{n})$ and $(\psi; d)\in\hat{\mathcal{C}}(\mathbf{q},\mathbf{m})$, define

$$(\phi; c)\circ(\psi; d) = \left(\phi\circ\psi;\ \mathop{\times}_{1\le j\le n} \gamma\left(c_j;\ \mathop{\times}_{\phi(i)=j} d_i\right)\sigma_j\right).$$

Here the d_i with $\phi(i)=j$ are ordered by the natural order on their indices i (between 1 and m) and σ_j is that permutation of $|(\phi\psi)^{-1}(j)|$ letters which converts the natural ordering of $(\phi\psi)^{-1}(j)$ as a subset of $\{1,\ldots,q\}$ to its ordering obtained by regarding it as

$\coprod_{\phi(i)=j} \psi^{-1}(i)$ so ordered that all elements of $\psi^{-1}(i)$ precede all elements of $\psi^{-1}(i')$ if $i < i'$ and that each $\psi^{-1}(i)$ has its natural ordering as a subset of $\{1, \ldots, q\}$. The associativity of composition follows readily from the definition of an operad. Define the augmentation ϵ: $\hat{\mathscr{C}} \to \mathscr{F}$ by $\epsilon(\phi; c) = \phi$ and observe that ϵ is an equivalence if each $\mathscr{C}(j)$ is contractible. Embed Π in $\hat{\mathscr{C}}$ by sending ϕ: $\mathbf{m} \to \mathbf{n}$ to $(\phi; c)$, where $c_j = 1$ if $j \in \operatorname{Im} \phi$ and $c_j = *$ if $j \notin \operatorname{Im} \phi$. This makes sense since $|\phi^{-1}(j)|$ is either 0 or 1. Certainly the identity $(1; 1^n)$ is a nondegenerate basepoint in $\hat{\mathscr{C}}(\mathbf{n}, \mathbf{n})$. If ϕ is an injection, then left composition by ϕ from $\hat{\mathscr{C}}_m$ to $\hat{\mathscr{C}}_n$ is an inclusion onto some of the components and is trivially a Σ_ϕ-equivariant cofibration.

Observe that this construction gives a functor from operads to categories of operators.

Definition 1.2 now specializes to give a notion of $\hat{\mathscr{C}}$-space, and the following observation shows that this is a natural generalization of the notion of $\mathscr{C}$-space [8, §1].

LEMMA 4.2. *A $\hat{\mathscr{C}}$-space with underlying Π-space RY determines and is determined by a $\mathscr{C}$-space structure on Y.*

Proof. An action of $\mathscr{C}$ on Y is a map of operads $\mathscr{C} \to \mathscr{E}_Y$ and so determines a map of categories of operators $\hat{\mathscr{C}} \to \hat{\mathscr{E}}_Y$, where $\mathscr{E}_Y$ is the endomorphism operad of Y [8, §1]. Since $\mathscr{E}_Y(j)$ is the space of based maps $Y^j \to Y$, a trivial reinterpretation allows us to regard $\hat{\mathscr{C}} \to \hat{\mathscr{E}}_Y$ as a $\hat{\mathscr{C}}$-space $\hat{\mathscr{C}} \to \mathscr{T}$ with underlying Π-space RY. For the converse, let $\phi_j \in \mathscr{F}(\mathbf{j}, \mathbf{1})$ be given by $\phi_j(i) = 1$ for $1 \le i \le j$, this being the canonical j-fold product in the context of $\mathscr{F}$-spaces, and think of $\mathscr{C}(j)$ as the component of ϕ_j in $\hat{\mathscr{C}}(\mathbf{j}, \mathbf{1})$. Given a $\hat{\mathscr{C}}$-space with underlying Π-space RY, restriction gives maps $\mathscr{C}(j) \to \mathscr{E}_Y(j)$ which define a morphism of operads, the maps γ and unit 1 being preserved since a $\hat{\mathscr{C}}$-space is a functor and the action of Σ_j being preserved since $(\phi_j; c\sigma) = (\phi_j; c)(\sigma; 1^j)$ for $\sigma \in \Sigma_j$.

The following philosophical remarks on the comparisons between $\mathscr{F}$-spaces, $\mathscr{C}$-spaces, and $\hat{\mathscr{C}}$-spaces may be illuminating.

Remarks 4.3. Let $\mathscr{N}$ be the operad such that each $\mathscr{N}(j)$ is a point. Clearly $\hat{\mathscr{N}}$ is precisely $\mathscr{F}$. An $\mathscr{N}$-space is the same thing as a commutative topological monoid hence, if connected, has the homotopy type of a product of Eilenberg–MacLane spaces. Thus, for $\mathscr{F}$-spaces, the higher homotopies essential to distinguish general infinite loop spaces from $K(\pi, n)$'s come entirely from the distinction between general Π-spaces and Π-spaces of the form RY. For $\mathscr{C}$-spaces, these homotopies come entirely from the contractibility (or lesser connectivity in the theory of n-fold loop spaces) of the spaces $\mathscr{C}(j)$. The notion of $\hat{\mathscr{C}}$-space allows both sources of higher homotopies and is the natural simultaneous generalization of the notions of Γ-space and E_∞ space. There are no known machines based on categories of operators which do not come from operads.

§5. OPERADS AND MONADS

The required generalization of May's machine from $\mathscr{C}$-spaces to $\hat{\mathscr{C}}$-spaces amounts to a change of ground categories from $\mathscr{T}$ to $\Pi[\mathscr{T}]$. The essential starting point is the fact that the functorial association of a monad to an operad of [8, §2] remains valid after this change. Modulo the key verification, which is deferred momentarily, the following construction gives the monad $(\hat{C}, \hat{\mu}, \hat{\eta})$ in $\Pi[\mathscr{T}]$ associated to an operad $\mathscr{C}$.

Construction 5.1. Let X: $\Pi \to \mathscr{T}$ be a functor. Construct a functor $\hat{C}X$: $\Pi \to \mathscr{T}$ and natural transformations $\hat{\eta}$: $X \to \hat{C}X$ and $\hat{\mu}$: $\hat{C}\hat{C}X \to \hat{C}X$ as follows. The nth space $\hat{C}_nX$ of $\hat{C}X$ is the "coend"

$$\int_\Pi \hat{\mathscr{C}}(\mathbf{m}, \mathbf{n}) \times X_m = \coprod_{m \ge 0} \hat{\mathscr{C}}(\mathbf{m}, \mathbf{n}) \times X_m/(\sim),$$

where the equivalence relation is specified by

$$((\phi;c)\psi,x)\sim((\phi;c),\psi x)$$

for $(\phi;c)\in\hat{\mathscr{C}}(\mathbf{m},\mathbf{n})$, $\psi\in\Pi(\mathbf{q},\mathbf{m})$, and $x\in X_q$. Composition on the left by maps $\mathbf{n}\to\mathbf{p}$ in Π gives maps $\hat{C}_nX\to\hat{C}_pX$ such that $\hat{C}X$ is a functor. The maps $\hat{\eta}$: $X_n\to\hat{C}_nX$ and $\hat{\mu}$: $\hat{C}_n\hat{C}X\to\hat{C}_nX$ are obtained by use of the identity maps and composition of the category $\hat{\mathscr{C}}$, and the fact that $\hat{\mathscr{C}}$ is a category implies the commutativity of the diagrams

$$\begin{tikzcd} \hat{C}X \arrow[r, "\hat{C}\hat{\eta}"] \arrow[dr, equal] & \hat{C}\hat{C}X \arrow[d, "\hat{\mu}"] & \hat{C}X \arrow[l, "\hat{\eta}"'] \arrow[dl, equal] \\ & \hat{C}X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \hat{C}\hat{C}\hat{C}X \arrow[r, "\hat{\mu}"] \arrow[d, "\hat{C}\hat{\mu}"'] & \hat{C}\hat{C}X \arrow[d, "\hat{\mu}"] \\ \hat{C}\hat{C}X \arrow[r, "\hat{\mu}"] & \hat{C}X \end{tikzcd}$$

Therefore $(\hat{C},\hat{\mu},\hat{\eta})$ is a monad in the category of functors $\Pi\to\mathscr{T}$.

To show that $(\hat{C},\hat{\mu},\hat{\eta})$ is a monad in $\Pi[\mathscr{T}]$, we must show that $\hat{C}X$ is a Π-space if X is a Π-space; that is, we must show that $\hat{C}X$ inherits properties (1)–(3) of Definition 1.2 from X. As the cofibration condition (3) is easily verified, this not depending on the corresponding condition for X, we concentrate on (1) and (2). We require some elementary facts about the morphisms of $\mathscr{F}$ in order to give an explicit description of the spaces $\hat{C}_nX$.

Notations 5.2. Say that a morphism π: $\mathbf{m}\to\mathbf{n}$ of $\mathscr{F}$ is a projection if $\pi^{-1}(j)$ has exactly one element for $1\le j\le n$. Say that a morphism ϵ of $\mathscr{F}$ is effective if $\epsilon^{-1}(0)=\{0\}$. The effective morphisms are to be thought of as those operations of $\mathscr{F}$ which genuinely involve all variables. Note that the effective projections are precisely the permutations and the effective morphisms in Π are precisely the injections. Say that an effective morphism ϵ: $\mathbf{m}\to\mathbf{n}$ is ordered if $\epsilon(i)<\epsilon(i')$ implies $i<i'$ and note that the unique effective morphism, ϕ_m, from $\mathbf{m}$ to $\mathbf{1}$ is ordered. Let $\mathscr{E}$ be the subcategory of $\mathscr{F}$ whose morphisms are effective and ordered.

LEMMA 5.3. *A morphism ϕ: $\mathbf{m}\to\mathbf{n}$ in $\mathscr{F}$ factors as a composite*

$$\mathbf{m}\xrightarrow{\pi}\mathbf{q}\xrightarrow{\epsilon}\mathbf{n},\ q=n-|\phi^{-1}(0)|,$$

where π is a projection and ϵ is effective. If $\phi=\epsilon'\pi'$ is another such factorization, then there is a unique permutation σ: $\mathbf{q}\to\mathbf{q}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathbf{q} \arrow[dr, "\epsilon"] \arrow[dd, "\sigma"] & \\ \mathbf{m} \arrow[ur, "\pi"] \arrow[dr, "\pi'"'] & & \mathbf{n} \\ & \mathbf{q} \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

LEMMA 5.4. *For any effective morphism ϵ: $\mathbf{m}\to\mathbf{n}$, there is a permutation τ: $\mathbf{m}\to\mathbf{n}$ such that $\epsilon\tau$ is ordered. If ϵ is ordered, then $\epsilon\tau$ is ordered if and only if $\tau\in\Sigma(\epsilon)$, where*

$$\Sigma(\epsilon)=\Sigma_{|\epsilon^{-1}(1)|}\times\cdots\times\Sigma_{|\epsilon^{-1}(n)|}\subset\Sigma_m.$$

These lemmas and the fact that the unique map $\mathbf{m}\to\mathbf{0}$ in $\mathscr{F}$ factors as $\mathbf{m}\to\mathbf{0}\to\mathbf{0}$ are easily seen to imply the following description of the $\hat{C}_nX$.

LEMMA 5.5. $\hat{C}_0X=X_0$. *For $n\ge1$, let $F_p\hat{C}_nX$ be the image of $\coprod_{m\le p}\hat{\mathscr{C}}(\mathbf{m},\mathbf{n})\times X_m$ in* $\hat{C}_nX$. *Then $\hat{C}_nX$ is the union of the $F_p\hat{C}_nX$. If X is a Π-space, then $F_p\hat{C}_nX$ is obtained*

from $F_{p-1}\hat{C}_n X$ by means of a pushout diagram

$$\begin{array}{ccc} \coprod_{\epsilon\in\mathscr{E}(p,n)} \prod_{1\le j\le n} \mathscr{C}(|\epsilon^{-1}(j)|)\times_{\Sigma(\epsilon)} sX_{p-1} & \xrightarrow{\nu} & F_{p-1}\hat{C}_n X, \\ \cap\downarrow & & \downarrow \\ \coprod_{\epsilon\in\mathscr{E}(p,n)} \prod_{1\le j\le n} \mathscr{C}(|\epsilon^{-1}(j)|)\times_{\Sigma(\epsilon)} X_p & \longrightarrow & F_p\hat{C}_n X \end{array}$$

where sX_{p-1} is the union of the images of X_{p-1} under the p ordered injections σ_i: $\mathbf{p-1}\to\mathbf{p}$ and ν is specified by

$$\nu((\epsilon;c),\sigma_i x) = ((\epsilon,c)\circ\sigma_i, x).$$

Moreover, the vertical arrows of the diagram are cofibrations.

Here the last statement is a consequence of results in [4, App. §2]. By standard facts about the homotopy invariance of pushouts and unions of cofibrations together with an inductive argument just like that in the proof of [9, A. 4], the lemma has the following consequence.

LEMMA 5.6. *Assume that each $\mathscr{C}(j)$ is Σ_j-free. If f: $X\to X'$ is an equivalence of Π-spaces, then each $\hat{C}_n f$: $\hat{C}_n X\to\hat{C}_n X'$ is an equivalence.*

The hypothesis on the $\mathscr{C}(j)$ is used to deduce that f induces an equivalence on the spaces in the left column of the diagram of Lemma 5.5 because it induces an equivalence on the corresponding spaces before passage to orbits with respect to the $\Sigma(\epsilon)$. Technically, we are using the homotopy exact sequences of the evident coverings and the fact that, in the presence of cofibrations, a pushout of (weak) equivalences is a (weak) equivalence [14, III.8.2].

We need another consequence of Lemma 5.5. Recall the functors L and R relating spaces to Π-spaces from Definition 1.3 and recall the construction of the monad (C,μ,η) in $\mathscr{T}$ associated to $\mathscr{C}$ from [8, 2.4]. An easy inspection gives the following result.

LEMMA 5.7. *Let $Y\in\mathscr{T}$. Then $L\hat{C}RY\equiv\hat{C}_1RY = CY$ and $\hat{C}RY$ coincides with the Π-space RCY.*

For a Π-space X, let δ: $X\to RLX$ be the natural equivalence (given by $X_n\to X_1{}^n$ on the nth space; compare Definition 1.3). We have the commutative diagram

$$\begin{array}{ccc} \hat{C}_n X & \xrightarrow{\hat{C}_n\delta} & \hat{C}_n RLX. \\ \downarrow & & \downarrow\cong \\ (\hat{C}_1 X)^n & \xrightarrow{(\hat{C}_1\delta)^n} & (\hat{C}_1 RLX)^n \end{array}$$

Thus the previous lemmas imply the following result.

PROPOSITION 5.8. *Assume that each $\mathscr{C}(j)$ is Σ_j-free. then $\hat{C}X$ is a Π-space if X is a Π-space, hence $(\hat{C},\hat{\mu},\hat{\eta})$ is a monad in the category of Π-spaces.*

§6. MAY'S MACHINE

May's construction of spectra from spaces acted upon by suitable operads is based on a categorical two-sided bar construction $B(F,C,X)$ defined when given a monad C in some ground category $\mathscr{W}$ which acts from the left on an object $X\in\mathscr{W}$ and from the right on a functor F: $\mathscr{W}\to\mathscr{V}$, where $\mathscr{V}$ is some category with underlying spaces.

(See [8, §9].) It was the freedom to use arbitrary ground categories which gave this approach its efficacy in multiplicative infinite loop space theory, and that freedom also makes it easy to generalize the construction from spaces to Π-spaces.

We shall apply the bar construction to monads $\hat{C}$ in $\Pi[\mathcal{T}]$ associated to operads $\mathscr{C}$ with each $\mathscr{C}(j)$ Σ_j-free, and we need a few preliminaries concerning the variables X and F.

Just as in [8, 2.8], we can replace $\hat{\mathscr{C}}$ spaces by $\hat{C}$-spaces, collections of action maps $\hat{\mathscr{C}}(m, n) \times X_m \to X_n$ being replaced by sequences of action maps $\hat{C}_n X \to X_n$.

LEMMA 6.1. *A $\hat{\mathscr{C}}$-space determines and is determined by a $\hat{C}$-space in such a way that the categories of $\hat{\mathscr{C}}$-spaces and $\hat{C}$-spaces are isomorphic.*

To obtain the appropriate $\hat{C}$-functors, we must elaborate Lemma 5.7 to a formal comparison between the monad (C, μ, η) on spaces and the monad $(\hat{C}, \hat{\mu}, \hat{\eta})$ on Π-spaces. Observe that the functor RCL on Π-spaces is a monad with unit and product induced from those of C via the maps

$$X \xrightarrow{\delta} RLX \xrightarrow{R\eta} RCLX \quad \text{and} \quad RCLRCLX = RCCLX \xrightarrow{R\mu} RCLX.$$

(Recall that $LX = X_1$, $RY = \{Y^i\}$, and thus $LRY = Y$ for a space Y.)

LEMMA 6.2. *The maps $\hat{C}\delta$: $\hat{C}X \to \hat{C}RLX \cong RCLX$ specify a morphism of monads in $\Pi[\mathcal{T}]$.*

If (F, λ) is a C-functor in $\mathscr{V}$, so that F is a functor $\mathcal{T} \to \mathscr{V}$ and λ: $FC \to F$ is a natural transformation such that $\lambda \circ F\eta = 1$: $F \to F$ and $\lambda \circ F\mu = \lambda \circ \lambda$: $FCC \to F$, then FL: $\Pi[\mathcal{T}] \to \mathscr{V}$ is an RCL-functor in $\mathscr{V}$ with action induced from λ via the maps

$$FLRCLX = FCLX \xrightarrow{\lambda} FLX.$$

Therefore, by pullback, the previous lemma has the following consequence.

LEMMA 6.3. *If (F, λ) is a C-functor in $\mathscr{V}$, then FL is a $\hat{C}$-functor in $\mathscr{V}$ with action given by the composites*

$$FL\hat{C}X \xrightarrow{FL\hat{C}\delta} FLRCLX = FCLX \xrightarrow{\lambda} FLX.$$

We can now prove the following theorem, which allows us to apply the uniqueness theorem to compare the May and Segal machines. For what it is worth, we note that the argument applies equally well to generalize May's recognition principle for n-fold loop spaces from $\mathscr{C}_n$-spaces to $\hat{\mathscr{C}}_n$-spaces, where $\mathscr{C}_n$ is the nth little cubes operad [4, II.6 and 8, §4].

THEOREM 6.4. *Let $\mathscr{C}$ be an operad such that each $\mathscr{C}(j)$ is contractible. Then the functor M from $\mathscr{C}$-spaces to spectra constructed in* [8] *and* [9] *extends to an infinite loop space machine M defined on $\hat{\mathscr{C}}$-spaces.*

Proof. The details of [8] and [9] go through with only the slightest of modifications. We follow the sketch in [9, §2]. Let $\mathscr{D}_n$ be the product operad $\mathscr{C} \times \mathscr{C}_n$. Since each $\mathscr{C}_n(j)$ is Σ_j-free, so is each $\hat{\mathscr{D}}(j)$. Let D_n and $\hat{D}_n$ be the monads in $\mathcal{T}$ and $\Pi[\mathcal{T}]$ associated to $\mathscr{D}_n$. For a $\hat{C}$-space X, we have spaces $B(\Sigma^n L, \hat{D}_n, X)$ and inclusions (which are not equivalences)

$$B(\Sigma^n L, \hat{D}_n, X) \longrightarrow B(\Omega\Sigma^{n+1} L, \hat{D}_{n+1}, X) \xrightarrow{\gamma} \Omega B(\Sigma^{n+1} L, \hat{D}_{n+1}, X),$$

the first map being induced by the natural map $Y \to \Omega\Sigma Y$ and the inclusion $\mathscr{D}_n \to \mathscr{D}_{n+1}$ and the map γ being the natural comparison between geometric realization on loops and loops on geometric realization. We define MX to be the spectrum obtained by

application of the functor Ω^∞ to this prespectrum, so that

$$M_iX = \operatorname{colim} \Omega^j B(\Sigma^{i+j}L, D_{i+j}, X).$$

Clearly M is a functor from $\hat{\mathscr{C}}$-spaces to spectra. It follows formally from the natural isomorphism $D_nR \cong RD_n$ of functors on $\mathscr{C}$-spaces that

$$B(\Sigma^n L, \hat{D}_n, RY) \cong B(\Sigma^n, D_n, Y)$$

for $\mathscr{C}$-spaces Y. Thus the functor MR from $\mathscr{C}$-spaces to spectra agrees with that obtained in [8] and [9].

The required natural group completion ι: $X_1 \to M_0X$ is defined by commutativity of the following diagram:

$$\begin{tikzcd}
X_1 = LX \arrow[r, shift left, "L\tau"] \arrow[d, "\iota"] & LB(\hat{D}_\infty, \hat{D}_\infty, X) = B(L\hat{D}_\infty, \hat{D}_\infty, X) \arrow[l, shift left, dashed, "L\epsilon"] \arrow[d, "B(L\hat{D}_\infty\delta)"] \\
M_0X & B(QL, \hat{D}_\infty, X) \arrow[l, "\gamma^\infty"'] \quad B(DL, \hat{D}_\infty, X) \arrow[l, "B(\alpha_\infty\pi)"']
\end{tikzcd}$$

Here ϵ is the usual natural equivalence with inverse τ [8, 9.10], γ^∞ is the limit over n of the natural comparisons

$$\gamma^n: B(\Omega^n\Sigma^n L, \hat{D}_n, X) \longrightarrow \Omega^n B(\Sigma^n L, \hat{D}_n, X)$$

and is an equivalence by [8, 12.3], $B(f)$ is short for $B(f, 1, 1)$, π is the projection $D_\infty \to C_\infty$, and α_∞ is the natural map $C_\infty \to Q$ of [8, 5.2]. $B(\alpha_\infty\pi)$ is a group completion by precisely the same argument as was given in [9, 2.3], the only caveat being that the proof of [9, 2.2] is incomplete but is completed in [12, VI.2.7(iv)], and $B(L\hat{D}_\infty\delta)$ is an equivalence by Lemma 5.6 and [9, A.4] since δ: $X \to RLX$ is an equivalence for any Π-space X. Thus ι is a composite of equivalences and a group completion and is thus a group completion as asserted. This completes the proof.

We close with a conjecture on the nature of the spaces LX for a $\hat{\mathscr{C}}$-space X. While we are quite confident of its truth, we have not attempted a proof. Its statement refers to Lada's theory of strong homotopy C-spaces [5, V].

Conjecture 6.5. If X is a $\hat{C}$-space, then LX is naturally a strong homotopy C-space.

Since Lada's theory provides a functor U from *sh* C-spaces to actual C-spaces, this would have the conceptual attractiveness of completing the triangle

$$\begin{tikzcd}
& C\text{-spaces} \arrow[dr, "R"] & \\
sh\ C\text{-spaces} \arrow[ur, "U"] & & \hat{C}\text{-spaces} \arrow[ll, dashed]
\end{tikzcd}$$

and would presumably lead to homotopy invariance theorems for $\hat{C}$-spaces.

APPENDIX A. PRETERNATURALITY AND THE UP AND ACROSS THEOREM

We aim to prove Theorem 3.9, the up and across theorem. This is in fact quite elementary, given a certain amount of folklore technique. There are two technical points involved. First, it is essential that we use the loop functor on spectra defined with a twist in the structural maps and not the translation desuspension Λ; the latter is defined by $(\Lambda E)_i = \Omega(E_i)$, with structural maps $\Omega\sigma_i$: $\Omega E_i \to \Omega\Omega E_{i+1}$. The second is that we want to come out with a natural equivalence in the category of spectra rather than in the category of spectra and weak maps. The difference between these categories (or rather, between appropriate derived categories) is the same as the difference between cohomology theories on spaces and cohomology theories on spectra and is discussed, for example, in [12, II §3]; use of weak maps amounts to neglect of certain $\lim^1$ terms, or phantom maps.

These points are not unrelated. If $d_i\colon \Omega E_i \to \Omega E_i$ is defined to be the identity map for even i and the negative of the identity interpreted as coordinate reversal, $d(g)(t) = g(1-t)$, for odd i, then $d\colon \Omega E \to \Lambda E$ is a weak map. However, the relevant homotopies are natural in E, and a folklore result, perhaps due to Boardman from whom one of us learned it, allows us to conclude that ΩE and ΛE are naturally equivalent in the category of spectra. We shall need this argument twice, so we make it precise in the following definition and lemma, in which preternaturality allows control over phantoms.

Definition A.1. Let D and D' be functors from any category $\mathcal{K}$ to the category of spectra. A preternatural transformation $d\colon D \to D'$ is a natural transformation d in the category of spectra and weak maps together with a natural choice of (based) homotopies. That is, for each $K \in \mathcal{K}$ and each $i \geq 0$, there is given a map $d_i\colon D_iK \to D_i'K$ and a homotopy $h_i\colon D_iK \times I \to \Omega D_{i+1}'K$ from $\Omega d_{i+1} \circ \sigma_i$ to $\sigma_i' \circ d_i$, both d_i and h_i being natural in K.

LEMMA A.2. *There is a functor T from spectra to spectra and a natural equivalence $TE \to E$ such that any preternatural transformation $d\colon D \to D'$ determines a natural transformation $\bar{d}\colon TD \to D'$ of spectra-valued functors in such a way that the diagram of weak maps*

$$\begin{tikzcd}[column sep=small] & TDK \arrow[dl] \arrow[dr, "\bar{d}"] & \\ DK \arrow[rr, "d"] & & D'K \end{tikzcd}$$

homotopy commutes for each K. If $d\colon DK \to D'K$ is an equivalence, then so $\bar{d}\colon TDK \to D'k$.

Proof. The last statement will follow from the diagram. The functor T is given by the iterated mapping cylinder construction considered in [7, Thm 4]; that T is a functor from spectra to spectra and that there is a natural equivalence $TE \to E$ follows from the proof of [7, Thm 4(ii) and (iv)]. From the construction of T and generalities about mapping cylinders, maps of spectra $TE \to E'$ correspond bijectively to data consisting of a weak map $e\colon E \to E'$ and a choice of homotopies $h_i\colon \Omega e_{i+1} \circ \sigma_i \simeq \sigma_i' \circ e_i$ for $i \geq 0$, this correspondence being natural in E and E'. The existence of $\bar{d}$ follows, and homotopy commutativity of the triangle holds by [7, Thm 4(iii)].

Proof of Theorem 3.9. Let F be a bispectrum, with structural equivalences $\sigma_{ij}\colon F_{ij} \to \Omega F_{i,j+1}$ and $\tau_{ij}\colon F_{ij} \to \Omega F_{i+1,j}$, and recall the diagram following Definition 3.8. A map $f\colon F \to F'$ of bispectra is a collection of maps $f_{ij}\colon F_{ij} \to F'_{ij}$ such that f_{i*} and f_{*j} are maps of spectra for all i and j. We must show that the functors from bispectra to spectra which send F to its edge spectra F_{0*} and F_{*0} are naturally equivalent. In fact, we shall construct "diagonal" functors D and D' from bispectra to spectra together with natural equivalences $e\colon F_{0*} \to DF$ and $e'\colon F_{*0} \to D'F$ and a preternatural equivalence $d\colon DF \to D'F$. By the previous lemma, this will suffice. Thus define

$$D_nF = \Omega^n F_{nn} = D_n'F,$$

the structural maps $\delta_n\colon D_nF \to \Omega D_{n+1}F$ and $\delta_n'\colon D_n'F \to \Omega D_{n+1}'F$ being the respective composites

$$\Omega^n F_{nn} \xrightarrow{\Omega^n \sigma_{nn}} \Omega^{n+1} F_{n,n+1} \xrightarrow{\mu_n} \Omega^{n+1} F_{n,n+1} \xrightarrow{\Omega^{n+1}\tau_{n,n+1}} \Omega^{n+2} F_{n+1,n+1}$$

and

$$\Omega^n F_{nn} \xrightarrow{\Omega^n \tau_{nn}} \Omega^{n+1} F_{n+1,n} \xrightarrow{\mu_n} \Omega^{n+1} F_{n+1,n} \xrightarrow{\Omega^{n+1}\sigma_{n,n+1}} \Omega^{n+2} F_{n+1,n+1},$$

where μ_n is given by twisting the last coordinate past the first n coordinates, $(\mu_n g)(s)(t) = g(t)(s)$ for $s \in S^n$ and $t \in S^1$. For a map $f\colon F \to F'$ of bispectra, D_nf and $D_n'f$ are both $\Omega^n f_{nn}$. By the definition of a bispectrum, the two structural maps $\Omega^n F_{nn} \to \Omega^{n+2} F_{n+1,n+1}$ differ by the interchange of the first and last coordinates of the functor Ω^{n+2}. Note that the map of spheres which induces this has degree minus one and define $d_n\colon \Omega^n F_{nn} \to \Omega^n F_{nn}$ to be the identity map for even n and the negative of the identity interpreted as reversal of the first coordinate for odd n. Then $d\colon DF \to D'F$ is a natural weak map, and we can use the coordinate spheres to obtain homotopies $h_n\colon \Omega d_{n+1} \circ \delta_n \simeq \delta_n' \circ d_n$ which are natural in F. Thus d is preternatural, and it remains to construct the natural equivalences e and e'. By symmetry, it suffices to construct $e\colon F_{0*} \to DF$. Define equivalences $e_{in}\colon F_{in} \to \Omega^{n-i} F_{nn}$ by $e_{nn} = 1$ and, inductively, by letting e_{in} for $i < n$ be the composite

$$F_{in} \xrightarrow{\tau_{in}} \Omega F_{i+1,n} \xrightarrow{\Omega e_{i+1,n}} \Omega^{n-i} F_{nn}.$$

By a diagram chase which uses induction on $n-i$, the definition of a bispectrum, the naturality of τ, and the fact that $\tau \circ \Omega\mu_{n-i-1} = \mu_{n-i}$, we find that for each fixed $i \geq 0$ the maps e_{in} for $n \geq i$ specify a map of spectra $\{F_{in}, \sigma_{in}\} \to \{\Omega^{n-i} F_{nn}, \chi_{in}\}$, where χ_{in} is the composite

$$\Omega^{n-i} F_{nn} \xrightarrow{\Omega^{n-i}\sigma_{nn}} \Omega^{n+1-i} F_{n,n+1} \xrightarrow{\mu_{n-i}} \Omega^{n+1-i} F_{n,n+1} \xrightarrow{\Omega^{n+1-i}\tau_{n,n+1}} \Omega^{n+2-i} F_{n+1,n+1}.$$

The case $i = 0$ gives the required equivalence e.

Remarks A.3. We reiterate that the idea behind the up and across theorem is due to Fiedorowicz[6]. However, he worked with weak maps and, at least in the preprint version of [6], there was ambiguity as to whether Ω stood for the loop or the translation desuspension. The first author wishes to point out that the same ambiguity crept into [12], where the loop of a spectrum is correctly defined in II.2.3 but Ω is implicitly the translation desuspension in VII.3.4 (since that result refers back to [8, p. 147], where $\Omega = \Lambda$). Of course, the arguments here show that the confusion causes no real mathematical difficulty. Our version of the up and across theorem implies that there is a version of Fiedorowicz' uniqueness theorem valid for appropriate functors from rings to our category of spectra.

APPENDIX B. WHISKERINGS OF $\hat{\mathscr{C}}$-SPACES

We aim to prove Proposition 1.6 and related results needed to complete the proof of the uniqueness theorem. In that proposition, "appropriate" categories of operators are those of the form $\hat{\mathscr{C}}$ for an operad $\mathscr{C}$. We do not assume that Σ_j acts freely on $\mathscr{C}(j)$, hence the operad $\mathscr{N}$ with $\hat{\mathscr{N}} = \mathscr{F}$ of Remarks 4.3 is allowed. Indeed, this is the only example required for the proof of the uniqueness theorem. Along the lines of the cited remarks, the construction here is the appropriate simultaneous generalization of the whiskering of $\mathscr{C}$-spaces considered in [8, App.] and the thickening of Γ-spaces considered in [15, App.].

Let X be an improper $\hat{\mathscr{C}}$-space. We shall construct a (proper) $\hat{\mathscr{C}}$-space WX and an equivalence π: $WX \to X$ of improper $\hat{\mathscr{C}}$-spaces in the following definition and lemmas.

Let $I = [0, 1]$ have basepoint 1 and the product specified by $(s, t) \to \min\{s, t\}$; it is essential to Lemma B.7 below that minima rather than products of real numbers be used. Then I is a topological Abelian monoid and so determines an $\mathscr{F}$-space I: $\mathscr{F} \to \mathscr{T}$ with nth space I^n. Explicitly, the map ϕ: $I^m \to I^n$ determined by a map ϕ: $\mathbf{m} \to \mathbf{n}$ in $\mathscr{F}$ is specified by

$$\phi(s_1, \ldots, s_m) = (t_1, \ldots, t_n), \quad \text{where} \quad t_j = \min_{\phi(i)=j} \{s_i\}.$$

Here the minimum of the empty set is to be interpreted as the basepoint 1. Regard I as a $\hat{\mathscr{C}}$-space via the augmentation $\hat{\mathscr{C}} \to \mathscr{F}$.

We have a product improper $\hat{\mathscr{C}}$-space $I \times X$ with nth space $I^n \times X_n$.

Definition B.1. Define W_nX to be the subspace of $I^n \times X_n$ which consists of those points (t, x), $t = (t_1, \ldots, t_n)$, such that $x \in \phi(X_m)$, where ϕ is that ordered injection such that $j \in \operatorname{Im} \phi$, $1 \le j \le n$, if and only if $t_j = 0$. Define π: $W_nX \to X_n$ by $\pi(t, x) = x$. Observe that X_n embeds in W_nX as the set of points $(0, \ldots, 0, x)$ and is an unbased deformation retract of W_nX via the deformation $h((t, x), r) = (rt, x)$, where $rt = (rt_1, \ldots, rt_n)$. Thus π is an equivalence.

Remarks B.2. W_nX should be thought of as obtained from X_n by growing a beard consisting of a thick whisker I^{n-m} attached to each point of $\phi(X_m)$ for each ordered injection ϕ: $\mathbf{m} \to \mathbf{n}$, but with identifications of partial whiskers corresponding to compositions of injections. When the Π-space X is RY for a based space Y, W_nX is precisely $(I \vee Y)^n$, where I is given the basepoint 0 in forming the wedge.

LEMMA B.3. *The action map $\hat{\mathscr{C}}(m, n) \times I^m \times X_m \to I^n \times X_n$ restricts to a map $\hat{\mathscr{C}}(m, n) \times W_mX \to W_nX$, hence WX is a subfunctor of $I \times X$.*

Proof. Let $(\phi; c_1, \ldots, c_n) \in \hat{\mathscr{C}}(\mathbf{m}, \mathbf{n})$, ϕ: $\mathbf{m} \to \mathbf{n}$ in $\mathscr{F}$ and $c_j \in \mathscr{C}(|\phi^{-1}(j)|)$ and let $(s, x) \in W_mX$. We must show that the element

$$(\phi; c)(s, x) = (t, (\phi; c)(x))$$

of $I^n \times X_n$ is in W_nX. Let ψ: $\mathbf{p} \to \mathbf{m}$ and ω: $\mathbf{q} \to \mathbf{n}$ be the ordered injections such that $i \in \operatorname{Im} \psi$ if and only if $s_i = 0$ and $j \in \operatorname{Im} \omega$ if and only if $t_j = 0$. We are given that $x \in \psi(X_p)$, say $x = \psi(x')$, and must show that $(\phi; c)(x) \in \omega(X_q)$. Since $t_j = \min_{\phi(i)=j} \{s_i\}$, $i \in \operatorname{Im} \psi$ implies $\phi(i) \in \operatorname{Im} \omega$. There is thus a unique map ζ: $\mathbf{p} \to \mathbf{q}$ in $\mathscr{F}$ such that $\omega \circ \zeta = \phi \circ \psi$. If $\hat{\mathscr{C}}$ were $\mathscr{F}$ we would be done since we would have $\phi(x) = \phi\psi(x') = \omega\zeta(x')$. For a general $\mathscr{C}$, Construction 4.1 gives that

$$(\phi; c_1, \ldots, c_n) \circ \psi = (\phi \circ \psi; \bar{c}_1, \ldots, \bar{c}_n)$$

for certain $\bar{c}_j$ and that $\bar{c}_j = * \in \mathscr{C}(0)$ if $j \in \mathbf{n} - \operatorname{Im} \omega$ since then $j \in \mathbf{n} - \operatorname{Im}(\phi\psi)$. It follows that

$$(\phi \circ \psi; \bar{c}_1, \ldots, \bar{c}_n) = \omega \circ (\zeta; d_1, \ldots, d_m)$$

for certain d_i. We therefore have that

$$(\phi; c)(x) = (\phi; c)\psi(x') = (\phi \circ \psi; \bar{c})(x') = \omega(\zeta; \bar{d})(x').$$

The following result completes the proof of Proposition 1.6.

LEMMA B.4. *WX is a $\hat{\mathscr{C}}$-space and π: $WX \to X$ is an equivalence of improper $\hat{\mathscr{C}}$-spaces.*

Proof. In view of Definition B.1 and Lemma B.3, we need only show that ϕ: $W_mX \to W_nX$ is a Σ_ϕ-equivariant cofibration if ϕ: $\mathbf{m} \to \mathbf{n}$ is an injection in Π. We claim first that

$$\phi(W_mX) = \{(t, x) | t_j = 1 \quad \text{if} \quad j \notin \operatorname{Im} \phi\} \subset W_nX.$$

We certainly have $t_j = 1$ if $j \notin \operatorname{Im} \phi$ for all points in this image (by the convention that the minimum of the empty set is 1). Conversely, consider (t, x) with $t_j = 1$ if $j \notin \operatorname{Im} \phi$. Let ψ: $\mathbf{p} \to \mathbf{n}$ be that ordered injection such that $j \in \operatorname{Im} \psi$ is and only if $t_j = 0$, so that $x \in \psi(X_p)$. Then $\operatorname{Im} \psi \subset \operatorname{Im} \phi$ and there is a unique map ζ: $\mathbf{p} \to \mathbf{m}$

in Π such that $\psi = \phi \circ \zeta$. Thus $x \in \phi(X_m)$ and $(t, x) \in \phi(W_m X)$. From this description, it is easy to see that $\phi(W_m X)$ is a deformation retract of its neighborhood $\{(t, x) | t_j > 0$ if $j \notin \operatorname{Im} \phi\}$ and, in fact, that $(W_n X, \phi(W_m X))$ is a $\Sigma_* - NDR$ pair as defined in [4, p. 232]. Thus ϕ is a Σ_*-cofibration by [4, App., 2.2].

Remarks B.5. Applied to the unique injection ϕ: $\mathbf{0} \to \mathbf{n}$, the result gives that ϕ: $X_0 = W_0 X \to W_n X$ is a cofibration. Thus if X_0 is nondegenerately based, then so is each $W_n X$ even if the X_n for $n > 0$ are not. The condition on X_0 can be eliminated by setting $W'_n X = W_n X / \phi(X_0)$ and observing that $W'X$ then inherits a structure of (proper) $\hat{\mathscr{C}}$-space from WX, with all basepoints nondegenerate regardless of such conditions on X. Here the diagram $X \leftarrow WX \to W'X$ displays a natural equivalence of improper $\hat{\mathscr{C}}$-spaces between X and $W'X$. We conclude that, in the definition of an improper $\hat{\mathscr{C}}$-space, we need not assume the X_n to be nondegenerately based.

For the uniqueness theorem, we need the following analog of Proposition 3.7.

LEMMA B.6. *For improper $\hat{\mathscr{C}}$-spaces X, there is a natural equivalence ζ: $W\Omega X \to \Omega WX$ such that the following diagram commutes up to natural homotopy*:

$$\begin{tikzcd} W\Omega X \arrow[rr, "\zeta"] \arrow[dr, "\pi"'] & & \Omega WX \arrow[dl, "\Omega\pi"] \\ & \Omega X & \end{tikzcd}$$

Proof. We would like to define ζ_n: $W_n \Omega X \to \Omega W_n X$ by $\zeta_n(t, g)(s) = (t, g(s))$. Here $g \in \Omega X$ is a based map $S^1 \to X_n$, but this equation does not define a based map $S^1 \to W_n X$ unless all coordinates of $t = (t_1, \ldots, t_n)$ are 1. We modify the definition to

$$\zeta_n(t, g)(s) = \begin{cases} (u(s), g(0)), & u_j(s) = 1 - 3s + 3st_j, \quad \text{if} \quad 0 \le s \le 1/3 \\ (t, g(3s - 1)) & \qquad\qquad\qquad\qquad\quad \text{if} \quad 1/3 \le s \le 2/3 \\ (v(s), g(1)), & v_j(s) = 3s - 2 + (3 - 3s)t_j, \quad \text{if} \quad 2/3 \le s \le 1 \end{cases}$$

and find that ζ_n is a well-defined natural map $W_n \Omega X \to \Omega W_n X$ such that the ζ_n determine a map of $\hat{\mathscr{C}}$-spaces $W\Omega X \to \Omega WX$. A natural homotopy h: $\pi \simeq \Omega\pi \circ \zeta$ is specified by

$$h((t, g), r)(s) = \begin{cases} g(0) & \text{if} \quad 0 \le s \le (1/3)r \\ g(3s - r/3 - 2r) & \text{if} \quad (1/3)r \le s \le 1 - (1/3)r. \\ g(1) & \text{if} \quad 1 - (1/3)r \le s \le 1 \end{cases}$$

Since π and $\Omega\pi$ are equivalences, so is ζ.

We return to the proof of the uniqueness theorem. Let E be an infinite loop space machine defined on $\mathscr{F}$-spaces. For an $\mathscr{F}$-space Y, Lemmas 3.2 and 3.3 and Construction 3.4 give improper $\mathscr{F}$-spaces $E_i \bar{Y}$ and natural maps of improper $\mathscr{F}$-spaces ι: $Y \to E_0 \bar{Y}$ and σ_i: $E_i \bar{Y} \to \Omega E_{i+1} \bar{Y}$. Define $F_i Y$ to be the Segal spectrum $SWE_i \bar{Y}$ and define τ_i: $F_i Y \to \Omega F_{i+1} Y$ to be the composite equivalence

$$SWE_i \bar{Y} \xrightarrow{SW\sigma_i} SW\Omega E_{i+1} \bar{Y} \xrightarrow{S\zeta} S\Omega WE_{i+1} \bar{Y} \xrightarrow{\epsilon} \Omega SWE_{i+1} \bar{Y}.$$

We have a bispectrum and thus a natural equivalence between $F_0 Y = F_{0*} Y$ and $F_{*0} Y$. We have natural equivalences

$$SY \xleftarrow{S\pi} SWY \xrightarrow{SW\iota} SWE_0 \bar{Y} = F_0 Y,$$

hence it suffices to prove that EY is naturally equivalent to $F_{*0} Y$. Introduce an auxiliary spectrum "WEY" with ith space $(WE_i \bar{Y})_1$ and ith structural equivalence the composite

$$(WE_i \bar{Y})_1 \xrightarrow{(W\sigma_i)_1} (W\Omega E_{i+1} \bar{Y})_1 \xrightarrow{\zeta_1} (\Omega WE_{i+1} \bar{Y})_1 = \Omega(WE_{i+1} \bar{Y})_1,$$

the subscripts referring to the first space of the relevant $\mathscr{F}$-spaces. By the naturality of ι and the diagram of Proposition 3.7, the maps

$$\iota: (WE_i \bar{Y})_1 \to S_0 WE_i \bar{Y} = F_{i0} Y$$

specify a natural equivalence of spectra $WEY \to F_{*0} Y$. By the naturality of π and the diagram of the previous lemma, the maps

$$\epsilon_1: (WE_i \bar{Y})_1 \to (E_i \bar{Y})_1 = E_i Y$$

specify a preternatural equivalence $WEY \to EY$. Thus EY is naturally equivalent to WEY by Lemma A.2.

Remarks B.7. The proof above could equally well be carried out with W replaced by the functor W' of Remarks B.5. Indeed, the equivalence ζ of Lemma B.6 induces an equivalence ζ': $W'\Omega X \to \Omega W'X$ such that the diagram

$$\begin{tikzcd} W\Omega X \arrow[r, "\zeta"] \arrow[d] & \Omega WX \arrow[d] \\ W'\Omega X \arrow[r, "\zeta'"] & \Omega W'X \end{tikzcd}$$

commutes, and we need only interpolate the equivalence $SY \leftarrow SWY \rightarrow SW'Y$ in showing that SY is equivalent to $SW'E_0\bar{Y}$ and interpolate the equivalence $EY \leftarrow WEY \rightarrow W'EY$ in showing that EY is equivalent to $S_0W'E\bar{Y}$. We therefore find that there is no need to assume that the spaces E_iX are nondegenerately based in the definition of an infinite loop space machine. In general, it is technically best not to require the basepoints of the component spaces of spectra to be nondegenerate since this condition seems not to be preserved by the functor Ω^∞. In particular, we have not verified that May's machine produces spaces with nondegenerate basepoints (although we believe that it does).

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University of Chicago
Chicago, Illinois

and

M.I.T.
Cambridge, Massachusetts