A remark on duality and the Segal conjecture

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The Segal conjecture, in its nonequivariant form, provides a spectacular example of the failure of duality for infinite complexes. The purpose of this note is to point out that the Segal conjecture, in its equivariant form, implies the validity of duality for certain infinite G-complexes in theories, such as equivariant K-theory, which enjoy the same kind of invariance property that cohomotopy enjoys.

To establish context, we give a quick review of duality theory. For based spaces X, Y, and Z, there is an evident natural map

$$\nu\colon F(X,Y)\wedge Z \longrightarrow F(X,Y\wedge Z).$$

Here $F(X,Y)$ is the function space of based maps $X \to Y$ and ν is specified by $\nu(f\wedge z)(x) = f(x)\wedge z$. Any up-to-date construction of the stable category comes equipped with an analogous function spectrum functor F and an analogous natural map ν defined for spectra X, Y, and Z. If either X or Z is a finite CW-spectrum, then ν is an equivalence. The dual of X is $DX = F(X,S)$, where S denotes the sphere spectrum. Replacing Z by the representing spectrum k of some theory of interest, we obtain $\nu\colon DX\wedge k \to F(X,k)$. On passage to π_q, this gives $\nu_*\colon k_q(DX) \to k^{-q}(X)$, and ν_* is an isomorphism if X is finite. Classical Spanier-Whitehead duality amounts to an identification of the homotopy type of $D\Sigma^\infty X$ when X is a polyhedron embedded in a sphere, where Σ^∞ denotes the suspension spectrum functor. This outline applies equally well equivariantly, with spectra replaced by G-spectra for a compact Lie group G. We need only remark that a map of G-spectra is an equivalence if and only it induces an isomorphism on passage to $\pi_q^H(?) = [G/H_+\wedge S^q,?]_G$ for all integers q and all closed subgroups H of G (where the $+$ denotes addition of a disjoint basepoint) and that homology and cohomology are specified by

$$k_q^G(X) = \pi_q^G(X\wedge k_G) \quad\text{and}\quad k_G^q(X) = \pi_{-q}^G(F(X,k_G))$$

for any G-spectra X and k_G. See [6] for details on all of this.

We restrict our discussion of the Segal conjecture to finite p-groups for a fixed prime p, and we agree once and for all to complete all spectra at p without change of notation. See [4] for a good discussion of completions of spectra. Completions of G-spectra work the same way (and have properties analogous to completions of G-spaces [7]). The nonequivariant formulation of the Segal conjecture [1,5,8] asserts that a certain map

$$\alpha\colon \bigvee \Sigma^{\infty} BWH_{+} \longrightarrow DBG_{+}$$

is an equivalence, where B denotes the classifying space functor and the wedge runs over the conjugacy classes of subgroups H of G. Since both the mod p homology of DBG_{+} and the mod p cohomology of BG_{+} are concentrated in non-negative degrees, we see that the duality map $\nu_{*}\colon H_{*}(DBG_{+}) \to H^{-*}(BG_{+})$ cannot possibly be an isomorphism. It is not much harder to see that the corresponding duality map in p-adic K-theory also fails to be an isomorphism.

As explained in [5], the map α above is obtained by passage to G-fixed point spectra from the map of G-spectra

$$\beta\colon S \cong F(S^{0},S) \longrightarrow F(EG_{+},S)$$

induced by the projection $EG \to pt$, where EG is a free contractible G-space. The equivariant form of the Segal conjecture asserts that β is an equivalence. More generally, the analogous map with EG_{+} replaced by its smash product with any based finite G-CW complex X is an equivalence. The crux of our observation is just the following naturality diagram, where k_G is any G-spectrum.

$$\begin{tikzcd} DX \wedge k_G \arrow[r, "\beta \wedge 1"] \arrow[d, "\nu"'] & D(EG_{+} \wedge X) \wedge k_G \arrow[d, "\nu"] \\ F(X,k_G) \arrow[r, "\beta"] & F(EG_{+} \wedge X, k_G) \end{tikzcd}$$

The left map ν is an equivalence since X is finite. The top map β, hence also $\beta\wedge 1$, is an equivalence by the Segal conjecture. If k_G^{*} carries G-maps which are nonequivariant homotopy equivalences to isomorphisms, then β on the bottom is an equivalence (as we see by replacing X with $G/H_{+}\wedge X$ for all $H \subset G$) and we can conclude that ν on the right is an equivalence. In particular, duality holds in k_G^{*}-theory for the infinite G-complex $EG_{+}\wedge X$; that is,

$$\nu_{*}\colon k_q^{G}(D(EG_{+} \wedge X)) \longrightarrow k_G^{-q}(EG_{+} \wedge X)$$

is an isomorphism. Of course, equivariant K-theory has the specified invariance property by the Atiyah-Segal completion theorem [3]. Equivariant cohomotopy with coefficients in any equivariant classifying space also has this property [5,8,9].

In the examples just mentioned, k_G and its underlying non-equivariant spectrum k (which represents ordinary K-theory or ordinary cohomotopy with coefficients in the relevant nonequivariant classifying space) are sufficiently nicely related that, for any free G-CW spectrum X,

$$k_G^*(X) \cong k^*(X/G) \quad \text{and} \quad k_*^G(X) \cong k_*(X/G).$$

(See [6,II].) With X replaced by $EG_+\wedge X$ for a finite G-CW complex X, this may appear to be suspiciously close to a contradiction to the failure of duality in non-equivariant K-theory cited above. The point is that the dual of a free finite G-CW spectrum is equivalent to a free finite G-CW spectrum [2,8.4; 5,III.2.12], but the dual of a free infinite G-CW spectrum need not be equivalent to a free G-CW spectrum, and in fact $\Sigma^\infty EG_+\wedge X$ provides a counterexample.

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