A reprint from

CONTEMPORARY MATHEMATICS

Published by

American Mathematical Society
Providence • Rhode Island

Contemporary Mathematics
Volume 12, 1982

EQUIVARIANT HOMOTOPY AND COHOMOLOGY THEORY

by J. P. MAY

I shall give a very brief overview of some of the main features of the rapidly evolving area of equivariant algebraic topology. The emphasis will be on the general shape and feel of the subject rather than on explicit results.

We shall be concerned with G-spaces X, where G is a compact Lie group. Little that we shall discuss requires restriction to finite groups, although such restriction sometimes allows considerable simplification of the proofs.

Of course, smooth G actions on manifolds have long been studied, and PL and topological actions are now also beginning to be analyzed. There is a basic dichotomy between concrete geometric spaces such as manifolds and large infinite dimensional spaces such as classifying spaces for bundle theories. The translation of problems about manifolds to homotopy theoretical questions by passage to bundles and classifying maps and use of the Pontryagin-Thom construction is one of the central themes of modern topology. Much of our motivation comes from the analogous equivariant theory, but our present concerns are more primitive. For such translations to be useful, we must first have a good understanding of the foundations of equivariant homotopy and cohomology theory.

To begin with, we must understand how G-spaces can be decomposed into atomic parts and reconstructed and how the sets $[X,Y]_G$ of homotopy classes of (based) G-maps $X \to Y$ can be computed. We study $[X,Y]_G$ either by atomizing X into cells or atomizing Y into cocells.

G-CW COMPLEXES

A G-CW complex is a G-space X which is the union of an expanding sequence of G-subspaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching a disjoint union of (n+1)-G-cells $G/H \times B^{n+1}$ along

0271-4132/81/0000-0273/$03.25

attaching G-maps $G/H \times S^n \to X^n$. (Throughout, subgroups are understood to be closed.) Note that X^n/X^{n-1} is then a wedge of based G-spheres

$$(G/H)^+ \wedge X^n = G/H \times S^n/G/H \times \{*\}.$$

A G-map $f: X \to Y$ is said to be a weak equivalence if each fixed point map $f^H: X^H \to Y^H$ is a weak equivalence in the usual nonequivariant sense.

With these notions, the basic features of cellular theory directly generalize to the equivariant context.

(1) The Whitehead theorem holds: a weak equivalence between G-CW complexes is an actual G-homotopy equivalence.

(2) The G-cellular approximation theorem holds.

(3) Arbitrary G-spaces can be approximated up to weak equivalence by G-CW complexes.

(4) Smooth G-manifolds are triangulable as G-CW complexes.

(5) Reasonable G-spaces have the G-homotopy type of G-CW complexes.

For (1) - (3) see [12, 20, 33]. For (4), see [14, 32]. In (5), reasonable G-spaces include metric G-ANR's [17] and, more deeply, G-ELC spaces [1, 33]. In particular, function G-spaces X^K are G-CW homotopy types when K is a compact G-space and X is a G-CW homotopy type.

COHOMOLOGY AND OBSTRUCTION THEORY

Let $\mathcal{O}$ denote the homotopy category of G-spaces G/H and G-maps $G/H \to G/J$; any such map is given by a subconjugacy relation $gHg^{-1} \subset J$. Define a coefficient system to be a contravariant functor $\mathcal{O} \to \mathrm{Ab}$. Generic examples are given by

$$\underline{k}(X)(G/H) = k(X^H)$$

for any G-space X and Abelian group valued covariant homotopy functor k on spaces. In particular, in view of our notion of a weak equivalence, the coefficient systems $\underline{\pi}_n(X)$ on based G-spaces X (with G-fixed basepoint) give the right notion of equivariant homotopy groups.

For a G-CW complex X, define $\underline{C}_n(X) = \underline{H}_n(X^n/X^{n-1};Z)$. The connecting homomorphisms of the triples of H-fixed point subspaces of (X^n, X^{n-1}, X^{n-2}) give a morphism $\partial: \underline{C}_n(X) \to \underline{C}_{n-1}(X)$ of coefficient systems. For a coefficient system M, define

$$C^n(X;M) = \mathrm{Hom}(\underline{C}_n(X), M),$$

Hom being taken in the Abelian category of coefficient systems. Then $C^*(X;M)$ is an ordinary cochain complex, and its homology groups are the Bredon cohomology groups $H_G^*(X;M)$. With this definition, classical obstruction theory directly generalizes to the equivariant context. See [4, 12, 21].

K(M,n)'s and Postnikov systems,

An Eilenberg-MacLane G-space $K(M,n)$ is a based G-CW complex with homotopy group systems

$$\underline{\pi}_q K(M,n) = \begin{cases} M & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases}$$

An explicit functorial construction is obtained in [9] as a special case of a general procedure for constructing global G-spaces from prescribed fixed point data. Bredon cohomology is represented in the form

$$\tilde{H}_G^n(X;M) = [X, K(M,n)]_G.$$

A G-space X is said to be simple if each X^H is connected and simple. Any such G-space can be approximated up to weak equivalence by a Postnikov tower $\lim X_n$, where $X_0 = \{*\}$ and X_{n+1} is the homotopy theoretical fibre of a k-invariant

$$k^{n+2}: X_n \to K(\underline{\pi}_{n+1}X; n+2).$$

There is a more general analog for nilpotent G-spaces. See [9, 29].

LOCALIZATION AND COMPLETION

Let T be a set of (rational) primes. A G-space Y is said to be T-local or T-complete if each Y^H is T-local or T-complete. A nilpotent G-space has a localization $\lambda: X \to X_T$ and a completion $\gamma: X \to X_T$. These are characterized by the usual universal property: a G-map from X to a T-local or T-complete G-space factors uniquely up to G-homotopy through λ or γ. Equivalently, each λ^H or γ^H is a localization or completion at T. Another characterization is that λ induces an isomorphism on Bredon cohomology with T-local coefficient systems and γ induces an isomorphism on Bredon cohomology with mod p coefficient systems for $p \in T$. See [22, 23].

ALGEBRAIZATION OF RATIONAL HOMOTOPY THEORY

At least for finite G, there is an equivariant analog of

Sullivan's theory of minimal models and a concomitant algebraic determination of $[X,Y]_G$ for rational G-spaces X and Y, Bredon cohomology again providing the basic tool. See [30].

In summary, we conclude that the basic foundational results about the generic homotopical structure of spaces all admit equivariant generalizations. However, this is no longer true of results that are intrinsically calculational. For example, a Hopf G-space is a based G-space X with a G-map $X \times X \to X$ for which the basepoint is a unit up to G-homotopy. It is not in general true that a rational Hopf G-space is equivalent to a product of Eilenberg-MacLane G-spaces. This is true if G is cyclic of prime power order but is false if $G = Z_p \times Z_q$. See [31].

POINCARÉ DUALITY AND RO(G)-GRADED COHOMOLOGY

The first theorem in algebraic topology is the Poincaré duality theorem. To obtain an equivariant analog, we need the notion of an orientation of a real G-vector bundle ξ, such as the tangent bundle of a smooth G-manifold. Let $T\xi$ be the Thom complex of ξ; it is obtained from the total space by one-point compactification of fibres and identification of the resulting points at ∞. If the base space of ξ is a point, then ξ is a real G-representation V and $T\xi$ is its one-point compactification S^V. More generally, if the base space is an orbit G/H, then $T\xi = G^+ \wedge_H S^W$ for an H-representation W.

Now let k_G^* be a ring-valued cohomology theory on G-spaces. The definition $\tilde{k}_H^*(Y) = \tilde{k}_G^*(G^+ \wedge_H Y)$ gives the associated cohomology theory on H-spaces Y. A k_G^*-orientation of ξ is a cohomology class $\mu \in k_G^*(T\xi)$ whose restrictions to the Thom complexes of base orbits G/H are "generators". For this definition to make sense, it must be the case that $\tilde{k}_H^*(S^W)$ is free on one generator over $\tilde{k}_H^*(S^0)$. For general Z-graded cohomology theories k_G^*, this is simply not true.

This difficulty is one of many motivations for the introduction of RO(G)-graded cohomology theories. These come with suspension isomorphisms

$$\tilde{k}_G^a(X) \cong \tilde{k}_G^{a+V}(\Sigma^V X)$$

for $a \in RO(G)$ and G-representations V, where $\Sigma^V X = X \wedge S^V$. More generally, one has an associated homology theory, and there are isomorphisms

$$\tilde{k}^a_G(G^+ \wedge_H \Sigma^W Y) \cong \tilde{k}^{a-w}_H(Y) \quad \text{and} \quad \tilde{k}^G_a(G^+ \wedge_H \Sigma^W Y) \cong \tilde{k}^H_{a-w-t}(Y)$$

for H-spaces Y and H-representations W, where T is the tangent H-representation of G/H at eH (and is thus 0 if G is finite). For such theories, the notion of a k^*_G-orientation of a real G-bundle ξ makes good sense, and a k^*_G-orientation of the tangent bundle of a smooth G-manifold implies Poincaré duality exactly as in the nonequivariant case. See [36, 19].

EXAMPLES OF RO(G)-GRADED COHOMOLOGY THEORIES

The first examples are real and complex equivariant K-theory. These arise from Bott periodicity which, in the real case, gives isomorphisms $\widetilde{KO}_G(S^V) \cong \widetilde{KO}_G(S^0)$ for Spin representations V of dimension $8n$. See [2, 27].

Equivariant stable cohomotopy theory is perhaps the most obvious example, and $\pi^0_G(pt)$ is canonically isomorphic to the Burnside ring $A(G)$. When G is finite, $A(G)$ is just the Grothendieck ring associated to the semi-ring of finite G-sets. See [28, 15, 8].

Cobordism theory gives other examples. In particular, the real unoriented bordism groups $N^G_n(X)$ give the Z-graded part of an RO(G)-graded homology theory, although the suspension by V axiom fails on the geometric bordism level. See [7, 16, 26, 35].

For finite G, equivariant infinite loop space theory gives rise to many more RO(G)-graded theories, notably theories associated to equivariant algebraic K-theory. See [10, 11].

ORDINARY RO(G)-GRADED COHOMOLOGY THEORY

Bredon cohomology is missing from the above list, and the Eilenberg-MacLane G-spaces $K(M,n)$ are not in general infinite loop G-spaces in the strong sense of admitting de-loopings by arbitrary G-representations V. A necessary condition is easily derived in terms of transfer.

Let $H \subset G$. Embed G/H as a G-subspace of a G-representation V. Let T be the tangent H-space of G/H at eH and write $V = T \oplus T^{\perp}$ as an H-space. The projection $G\times_H T^{\perp} \to G/H$ is the normal G-bundle of the embedding $G/H \subset V$, and $G\times_H T^{\perp}$ may be embedded as a normal tube in V. Applying the Pontryagin-Thom construction, we obtain a G-map

$$t\colon S^V \to G^+ \wedge_H S^{t^\perp} \subset G^+ \wedge_H S^V \simeq (G/H)^+ \wedge S^V.$$

A theorem of Hopf implies that the degree of $\pi \circ t\colon S^V \to S^V$ is the Euler characteristic $\chi(G/H)$, where $\pi\colon (G/H)^+ \wedge S^V \to S^V$ is the projection [3, 2.4].

Suppose that $H_G^*(?;M)$ extends to an RO(G)-graded cohomology theory. We then obtain

$$\begin{array}{ccc} M(G/H) = \tilde{H}_G^0((G/H)^+;M) & \cong & \tilde{H}_G^V((G/H)^+ \wedge S^V;M) \\ & & \downarrow t^* \\ M(G/G) = \tilde{H}_G^0(S^0;M) & \cong & \tilde{H}_G^V(S^V;M). \end{array}$$

A similar construction shows that the coefficient system M must admit a transfer homomorphism $M(G/K) \to M(G/H)$ when $K \subset H \subset G$. Essentially, up to formalism (the notion of a "Mackey functor"), this necessary condition is sufficient for the extension of $H_G^*(?;M)$ to an RO(G)-graded cohomology theory. See [18, 19, 34].

OLIVER TRANSFER AND THE CONNOR CONJECTURE

While the philosophical importance of RO(G)-graded ordinary cohomology should be clear from the discussion above, like any new machinery it should justify itself by an application expressible without reference to that machinery. We have the following serendipitous corollary, repeated (and generalized) from [18].

Oliver transfer. Let X be an arbitrary G-space, let H be a closed subgroup of G, and let $\pi\colon X/H \to X/G$ be the natural projection. Then for any coefficient group R there exists a natural transfer homomorphism

$$\tau\colon H^n(X/H;R) \to H^n(X/G;R),\ n \geq 0,$$

such that $\tau \circ \pi^*$ is multiplication by $\chi(G/H)$.

Proof. Since X is weakly equivalent to a G-CW complex and a weak equivalence of G-spaces induces cohomology isomorphisms on passage to orbit spaces [22, Prop. 2], we may as well assume that X is a G-CW complex. The constant coefficient system $\underline{R}$ admits transfer homomorphisms [19], and we use the following diagram to construct τ:

$$\begin{array}{ccccc} H^n(X/G;R) & \cong & \tilde{H}_G^n(X^+;\underline{R}) & \cong & \tilde{H}_G^{n+V}(X^+ \wedge S^V;\underline{R}) \\ \pi^* \downarrow & & \pi^* \downarrow & & \pi^* \downarrow \uparrow \tau = (1 \wedge t)^* \\ H^n(X/H;R) & \cong & \tilde{H}_G^n(X^+ \wedge (G/H)^+;\underline{R}) & \cong & \tilde{H}_G^{n+V}(X^+ \wedge (G/H)^+ \wedge S^V;\underline{R}) \end{array}$$

The top left isomorphism is an obvious comparison between two Z-graded ordinary cohomology theories with the same values on orbits, and the bottom left isomorphism results by change of groups. The Euler characteristic formula follows from Hopf's theorem noted above.

This was first proved by Oliver [25] (for Čech cohomology and suitably restricted X) using totally different techniques. See also [5]. It has the following consequence, which was also first proven by Oliver using different techniques. See [6, 24].

Connor conjecture. Let X be finite dimensional and have finitely many orbit types. Then $\tilde{H}^*(X/G;R) = 0$ if $\tilde{H}^*(X;R) = 0$.

Proof. If N is the normalizer of a maximal torus in G, then $\chi(G/N) = 1$. Therefore $\pi^*: H^*(X/G;R) \to H^*(X/N;R)$ is a split monomorphism. Since N is a finite extension of a torus, $\tilde{H}^*(X/N;R) = 0$ by Smith theory and induction; see [6].

COMPUTATIONS OF ORDINARY RO(G)-GRADED COHOMOLOGY

The application just given was an immediate consequence of the mere existence of ordinary RO(G)-graded cohomology. While other applications are on hand (and will appear in [19]), it is clear that calculations are essential if these theories are to become as important in practice as their philosophical centrality would lead one to expect. The most basic coefficient system is that given by Burnside rings, $\underline{A}(G/H) = A(H)$, and $H_G^*(X;\underline{A})$ is the equivariant analog of ordinary integral cohomology. We pose the following paradoxical open question.

Problem. Compute the ordinary cohomology of a point. That is, compute the RO(G)-graded ring $H_G^*(pt;\underline{A})$.

For non-finite compact Lie groups virtually nothing is known. For finite G, it is convenient to localize the problem at a given prime p. There are general procedures for working out the answer when p is prime to the order of G and for reducing the problem to one about p-groups when p does divide the order of G [8, 15, 19]. These general procedures say nothing whatever about calculations for p-groups at the prime p.

As a first step, Stong [unpublished] has computed $H_G^*(pt;\underline{A})$ when G is cyclic of order p. The answer is already fairly complicated, and it remains to be seen whether or not $H_G^*(pt;\underline{A})$ is a tractable algebraic functor of G.

BIBLIOGRAPHY

1. S. Araki and M. Murayama. G-homotopy types of G-complexes and representation of G-cohomology theories. Publ. RIMS Kyoto Univ. 14 (1978), 203-222.

2. M. F. Atiyah. Bott periodicity and the index of elliptic operators. Quart. J. Math. Oxford 74 (1968), 113-140.

3. J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. Topology 14 (1975), 1-12.

4. G. Bredon. Equivariant cohomology theories. Springer Lecture Notes in Mathematics Vol 34. 1967.

5. G. Brumfiel. The homology transfer problem for compact group actions. Preprint.

6. P. Connor. Retraction properties of the orbit space of a compact topological transformation group. Duke Math. J. 27 (1960), 341-357.

7. T. tom Dieck. Bordism of G-manifolds and integrality theorems. Topology 9 (1970), 349-358.

8. T. tom Dieck. Transformation groups and representation theory. Springer Lecture Notes in Mathematics Vol 766. 1979.

9. A. Elmendorf. Equivariant homotopy and fixed point sets. Preprint.

10. Z. Fiedorowicz, H. Hauschild, and J. P. May. Equivariant algebraic K-theory. Proceedings Oberwolfach, 1980, conference in algebraic K-theory. To appear.

11. H. Hauschild, J. P. May, and S. Waner. Equivariant infinite loop space theory. In preparation.

12. S. Illman. Equivariant singular homology and cohomology for actions of compact Lie groups. Springer Lecture Notes in Mathematics Vol 298, 1972, 403-415.

13. S. Illman. Equivariant singular homology and cohomology I. Memoirs Amer. Math. Soc. 156, 1975.

14. S. Illman. Smooth equivariant triangulations of G-manifolds for G a finite group. Math. Ann. 233 (1978), 199-220.

15. C. Kosniowski. Equivariant cohomology and stable cohomology. Math. Ann. 210 (1974), 83-104.

16. C. Kosniowski. A note on RO(G) graded G bordism theory. Quart. J. Math. Oxford (3) 26 (1975), 411-419.

17. S. Kwasik. On equivariant homotopy type of G-ANR's. Proc. Amer. Math. Soc. To appear.

18. L. G. Lewis, J. P. May, and J. E. McClure. Ordinary RO(G)-graded cohomology. Bull. Amer. Math. Soc. 4 (1981), 208-212.

19. L. G. Lewis, J. P. May, J. E. McClure, and S. Waner. Equivariant cohomology theory. In preparation.

20. T. Matumoto. On G-CW complexes and a theorem of J.H.C. Whitehead. J. Fac. Sci. Tokyo 18 (1971), 363-374.

21. T. Matumoto. Equivariant cohomology theories on G-CW complexes. Osaka J. Math. 10 (1973), 51-68.

22. J. P. May, J. E. McClure, and G. Triantafillou. Equivariant localization. Bull. London Math. Soc. To appear.

23. J. P. May. Equivariant completion. Bull. London Math. Soc. To appear.

24. R. Oliver. A proof of the Connor conjecture. Annals Math 103 (1976), 637-644.

25. R. Oliver. A transfer homomorphism for compact Lie group actions. Aarhus preprint series 1981/82 No 1.

26. W. Pulikowski. RO(G)-graded G bordism theory. Bull. Acad. Polon Sci. 21 (1973), 991-995.

27. G. Segal. Equivariant K-theory. Inst. Hautes Etudes Sci. Publ. Math. 34 (1968), 129-151.

28. G. Segal. Equivariant stable homotopy theory. Actes Congres Intern. Math. 1970 Tome 2, 59-63.

29. G. Triantafillou. Aquivariante rationale Homotopietheorie Bonner Mathematische Schriften Nr 110, 1978.

30. G. Triantafillou. Equivariant minimal models. Trans. Amer. Math. Soc. To appear.

31. G. Triantafillou. Rationalization of Hopf G-spaces. Preprint.

32. A. Verona. Triangulation of stratified fibre bundles. Manuscripta Math. 30 (1980), 425-445.

33. S. Waner. Equivariant homotopy theory and Milnor's theorem. Trans. Amer. Math. Soc. 258 (1980), 351-368.

34. S. Waner. Equivariant RO(G)-graded singular cohomology. Preprint.

35. S. Waner. Unoriented equivariant RO(G)-graded bordism. Preprint.

36. K. Wirthmuller. Equivariant homology and duality. Manuscripta Math 11 (1974), 373-390.