THE BOCKSTEIN AND THE ADAMS SPECTRAL SEQUENCES

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ABSTRACT. We show that, above the appropriate "vanishing line", the Adams spectral sequence of a connective spectrum can be read off from its Bockstein spectral sequence.

In this short note, we prove a basic folklore theorem which relates the mod $p$ homology Bockstein spectral sequence of $X$ to the Adams spectral sequence $\{E^r X\}$ converging from $E_2 X = \text{Ext}_A(H^* X, Z_p)$ to $\pi_* X$ where $X$ is a bounded below spectrum with integral homology of finite type. As usual, we grade $\{E^r X\}$ so that

$$E_2^{s,t} X = \text{Ext}_A(H^* X, Z_p),$$

with $d_r : E_r^{s,t} X \to E_r^{s+r, t+r-1} X$, the total degree being $t - s$. We have a natural homomorphism $E_2^{0,s} X \to H_* X$ which factors the mod $p$ Hurewicz homomorphism, and we shall sometimes identify elements of $H_* X$ with their inverse images in $E_2^{0,s} X$. We have a pairing of spectral sequences $E_r S \otimes E_r X \to E_r X$, where $S$ is the sphere spectrum. Finally, we have an infinite cycle $a_0 \in E_2^{1,1} S$ such that if $x \in E_\infty^{s,t} X$ and if $y \in F_{s-r} \pi_{t-s} X$ projects to $x$, then $py \in F_{s+1} \pi_{t-s} X$ and $py$ projects to $a_0 x$.

Our main theorem will be a consequence of the following vanishing theorem, which is due to Adams [1] when $p = 2$ and to Liulevicius [4] when $p > 2$. Let $A_0 = E(\beta) \subset A$ and recall that an $A_0$-module $M$ is free if and only if $H(M; \beta) = 0$.

**THEOREM 1.** Let $M$ be an $(m+1)$-connected $A_0$-free $A$-module. Then

$$\text{Ext}_A^s(M, Z_p) = 0 \text{ for } s > 1 \text{ and } t - s < m + f(s),$$

where $f(s) = 2(p - 1)s$ if $p > 2$ and, if $p = 2$, $f(4k) = 8k + 1$, $f(4k + 1) = 8k + 2$, $f(4k + 2) = 8k + 3$, and $f(4k + 3) = 8k + 5$.

**DEFINITION 2.** Let $M$ be an $A$-module. We say that $x \in \text{Ext}_A^s(M, Z_p)$ generates a spike if $x$ is not of the form $a_0 x'$ and if $a_0^i x \neq 0$ for all $i$. The set of spikes in $\text{Ext}_A^s(M, Z_p)$ has its evident meaning. The same language will be applied to each $E_r X$.

Let $K(R, n)$ denote the $n$th Eilenberg-Mac Lane spectrum of $R$ and abbreviate $H R = K(R, 0)$. Let $y$ denote the canonical generator of $H_0(HZ_p)$ and let $\beta$, denote the $r$th mod $p$ Bockstein (in homology or cohomology according to context); let $y = \beta_r z$. 

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Received by the editors September 23, 1980.

1980 Mathematics Subject Classification. Primary 55T15; Secondary 55P42.
LEMMA 3. (i) $E_2^*HZ_p = E_\infty^*HZ_p$ is $Z_p$ in bidegree $(0, 0)$.
(ii) For $r \geq 2$, $E_2^*HZ_p$, is the sum of a spike generated by $y \in E_2^0HZ_p$, and a spike
generated by $z \in E_2^1HZ_p$; moreover, $d_r(a^s_y) = a^{s+r}y$ and $E_\infty^*HZ_p$ has basis
\[ \{a^i_0y^i \mid 0 \leq i < r\}. \]
(iii) $E_2^*HZ = E_\infty^*HZ$ is a spike generated by $y \in E_2^0HZ$.

PROOF. $H^*(HZ_p) = A \cdot \mathbb{1}$, $H^*(HZ_p^r) = (A/A\beta) \cdot \mathbb{1}$ for $r \geq 2$, and
$H^*(HZ) = (A/A\beta) \cdot \mathbb{1}$, where $\mathbb{1}$ denotes the fundamental class. The calculation
of the specified $E_2$ terms is immediate by change of rings [2, VI.4.13], and the
differentials in (ii) follow (up to a nonzero constant) by convergence. That the
constant is $1$ can be checked by a comparison of the constructions of the Adams
and the Bockstein spectral sequences.

We also record the following triviality.

LEMMA 4. $E_r(X \vee Y) = E_rX \oplus E_rY$, with $d_r = d_r \oplus d_r$.

Here, now, is the main result. Its proof derives from a discussion of the edge
theorem one of us had with Mark Mahowald many years ago.

THEOREM 5. Let $X$ be an $(m - 1)$-connected spectrum with integral homology of
finite type. Let $C_r$, $r \geq 1$, be a basis for the $r$th term $E^*X$ of the mod $p$ homology
Bockstein spectral sequence of $X$ and assume the $C_r$, chosen so that
\[ C_r = D_r \cup \beta_rD_r \cup C_{r+1}, \]
where $D_r$, $\beta_rD_r$, and $C_{r+1}$ are disjoint linearly independent subsets of $E^*X$ such that
$\beta_rD_r = \{ \beta_r d \mid d \in D_r \}$ and $C_{r+1}$ is a set of cycles under $\beta_r$ which projects to the
chosen basis for $E^{r+1}X$.

(i) The set of spikes in $E_rX$, $2 \leq r$ and $r = \infty$, is in one-to-one correspondence with
$C_r$; if $c \in C_r$ has degree $q$ and $\gamma \in E^*X$ generates the corresponding spike, then
\[ f(s) + m < q = t - s. \]
(ii) If $d \in D_r$ and if $\delta \in E^*X$ and $e \in E^{m+1}X$, $v - u = t - s - 1$, generate the
spikes corresponding to $d$ and to $\beta_r e$, then
\[ d_r(a^i_0\delta) = a^{i+r+s-1}_0 e \]
provided that $m + f(i + s) > t - s$.

PROOF. Modulo torsion prime to $p$, $H_*^a(X; Z)$ is the direct sum of cyclic groups
of order $p^r$ whose generators reduce mod $p$ to the elements of $\beta_rD_r$ and of infinite
cyclic groups whose generators reduce mod $p$ to the elements of $C_\infty$. By exploiting
the universal coefficients theorem and the representability of integral and mod $p^r$
cohomology, we can use this decomposition to construct maps
\[ \phi_i : X \to K(H^a(X; Z), i) \]
which induce isomorphisms (modulo torsion prime to $p$) on integral homology in
degree $i$. The map
\[ \phi = \sum \phi_i : X \to \bigvee_i K(H^a(X; Z), i) \equiv Y \]
induces a monomorphism on mod $p^r$ homology for all $r$. In particular, we have a short exact sequence

\[ 0 \to H_\bullet(X; \mathbb{Z}_p) \to H_\bullet(Y; \mathbb{Z}_p) \to M_\bullet \to 0, \]

and closer inspection of the construction of the $\phi_i$ shows that

\[ M_\bullet \cong \sum_{q > i + 2} H_q(K(H_i(X; \mathbb{Z}); i); \mathbb{Z}_p). \]

Since $H(A/A\beta; \beta) = 0$, we find (from the proof of Lemma 3) that the dual $M$ of $M_\bullet$ is $A_0$-free and $(m + 1)$-connected. The exact sequence (\ast) gives rise to a long exact sequence

\[ \cdots \to \Ext_A^{i-1}(M, \mathbb{Z}_p) \to E^{s+t}X \to E^{s+t}Y \to \Ext_A^i(M, \mathbb{Z}_p) \to \cdots. \]

By Theorem 1, $E^{s+t}X \to E^{s+t}Y$ is an epimorphism if $s > 1$ and $t - s < m + f(s)$ and is an isomorphism if $s > 2$ and $t - s < m + f(s - 1)$. The conclusions follow directly from Lemmas 3 and 4, by naturality.

**Remark 6.** Spikes of $E_2X$ can be generated by elements lying in lower filtration degree than the range of isomorphism. Such generators can have nontrivial differentials earlier than predicted by the theorem (hitting classes annihilated by appropriate powers of $a_0$); in particular, such differentials can occur on the bottoms of spikes the top parts of which survive to $E_\infty X$. When $X$ is a ring spectrum, such anomalous behavior is sometimes prevented by the relationship between the algebra structure of $H_*X$ and its Bockstein spectral sequence.

We shall apply Theorem 5 to the study of the Adams spectral sequence converging to $\pi_\bullet MS^{top}$ in [3]. As will be illustrated there, the result can be a powerful tool for the computation of differentials in the Adams spectral sequence.

**Bibliography**


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