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HOMOLOGY OPERATIONS REVISITED

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There are two principal kinds of input data for infinite loop space theory, namely E_∞ spaces à la Boardman-Vogt [**3**] and May [**7**] and Γ-spaces à la Segal [**14**]. May and Thomason [**13**] introduced a common generalization and used it to prove the equivalence of the output obtained from these two kinds of input.

This suggests that any invariants of one kind of input should have analogs for the other. Homology operations are among the most basic invariants of E_∞ spaces, and we here establish the analogous invariants for Γ-spaces. The definition is transparently obvious from the point of view of the common generalization but is at first sight rather surprising and unnatural from the point of view of Γ-spaces alone. Probably for this reason, there is no hint of the possibility of a direct definition of homology operations for Γ-spaces in the literature.

We shall explain the philosophy and details of the definition in Section 1, deferring some proofs until Section 4. We shall give a number of consistency results which are essential for the new operations to be of calculational utility in Section 2. Our motivation largely comes from the study of bipermutative categories and multiplicative infinite loop space theory, where a combination of the two kinds of input plays a central role, and these topics will be discussed in Section 3.

1. The construction of the operations. We begin by recalling from [**13**] the generalized domain data for infinite loop space theory. Let $\mathscr{F}$ denote the category of finite based sets $\mathbf{n} = \{0, 1, \ldots, n\}$ (with basepoint 0) and based functions. Let Π denote the subcategory of $\mathscr{F}$ consisting of all morphisms $\phi : \mathbf{m} \to \mathbf{n}$ such that $\phi^{-1}(j)$ has at most one element for $1 \leqq j \leqq n$ ($\phi^{-1}(0)$ may have more than one element). Let $\mathscr{C}$ be an operad [**7**, p. 1]. Thus $\mathscr{C} = \{\mathscr{C}(j) | j \geqq 0\}$ is a collection of (unbased) spaces with suitable structure. As is made precise in [**13**, p. 215], this structure leads to a small topological category $\hat{\mathscr{C}}$ which contains Π and augments to $\mathscr{F}$ via a functor $\epsilon : \hat{\mathscr{C}} \to \mathscr{F}$. The maps of morphism spaces $\epsilon : \hat{\mathscr{C}}(\mathbf{m}, \mathbf{n}) \to \mathscr{F}(\mathbf{m}, \mathbf{n})$ are homotopy equivalences if and only if all $\mathscr{C}(j)$ are contractible, and we shall concentrate on such spacewise contractible operads in this paper.

Let $\mathscr{T}$ be the category of (well-behaved) based spaces. A $\hat{\mathscr{C}}$-space is a continuous functor $X : \hat{\mathscr{C}} \to \mathscr{T}$, written $\mathbf{n} \to X_n$ on objects, such that

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the natural maps $\delta : X_n \to X_1{}^n$ are weak homotopy equivalences. Here δ has ith coordinate induced by the morphism $\delta_i : \mathbf{n} \to \mathbf{1}$ in Π such that $\delta_i(j) = 1$ if $i = j$ and $\delta_i(j) = 0$ otherwise. In particular, $X_0 \simeq \{*\}$. We also require the cofibration condition specified in [**13**, p. 206], but we won't have to make explicit use of it.

We regard $\hat{\mathscr{C}}$-spaces as underlying Π-spaces with extra structure. Note that a space $Y \in \mathscr{T}$ determines a Π-space RY with nth space Y^n. The functor R is right adjoint to the functor L which assigns X_1 to a Π-space X. Loosely, we think of a $\hat{\mathscr{C}}$-space X as providing X_1 with a structure of H-space which is commutative and associative up to all higher coherence homotopies. We refer to [**1**] for an excellent intuitive discussion of what we mean by this.

Our $\mathscr{F}$-spaces are essentially just the same thing as Segal's Γ-spaces (see [**13**, 1.4]). When $X = RY$, our $\hat{\mathscr{C}}$-spaces X are essentially just the same thing as May's $\mathscr{C}$-spaces Y (see [**13**, 4.2]). There is an operad $\mathscr{N}$ with each $\mathscr{N}(j)$ a point; $\hat{\mathscr{N}} = \mathscr{F}$ and $\epsilon : \hat{\mathscr{C}} \to \mathscr{F}$ is induced by the obvious augmentation $\epsilon : \mathscr{C} \to \mathscr{N}$. An $\mathscr{N}$-space is the same thing as a commutative monoid, and we have inclusions of categories

$$\begin{tikzcd}
 & \mathscr{N}\text{-spaces} \arrow[dl, "R"'] \arrow[dr, "\epsilon^*"] & \\
\mathscr{F}\text{-spaces} \arrow[dr, "\epsilon^*"'] & & \mathscr{C}\text{-spaces} \arrow[dl, "R"] \\
 & \hat{\mathscr{C}}\text{-spaces} &
\end{tikzcd}$$

This satisfactorily outlines the input data for infinite loop space theory. However, for the discussion of invariants of these data, the essential ingredient has not yet been mentioned: it is the (right) action of the symmetric groups Σ_j on the spaces $\mathscr{C}(j)$. For example, we shall prove the following sharpening of [**7**, 3.6] in Section 4.

PROPOSITION 1.1. *Let $\mathscr{C}$ be an operad such that each $\mathscr{C}(j)$ is Σ_j-equivariantly contractible. Then there is a functor U from $\mathscr{C}$-spaces to $\mathscr{C}$-spaces, a functor W from $\mathscr{C}$-spaces to $\mathscr{N}$-spaces, and a pair of natural maps of $\mathscr{C}$-spaces*

$$Y \xleftarrow{\epsilon} UY \xrightarrow{\epsilon'} WY$$

such that both ϵ and ϵ' are homotopy equivalences. That is, Y is weakly equivalent as a $\mathscr{C}$-space to the commutative monoid WY. Therefore, if $\pi_0 Y$ is a group, then Y has the homotopy type of a product of Eilenberg-MacLane spaces.

It may well be objected that no such Σ_j-equivariance phenomenon is visible in the notion of an $\mathscr{F}$-space. However, we shall also prove the following analog of the previous result in Section 4. For a $\hat{\mathscr{C}}$-space X, each $\delta : X_n \to X_1{}^n$ is a Σ_n-equivariant map and a weak homotopy equivalence. Let us say that X is a $\Sigma\hat{\mathscr{C}}$-space if each δ is a Σ_n-equivariant homotopy equivalence.

PROPOSITION 1.2. *Let $\mathscr{C}$ be any operad. Then there is a functor U from $\Sigma\hat{\mathscr{C}}$-spaces to $\Sigma\hat{\mathscr{C}}$-spaces, a functor V from $\Sigma\hat{\mathscr{C}}$-spaces to $\mathscr{C}$-spaces, and a pair of natural maps of $\Sigma\hat{\mathscr{C}}$-spaces*

$$X \xleftarrow{\epsilon} UX \xrightarrow{\delta} RVX$$

such that both ϵ and δ are spacewise homotopy equivalences. In particular, with $\mathscr{C} = \mathscr{N}$ and thus $\hat{\mathscr{C}} = \mathscr{F}$, a $\Sigma\mathscr{F}$-space X is weakly equivalent as an $\mathscr{F}$-space to RVX, where VX is a commutative monoid. Therefore, if $\pi_0 X_1$ is a group, then X_1 has the homotopy type of a product of Eilenberg-MacLane spaces.

Both propositions may be viewed as fattened versions of the assertion that an $\mathscr{N}$-space is a commutative monoid. To see what is really going on, recall that the spaces $\mathscr{C}(p)$ are to be thought of as parameter spaces for p-fold multiplications. If Y is a $\mathscr{C}$-space, then, via the maps

(1) $\quad \mathscr{C}(p) \times Y^p \to Y,$

each $c \in \mathscr{C}(p)$ specifies a p-fold product on Y. Since $\mathscr{C}(p)$ is contractible, all of these products are homotopic. Again, if X is an $\mathscr{F}$-space, then we have a diagram

(2) $\quad X_1{}^p \xleftarrow{\delta} X_p \xrightarrow{\phi_p} X_1,$

where ϕ_p is the canonical p-fold multiplication induced by the morphism $\phi_p : \mathbf{p} \to \mathbf{1}$ in $\mathscr{F}$ which sends all $j \geqq 1$ to 1. If δ is an actual homotopy equivalence, then any choice of homotopy inverse for δ specifies a p-fold product on X_1, and all of these products are homotopic. In the previous propositions, these products were Σ_p-equivariantly homotopy commutative in the sense that, up to homotopy, they factored through the orbit space Y^p/Σ_p or $X_1{}^p/\Sigma_p$. Homology operations may be viewed as obstructions to such Σ_p-equivariant homotopy commutativity.

An operad $\mathscr{C}$ is said to be Σ-*free* if Σ_j acts freely on $\mathscr{C}(j)$ for each j; $\mathscr{C}$ is said to be an E_∞ *operad* if it is also spacewise contractible. An E_∞ space is a $\mathscr{C}$-space over any E_∞ operad $\mathscr{C}$. For an E_∞ space Y and a prime p, we can pass to orbits in (1) and then pass to mod p homology to obtain

(3) $\quad H_*(\mathscr{C}(p) \times_{\Sigma_p} Y^p) \to H_* Y.$

We can then define homology operations on $H_* Y$ by a process precisely analogous to the construction of Steenrod operations on the cohomology of spaces (e.g. [**4**, I §1]).

The distinction between spacewise contractible operads and E_∞ operads is effectively finessed by the simple trick of taking products. Given a spacewise contractible operad $\hat{\mathscr{C}}'$, one replaces it by $\mathscr{C} = \mathscr{C}' \times \mathscr{C}_\infty$, where $\mathscr{C}_\infty$ is the little cubes E_∞ operad (or any other convenient fixed chosen E_∞ operad). A $\hat{\mathscr{C}}'$-space is then a $\hat{\mathscr{C}}$-space by pullback. The example to keep in mind is $\mathscr{C}' = \mathscr{N}$, when $\mathscr{C} = \mathscr{C}_\infty$. We can define homology operations for $\mathscr{N}$-spaces this way, but they are obviously trivial. Nevertheless, this simple trick works to define non-trivial operations on H_*X_1 for general $\mathscr{F}$-spaces X. That is, we shall exploit the diagram

$$\text{(4)}\quad H_*(\mathscr{C}(p) \times_{\Sigma_p} X_1^p) \xleftarrow{(1 \times \delta)_*} H_*(\mathscr{C}(p) \times_{\Sigma_p} X_p) \xrightarrow{(\epsilon \times \phi_p)_*} H_*(X_1).$$

Here $(1 \times \delta)_*$ is an isomorphism since $1 \times \delta$ is a weak homotopy equivalence, as one sees by comparing the covering projections

$$\mathscr{C}(p) \times X_1^p \to \mathscr{C}(p) \times_{\Sigma_p} X_1^p \quad \text{and}$$
$$\mathscr{C}(p) \times X_p \to \mathscr{C}(p) \times_{\Sigma_p} X_p.$$

At first glance, the use of $\epsilon : \mathscr{C}(p) \to \mathscr{N}(p) = \{*\}$ and the single multiplication ϕ_p might lead one to believe that only trivial operations could be obtained this way. However, Proposition 1.2 puts things in perspective: The resulting homology operations may be viewed as measuring the deviation of $\delta : X_p \to X_1^p$ from being a Σ_p-equivariant homotopy equivalence.

It is best to view the operations for $\mathscr{F}$-spaces as obtained by specialization of operations constructed for general $\hat{\mathscr{C}}$-spaces. Precisely, we have the following result.

THEOREM 1.3. *Let $\mathscr{C}$ be an E_∞ operad and consider $\hat{\mathscr{C}}$-spaces X. There are natural homomorphisms $Q^s : H_*X_1 \to H_*X_1$ which satisfy all of the standard properties valid for the homology operations Q^s on $\mathscr{C}$-spaces Y. If $X = RY$, then the new operations coincide with the old operations on H_*Y.*

The "standard properties" are those listed in Theorem 1.1 of [**4**, p. 5]. They include the Cartan formula, Adem relations, and stability with respect to $\sigma_* : H_*\Omega X_1 \to H_*X_1$, where $\Omega X = \{\Omega X_n\}$ is a $\hat{\mathscr{C}}$-space by composition of the functors $X : \hat{\mathscr{C}} \to \mathscr{T}$ and $\Omega : \mathscr{T} \to \mathscr{T}$. The proof proceeds as follows. By the construction of $\hat{\mathscr{C}}$ from $\mathscr{C}$, the space $\mathscr{C}(p)$ may be identified with the subspace $\epsilon^{-1}(\phi_p)$ of $\hat{\mathscr{C}}(\mathbf{p}, \mathbf{1})$. We thus have an evaluation (or action) map

$$\Theta : \mathscr{C}(p) \times X_p \to X_1.$$

When X is an $\mathscr{F}$-space regarded as a $\hat{\mathscr{C}}$-space by pullback along ϵ, Θ is just $\epsilon \times \phi_p$ as in (4). We therefore have the homomorphism

$$\theta_*(1 \times \delta)_*^{-1} : H_*(\mathscr{C}(p) \times_{\Sigma_p} X_1{}^p) \to H_*X_1.$$

Let π be the cyclic group of order p embedded in Σ_p in the obvious way and let W be the standard π-free resolution of Z_p. As explained in detail in [**4**, p. 7], we have a standard homomorphism

$$H_*(W \otimes_\pi (H_*X_1)^p) \cong H_*(\mathscr{C}(p) \times_\pi X_1{}^p) \to H_*(\mathscr{C}(p) \times_{\Sigma_p} X_1{}^p).$$

We compose these and define the operations $Q^s x$ in terms of the usual classes $e_i \otimes x^p$ exactly as in [**4**, p. 7]. The verification of most of the properties of the operations is no more complicated than for actual $\mathscr{C}$-spaces. In particular, by naturality diagrams, stability presents no difficulty. The Cartan formula and Adem relations can be derived by a slight elaboration of the earlier proofs and exploitation of the following result, which is the appropriate version for $\hat{\mathscr{C}}$-spaces of the diagram of [**7**, p. 5] that was the heart of the definition of $\mathscr{C}$-spaces.

LEMMA 1.4. *Let X be a $\hat{\mathscr{C}}$-space. Then the following diagram commutes, where $j = j_1 + \ldots + j_k$ and the μ are shuffle homeomorphisms.*

$$\begin{tikzcd}[column sep=small]
\mathscr{C}(j) \times X_1{}^j & & \mathscr{C}(j) \times X_j \arrow[ll, "1 \times \delta"', "\simeq"] \arrow[rr, "\theta"] & & X_1 \\
& & \mathscr{C}(k) \times \left[\mathop{\times}\limits_{s=1}^{k} \mathscr{C}(j_s)\right] \times X_j \arrow[u, "\gamma \times 1"'] \arrow[dll, "1 \times \delta", "\simeq"'] \arrow[drr, "1 \times \theta"] \arrow[dd, "1 \times \delta", "\simeq"'] & & \\
\mathscr{C}(k) \times \left[\mathop{\times}\limits_{s=1}^{k} \mathscr{C}(j_s)\right] \times X_1{}^j \arrow[uu, "\gamma \times 1"] \arrow[dd, "1 \times \mu", "\simeq"'] & & & & \mathscr{C}(k) \times X_k \arrow[uu, "\theta"'] \arrow[dd, "1 \times \delta", "\simeq"'] \\
& & \mathscr{C}(k) \times \left[\mathop{\times}\limits_{s=1}^{k} \mathscr{C}(j_s)\right] \times \left[\mathop{\times}\limits_{s=1}^{k} X_{j_s}\right] \arrow[ull, "1 \times \delta^j", "\simeq"'] \arrow[d, "1 \times \mu", "\simeq"'] & & \\
\mathscr{C}(k) \times \left[\mathop{\times}\limits_{s=1}^{k} \mathscr{C}(j_s) \times X_1{}^{j_s}\right] & & \mathscr{C}(k) \times \left[\mathop{\times}\limits_{s=1}^{k} \mathscr{C}(j_s) \times X_{j_s}\right] \arrow[ll, "1 \times (1 \times \delta)^k", "\simeq"'] \arrow[rr, "1 \times \theta^k"] & & \mathscr{C}(k) \times X_1{}^k
\end{tikzcd}$$

Here $\delta : X_j \to \mathop{\times}_{s=1}^{k} X_{j_s}$ has sth coordinate induced by the projection $\mathbf{j}_1 \vee \ldots \vee \mathbf{j}_k \to \mathbf{j}_s$ in Π and $\mathop{\times}_{s=1}^{k} \mathscr{C}(j_s)$ in the middle is identified with the component $\epsilon^{-1}(\phi_{j_1} \vee \ldots \vee \phi_{j_k})$ of the space $\hat{\mathscr{C}}(\mathbf{j}, \mathbf{k})$.

The proof is immediate from the definitions. However, those who dislike diagrams may be relieved to learn that we actually do not have to give direct proofs of the properties of the operations. In view of the following replacement result, to be proven in Section 4, they are immediate consequences of the known properties in the case of actual $\mathscr{C}$-spaces.

PROPOSITION 1.5. *Let $\mathscr{C}$ be any Σ-free operad. Then there is a functor U from $\hat{\mathscr{C}}$-spaces to $\hat{\mathscr{C}}$-spaces, a functor V from $\hat{\mathscr{C}}$-spaces to $\mathscr{C}$-spaces, and a pair of natural maps of $\hat{\mathscr{C}}$-spaces*

$$X \xleftarrow{\epsilon} UX \xrightarrow{\delta} RVX$$

such that ϵ is a spacewise equivalence and δ is a spacewise weak equivalence.

That is, X is weakly equivalent as a $\hat{\mathscr{C}}$-space to RVX. Therefore, the new homology operations on H_*X_1 agree under the induced isomorphism with the old homology operations on H_*VX. In particular, we can obtain the new operations for $\mathscr{F}$-spaces by first regarding $\mathscr{F}$-spaces as $\hat{\mathscr{C}}$-spaces by pullback and then replacing them by weakly equivalent $\mathscr{C}$-spaces, where $\mathscr{C}$ is any E_∞ operad.

Remarks 1.6. (i) If $\mathscr{C} \to \mathscr{C}'$ is a map of E_∞ operads, then

$$\mathscr{C}(p) \times_{\Sigma_p} Z \to \mathscr{C}'(p) \times_{\Sigma_p} Z$$

is an equivalence for any Σ_p-space Z. Therefore the homology operations for $\hat{\mathscr{C}}'$-spaces are the same when constructed using $\mathscr{C}'$ or when using $\mathscr{C}$. In particular, if $\mathscr{C}$ and $\mathscr{D}$ are E_∞ operads, then this observation applies to the projections from $\mathscr{C} \times \mathscr{D}$ to $\mathscr{C}$ and to $\mathscr{D}$. Therefore the homology operations for $\mathscr{F}$-spaces are independent of the choice of E_∞ operad $\mathscr{C}$ used in their construction. In fact, we could replace $\mathscr{C}(p)$ by any Σ_p-free contractible space in the construction.

(ii) Instead of starting with a spacewise contractible operad, we could have started a bit more generally with an arbitrary category of operators $\mathscr{G}$ (in the sense of [**13**, 1.1]) such that $\epsilon : \mathscr{G} \to \mathscr{F}$ is an equivalence. One then defines homology operations on H_*X_1, where X is a $\mathscr{G}$-space, as above using the diagram

$$(E \times \epsilon^{-1}(\phi_p)) \times_{\Sigma_p} X_1{}^p \xleftarrow{1 \times \delta} (E \times \epsilon^{-1}(\phi_p)) \times_{\Sigma_p} X_p \xrightarrow{\epsilon \circ \pi \times \phi_p} X_1$$

where E is any contractible space on which Σ_p acts freely. The resulting homology operations are natural with respect to maps of $\mathscr{G}$-spaces and satisfy all the standard properties. This follows by naturality and the existence of the sequence of weak equivalences of $\mathscr{G}$-spaces

$$X \leftarrow 1_*X \to \epsilon^*\epsilon_*X$$

where ϵ_*X is an $\mathscr{F}$-space (cf. [**13**, 1.8]).

2. Operations on infinite loop spaces. We turn next to consistency statements relating the new homology operations on $\hat{\mathscr{C}}$-spaces to the known homology operations on infinite loop spaces. As before, we start with a spacewise contractible operad $\mathscr{C}'$ and set $\mathscr{C} = \mathscr{C}' \times \mathscr{C}_\infty$, where $\mathscr{C}_\infty$ is the little cubes E_∞ operad.

By a spectrum E, we understand a sequence of spaces E_i and homeomorphisms

$$\tilde{\sigma}_i : E_i \to \Omega E_{i+1}.$$

By [**7**, 5.1], E_0 is naturally a $\mathscr{C}_\infty$-space. This structure gives canonical homology operations on infinite loop spaces. The same operations are obtained by regarding E_0 as a $\mathscr{C}$-space by pullback along the projection $\mathscr{C} \to \mathscr{C}_\infty$.

Now [**13**, § 6] associates to a $\hat{\mathscr{C}}$-space X a spectrum MX and a map $\iota : X_1 \to M_0X$ which induces group completion on homology. Moreover, with the notation of Proposition 1.5, we have a diagram

$$X \xleftarrow{\epsilon} UX \xrightarrow{\delta} RVX \xrightarrow{R\gamma} RM_0X$$

of $\hat{\mathscr{C}}$-spaces in which the $\mathscr{C}$-map $\gamma : VX \to M_0X$ is a group completion. Further $\iota = \gamma \circ \delta_1 \circ \tau_1$, where $\tau : X \to UX$ is a natural map of Π-spaces homotopy inverse to ϵ. Granting these statements from [**13**, § 6] and [**12**, 4.6], some details of which will be recalled in Section 4, we conclude the following result.

PROPOSITION 2.1. *For a $\hat{\mathscr{C}}$-space X, the homomorphism*

$$\iota_* : H_*X_1 \to H_*M_0X$$

preserves homology operations.

This proof depends on use of the "May machine" for the passage from input to output. In fact, as discussed in [**7**, p. 154–155], it was the need for just such a consistency result that originally led to the invention of the May machine. However, we can use the uniqueness theorem of [**13**] to transport the conclusion to the Segal or any other machine. Recall from [**13**, 2.1] that, for a category of operators $\mathscr{G}$ such that $\epsilon : \mathscr{G} \to \mathscr{F}$ is an equivalence, an infinite loop space machine defined on $\mathscr{G}$-spaces is a functor E from $\mathscr{G}$-spaces to connective spectra together with a natural group completion $\iota : X_1 \to E_0X$. As in Remarks 1.6 (ii), we may as well restrict attention to the case $\mathscr{G} = \mathscr{F}$. With $\mathscr{C}' = \mathscr{N}$ and X restricted to be an $\mathscr{F}$-space regarded as a $\hat{\mathscr{C}}_\infty$-space by pullback, we have the May machine M described above. For any other machine E, the uniqueness theorem of [**13**] provides an equivalence of spectra $\zeta : MX \to EX$. As should have been, but wasn't, explicitly stated in [**13**], the following diagram commutes.

$$\begin{tikzcd} & M_0X \arrow[dd, "\zeta_0"] \\ X_1 \arrow[ur, "\iota"] \arrow[dr, "\iota"'] & \\ & E_0X \end{tikzcd}$$

Strictly speaking, ζ is actually not a map but a chain of (weak) equivalences of spectra with suitable compatibility triangles as above on the zeroth space level; compare [**13**, p. 215 and 221]. Since ζ_0 preserves homology operations by naturality, we have the following immediate consequence of the preceding result.

PROPOSITION 2.2. *For an $\mathscr{F}$-space X and an infinite loop space machine E, the homomorphism $\iota_* : H_*X_1 \to H_*E_0X$ preserves homology operations.*

In fact, the homology operations on $\mathscr{F}$-spaces are characterized by this compatibility assertion for any given E.

PROPOSITION 2.3. *Let E be an infinite loop space machine. Suppose given natural homology operations $\{Q_E{}^s\}$ on H_*X_1 for $\mathscr{F}$-spaces X and suppose that these operations are mapped under ι_* to the canonical homology operations on H_*E_0X. Then $\{Q_E{}^s\}$ coincides with the set $\{Q^s\}$ of homology operations on $\mathscr{F}$-spaces constructed in the previous section.*

Proof. By hypothesis and by the previous proposition, ι_* carries both sets of operations to the canonical operations. Therefore these operations certainly agree whenever ι_* is a monomorphism. While ι_* need not be a monomorphism in general, we can nevertheless reduce the general case to this special case. By Proposition 1.5, for $\mathscr{F}$-spaces X we have a natural weak equivalence of $\mathscr{C}_\infty$-spaces

$$\epsilon^*X \leftarrow UX \to RVX,$$

where VX is a $\mathscr{C}_\infty$-space. By [**13**, 1.8] ,we have a derived natural weak equivalence of $\mathscr{F}$-spaces

$$X \leftarrow \epsilon_*\epsilon^*X \to \epsilon_*UX \to \epsilon_*RVX.$$

Thus we may assume without loss of generality that $X = \epsilon_*RY$ for a $\mathscr{C}_\infty$-space Y. The action of $\mathscr{C}_\infty$ on Y is given by a $\mathscr{C}_\infty$-map $\Theta : C_\infty Y \to Y$, where C_∞ is the monad in $\mathscr{T}$ associated to $\mathscr{C}_\infty$ (so that $C_\infty Y$ is the free $\mathscr{C}_\infty$-space generated by Y; see [**7**, § 2]). By [**13**, 1.8] again, we have the following commutative diagram of $\mathscr{C}_\infty$-spaces in which the horizontal arrows are equivalences:

$$\begin{tikzcd} RC_\infty Y \arrow[d, "R\Theta"'] & 1_*RC_\infty Y \arrow[l] \arrow[r] \arrow[d, "1_*R\Theta"] & \epsilon^*\epsilon_*RC_\infty Y \arrow[d, "\epsilon^*\epsilon_*R\Theta"] \\ RY & 1_*RY \arrow[l] \arrow[r] & \epsilon^*\epsilon_*RY \end{tikzcd}$$

By comparison with known properties of the May machine (namely the version of the Barratt-Quillen Theorem given by [**7**, 14.4 (vii)] or [**9**, VII 3.3]), it follows from the top row that the group completion

$$\iota : (\epsilon_*RC_\infty Y)_1 \to E_0\epsilon_*RC_\infty Y$$

is equivalent to the natural group completion

$$\alpha_\infty : C_\infty Y \to QY$$

and therefore induces a monomorphism in homology by the explicit calculations of [**4**, I § 4]. Thus $Q_E^s = Q^s$ on $H_*(\epsilon_*RC_\infty Y)_1$. Since Θ is a retraction, it induces an epimorphism on homology. Therefore, by the diagram, $(\epsilon_*R\Theta)_1$ also induces an epimorphism on homology. Since $\epsilon_*R\Theta$ is an $\mathscr{F}$-map, $(\epsilon_*R\Theta)_1$ preserves both sets of homology operations. It follows that $Q_E^s = Q^s$ on $H_*(\epsilon_*RY)_1$, as was to be shown.

This result is a uniqueness assertion for homology operations on $\mathscr{F}$-spaces modulo the question of the uniqueness of homology operations on infinite loop spaces. We remark parenthetically that the second author tried quite hard, without success, to prove an algebraic axiomatization of the homology operations on the homology of infinite loop spaces analogous to the standard axiomatization of the Steenrod operations on the cohomology of spaces. (Incidentally, as far as we know, nobody has bothered to work out the comparison of geometric constructions that would be needed to check that our canonical homology operations agree with the operations originally defined by Araki and Kudo [**2**] or by Dyer and Lashof [**5**].)

This leads to one further consistency question. In [**14**, 3.3], Segal constructed a functor A from connective spectra to $\mathscr{F}$-spaces such that $(AE)_1 = E_0$. There result new homology operations on infinite loop spaces coming from the $\mathscr{F}$-space structure, and the latter bears little geometric similarity to the $\mathscr{C}_\infty$-space structure used above.

PROPOSITION 2.4. *For a connective spectrum E, the $\mathscr{C}_\infty$-space operations and $\mathscr{F}$-space operations on H_*E_0 coincide.*

Proof. Consider the Segal infinite loop space machine S with its natural group completion ι. By [**14**, 3.3], there is an equivalence of spectra $\xi : SAE \to E$ such that $\xi_0 \circ \iota$ is the identity map of E_0. By Proposition

2.2, ι carries the $\mathscr{F}$-space operations to the $\mathscr{C}_\infty$-space operations. By naturality, ξ_0 preserves the $\mathscr{C}_\infty$-space operations.

3. Permutative and bipermutative categories. The passage from permutative categories to spectra was axiomatized in [**10**]. The axiomatization was needed because there are two quite different ways of passing from permutative categories $\mathscr{A}$ to the input data of infinite loop space theory. On the one hand, the classifying space $B\mathscr{A}$ is a $\mathscr{D}$-space for a certain E_∞ operad $\mathscr{D}$ (see [8, § 3]). On the other hand, $B\mathscr{A}$ is $B\overline{\mathscr{A}}_1$ where $\overline{\mathscr{A}} : \mathscr{F} \to \mathrm{Cat}$ is a functor such that $B\overline{\mathscr{A}}$ is an $\mathscr{F}$-space (see [**14**, § 2] and [**10**, Const. 10]). Now both kinds of data have homology operations, and these operations are carried to the canonical operations by $\iota : B\mathscr{A} \to E_0\mathscr{A}$ for an infinite loop space machine E defined on either sort of data. Again, ι_* need not be a monomorphism, and we need a little argument to prove the following result.

PROPOSITION 3.1. *For a permutative category $\mathscr{A}$, the $\mathscr{D}$-space operations and $\mathscr{F}$-space operations on $H_*B\mathscr{A}$ coincide.*

Proof. $\mathscr{D}(p) = B\widetilde{\Sigma}_p$, where $\widetilde{\Sigma}_p$ is the translation category of Σ_p, and the action of $\mathscr{D}$ on $B\mathscr{A}$ is induced by passage to classifying spaces from the functor

$$c_p : \widetilde{\Sigma}_p \times \mathscr{A}^p \to \mathscr{A}$$

specified on [8, p. 81]. With ϵ the projection from $\widetilde{\Sigma}_p$ to the trivial category, it suffices to prove that the following diagram of functors commutes up to Σ_p-equivariant natural transformation:

$$\begin{tikzcd} \widetilde{\Sigma}_p \times \mathscr{A}^p \arrow[d, "c_p"'] & \widetilde{\Sigma}_p \times \overline{\mathscr{A}}_p. \arrow[l, "1 \times \delta"'] \arrow[d, "\epsilon \times \phi_p"] \\ \mathscr{A} \arrow[r, equal] & \mathscr{A} \end{tikzcd}$$

With the notations of [**10**, Const. 10], for objects

$$\sigma \in \widetilde{\Sigma}_p \quad \text{and} \quad \langle A_s, i_{(s,t)} \rangle \in \overline{\mathscr{A}}_p,$$

we have

$$c_p \circ (1 \times \delta)(\sigma, \langle A_s, i_{(s,t)} \rangle) = A_{\sigma^{-1}(1)} \square \cdots \square A_{\sigma^{-1}(p)}$$

and

$$(\epsilon \times \phi_p)(\sigma, \langle A_s, i_{(s,t)} \rangle) = A_{\mathbf{p}}.$$

The $\iota_{(s,t)}$ determine the required isomorphism

$$A_{\mathbf{p}} \to A_{\sigma^{-1}(1)} \square \dots \square A_{\sigma^{-1}(p)},$$

and nothing is changed upon replacement of $(\sigma, \langle A_s, i_{(s,t)}\rangle)$ by

$$(\sigma\tau, \langle A_s, i_{(s,t)}\rangle\tau) \quad \text{for } \tau \in \Sigma_p.$$

Remark 3.2. Application of "Street's first construction" gives a different and in many ways preferable functor $\tilde{\mathscr{A}}: \tilde{\mathscr{F}} \to \mathrm{Cat}$ with $B\tilde{\mathscr{A}}$ an $\tilde{\mathscr{F}}$-space; here $B\tilde{\mathscr{A}}_1$ is equivalent rather than equal to $B\mathscr{A}$ (see [**15**] or [**11**, § 3]). By [**11**, A.4], there is a natural map $B\tilde{\mathscr{A}} \to B\bar{\mathscr{A}}$ of $\tilde{\mathscr{F}}$-spaces which realizes the equivalence $B\tilde{\mathscr{A}}_1 \to B\bar{\mathscr{A}}_1 = B\mathscr{A}$, hence this approach also leads to the same homology operations.

We turn next to a discussion of homology operations in the context of E_∞ ring spaces [**9**] and their generalizations of [**12**]. Here we start out with a suitably related pair of operads $(\mathscr{C}, \mathscr{G})$, such as the pair $(\mathscr{N}, \mathscr{N})$, and construct from it a certain wreath product category $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$ which contains $\Pi \int \Pi$ and is augmented over $\mathscr{F} \int \mathscr{F}$. Where in the additive theory we had $\mathscr{C}$-spaces Y defined in terms of powers of Y and $\hat{\mathscr{C}}$-spaces X defined in terms of Π-spaces $X = \{X_n\}$, the multiplicative theory deals with $(\mathscr{C}, \mathscr{G})$-spaces Z, $\mathscr{J}$-spaces X, and an intermediate category of $(\hat{\mathscr{C}}, \mathscr{G})$-spaces Y. Here X has an underlying $\Pi \int \Pi$-space which is a collection of spaces

$$X(n; \mathbf{s}_1, \dots, \mathbf{s}_n) \simeq X(1; \mathbf{1})^{s_1 + \dots + s_n}.$$

For a $(\hat{\mathscr{C}}, \mathscr{G})$-space Y, Y is a Π-space and determines a $\mathscr{J}$-space $R''Y$ whose underlying $\Pi \int \Pi$-space is given by

$$(R''Y)(n; \mathbf{s}_1, \dots, \mathbf{s}_n) = Y_{s_1} \times \dots \times Y_{s_n}.$$

For a $(\hat{\mathscr{C}}, \mathscr{G})$-space Z, Z is a space and determines a $(\hat{\mathscr{C}}, \mathscr{G})$-space $R'Z$ with $(R'Z)_s = Z^s$ and thus determines a $\mathscr{J}$-space $RZ = R''R'Z$. The category $\mathscr{J}$ contains both $\hat{\mathscr{C}}$ and $\hat{\mathscr{G}}$, and a $\hat{\mathscr{G}} \int \hat{\mathscr{C}}$-space X restricts to a $\hat{\mathscr{C}}$-space $X_\oplus$ with underlying Π-space $\{X(1; \mathbf{n})\}$ and to a $\hat{\mathscr{G}}$-space $X_\otimes$ with underlying Π-space $\{X(n; \mathbf{1}^n)\}$. Therefore, by Theorem 1.3, $H_*X(1; \mathbf{1})$ has "additive" operations Q^s coming from the action of $\hat{\mathscr{C}}$ and "multiplicative" operations $\tilde{Q}^s$ coming from the action of $\hat{\mathscr{G}}$ when $\mathscr{C}$ and $\mathscr{G}$ are E_∞ operads. For $(\mathscr{C}, \mathscr{G})$-spaces Z, these operations were studied in [**4**, II]. We have the following multiplicative analog of Theorem 1.3.

THEOREM 3.3. *Let $(\mathscr{C}, \mathscr{G})$ be an E_∞ operad pair and consider $\mathscr{J}$-spaces X, where $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$. The operations Q^s and $\tilde{Q}^s$ on $H_*X(1; \mathbf{1})$ satisfy all of the algebraic properties valid for the operations on $(\mathscr{C}, \mathscr{G})$-spaces Z.*

The properties in question are the "mixed Cartan formula", the "mixed Adem relations", and various other formulas derived in [**4**, II § 1–3]. As in the additive theory, two proofs are possible. The first consists of slight elaborations of the earlier arguments and is based on the following analog of Lemma 1.4, which gives the appropriate version for $\mathscr{J}$-spaces of the diagram of [**4**, p. 77] that was the heart of the definition of $(\mathscr{C}, \mathscr{G})$-spaces.

LEMMA 3.4. *Let $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$, where $(\mathscr{C}, \mathscr{G})$ is an operad pair. Let X be a $\mathscr{J}$-space and abbreviate $Z = X(1; \mathbf{1})$. Then the following diagram commutes, where $J = (\mathbf{j}_1, \ldots, \mathbf{j}_k)$ and $j = j_1 \ldots j_k$:*

$$\begin{tikzcd}
\mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r) \times Z^{j_r}\right) \arrow[d, "\cong" description, no head] & \mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r) \times X(1; \mathbf{j}_r)\right) \arrow[l, "\simeq"'] \arrow[r, "1 \times \left(\bigtimes_{r=1}^{k} \theta\right)"] \arrow[d, "\cong" description, no head] & \mathscr{G}(k) \times Z^k \\
\mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r)\right) \times \left(\bigtimes_{r=1}^{k} Z^{j_r}\right) \arrow[d, "\alpha"'] & \mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r)\right) \times \left(\bigtimes_{r=1}^{k} X(1; \mathbf{j}_r)\right) \arrow[l, "\simeq"'] & \\
\mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r)\right) \times (\mathscr{G}(k) \times Z^k)^j & \mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r)\right) \times X(k; J) \arrow[u, "\simeq"'] \arrow[r, "1 \times \theta"] \arrow[d, "(1 \times \iota \times 1)(\Delta \times 1 \times 1)"] & \mathscr{G}(k) \times X(k; \mathbf{1}^k) \arrow[uu, "\simeq"'] \arrow[dd, "\xi"] \\
\mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r)\right) \times (\mathscr{G}(k) \times X(k; \mathbf{1}^k))^j \arrow[u, "\simeq"] \arrow[d, "\lambda \times \xi^j"'] & \mathscr{G}(k) \times \left(\bigtimes_{r=1}^{k} \mathscr{C}(j_r)\right) \times \mathscr{G}(k) \times X(k; J) \arrow[l, "1 \times 1 \times \beta"'] \arrow[d, "\lambda \times \xi"] & \\
\mathscr{C}(j) \times Z^j & \mathscr{C}(j) \times X(1; \mathbf{j}) \arrow[l, "\simeq"'] \arrow[r, "\theta"] & Z
\end{tikzcd}$$

Here the θ and ξ are evaluation maps of the functor X, the arrows labeled $\simeq$ are equivalences given by the definition of a $\mathscr{J}$-space, and the arrows labeled $\cong$ are shuffle homeomorphisms. The map α is given by

$$\alpha(g, c, y) = \left(g, c, \bigtimes_{Q} (g, y_Q)\right)$$

for $g \in \mathscr{G}(k)$,

$$c \in \bigtimes_{r=1}^{k} \mathscr{C}(j_r) \quad \text{and} \quad y = \bigtimes_{r=1}^{k} \bigtimes_{q=1}^{j_r} z_{r,q}, \quad z_{r,q} \in Z,$$

where

$$y_Q = \bigtimes_{r=1}^{k} z_{r,q_r}$$

for sequences $Q = (q_1, \ldots, q_k)$ with $1 \leqq q_r \leqq j_r$; the map

$$\beta : \mathscr{G}(k) \times X(k; J) \to (\mathscr{G}(k) \times X(k; \mathbf{1}^k))^j$$

has Qth coordinate $(1, \beta_Q)$, where

$$\beta_Q = (1; \mathbf{k} \to \mathbf{k}; \delta_{q_1}, \ldots, \delta_{q_k}) : X(k; J) \to X(k; \mathbf{1}^k).$$

The proof is an exercise in the interpretation of the definitions of [**12**, § 1–2], the commutation formula [**12**, 1.6] being the crux of the matter.

Rather than exploit this diagram, we can appeal to the following analog of Proposition 1.5, which is proven in [**12**]. We write $U\oplus$ and $V\oplus$ for the functors of Proposition 1.5 in order to emphasize that they were defined solely in terms of the underlying additive structure.

THEOREM 3.5. *Let* $\mathscr{J} \int = \hat{\mathscr{G}}\ \hat{\mathscr{C}}$, *where* $(\mathscr{C}, \mathscr{G})$ *is a* Σ*-free operad pair.*

(i) *There is a functor* U *from* $\mathscr{J}$*-spaces to* $\mathscr{J}$*-spaces, a functor* V *from* $\mathscr{J}$*-spaces to* $(\mathscr{C}, \mathscr{G})$*-spaces, and a pair of natural maps of* $\mathscr{J}$*-spaces*

$$X \xleftarrow{\epsilon} UX \xrightarrow{\delta} R''VX$$

such that ϵ *is a spacewise equivalence and* δ *is a spacewise weak equivalence.*

(ii) *If* Y *is a* $(\hat{\mathscr{C}}, \mathscr{G})$*-space, then* $U\oplus Y$ *is a* $(\hat{\mathscr{C}}, \mathscr{G})$*-space,* $V\oplus Y$ *is a* $(\mathscr{C}, \mathscr{G})$*-space, and the natural maps*

$$Y \xleftarrow{\epsilon} U\oplus Y \xrightarrow{\delta} R'V\oplus Y$$

are maps of $(\hat{\mathscr{C}}, \mathscr{G})$*-spaces.*

Thus a $\mathscr{J}$-space X is weakly equivalent as a $\mathscr{J}$-space to $RV\oplus VX$, where $V\oplus VX$ is a $(\mathscr{C}, \mathscr{G})$-space. Hence the algebraic properties of the homology operations on $H_*X(1; \mathbf{1})$ are immediate consequences of their properties on $H_* V\oplus VX$.

Now let

$$(\mathscr{C}, \mathscr{G}) = (\mathscr{C}' \times \mathscr{K}_\infty, \mathscr{G}' \times \mathscr{L}),$$

where $(\mathscr{C}', \mathscr{G}')$ is a spacewise contractible operad pair and $(\mathscr{K}_\infty, \mathscr{L})$ is the canonical E_∞ operad pair used in [**12**]; $\mathscr{K}_\infty$ can be used instead of $\mathscr{C}_\infty$ in the purely additive theory above. There is a notion of a $\mathscr{G}$-spectrum, and the zeroth space of a $\mathscr{G}$-spectrum is a $(\mathscr{C}, \mathscr{G})$-space [**9**, IV. 1.1 and VII. 2.4]. If Y is a $(\hat{\mathscr{C}}, \mathscr{G})$-space, then MY is a $\mathscr{G}$-spectrum and

$$\gamma : V\oplus Y \to M_0 Y$$

is a map of $(\mathscr{C}, \mathscr{G})$-spaces [**12**, 4.4 and 4.8]. We therefore have the following analog of Proposition 2.1.

PROPOSITION 3.7. *Let* $\mathscr{J} = \hat{\mathscr{G}} \int \hat{\mathscr{C}}$ *with* $(\mathscr{C}, \mathscr{G})$ *as above. For a* $\mathscr{J}$*-space* X,

$$H_*X(1; \mathbf{1}) \cong H_*(VX)_1 \xrightarrow{\iota_*} H_* M_0 VX$$

preserves both additive and multiplicative homology operations.

Here MVX is equivalent as an additive spectrum to $MX_{\oplus}$, but $MX_{\oplus}$ itself is not a $\mathscr{G}$-spectrum.

Woolfson [16] has described a Segal style multiplicative infinite loop space machine. As explained in [12, Appendix D], it accepts our $\mathscr{F}\int\mathscr{F}$-spaces as input but has considerably more complicated up to homotopy versions of our $\mathscr{L}$-spectra as output. In particular, it is far from clear precisely what structure the 0th spaces of his output spectra carry, the most optimistic guess being that they are $(1;\mathbf{1})$th spaces of $\mathscr{F}\int\mathscr{F}$-spaces. Since the extra complexity appears to offer no compensatory advantages, we have not tried to compare machines.

Need for the full generality of $\mathscr{J}$-spaces arises in the study of bipermutative categories. These are categories $\mathscr{A}$ which are permutative under both a sum $\oplus$ and product $\otimes$ and which satisfy appropriate distributivity and nullity of zero laws; see [9, VI § 3]. We write $\mathscr{A}_{\oplus}$ or $\mathscr{A}_{\otimes}$ for $\mathscr{A}$ regarded just as an additive or multiplicative permutative category. As sketched in [16] in a special case and explained in detail in [12, D.6], $B\mathscr{A} = B\bar{\mathscr{A}}(1;\mathbf{1})$, where $\bar{\mathscr{A}}:\mathscr{F}\int\mathscr{F}\rightarrow \mathrm{Cat}$ is a functor such that $B\bar{\mathscr{A}}$ is an $\mathscr{F}\int\mathscr{F}$-space. A simple comparison of [12, D.6] and [10, Const. 10] gives the following consistency statement.

LEMMA 3.8. *The restrictions of* $\bar{\mathscr{A}}:\mathscr{F}\int\mathscr{F}\rightarrow \mathrm{Cat}$ *to the additive and multiplicative copies of* $\mathscr{F}$ *contained in* $\mathscr{F}\int\mathscr{F}$ *coincide with the functors* $\bar{\mathscr{A}}_{\oplus}$ *and* $\bar{\mathscr{A}}_{\otimes}$ *defined solely in terms of the underlying additive and multiplicative permutative categories of* $\mathscr{A}$.

Regarding $\mathscr{F}\int\mathscr{F}$-spaces as $\hat{\mathscr{G}}\int\hat{\mathscr{C}}$-spaces by pullback, where $(\mathscr{C},\mathscr{G})$ is an E_∞ operad pair such as $(\mathscr{K}_\infty,\mathscr{L})$,we obtain additive and multiplicative homology operations Q^s and $\tilde{Q}^s$ on $H_*B\mathscr{A}$ which satisfy the conclusion of Theorem 3.3. By the lemma, these operations coincide with the $\mathscr{F}$-space operations on $H_*B\mathscr{A}_{\oplus}$ and on $H_*B\mathscr{A}_{\otimes}$. Therefore, by Proposition 3.1, they also coincide with the $\mathscr{D}$-space homology operations on $H_*B\mathscr{A}_{\oplus}$ and on $H_*B\mathscr{A}_{\otimes}$.

It was asserted in [9, VI] that $(\mathscr{D},\mathscr{D})$ is an operad pair and $B\mathscr{A}$ is a $(\mathscr{D},\mathscr{D})$-space. For reasons explained in [12, Appendix A], these assertions are false. However, by the previous paragraph, the operations Q^s and $\tilde{Q}^s$ obtained from the two $\mathscr{D}$-space structures on $B\mathscr{A}$ satisfy the mixed Cartan formula, mixed Adem relations, and so forth. This is fortunate since, as was exploited in [4, II] and [6, Appendix C], the concrete specification of the actions by $\mathscr{D}$ in terms of wreath products makes them very convenient for purposes of explicit calculation. The following remark validates the discussion of this point given in [9, p. 207–208] and [4, p. 142–143].

Remark 3.9. The arguments above imply that if $\Gamma B\mathscr{A}$ is interpreted as $M_0 VB\mathscr{A}$ and $\iota : B\mathscr{A} \to \Gamma B\mathscr{A}$ is interpreted as the composite group completion

$$B\mathscr{A} \xrightarrow{\delta\epsilon^{-1}} (VB\mathscr{A})_1 \xrightarrow{\iota} M_0 VB\mathscr{A},$$

then the diagrams of [**9**, VIII. 1.2] are homotopy commutative. (This verification was promised in [**12**, A.2 (21)] and completes the check that nothing of consequence in [**9**] is affected by the errors noted above.)

Remark 3.10. By [**12**, § 3], application of "Street's first construction" gives a different and in many ways preferable functor $\tilde{\mathscr{A}} : \mathscr{F}\int\mathscr{F} \to \mathrm{Cat}$ with $B\tilde{\mathscr{A}}$ an $\mathscr{F}\int\mathscr{F}$-space; here $B\tilde{\mathscr{A}}(1; \mathbf{1})$ is equivalent rather than equal to $B\mathscr{A}$. By [**12**, D.8] there is a natural map $B\tilde{\mathscr{A}} \to B\bar{\mathscr{A}}$ of $\mathscr{F}\int\mathscr{F}$-spaces which realizes the equivalence, hence this approach leads to the same homology operations Q^s and $\tilde{Q}^s$ on $H_* B\mathscr{A}$. We have used the less general and conceptual $\mathscr{A}$ constructions only because Lemma 3.8 is simpler and more precise than its counterpart for $\tilde{\mathscr{A}}$ given in [**12**, 3.2].

Remark 3.11. More generally we can define additive and multiplicative homology operations directly on $H_* X(1; \mathbf{1})$, when X is a $\mathscr{J}$-space and $\mathscr{J}$ is an arbitrary category of ring operators such that $\epsilon : \mathscr{J} \to \mathscr{F}\int\mathscr{F}$ is an equivalence (cf. [**14**, 1.7]). For we can associate to $\mathscr{J}$ two categories of operators $\mathscr{J}_\oplus$ and $\mathscr{J}_\otimes$ by means of the pullback diagrams

$$\begin{tikzcd} \mathscr{J}_\oplus \arrow[r] \arrow[d, "\epsilon"] & \mathscr{J} \arrow[d, "\epsilon"] \\ \mathscr{F} \arrow[r, "i_\oplus"] & \mathscr{F}\int\mathscr{F} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{J}_\otimes \arrow[r] \arrow[d, "\epsilon"] & \mathscr{J} \arrow[d, "\epsilon"] \\ \mathscr{F} \arrow[r, "i_\otimes"] & \mathscr{F}\int\mathscr{F} \end{tikzcd}$$

where $i_\oplus$ and $i_\otimes$ are the additive and multiplicative embeddings of $\mathscr{F}$ in $\mathscr{F}\int\mathscr{F}$ (cf. [**14**, 1.5]). Then $\epsilon : \mathscr{J}_\oplus \to \mathscr{F}$ and $\epsilon : \mathscr{J}_\otimes \to \mathscr{F}$ are equivalences and we may use the procedure of Remark 1.6 (ii) to define additive and multiplicative operations on $H_* X(1; \mathbf{1})$ satisfying all the standard properties. This follows by naturality and the existence of the sequence of weak equivalences of $\mathscr{J}$-spaces

$$X \leftarrow 1_* X \to \epsilon^* \epsilon_* X$$

where $\epsilon_* X$ is an $\mathscr{F}\int\mathscr{F}$-space (cf. [**14**, 2.9]).

4. Proofs of structured weak equivalences. The proofs of Propositions 1.1, 1.2 and 1.5 are all simple exercises in the use of the monadic two-sided bar construction introduced in [**7**, § 9]. In fact, the replacement of A_∞ spaces by weakly equivalent topological monoids given in [**7**, 13.5] provides a paradigmatic example of this type of proof.

Proof of Proposition 1.1. For a $\mathscr{C}$-space Y, define

$$UY = B(C, C, Y) \quad \text{and} \quad WY = B(N, C, Y).$$

Let $\epsilon : UY \to Y$ be the standard natural homotopy equivalence [**7**, 9.8 and 11.10]. Let $\epsilon' : UY \to WY$ be $B(\epsilon, 1, 1)$, where ϵ here is the morphism of monads $C \to N$ induced by the augmentation $\epsilon : \mathscr{C} \to \mathscr{N}$. By hypothesis, each $\epsilon : \mathscr{C}(j) \to \mathscr{N}(j)$ is a Σ_j-equivariant homotopy equivalence. Therefore, by [**8**, A.2 (ii)], $\epsilon : CZ \to NZ$ is a homotopy equivalence for any space Z. By [**8**, A.4 (ii)], it follows that $\epsilon' : UY \to WY$ is a homotopy equivalence. As the realization of a simplicial $\mathscr{N}$-space, WY is an $\mathscr{N}$-space; as realizations of maps of simplicial $\mathscr{C}$-spaces, ϵ and ϵ' are maps of $\mathscr{C}$-spaces (see [**7**, 12.2]).

The essential point for the remaining propositions is the following invariance result, the second part of which was already noted in [**13**, 5.6]. Recall that [**13**, 5.1] associates a monad $\hat{C}$ in the category of functors $\Pi \to \mathscr{T}$ to an operad $\mathscr{C}$.

LEMMA 4.1. *Let $\mathscr{C}$ be an operad and let $f : X \to X'$ be a natural transformation of functors $\Pi \to \mathscr{T}$.*

(i) *If each $f_n : X_n \to X_n'$ is a Σ_n-equivariant homotopy equivalence, then each $\hat{C}_n f : \hat{C}_n X \to \hat{C}_n X'$ is a Σ_n-equivariant homotopy equivalence.*

(ii) *If each $\mathscr{C}(j)$ is Σ_j-free and each f_n is a weak homotopy equivalence, then each $\hat{C}_n f$ is a weak homotopy equivalence.*

These are both consequences of the specification of $\hat{C}_n X$ as a filtered space given in [**13**, 5.5]. For (ii), one must use standard results about equivariant cofibrations, as in [**3**, Appendix 2.7, 4.4, and 4.6], together with a somewhat tedious inspection of the Σ_n-actions on the spaces involved.

As observed in [**13**, 5.7], inspection of the construction of $\hat{C}$ shows that $\hat{C}\{Y^n\} = \{(CY)^n\}$ for a space Y; in particular,

$$\hat{C}_1\{Y^n\} = CY.$$

In functor notation, these equalities read $\hat{C}R = RC$ and $L\hat{C}R = C$. For a functor $X : \Pi \to \mathscr{T}$, the maps $\delta : X_n \to X_1^n$ specify a natural transformation $\delta : X \to RLX$ (the unit of the adjunction between L and R), and the following diagram commutes by naturality:

$$\begin{tikzcd}
\hat{C}_n X \arrow[r, "\hat{C}_n\delta"] \arrow[d, "\delta"'] \arrow[dr, dashed, "\bar{\delta}"] & \hat{C}_n RLX \arrow[d, "\delta", "\cong"'] \\
(\hat{C}_1 X)^n \arrow[r, "(\hat{C}_1\delta)^n"] & (\hat{C}_1 RLX)^n = (CX_1)^n.
\end{tikzcd}$$

Therefore the lemma has the following immediate consequence.

PROPOSITION 4.2. *For any operad* $\mathscr{C}$, $\hat{C}$ *restricts to a monad in the category of* $\Sigma\Pi$*-spaces. For a* Σ*-free operad* $\mathscr{C}$, $\hat{C}$ *restricts to a monad in the category of* Π*-spaces.*

By [**13**, 6.2] ,the diagonal arrows $\bar{\delta}$ in the previous diagram specify a morphism of monads $\hat{C} \to RCL$. (See [**12**, § 5] for a conceptual discussion of this fact and of other formal parts of the following proof.)

Proof of Propositions 1.2 *and* 1.5. Modulo Proposition 4.2, these two results have identical proofs. For a $\hat{C}$-space X, define

$$UX = B(\hat{C}, \hat{C}, X) \quad \text{and} \quad VX = B(CL, \hat{C}, X).$$

Then $RVX = B(RCL, \hat{C}, X)$. Let $\epsilon : UX \to X$ be the natural equivalence of Π-spaces derived from [**7**, 9.8 and 11.10] and let $\delta = B(\bar{\delta}, 1,1) : UX \to RVX$. According to cases, δ is a spacewise equivalence or a spacewise weak equivalence because $\bar{\delta}$ is so. The rest follows as in the proof of Proposition 1.1.

When $\mathscr{C} = \mathscr{C}' \times \mathscr{C}_\infty$, the group completion $\gamma : VX \to M_0X$ is the composite

$$B(CL, \hat{C}, X) \xrightarrow{B(\alpha_\infty \pi, 1, 1)} B(QL, \hat{C}, X) \xrightarrow{\gamma^\infty} \operatorname{colim} \Omega^n B(\Sigma^n L, \hat{C}_n, X).$$

Here $\pi : C \to C_\infty$ and $\alpha_\infty : C_\infty \to Q$ are the morphisms of monads given by the projection and by [**7**, 5.2], C_n denotes (abusively) the monad associated to $\mathscr{C}' \times \mathscr{C}_n$, and γ^∞ is the colimit over n of comparison maps

$$B(\Omega^n\Sigma^n L, \hat{C}_n, X) \to \Omega^n B(\Sigma^n L, \hat{C}_n, X)$$

obtained by inductive use of the natural map $|\Omega Y| \to \Omega |Y|$ for simplicial spaces Y. See [**13**, 6.4] for details; the requisite constructions and proofs are just slight elaborations of those originally given for C-spaces in [**7**, § 14] and [**8**, § 2].

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