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The spectra associated to $\mathscr{I}$-monoids

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In this final sequel to (9), I shall prove a general consistency statement which seems to me to complete the foundations of infinite loop space theory. In particular, this result will specialize to yield the last step of the proof of the following theorem about the stable classifying spaces of geometric topology.

THEOREM. *When localized at an odd prime p, the natural braid of infinite loop fibrations*

$$\begin{tikzcd}
\mathrm{Top}/O \arrow[rr, bend left] \arrow[dr] & & BO \arrow[rr, bend left] \arrow[dr] & & BF \\
& F/O \arrow[ur] \arrow[dr] & & B\,\mathrm{Top} \arrow[ur] \arrow[dr] & \\
SF \arrow[ur] \arrow[rr, bend right] & & F/\mathrm{Top} \arrow[ur] \arrow[rr, bend right] & & B(\mathrm{Top}/O)
\end{tikzcd}$$

splits, as a commutative diagram of infinite loop spaces and infinite loop maps, as the product of the p-local braids

$$\begin{tikzcd}
\mathrm{Coker}\,J \arrow[rr, bend left] \arrow[dr, equal] & & * \arrow[rr, bend left] \arrow[dr] & & B\,\mathrm{Coker} \\
& \mathrm{Coker}\,J \arrow[ur] \arrow[dr] & & B\,\mathrm{Coker}\,J \arrow[ur, equal] \arrow[dr, equal] & \\
\mathrm{Coker\,J} \arrow[ur, equal] \arrow[rr, bend right] & & * \arrow[ur] \arrow[rr, bend right] & & B\,\mathrm{Coker\,J}
\end{tikzcd}$$

and

$$\begin{tikzcd}
N \arrow[rr, bend left] \arrow[dr] & & BO \arrow[dr, "\theta^r"] \arrow[rr, bend left] & & BJ \\
& BO \arrow[ur, "\psi^r - 1"] \arrow[dr, "\theta^r"'] & & BO_{\otimes} \arrow[ur] \arrow[dr] & \\
J \arrow[ur] \arrow[rr, bend right] & & BO_{\otimes} \arrow[ur, "\psi^r/1"'] \arrow[rr, bend right] & & BN
\end{tikzcd}$$

Here r reduces mod p^2 to a generator of the group of units, ψ^r is the Adams operation, θ^r is the Sullivan cannibalistic class, and J and N are the (homotopy) fibres of $\psi^r - 1$ and θ^r, J also being the fibre of $\psi^r/1$.

That is, we have displayed the zeroth space level of a corresponding splitting of spectra in the stable homotopy category. Thus the associated cohomology theories decompose in terms of reasonably well understood variants of real connective K-theory and a mysterious Coker J theory. A thorough discussion of the fibration theory classified by $B\,\mathrm{Coker}\,J$ may be found in (7), V, § 5.

On the space level the result is due to Adams, Quillen, and especially Sullivan (11). On the infinite loop space level, beyond the foundational work of Boardman and Vogt (2), Segal (10), and myself (5)–(7), the essential contributions are those of Adams and Priddy (1), Madsen, Snaith, and Tornehave (4), Quinn, Ray, Tornehave, and myself (7), and Friedlander and Seymour (3).

A diagram displaying all of the requisite splitting maps is given in (7), p. 125, and the space level splittings are all explained in (7), pp. 113 and 127. Using the contributions of all but the last two authors named, I proved in (7), V, § 7, that the cited diagram (and the analogous J-theory diagram of (7), p. 107, at $p > 2$) would be commutative diagrams of infinite loop spaces and infinite loop maps, and in particular the theorem would hold, provided only that the infinite loop complex Adams conjecture were true. This asserts that the composite

$$BU \xrightarrow{\psi^r - 1} BU \to BSF[1/r]$$

is null homotopic on the infinite loop space level. Since Friedlander and Seymour have proved this assertion,† independently, it would appear that our theorem is already demonstrated. However, it is vital to both of their proofs that the infinite loop space structure on BF come from the permutative category $\amalg F(n)$ by the procedure axiomatized in (8). On the other hand, the pivotal space in the derivation of our theorem is the classifying space $B(SF; kO)$ for kO-oriented stable spherical fibrations. Indeed, at the prime p (the case $p = 2$ also being allowed), $B\,\mathrm{Coker}\,J$ is best defined as the fibre of

† (Added in proof.) At this writing, Seymour's proof contains a gap.

the universal cannibalistic class $c(\psi^r): B(SF; kO) \to B\mathrm{Spin}_{\otimes}$ while, away from 2, BTop is equivalent to $B(SF; kO)$. All of the machinery introduced in (7) was necessary to the proof that $B(SF; kO)$ is an infinite loop space and that $c(\psi^r)$ and the equivalence $B\mathrm{Top} \to B(SF; kO)$ are infinite loop maps. It is vital to these proofs that the infinite loop space structures on BTop and BF come from the actions on these spaces of the linear isometries E_∞ operad $\mathscr{L}$ by the procedure axiomatized in (9).

Thus to prove the theorem it remains to show that these two natural ways, via permutative categories and $\mathscr{L}$-spaces, of giving BTop and BF infinite loop space structures yield equivalent 0-connected spectra. The analogous assertion for O, U and Sp is known, by comparison of (7), I, §1 and VIII, §1, but the cited proofs make essential use of Bott periodicity. In general, the requisite consistency statement is not at all obvious. When regarding $\amalg F(n)$ as a permutative category, one is throwing away all information about actions by isometries and using a procedure equally applicable to discrete examples such as $\amalg\Sigma_n$ or $\amalg GL(n, A)$. When regarding $F = \operatorname{colim} F(n)$ as an $\mathscr{L}$-space, one is using a limit procedure which throws away the permutative structure, being analogous to passage to Σ_∞ or $GL(\infty, A)$.

We give a precise formulation of our consistency theorem, discuss examples, and outline the proof in Section 1. The details of the requisite constructions are given in Section 2. At least one of the new concepts needed for the proof, that of a 'partial permutative category', should be of independent interest.

1. *$\mathscr{I}$-monoids and permutative categories.* We first generalize some definitions due to Boardman and Vogt (2), VI, §5. Let $\mathscr{I}$ denote the category of finite or countably infinite dimensional real inner product spaces and their linear isometries (not necessarily isomorphisms). Let $\mathscr{I}_*$ denote the subcategory of $\mathscr{I}$ consisting of the finite dimensional spaces and their isometric isomorphisms. In the following and a later definition, a 'suitable' category is one with underlying based spaces, continuous composition, and whatever extra structure is needed to make sense; we have enough examples in mind to justify the generality but not enough to justify further pedantic details.

Definitions 1·1. An $\mathscr{I}_*$-object in a suitable category $\mathscr{C}$ is a continuous functor T: $\mathscr{I}_* \to \mathscr{C}$ together with a (coherently) commutative and associative continuous natural transformation $\omega: T \times T \to T \circ \oplus$ (of functors $\mathscr{I}_* \times \mathscr{I}_* \to \mathscr{C}$) such that

(1) The composite $TV \times * \to TV \times T\{0\} \xrightarrow{\omega} T(V \oplus \{0\})$ is the identity of TV.

(2) The composite $TV \times * \to TV \times TW \xrightarrow{\omega} T(V \oplus W)$ is a closed inclusion.

An $\mathscr{I}$-object in $\mathscr{C}$ is defined in the same way but with the additional requirement

(3) $TV = \operatorname{colim} TV'$, where V' runs over the finite dimensional sub inner product spaces of V.

I called $\mathscr{I}$-spaces $\mathscr{I}$-functors in (7), I, §1, to which the reader is referred for topological details. The following generalization of (7), I, 1·9, admits the same proof.

LEMMA 1·2. *By passage to colimits, an $\mathscr{I}_*$-object in $\mathscr{C}$ admits a unique extension to an $\mathscr{I}$-object in $\mathscr{C}$.*

Our basic objects of study are $\mathscr{I}$-monoids, such as F, Top, and others displayed in Examples 1·4 below. For an $\mathscr{I}$-monoid G, $G(R^\infty)$ is an $\mathscr{L}$-monoid and $BG(R^\infty)$ is an

$\mathscr{L}$-space, where $\mathscr{L}$ is the E_∞ operad with $\mathscr{L}(q) = \mathscr{I}(R^{\infty q}, R^\infty)$ and B is the standard classifying space functor (6), 3·5, and (7), 1·2, 1·6, 2·2. By abuse, we generally write G both for a given $\mathscr{I}$-monoid and for the derived $\mathscr{L}$-monoid $G(R^\infty)$.

Let E be any infinite loop space machine defined on $\hat{\mathscr{L}}$-spaces (9), 2·1, where $\hat{\mathscr{L}}$ is the category of operators associated to $\mathscr{L}$ (9), 4·1. Although the arguments above give $\mathscr{L}$-spaces, our proofs below will actually require use of more general $\hat{\mathscr{L}}$-spaces. For an $\mathscr{I}$-monoid G, we have a spectrum EBG. Since BG is connected, the group completion $\iota: BG \to E_0 BG$ is an equivalence and EBG is 0-connected.

To obtain permutative categories from $\mathscr{I}_*$-monoids, we need a slightly more restrictive notion. Recall that a symmetric monoidal category is a (topological) category with a coherently associative, unital, and commutative sum; $\mathscr{I}_*$ and $\mathscr{I}$ are examples. A map F: $\mathscr{A} \to \mathscr{A}'$ of symmetric monoidal categories is a (continuous) functor F together with a natural isomorphism $FA \oplus FB \to F(A \oplus B)$ compatible with the associativity, unity, and commutativity isomorphisms.

Definition 1·3. A symmetric monoidal category $\mathbf{G}$ is said to lie under $\mathscr{I}_*$ if it comes with a map $\mathscr{I}_* \to \mathbf{G}$ of symmetric monoidal categories which is the identity on objects and which satisfies (1) and (2) of Definition 1·1, where $GV = \mathbf{G}(V, V)$ and

$$\omega: GV \times GW \to G(V \oplus W)$$

is given by the categorical sum on morphisms. By abuse, continue to write f: $V \to W$ for the image in $\mathbf{G}$ of an isometry f. With Gf: $GV \to GW$ specified by

$$(Gf)(g) = f \circ g \circ f^{-1}: W \to W \quad \text{for} \quad g \in GV,$$

$\mathbf{G}$ determines an $\mathscr{I}_*$-monoid G by neglect of structure.

The point of the definition is to incorporate the displayed categorical description of the isomorphisms $GV \cong GW$ given by isometries which are present in arbitrary $\mathscr{I}_*$-monoids. The following examples should help fix ideas.

Examples 1·4. Let tV be the one-point compactification of V and note that

$$t(V \oplus V') \cong tV \wedge tV'.$$

We have the following symmetric monoidal categories under $\mathscr{I}_*$. In each case, the sum is given in an evident way by the smash product of maps.

(i) **F, Top, O**. Let $\mathbf{F}(V, W)$ be the space of homotopy equivalences $tV \to tW$ and let $\mathbf{Top}(V, W)$ and $\mathbf{O}(V, W)$ be the subspaces of one-point compactifications of homeomorphisms and linear isometries $V \to W$. Here tV could be replaced by $t\tau V$ for any non-trivial additive functor $\tau: \mathscr{I}_* \to \mathscr{I}_*$.

(ii) **U, Sp**. Use one-point compactifications of complexifications and quaternionifications of real inner product spaces.

(iii) **SF, STop, SO**. Restriction to degree one maps $tV \to tW$ fails to make sense. Note that $tV \wedge tV$ has a canonical fundamental class, namely the image of

$$i \otimes i \in H_n tV \otimes H_n tV, \quad \dim V = n,$$

for either choice of fundamental class $i \in H_n tV$. Let $\mathbf{SF}(V, W)$ be the space of degree 1 maps $t(V \oplus V) \to t(W \oplus W)$ and similarly for **STop** and **SO**; for f: $V \to W$, $t(f \oplus f)$ is the image of f in any of these categories.

(iv) $\mathbf{F}_M$. Let M be a non-trivial multiplicative submonoid of the positive integers and let $\mathbf{F}_M(V, W)$ be the space of maps $t(V \otimes V) \to t(W \oplus W)$ with degree in M. By (7), VII, 5·3, the 1-connected cover of EBF_M is equivalent to the localization of $EBSF$ at M (obtained by inverting the primes which divide elements of M).

The last example plays a key role in Seymour's proof of the infinite loop complex Adams conjecture. Of course, for such applications, $V \oplus V$ should be regarded as the realification of the complexification of V.

Henceforward, assume given a fixed symmetric monoidal category $\mathbf{G}$ under $\mathscr{I}_*$. Write $G(n)$ for $G(R^n)$.

Definition 1·5. Define the associated permutative category $\mathscr{G}$ of $\mathbf{G}$ to have objects $\{n | n \geq 0\}$ and morphisms $\amalg G(n)$, where $G(n)$ maps n to n and there are no maps $m \to n$ for $m \neq n$. Define $\oplus : G(m) \times G(n) \to G(m+n)$ to be the composite

$$G(m) \times G(n) \xrightarrow{\omega} G(R^m \oplus R^n) \xrightarrow{G(i_{mn})} G(m+n),$$

where $i_{mn}: R^m \oplus R^n \to R^{m+n}$ is the standard isomorphism. Then $\oplus$ is strictly associative and unital and is commutative up to the coherent natural isomorphism given by the images in $\mathbf{G}$ of the interchange maps $c_{mn}: R^{m+n} \to R^{m+n}$.

With the notations of (8), we have the Segal spectrum $SB\overline{\mathscr{G}}$ associated to $\mathscr{G}$. Here the group completion $\iota : B\mathscr{G} = \amalg BG(n) \to S_0 B\overline{\mathscr{G}}$ and the fact that BG is a simple space (because it is an $\mathscr{L}$-space and thus an H-space) imply that $S_0 B\overline{\mathscr{G}}$ is equivalent to $BG \times Z$. We can now state our main result.

THEOREM 1·6. *For any infinite loop space machine E defined on $\hat{\mathscr{L}}$-spaces, there is a natural equivalence of spectra between EBG and the* 0*-connected cover of $SB\overline{\mathscr{G}}$.*

Our conventions on spectra are explained in (9), §2. By (8), $SB\overline{\mathscr{G}}$ is naturally equivalent to $E'\mathscr{G}$ for any infinite loop space machine E' defined on permutative categories.

Proof. The notion of an $\mathscr{F}\hat{\mathscr{L}}$-space was defined in (9), 3·1. In the next section, we shall construct a certain $\mathscr{F}\hat{\mathscr{L}}$-space $B\overline{\mathscr{H}}(R^\infty)_*$ with first $\hat{\mathscr{L}}$-space denoted $B\mathscr{H}(R^\infty)_*$ and first $\mathscr{F}$-space (which will be an $\mathscr{F}$-space and not just an improper $\mathscr{F}$-space) denoted $B\overline{\mathscr{H}}(R^\infty)$. We shall prove that EBG is the 0-connected cover of $EB\mathscr{H}(R^\infty)_*$ and that $SB\overline{\mathscr{G}}$ is equivalent to $SB\overline{\mathscr{H}}(R^\infty)$. Given this much, the rest of the proof proceeds precisely as in (9). If the improper $\mathscr{F}$-spaces $E_i B\overline{\mathscr{H}}(R^\infty)_*$ obtained by use of (9), 3·2, are proper, then we have a bispectrum $\{SE_i B\overline{\mathscr{H}}(R^\infty)_*, \tau_i\}$ with τ_i the composite equivalence

$$SE_i B\overline{\mathscr{H}}(R^\infty)_* \xrightarrow{S\sigma_i} S\Omega E_{i+1} B\overline{\mathscr{H}}(R^\infty)_* \xrightarrow{\xi} \Omega SE_{i+1} B\overline{\mathscr{H}}(R^\infty)_*$$

(9), 3·3, 3·7, 3·8. By (9), 3·9, we conclude that the spectra

$$SE_0 B\overline{\mathscr{H}}(R^\infty)_* \quad \text{and} \quad S_0 EB\overline{\mathscr{H}}(R^\infty)_*$$

are equivalent. By (9), 2·3, 3·3, 3·7, we have further equivalences

$$S\iota : SB\overline{\mathscr{H}}(R^\infty) \to SE_0 B\overline{\mathscr{H}}(R^\infty)_* \quad \text{and} \quad \iota : EB\mathscr{H}(R^\infty)_* \to S_0 EB\overline{\mathscr{H}}(R^\infty)_*,$$

and the proof is complete. The modifications necessary to account for possible improper $\mathscr{F}$-spaces are the same as in (9), appendix B.

2. *The construction of* $B\bar{\mathscr{H}}(R^\infty)_*$. The idea of the construction is to apply the functors from $\mathscr{I}_*$-monoids to $\mathscr{I}$-monoids to $\mathscr{L}$-monoids to $\mathscr{L}$-spaces obtained by passage to colimits, restriction to R^∞, and application of B but with monoids replaced by permutative categories and with the passage from permutative categories to $\mathscr{F}$-permutative categories of (8), construction 10, woven in. We begin with the following definition. Recall that we are given $\mathbf{G}$ as in Definition 1·3.

Definition 2·1. Define a functor $\mathscr{I}_* \to \mathscr{P}\mathscr{C}$ as follows. For $V \in \mathscr{I}_*$, let $\mathscr{G}(V)$ be the category with objects $\{n \mid n \geq 0\}$ and morphisms $G(V \otimes R^n)$ from n to n; its permutative structure is specified just as in Definition 1·5. For $f\colon V \to W$ in $\mathscr{I}_*$, $\mathscr{G}(f)\colon \mathscr{G}(V) \to \mathscr{G}(W)$ is given by $G(f \otimes 1)$ on morphisms. Observe that the permutative category $\mathscr{G}$ may be identified with $\mathscr{G}(R^1)$.

At this point, we encounter a problem. We would like to say that $\mathscr{G}(?)$ is an $\mathscr{I}_*$-permutative category in the sense of Definition 1·1. For this we require maps

$$\omega\colon \mathscr{G}(V) \times \mathscr{G}(W) \to \mathscr{G}(V \oplus W)$$

of permutative categories. It is easy to define ω, but only as a map of symmetric monoidal categories (compare (7), VI, 3·1); ω also fails to be commutative. As the experts will long since have anticipated, the source of trouble is interchange of coordinates. We shall overcome this difficulty by use of the following general notions, analogues and special cases of which have been exploited elsewhere by Boardman, Segal, and myself.

Recall the definition of the category Π from (9), above 1·1. We can define the notion of a Π-object in a suitable category $\mathscr{C}$ precisely as Π-spaces were defined in (9), 1·2. These are sequences of objects X_q of $\mathscr{C}$ which behave up to equivalence like the iterated powers of X_1. In particular, as in (9), 1·3, the powers of an object X determine a Π-object RX.

Definition 2·2. A structural domain X_* of an object X in a suitable category $\mathscr{C}$ is a sub Π-object of RX such that $X_1 = X$. It follows that X_0 is the zero object and that the inclusions $X_q \to X^q$ are equivalences for $q \geq 2$. For any type of mathematical structure defined in terms of maps $X^q \to X$ in $\mathscr{C}$, define a partial structure to be a structural domain X_* of X together with maps $X_q \to X$ which satisfy the same formal properties as the given type of structure. Maps of partial structures are defined in the evident way.

In particular, we have the notion of a partial permutative category, or *PPC*, which is a category $\mathscr{A}$ (not a partial category) with equivalent subcategories $\mathscr{A}_q$ of $\mathscr{A}^q$ and unital, associative, and coherently commutative sums $\mathscr{A}_q \to \mathscr{A}$ as in (8), definition 1. An immediate verification shows that everything in (8) applies verbatim with permutative categories replaced by partial permutative categories. The only significant point is that $\delta\colon \bar{\mathscr{A}}_n \to \mathscr{A}^n$ of (8), step 2 of construction 10, now factors through an equivalence $\bar{\mathscr{A}}_n \to \mathscr{A}_n$. Thus we have a well-defined functor, written $\mathscr{A} \to \bar{\mathscr{A}}$ on objects, from $\mathscr{P}\mathscr{P}\mathscr{C}$ to $\mathscr{F}\mathscr{P}\mathscr{P}\mathscr{C}$, where $\mathscr{P}\mathscr{P}\mathscr{C}$ is the category of partial permutative categories.

In a similar spirit (and as can be viewed via functor categories as another special case), we have the notion of a partial $\mathscr{I}_*$-category or partial $\mathscr{I}$-category $\mathscr{A}(?)$, for which the q-fold Whitney sum ω is only required to be defined on appropriate equivalent subcategories of the product categories $\mathscr{A}(V_1) \times \ldots \times \mathscr{A}(V_q)$. With ω and the maps $\mathscr{A}(f)$

induced by isometries required to be maps of *PPC*'s, we arrive at the combined notion of a partial $\mathscr{I}_*$-*PPC* or partial $\mathscr{I}$-*PPC*.

We return to our original idea and find that partial structures allow the definition of strictly commutative sums. All *PPC*'s we shall consider will have the space of objects specified in the following definition.

Definition 2·3. Let R^∞ have its standard ordered basis $\{e_i \mid i \geq 1\}$. Let $\mathscr{O}$ denote the set (regarded as a discrete space) of subspaces of R^∞ spanned by finite subsets of $\{e_i\}$. For objects A and A' of $\mathscr{O}$ of the same dimension, their given ordered bases specify a canonical isomorphism $j: A \to A'$. Let $\mathscr{O}_q \subset \mathscr{O}^q$ be the set of q-tuples $(A_1, \ldots, A_q)$ such that the A_p are pairwise orthogonal and define the sum map $+ : \mathscr{O}_q \to \mathscr{O}$ by

$$(A_1, \ldots, A_q) \to \Sigma A_p,$$

the internal direct sum of the $A_p \subset R^\infty$.

Definition 2·4. Define a partial $\mathscr{I}_*$-*PPC* $\mathscr{H}(?)$ as follows. For $V \in \mathscr{I}_*$, let $\mathscr{H}(V)$ be the category with object space $\mathscr{O}$ and morphism space $\amalg \mathbf{G}(V \otimes A, V \otimes A')$, where $\dim A = \dim A'$. The sum $+$ on $\mathscr{H}(V)$ has structural domain given by the full subcategories $\mathscr{H}(V)_q$ of $\mathscr{H}(V)^q$ with object spaces $\mathscr{O}_q$ and is induced on morphisms by the sum $\oplus$ in $\mathbf{G}$ and the evident isomorphisms

$$i: \bigoplus_{p=1}^{q} A_p \to \sum_{p=1}^{q} A_p$$

from external to internal direct sums. It is strictly unital, associative, and commutative. For $f: V \to W$ in $\mathscr{I}_*$, define $\mathscr{H}(f): \mathscr{H}(V) \to \mathscr{H}(W)$ on morphisms by

$$\mathscr{H}(f)(g) = (f \otimes 1) g (f^{-1} \otimes 1).$$

For objects $V_1, \ldots V_q$ of $\mathscr{I}_*$, the domain of ω is again given by the appropriate full subcategories with object spaces $\mathscr{O}_q$, and ω is given on morphisms by the composites

$$\mathop{\times}_{p=1}^{q} \mathbf{G}(V_p \otimes A_p, V_p \otimes A'_p) \xrightarrow{\oplus} \mathbf{G}\left(\bigoplus_{p=1}^{q} V_p \otimes A_p, \bigoplus_{p=1}^{q} V_p \otimes A'_p\right) \to \mathbf{G}(W \otimes A, W \otimes A'),$$

where $$W = \bigotimes_{p=1}^{q} V_p, \quad A = \sum_{p=1}^{q} A_p, \quad A' = \sum_{p=1}^{q} A'_p,$$

and the second map is obtained by summing with the maps in $\mathbf{G}$ induced by the canonical isomorphisms $1 \otimes j: V_p \otimes A_r \to V_p \otimes A'_r$ for $p \neq r$.

The conditions on ω set out in Definition 1·1 are trivial to verify, the object $\{0\} \in \mathscr{O}$ determining the map from the trivial category to $\mathscr{H}(W)$ needed to make sense of (1) and (2). As in Lemma 1·2, $\mathscr{H}(?)$ extends uniquely to a partial $\mathscr{I}$-*PPC*.

By analogy with the definition of an $\mathscr{L}$-space (5), 1·2 and 1·4, we have the notion of an $\mathscr{L}$-category $\mathscr{A}$. For $f \in \mathscr{L}(q)$, we require a functor $\theta(f): \mathscr{A}^q \to \mathscr{A}$, these being subject to the evident continuity, unity, equivariance, and associativity conditions. By allowing the $\theta(f)$ to be defined on appropriate equivalent subcategories $\mathscr{A}_q$ of the product categories $\mathscr{A}^q$, we obtain the notion of a partial $\mathscr{L}$-category. We write $\mathscr{A}_*$ for such an object. With the $\theta(f)$ required to be maps of *PPC*'s, we have the combined notion of a partial $\mathscr{L}$-*PPC*. Clearly $\mathscr{H}(R^\infty)_*$ is an example, the requisite functors $\theta(f)$ being the composites

$$\mathscr{H}(R^\infty)_q \xrightarrow{\omega} \mathscr{H}(R^{\infty q}) \xrightarrow{\mathscr{H}(f)} \mathscr{H}(R^\infty).$$

An $\mathscr{L}$-monoid is an $\mathscr{L}$-category with a single object, and the classifying space functor carries $\mathscr{L}$-categories to $\mathscr{L}$-spaces by an immediate generalization of (6), 3·5 and (7), pp. 20–21. If we start out with a partial $\mathscr{L}$-category, then we arrive at a partial $\mathscr{L}$-space, the latter notion being obtained by specialization of Definition 2·2. Moreover, by an obvious generalization of (9), 4·2, partial $\mathscr{L}$-spaces are perfectly good examples of $\hat{\mathscr{L}}$-spaces.

In sum, we now have a well-defined $\hat{\mathscr{L}}$-space $B\mathscr{H}(R^\infty)_*$ derived from the partial $\mathscr{I}_*$-*PPC* $\mathscr{H}(?)$ by application of the functors from partial $\mathscr{I}_*$-categories to partial $\mathscr{I}$-categories to partial $\mathscr{L}$-categories to $\hat{\mathscr{L}}$-spaces obtained by passage to colimits, restriction to R^∞, and application of B. Further, by analogy with (9), 3·1, and (8), definition 7, we have notions of $\mathscr{F}$-objects in each of these categories, and the specified functors clearly carry $\mathscr{F}$-objects to $\mathscr{F}$-objects. In view of the functoriality of the passage from *PPC*'s $\mathscr{A}$ to $\mathscr{F}$-*PPC*'s $\bar{\mathscr{A}}$ (together with an easy check of continuity), the $\mathscr{I}_*$-*PPC* $\mathscr{H}(?)$ induces an $\mathscr{F}\mathscr{I}_*$-*PPC* $\bar{\mathscr{H}}(?)$ with nth $\mathscr{I}_*$-*PPC* $\bar{\mathscr{H}}_n(?)$.

Applying our functors, we arrive at the required $\mathscr{F}\hat{\mathscr{L}}$-space $B\bar{\mathscr{H}}(R^\infty)_*$.

It remains to examine the spectra $SB\bar{\mathscr{H}}(R^\infty)$ and $EB\mathscr{H}(R^\infty)_*$, and we need only pay attention to the permutative structure when studying $\bar{\mathscr{H}}(R^\infty)$ while we need only pay attention to the isometries when studying $\mathscr{H}(R^\infty)_*$.

The functorial passage from symmetric monoidal categories to permutative categories explained in (6), 4·2 and (7), VI, 3·2, extends readily to the partial context and, together with the following result, implies that there are equivalences of *PPC*'s

$$\mathscr{G}(V) \xleftarrow{\pi} \Phi\mathscr{G}(V) \xleftarrow{\Phi F} \Phi\mathscr{H}(V) \xrightarrow{\pi} \mathscr{H}(V).$$

LEMMA 2·5. *For each $V \in \mathscr{I}_*$, there is an equivalence of partial symmetric monoidal categories $F: \mathscr{H}(V) \to \mathscr{G}(V)$.*

Proof. Define F on objects by $FA = \dim A$ and on morphisms by

$$Fg = (1 \otimes j) g (1 \otimes j^{-1}) \colon V \otimes R^n \to V \otimes R^n \quad \text{for} \quad g\colon V \otimes A \to V \otimes A', \quad \dim A = n.$$

Certainly F is an equivalence of categories. It fails to be a map of *PPC*'s, but the maps

$$\phi = \phi(A,B)\colon R^{m+n} \to R^{m+n}, \quad \dim A = m \quad \text{and} \quad \dim B = n,$$

induced by the composites

$$R^{m+n} \xrightarrow{i_{mn}^{-1}} R^m \oplus R^n \xrightarrow{j^{-1} \oplus j^{-1}} A \oplus B \xrightarrow{i} A + B \xrightarrow{j} R^{m+n}$$

are easily verified to specify a natural isomorphism $FA \oplus FB \to F(A+B)$ which satisfies the requirements specified in (7), VI, 3·1, for a map of (partial) symmetric monoidal categories.

With the following lemma, this implies that the spectra $SB\bar{\mathscr{G}}$ and $SB\bar{\mathscr{H}}(R^\infty)$ are equivalent (since $\mathscr{G} = \mathscr{G}(R^1)$).

LEMMA 2·6. *If $W = V \oplus V'$ in $\mathscr{I}$, then the map of PPC's*

$$\mathscr{H}(V) = \mathscr{H}(V) \times * \rightarrow \mathscr{H}(V) \times \mathscr{H}(V') \xrightarrow{\omega} \mathscr{H}(W)$$

induces an equivalence of spectra $SB\overline{\mathscr{H}}(V) \rightarrow SB\overline{\mathscr{H}}(W)$.

Proof. We have a commutative diagram

$$\begin{tikzcd} B\mathscr{H}(V) \arrow[r] \arrow[d, "\iota"'] & B\mathscr{H}(W) \arrow[d, "\iota"] \\ S_0B\overline{\mathscr{H}}(V) \arrow[r] & S_0B\overline{\mathscr{H}}(W). \end{tikzcd}$$

Since the maps ι are group completions, the bottom map is equivalent to the map

$$BG(V \otimes R^\infty) \times Z = \operatorname{colim} BG(V \otimes A) \times Z \rightarrow \operatorname{colim} BG(W \otimes A) \times Z = BG(W \otimes R^\infty) \times Z$$

induced by the inclusion of V in W. Clearly this map is an equivalence.

To analyse $EB\mathscr{H}(R^\infty)_*$ we need another construction.

Definition 2·7. Define a partial $\mathscr{I}_*$-category $\mathscr{J}(?)$ as follows. For $V \in \mathscr{I}_*$, $\mathscr{J}(V)$ has object space $\mathscr{O}$ and morphism space

$$\amalg \mathbf{G}((V \otimes A) \oplus V, (V \otimes A') \oplus V), \quad \dim A = \dim A'.$$

The maps $\mathscr{J}(f): \mathscr{J}(V) \rightarrow \mathscr{J}(W)$ for $f: V \rightarrow W$ and the q-fold Whitney sum ω are defined as in Definition 2·4, but with the evident use of f and the sum on the extra copy of V.

The $\mathscr{I}_*$-monoid $G(?)$ may be regarded as an $\mathscr{I}_*$-category with each $G(V)$ having only one object 0.

LEMMA 2·8. *There are maps of partial $\mathscr{I}_*$-categories*

$$\mathscr{H}(?) \rightarrow \mathscr{J}(?) \leftarrow G(?).$$

By passage to colimits, restriction to R^∞, and application of B, these induce maps of $\hat{\mathscr{L}}$-spaces

$$B\mathscr{H}(R^\infty)_* \rightarrow B\mathscr{J}(R^\infty)_* \leftarrow BG(R^\infty),$$

the first of which is an equivalence and the second of which is an equivalence onto the component of the basepoint.

Proof. $\mathscr{H}(V) \rightarrow \mathscr{J}(V)$ is the identity on objects and is obtained on morphisms by summing with the identity map of V. $G(V) \rightarrow \mathscr{J}(V)$ is given by $0 \rightarrow \{0\}$ on objects and the obvious identification on morphisms. On passage to R^∞, $\mathscr{H}(R^\infty)_* \rightarrow \mathscr{J}(R^\infty)_*$ clearly induces an equivalence upon application of B, while $G(R^\infty) \rightarrow \mathscr{J}(R^\infty)$ is clearly an isomorphism onto the full subcategory with object $\{0\}$.

It follows that EBG is the 0-connected cover of $EB\mathscr{H}(R^\infty)_*$, and this completes the proof of Theorem 1·6.

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