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CHARACTERISTIC CLASSES IN BOREL COHOMOLOGY

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Let G be a topological group and let EG be a free contractible G-space. The Borel construction on a G-space X is the orbit space $X_G = EG \times_G X$. When asked what equivariant cohomology is, most people would answer Borel cohomology, namely $H_G^*(X) = H^*(X_G)$. This theory has the claim of priority and the merit of ready computability, and many very beautiful results have been proven with it. However, it suffers from the defects of its virtues. Precisely, it is 'invariant', in the sense that a G-map $f: X \to Y$ which is a nonequivariant homotopy equivalence induces an isomorphism on Borel cohomology. A quick way to see this is to observe that $1 \times f: EG \times X \to EG \times Y$ is a map of principal G-bundles with base map $1 \times_G f = f_G : X_G \to Y_G$, so that f_G is a weak homotopy equivalence. As explained in [1], this invariance property is the crudest of a hierarchy of such properties that a theory might have. We shall show how to compute all characteristic classes in any invariant equivariant cohomology theory, the conclusion being that no such theory is powerful enough to support a very useful theory of characteristic classes. As explained in [2], a less crude invariance property can sometimes be exploited to obtain a calculation of equivariant characteristic classes in more powerful theories, such as equivariant K-theory.

To establish context, consider a closed normal subgroup Π of a topological group Γ with quotient group G. If Y is a Π-free Γ-space, we think of the orbit projection $q: Y \to Y/\Pi$ as a particular kind of equivariant bundle. It is a principal Π-bundle in the usual sense, its base space is a G-space and thus a Π-trivial Γ-space, and its projection is a Γ-map. The classical case is $\Gamma = G \times \Pi$. Here q is called a principal (G, Π)-bundle, and there is a theory of associated (G, Π)-bundles exactly as in the nonequivariant case. For example, if G is a compact Lie group acting smoothly on a differentiable manifold M^n, then the tangent n-plane bundle of M is a $(G, \mathrm{O}(n))$-bundle and is determined by its associated principal $(G, \mathrm{O}(n))$-bundle. See e.g. [4; 5; 7, V §1] for background.

There is a universal bundle $E(\Pi;\Gamma) \to B(\Pi;\Gamma) = E(\Pi;\Gamma)/\Pi$ of the sort just specified. Its total space $E(\Pi;\Gamma)$ is a Π-free Γ-CW complex such that the Λ-fixed point space $E(\Pi;\Gamma)^\Lambda$ is contractible for any closed subgroup Λ of Γ such that

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$\Lambda \cap \Pi = e$. Of course, since Π acts freely, $E(\Pi;\Gamma)^\Lambda$ is empty if $\Lambda \cap \Pi \neq e$. See Elmendorf [3] for a nice conceptual construction of $E(\Pi;\Gamma)$. Given any Π-free Γ-CW complex Y, there is one and, up to Γ-homotopy, only one Γ-map $Y \to E(\Pi;\Gamma)$. In particular, we have a Γ-map $\theta : E\Gamma \to E(\Pi;\Gamma)$. Since $E\Gamma$ and $E(\Pi;\Gamma)$ are both Π-free and contractible, θ is a Π-homotopy equivalence. Therefore, for any Γ-space X, the map

$$f = \theta \times_\Pi 1 : E\Gamma \times_\Pi X \to E(\Pi;\Gamma) \times_\Pi X$$

is a G-map which is a nonequivariant homotopy equivalence. This already proves the following result.

Proposition 1. *Let $G = \Gamma/\Pi$. For any invariant cohomology theory H_G^* and any Γ-space X,*

$$f^* : H_G^*(E(\Pi;\Gamma) \times_\Pi X) \to H_G^*(E\Gamma \times_\Pi X)$$

is an isomorphism. In particular, with X a point,

$$H_G^*(B(\Pi;\Gamma)) = H_G^*((E\Gamma/\Pi).$$

Of course, $(E\Gamma)/\Pi$ is a G-space of the same underlying homotopy type as $B\Pi$. Note next that $EG \times E\Gamma$ is a free contractible Γ-space, so that its projection to $E\Gamma$ is a Γ-homotopy equivalence. Therefore

$$(E\Gamma \times_\Pi X)_G = EG \times_G (E\Gamma \times_\Pi X) = (EG \times E\Gamma) \times_\Gamma X \simeq E\Gamma \times_\Gamma X = X_\Gamma.$$

In particular, with X a point, $(E\Gamma/\Pi)_G$ is homotopy equivalent to $B\Gamma$. Thus the proposition specializes as follows to Borel cohomology.

Corollary 2. *Let H^* be any nonequivariant cohomology theory (not necessarily ordinary) and let H_G^* and H_Γ^* be the invariant theories on G-spaces and Γ-spaces obtained by applying H^* to the respective Borel constructions. Then, for any Γ-space X,*

$$H_G^*((E(\Pi;\Gamma) \times_\Pi X) = H_\Gamma^*(X).$$

In particular, $H_G^*(B(\Pi;\Gamma)) = H^*(B\Gamma)$.

Now specialize to the classical case $\Gamma = G \times \Pi$ and change notations by setting $E(G, \Pi) = E(\Pi; G \times \Pi)$ and $B(G, \Pi) = B(\Pi; G \times \Pi)$. Here $E\Gamma = EG \times E\Pi$, the projection $EG \times E\Pi \to E\Pi$ is a $(G \times \Pi)$-map which is a Π-homotopy equivalence, and Proposition 1 has the following specialization.

Corollary 3. *For any invariant cohomology theory H_G^* and any $(G \times \Pi)$-space X,*

$$H_G^*(E(G, \Pi) \times_\Pi X) = H_G^*(E\Pi \times_\Pi X).$$

In particular, $H_G^*(B(G, \Pi)) = H_G^*(B\Pi)$.

In other words, an invariant cohomology theory can't tell the difference between the equivariant classifying space $B(G, \Pi)$ and the nonequivariant classifying space $B\Pi$ regarded as a G-trivial G-space. The situation becomes particularly clear in ordinary Borel cohomology. The Borel construction on $B(G, \Pi)$ is homotopy equivalent to $BG \times B\Pi$, and the Künneth theorem gives the following conclusion.

Corollary 4. *Let H^* be ordinary cohomology with coefficients in a field and let H_G^* be the associated Borel cohomology theory. Then*

$$H_G^*(B(G, \Pi)) = H^*(BG) \otimes H^*(B\Pi)$$

as an $H^*(BG)$-module.

If we apply the Borel construction to a principal (G, Π)-bundle over a G-space X, we obtain a principal Π-bundle over X_G. This construction generally loses information, although it induces a bijection on equivalence classes of bundles when G and Π are compact Lie groups with Π abelian [6]. In terms of classification theory, the assignment of Borel cohomology characteristic classes to (G, Π)-bundles amounts to the composite

$$\begin{aligned}[X, B(G, \Pi)]_G &\to [X_G, B(G, \Pi)_G] \\ &= [X_G, BG \times B\Pi] \to \mathrm{Hom}(H^*BG \otimes H^*B\Pi, H^*X_G).\end{aligned}$$

The interpretation is that all characteristic classes of (G, Π)-bundles over X are determined by the $H^*(BG)$-module structure on $H^*(X_G)$ and the nonequivariant characteristic classes of the Π-bundles obtained by application of the Borel construction.

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